The Stokes theorem.

Theorem: (Stokes theorem)

Let $\mathbf{f} : D \to \mathbb{R}^3$ be a $\mathcal{C}^1$–vector field on a domain $D \subset \mathbb{R}^3$.

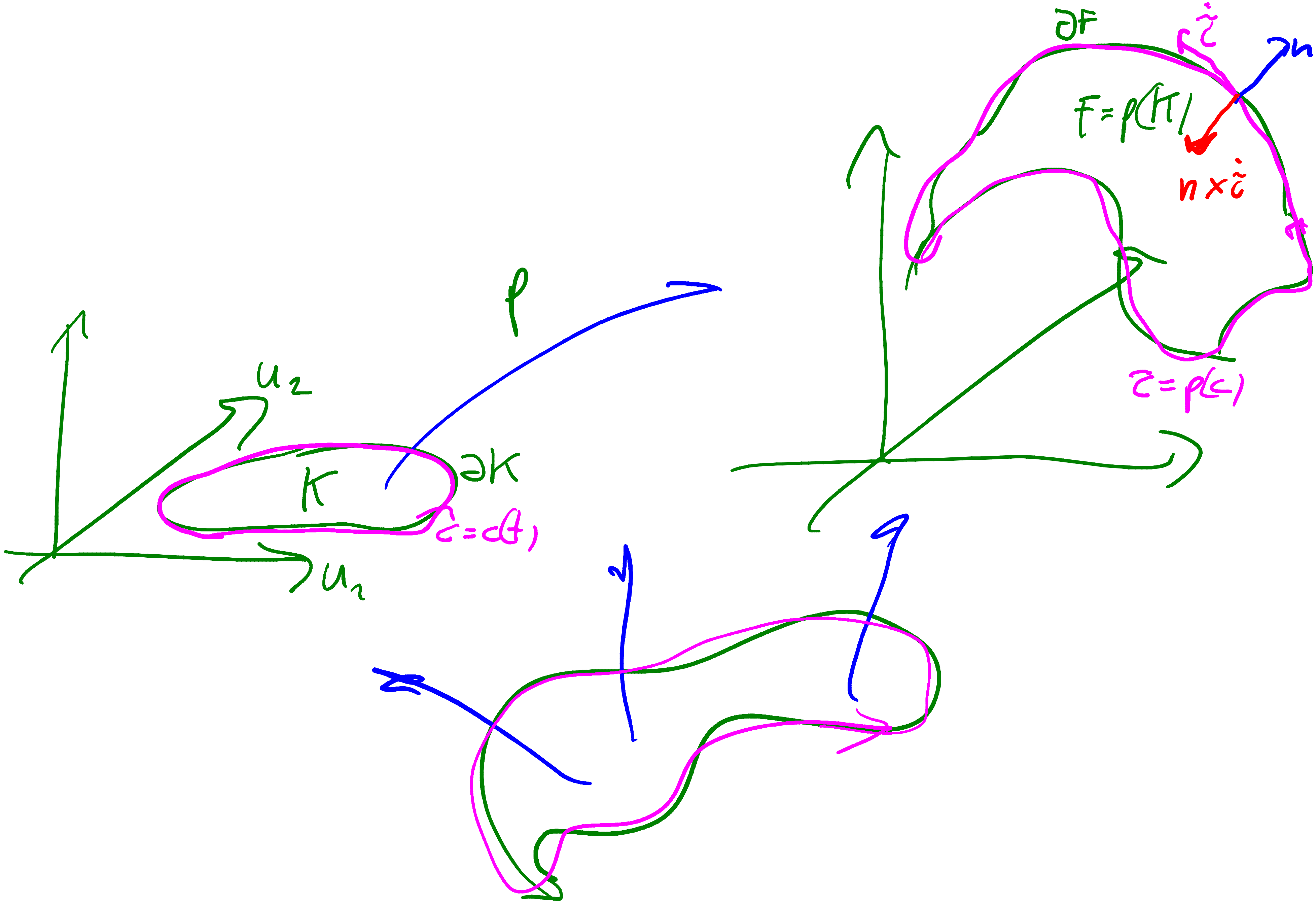
Let $\boxed{F = \mathbf{p}(K)}$ be a surface in D, $F \subset D$, with parameterisation $\boxed{\mathbf{x} = \mathbf{p}(\mathbf{u})}$, $\mathbf{u} \in \mathbb{R}^2$. Let $K \subset \mathbb{R}^2$ be a Green area.

The boundary ∂K is parameterised by a piecewise smooth $\mathcal{C}^1$–curve $\mathbf{c}$ and the image $\tilde{\mathbf{c}}(t) := p(\mathbf{c}(t))$ parameterises the boundary ∂F of the surface F.

The orientation of the boundary curve $\tilde{\mathbf{c}}(t)$ is chosen such that $\mathbf{n}(\tilde{\mathbf{c}}(t)) \times \dot{\tilde{\mathbf{c}}}(t)$ points in the direction of the surface.

Then we have

$$\int_F \operatorname{curl} \mathbf{f}(\mathbf{x})\, do = \oint_{\partial F} \mathbf{f}(\mathbf{x})\, d\mathbf{x}$$

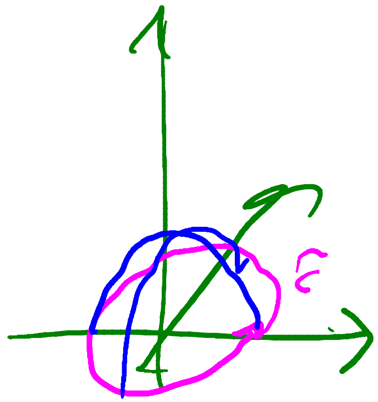

u_2
u_1
K
∂K
ċ = c(t)
p
∂F
F = p(K)
n
n × c̃̇
c̃̇
c̃ = p(c)

Example.

Given the vector field

$\operatorname{curl} f = \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ -y & x & -z \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

$$\mathbf{f}(x, y, z) = (-y, x, -z)^T$$



and let the closed curve $\tilde{\mathbf{c}} : [0, 2\pi] \to \mathbb{R}^3$ be parameterised by

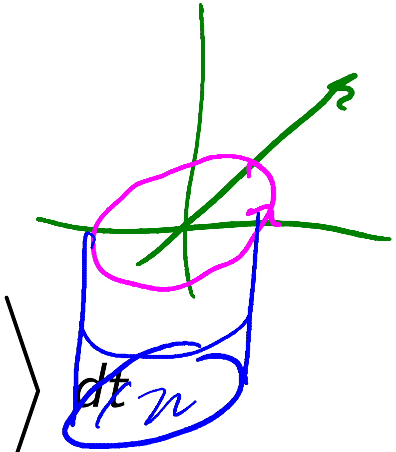
$$\tilde{\mathbf{c}}(t) = (\cos t, \sin t, 0)^T \qquad \text{für } 0 \le t \le 2\pi$$

Then:

$f(c(t)) = (-\sin t, \cos t, 0)$

$$\oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_0^{2\pi} \langle \mathbf{f}(\mathbf{c}(t)), \dot{\mathbf{c}}(t) \rangle \, dt$$

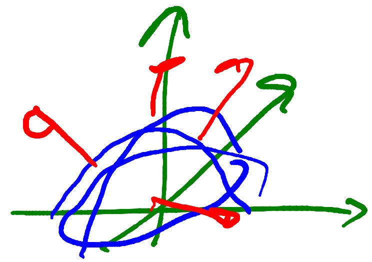
$$= \int_0^{2\pi} \left\langle \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \\ 0 \end{pmatrix} \right\rangle dt$$



$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi$$

Continuation of the example.

We define a surface $F \subset \mathbb{R}^3$, bounded by the curve $\mathbf{c}(t)$:



$r=1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix} =: \mathbf{p}(\varphi,\psi)$$

with $(\varphi,\psi) \in \underbrace{K = [0,2\pi] \times [0,\pi/2]}_{\text{upper half sphere}}$, i.e. the surface F is the upper half sphere.

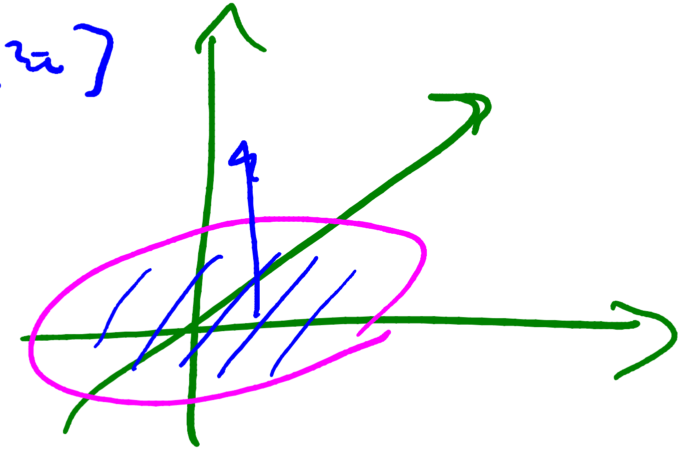
Stokes theorem tells us:

$$\int_F \operatorname{curl} \mathbf{f}(\mathbf{x})\, do = \oint_{\mathbf{c}=\partial F} \mathbf{f}(\mathbf{x})\, d\mathbf{x} \quad = 2\pi$$

We have already calculated the right side, a **surface integral of a vector field**:

$$\oint_{\mathbf{c}=\partial F} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = 2\pi$$

$$p(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} \qquad (r,\varphi) = [0,1]\times[0,2\pi]$$



$$n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\int \operatorname{curl} f \, d\sigma = \iint \left\langle \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle r\, dr\, d\varphi = 2\pi$$

Completion of the example.

It remains a **surface integral of a vector field**:

$$\int_F \overset{\text{curl}}{\cancel{\text{rot}}}\,\mathbf{f}(\mathbf{x})\,do = \int_K \left\langle \overset{\text{curl}}{\cancel{\text{rot}}}\,\mathbf{f}(\mathbf{p}(\varphi,\psi)), \frac{\partial \mathbf{p}}{\partial \varphi} \times \frac{\partial \mathbf{p}}{\partial \psi} \right\rangle d\varphi d\psi$$

Attention: the right hand side is an **intergal over a domain**.

We have $\operatorname{curl} \mathbf{f}(\mathbf{x}) = (0,0,2)^T$ and

$$\frac{\partial \mathbf{p}}{\partial \varphi} \times \frac{\partial \mathbf{p}}{\partial \psi} = \begin{pmatrix} \cos\varphi\cos^2\psi \\ \sin\varphi\cos^2\psi \\ \sin\psi\cos\psi \end{pmatrix}$$

$\eta:1$

$$n = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\psi \\ \sin\varphi\cos\psi \\ \sin\psi \end{pmatrix}$$

Thus:

$$= \int_0^{\pi/2}\int_0^{2\pi} \left\langle \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}, n \right\rangle \cos\psi \, d\varphi d\psi = \iint 2 \sin\psi\cos\psi \, d\psi d\varphi =$$

$$\int_F \operatorname{curl} \mathbf{f}(\mathbf{x})\,do = \int_0^{\pi/2}\int_0^{2\pi} 2\sin\psi\cos\psi\,d\varphi d\psi = 2\pi\int_0^{\pi/2} \sin(2\psi)\,d\psi = 2\pi$$