3.3 Surface integrals

Definition: Let $D \subset \mathbb{R}^2$ be a domain and $\mathbf{p} : D \to \mathbb{R}^3$ a $\mathcal{C}^1$–map

$$\mathbf{x} = \mathbf{p}(\mathbf{u}) \quad \text{with } \mathbf{x} \in \mathbb{R}^3 \text{ and } \mathbf{u} = (u_1, u_2)^T \in D \subset \mathbb{R}^2$$

$$p(u) = \begin{pmatrix} p_1(u_1, u_2) \\ p_2(u_1, u_2) \\ p_3(u_1, u_2) \end{pmatrix}$$

If for all $\mathbf{u} \in D$ the two vectors

$$\frac{\partial \mathbf{p}}{\partial u_1} \quad \text{and} \quad \frac{\partial \mathbf{p}}{\partial u_2}$$

$c = c(t) \quad t \in D \subset \mathbb{R}^1$

$\frac{dc}{dt}$ tangent vector

are linear independent, we call

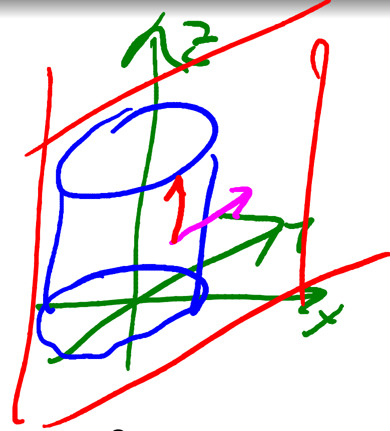
$$F := \{\mathbf{p}(\mathbf{u}) \mid \mathbf{u} \in D\}$$

a surface or a piece o surface. The map $\mathbf{x} = \mathbf{p}(\mathbf{u})$ is called a parameterisation or parameter representation of the surface F.

Example I.

We consider for a given $r > 0$ the map

R fixed



$$\mathbf{p}(\varphi, z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \text{for } (\varphi, z) \in \mathbb{R}^2.$$

The corresponding parameterized surface is an unbounded cylinder in $\mathbb{R}^3$.

If we restrict the area of definition, e.g.

$$(\varphi, z) \in K := [0, 2\pi] \times [0, H] \subset \mathbb{R}^2$$

we obtain a bounded cylinder of height H.

The partial derivatives

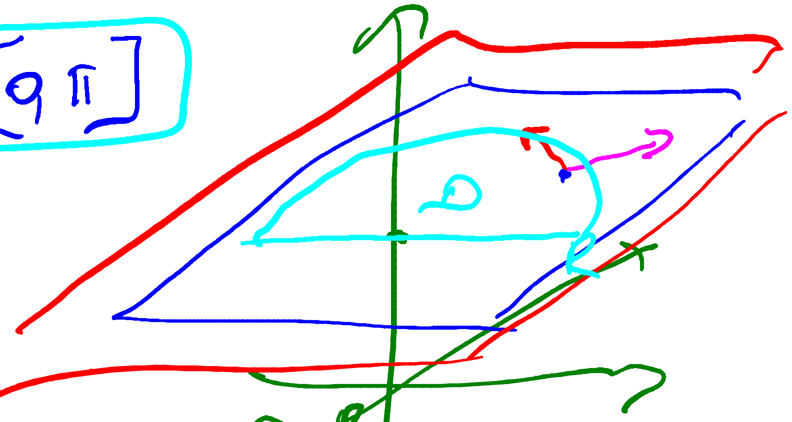
$$\frac{\partial \mathbf{p}}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}, \qquad \frac{\partial \mathbf{p}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

of $\mathbf{p}(\varphi, z)$ are linearly independent on $\mathbb{R}^2$.

$$\rho(r,\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

z fixed

$$\boxed{D = [0,R]\times[0,\pi]}$$



$$\frac{\partial\rho}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \qquad \frac{\partial\rho}{\partial\varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}$$

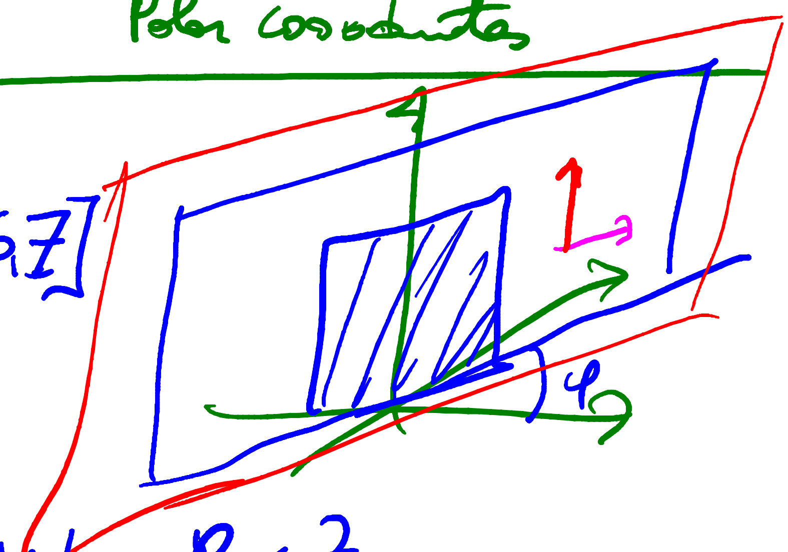
$$O(D) = \int_D d\sigma = \int \left\| \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \right\| d(r,\varphi) = \int \left\| \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right\| dr\,d\varphi = \int_0^{\pi}\int_0^{R} r\,dr\,d\varphi = \frac{R^2}{2}\pi$$

Polar coordinates

$$\rho(r,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

φ fixed

$$D = [0,R]\times[0,Z]$$



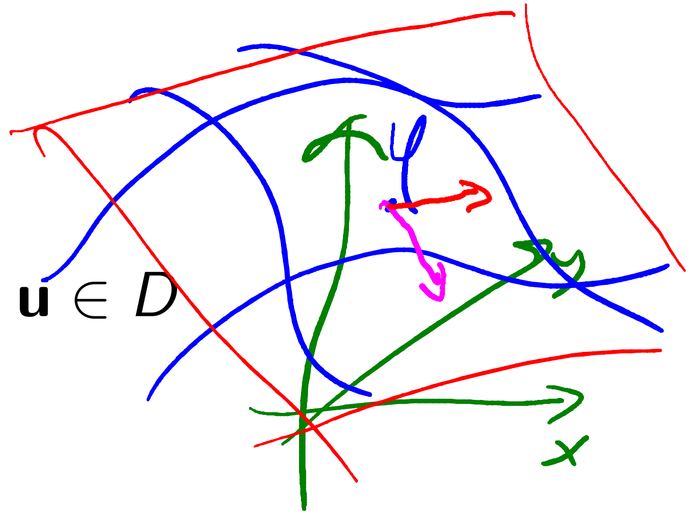
$$\frac{\partial\rho}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \qquad \frac{\partial\rho}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$O(D) = \int d\sigma = \int \left\| \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| dr\,dz = \int_0^{R}\int_0^{Z} \underbrace{\left\| \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix} \right\|}_{1} dr\,dz = R \times Z$$

Example II.

special case

The graph of a scalar $\mathcal{C}^1$–function $\varphi : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$, is a surface.



A parametrisation is given by

$$\mathbf{p}(u_1, u_2) := \begin{pmatrix} u_1 \\ u_2 \\ \varphi(u_1, u_2) \end{pmatrix} \qquad \text{for } \mathbf{u} \in D$$

The partial derivatives

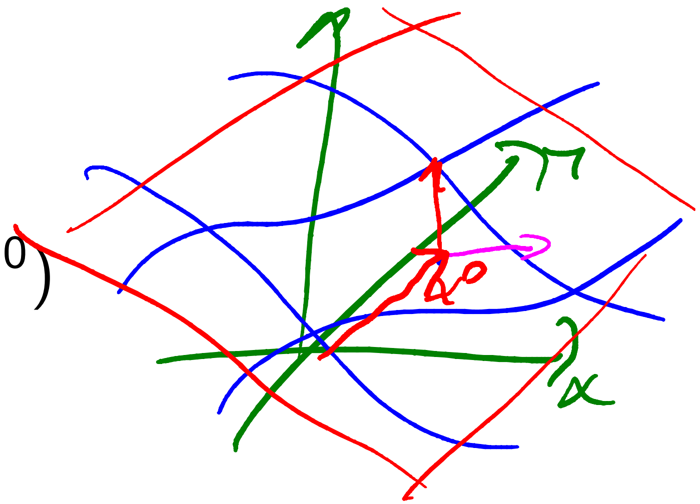
$$\frac{\partial \mathbf{p}}{\partial u_1} = \begin{pmatrix} 1 \\ 0 \\ \varphi_{u_1} \end{pmatrix}, \qquad \frac{\partial \mathbf{p}}{\partial u_2} = \begin{pmatrix} 0 \\ 1 \\ \varphi_{u_2} \end{pmatrix}$$

are linear independent.

The tangential plane on a surface.

The two linear independent vectors

$$\frac{\partial \mathbf{p}}{\partial u_1}(\mathbf{u}^0) \qquad \text{und} \qquad \frac{\partial \mathbf{p}}{\partial u_2}(\mathbf{u}^0)$$



are tangential on the surface F.

The two vectore span the tangential plane $T_{\mathbf{x}^0}F$ of the surface F at the point $\mathbf{x}^0 = \mathbf{p}(\mathbf{u})$.

curve tangent

$x = x^0 + \lambda \dot{c}(t)$

The tangential plane has a parameter representation

$$T_{\mathbf{x}^0}F \ : \boxed{\mathbf{x} = \mathbf{x}^0 + \lambda \frac{\partial \mathbf{p}}{\partial u_1}(\mathbf{u}^0) + \mu \frac{\partial \mathbf{p}}{\partial u_2}(\mathbf{u}^0)} \qquad \text{for } \lambda, \mu \in \mathbb{R}.$$

arbitrary

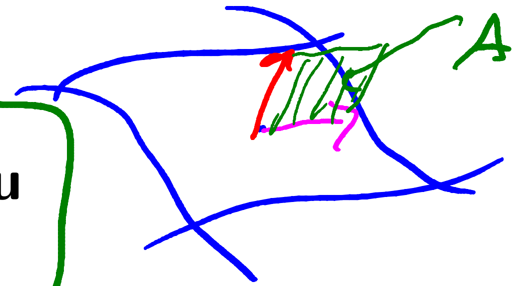
Question: How can wie calculate the size of a given surface F?

The surface integral of a piece of surface.

$c = c(t)$ $\qquad ds = \|\dot{c}(t)\|\,dt \qquad ds = \|\dot{c}(t)\|\,dt$

Definition: Let $\mathbf{p} : D \to \mathbb{R}^3$ be a parameterisation of a surface, and let $K \subset D$ be compact, measurable and connected. Then the "content" of $\mathbf{p}(K)$ is defined by the surface integral

$$\boxed{\int_{\mathbf{p}(K)} do := \int_K \underbrace{\left\| \frac{\partial \mathbf{p}}{\partial u_1}(\mathbf{u}) \times \frac{\partial \mathbf{p}}{\partial u_2}(\mathbf{u}) \right\|}_{A} \, d\mathbf{u}}$$



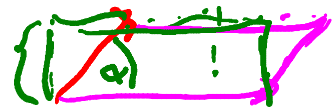
We call

$$do := \left\| \frac{\partial \mathbf{p}}{\partial u_1}(\mathbf{u}) \times \frac{\partial \mathbf{p}}{\partial u_2}(\mathbf{u}) \right\| \, d\mathbf{u}$$

the surface element of the surface $\mathbf{x} = \mathbf{p}(\mathbf{u})$.

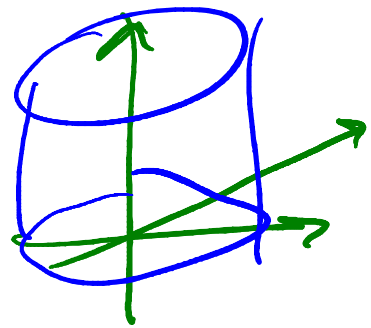
Remark: The surface integral is **independent** of the particular parameterisation of the surface. This follows from the theorem of transformation.

$$\text{Area} = \left\|\frac{\partial p}{\partial u_1}\right\| \left\|\frac{\partial p}{\partial u_2}\right\| \sin\alpha = \left\| \frac{\partial p}{\partial u_1} \times \frac{\partial p}{\partial u_2} \right\|$$

$\sin\alpha$ 

Example.

For the lateral surface of a cylinder $Z = \mathbf{p}(K)$ with



$$K := [0, 2\pi] \times [0, H] \subset \mathbb{R}^2$$

and

$$\mathbf{x} = \mathbf{p}(\varphi, z) := \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \text{for } (\varphi, z) \in \mathbb{R}^2$$

we obtain

$$\left\| \frac{\partial \mathbf{p}}{\partial \varphi} \times \frac{\partial \mathbf{p}}{\partial z} \right\| = r = \left\| \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ 0 \end{pmatrix} \right\| = \underline{\underline{r}}$$

the value

$$O(Z) = \int_Z do = \int_K r d(\varphi, z) = \int_{\underset{\varphi}{0}}^{2\pi} \int_{\underset{z}{0}}^{H} r dz d\varphi = \boxed{2\pi r H}$$

Example.

If the surface is the graph of a scalar function, i.e. $x_3 = \varphi(x_1, x_2)$, the for the related tangential vectors we have

$$\frac{\partial \mathbf{p}}{\partial x_1} \times \frac{\partial \mathbf{p}}{\partial x_2} = \begin{pmatrix} 1 \\ 0 \\ \varphi_{x_1} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \varphi_{x_2} \end{pmatrix} = \begin{pmatrix} -\varphi_{x_1} \\ -\varphi_{x_2} \\ 1 \end{pmatrix}$$

Thus we obtain

$$\left\| \frac{\partial \mathbf{p}}{\partial x_1} \times \frac{\partial \mathbf{p}}{\partial x_2} \right\| = \sqrt{1 + \varphi_{x_1}^2 + \varphi_{x_2}^2}$$

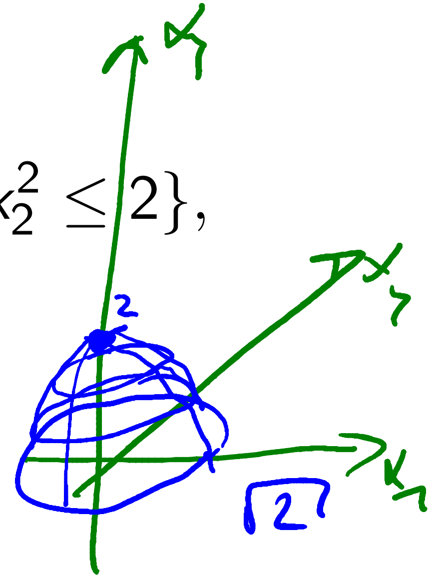
and

$$\begin{aligned} O(\mathbf{p}(K)) &= \int_{\mathbf{p}(K)} do \\ &= \int_K \sqrt{1 + \varphi_{x_1}^2 + \varphi_{x_2}^2}\, d(x_1, x_2) \end{aligned}$$

Example.

For the surface of the parabloid P, given by

$$P := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_3 = 2 - x_1^2 - x_2^2,\ x_1^2 + x_2^2 \leq 2\},$$



we have $\varphi(x_1,x_2) = 2 - x_1^2 - x_2^2$ $\quad \varphi_{x_1} = -2x_1$, $\varphi_{x_2} = -2x_2$

$$O(P) = \int_{x_1^2+x_2^2 \leq 2} \sqrt{1 + \underbrace{4x_1^2}_{\varphi_{x_1}^2} + \underbrace{x_2^2}_{\varphi_{x_2}^2}}\, d(x_1, x_2)$$

Polar coordinates
$x_1^2 + x_2^2 \leq 2$
$(r,\varphi) \in [0,\sqrt{2}] \times [0, 2\pi]$

$r^2 = s$
$2r\,dr = ds$

$$= \int_0^{\sqrt{2}} \int_0^{2\pi} \sqrt{1 + 4r^2}\, r \, d\varphi\, dr = \pi \int_0^2 \sqrt{1 + 4s}\, ds$$

$$= \pi \left[\frac{1}{6}(1 + 4s)^{3/2}\right]_0^2 = \pi \left(\frac{1}{6}(27 - 1)\right) = \frac{13}{3}\pi$$

Remark.

For the vector product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ we have

$$\|a\|^2\|b\|^2\sin^2\varphi = \|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - \langle \mathbf{a}, \mathbf{b} \rangle^2$$

$$= \|a\|^2\|b\|^2 - \|a\|^2\|b\|^2\cos^2\varphi = \|a\|^2\|b\|^2(1-\cos^2\varphi) = \|a\|^2\|b\|^2\sin^2\varphi$$

Thus we have

$$\underline{\left\| \frac{\partial \mathbf{p}}{\partial x_1} \times \frac{\partial \mathbf{p}}{\partial x_2} \right\|^2} = \left\| \frac{\partial \mathbf{p}}{\partial x_1} \right\|^2 \left\| \frac{\partial \mathbf{p}}{\partial x_2} \right\|^2 - \left\langle \frac{\partial \mathbf{p}}{\partial x_1}, \frac{\partial \mathbf{p}}{\partial x_2} \right\rangle^2$$

If we define

$$E := \left\| \frac{\partial \mathbf{p}}{\partial x_1} \right\|^2, \quad F := \langle \frac{\partial \mathbf{p}}{\partial x_1}, \frac{\partial \mathbf{p}}{\partial x_2} \rangle^2, \quad G := \left\| \frac{\partial \mathbf{p}}{\partial x_2} \right\|^2,$$

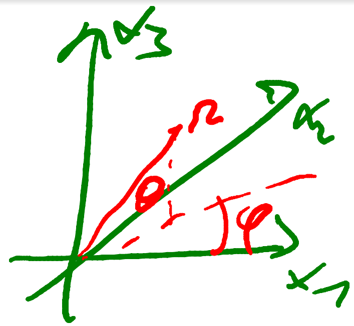
we obtain the relation

$$do = \sqrt{EG - F^2}\, d(u_1, u_2)$$

Example.

For the surface element of the sphere

r fixed



$$S_r^2 = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = r^2\}$$

we obtain using the parameterisation via spherical coordinates

$$p(\varphi,\theta) = \boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = r \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}} \quad \text{für } (\varphi,\theta) \in [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

the relations

$$\frac{\partial \mathbf{p}}{\partial \varphi} = r \begin{pmatrix} -\sin\varphi\cos\theta \\ \cos\varphi\cos\theta \\ 0 \end{pmatrix} \quad \text{und} \quad \frac{\partial \mathbf{p}}{\partial \theta} = r \begin{pmatrix} -\cos\varphi\sin\theta \\ -\sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}$$

Thus we have

$$\left\|\frac{\partial p}{\partial \varphi}\right\|^2 = E = r^2\cos^2\theta, \quad F \overset{\left\langle \frac{\partial p}{\partial \varphi}, \frac{\partial p}{\partial \theta}\right\rangle}{=} 0, \quad G = r^2 = \left\|\frac{\partial p}{\partial \theta}\right\|^2$$

Continuation of the examples.

With

$$E = r^2 \cos^2\theta, \quad F = 0, \quad G = r^2$$

we obtain the relation

$$\boxed{do = \sqrt{EG - F^2}\, d(u_1, u_2)} = \sqrt{R^4 \cos^2\theta}$$

and therefore

$$do = \underline{\underline{r^2 \cos\theta}}\, d(\varphi, \theta) \qquad \text{für } (\varphi, \theta) \in [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We can calculate the surface of the sphere as follows

$$\begin{aligned} O &= \int_{S_r^2} do = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} r^2 \cos\theta\, d\varphi\, d\theta \\ &= 2\pi r^2 \sin\theta \Big|_{-\pi/2}^{\pi/2} = \underline{\underline{4\pi r^2}} \end{aligned}$$

Surface integrals of scalar and vector fields.

Definition: Let $\mathbf{x} = \mathbf{p}(\mathbf{u})$ be a $\mathcal{C}^1$–parametrisation of a surface $F = \mathbf{p}(K)$, where $K \subset D$ is compact, measurable and connected.

- For a continuous function $f : F \to \mathbb{R}$ the surface integral of a scalar field is defined as

$$\boxed{\int f(c(t))\,\|\dot{c}(t)\|\,dt}$$

$$\int_F f(\mathbf{x})\,do := \int_K f(\mathbf{p}(\mathbf{u}))\,\left\|\frac{\partial \mathbf{p}}{\partial u_1} \times \frac{\partial \mathbf{p}}{\partial u_2}\right\|\,d\mathbf{u}$$

- For a continuous vector field $\mathbf{f} : F \to \mathbb{R}^3$ the surface integral of a vector field is defined as

$$\boxed{\int \langle f(c(t)), N \rangle\,dt}$$

Normal vector

$$n = \frac{N}{\|N\|}$$

$$\int_F \mathbf{f}(\mathbf{x})\,do := \int_K \left\langle \mathbf{f}(\mathbf{p}(\mathbf{u})), \frac{\partial \mathbf{p}}{\partial u_1} \times \frac{\partial \mathbf{p}}{\partial u_2} \right\rangle\,d\mathbf{u}$$

Othere representations of surface integrals of vector fields

The unit normal vector $\mathbf{n}(\mathbf{x})$ on a surface F is given by

$$\mathbf{n}(\mathbf{x}) = \mathbf{n}(\mathbf{p}(\mathbf{u})) = \frac{\dfrac{\partial \mathbf{p}}{\partial u_1} \times \dfrac{\partial \mathbf{p}}{\partial u_2}}{\left\| \dfrac{\partial \mathbf{p}}{\partial u_1} \times \dfrac{\partial \mathbf{p}}{\partial u_2} \right\|}$$

$$\int \langle f(c(t)), \dot{c}(t) \rangle dt = \int \left\langle f(c(t)), \frac{\dot{c}(t)}{\|\dot{c}(t)\|} \right\rangle \|\dot{c}(t)\| dt$$

Therefore we can write

$$\begin{aligned}
\int_F \mathbf{f}(\mathbf{x})\, do &= \int_K \left\langle \mathbf{f}(\mathbf{p}(\mathbf{u})), \frac{\partial \mathbf{p}}{\partial u_1} \times \frac{\partial \mathbf{p}}{\partial u_2} \right\rangle d\mathbf{u} \\
&= \int_K \langle \mathbf{f}(\mathbf{p}(\mathbf{u})), \mathbf{n}(\mathbf{p}(\mathbf{u})) \rangle \left\| \frac{\partial \mathbf{p}}{\partial u_1} \times \frac{\partial \mathbf{p}}{\partial u_2} \right\| d\mathbf{u} \\
&= \int_F \langle \mathbf{f}(\mathbf{x}), \mathbf{n}(\mathbf{x}) \rangle \, do
\end{aligned}$$

vector

scalar

Interpretation of surface integrals.

Remark:

- If $f(\mathbf{x})$ is the mass density of a surface with a mass distribution, the the surface integral of the scalar field (mass density) gives the total mass of the surface.

  scalar

- If $\mathbf{f}(\mathbf{x})$ is the velocity field of a stationary flow, then the surface integral of the vector field (velocity field) gives the amount of flow which passes the surface F per time unit, i.e. the flow of $\mathbf{f}(\mathbf{x})$ through the surface F.

- If F is a closed surface, i.e. surface (boundary) of a compact and simply connected region (body) in $\mathbb{R}^3$, we write

$$\oint_F f(\mathbf{x})\, do \qquad \text{bzw.} \qquad \oint_F \mathbf{f}(\mathbf{x})\, do$$

  The parameterisation is chosen such that the unit normal vector $\mathbf{n}(\mathbf{x})$ is pointing outwards.

The divergence theorem (Gauß theorem).

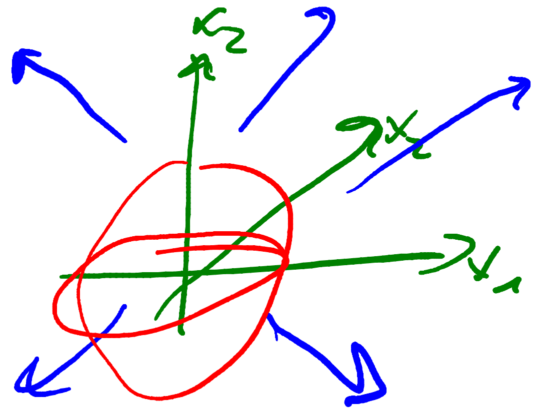
Theorem: (divergence theorem/Gauß theorem) Let $G \subset \mathbb{R}^3$ a compact and measurable standard domain, i.e. G is projectable with respect to all coordinates. The boundary ∂G consists of finite many smooth surfaces with outer normal vector $\mathbf{n}(\mathbf{x})$.
If $\mathbf{f} : D \to \mathbb{R}^3$ is a $\mathcal{C}^1$–vector field with $G \subset D$, then

$$\int_G \operatorname{div} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \oint_{\partial G} \mathbf{f}(\mathbf{x})\, do$$

Interpretation of the Gauß theorem: The left side is an integral of the scalar function $g(\mathbf{x}) := \operatorname{div} \mathbf{f}(\mathbf{x})$ over G. The right hand side is a surface integral of the vector field $\mathbf{f}(\mathbf{x})$. If $\mathbf{f}(\mathbf{x})$ is the vectorfield of an incompressible flow, then $\operatorname{div} \mathbf{f}(\mathbf{x}) = 0$ and therefore

$$\oint_{\partial G} \mathbf{f}(\mathbf{x})\, do = 0$$

Example.



Consider the vector field

$$\mathbf{f}(\mathbf{x}) = \mathbf{x} = (x_1, x_2, x_3)^T$$

and the sphere K:

$$K := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$$

We have

$$\operatorname{div} \mathbf{f}(\mathbf{x}) = 3$$

$$\oint_{\partial K} f\,d\sigma = \int \underbrace{\left\langle \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle}_{r^2} r^2 \cos\theta \, d\theta \, d\varphi = \oint_{\partial K} r^2 \cos\theta \, d\theta \, d\varphi = 4\pi$$

and thus

$$\oint_{\partial K} f\,d\sigma = \int_K \operatorname{div} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = 3 \cdot \operatorname{vol}(K) = 4\pi = 3 \cdot \frac{4\pi}{3}$$

The related surface integral can be calculated easily using spherical coordinates.

The Green formulas.

Theorem: (Green formulas) Let the set $G \subset \mathbb{R}^3$ satisfy the prerequisites of the Gauß theorem. For $\mathcal{C}^2$–functions $f, g : D \to \mathbb{R}$, $G \subset D$ we have the relations:

$$\boxed{\int_G (f \Delta g + \langle \nabla f, \nabla g \rangle)\, d\mathbf{x} = \oint_{\partial G} f \frac{\partial g}{\partial \mathbf{n}}\, do}$$

$$\int g \Delta f + \langle \nabla g, \nabla f \rangle\, dx = \int g \frac{\partial f}{\partial n}\, do$$

$$\int_G (f \Delta g - g \Delta f)\, d\mathbf{x} = \oint_{\partial G} \left(f \frac{\partial g}{\partial \mathbf{n}} - g \frac{\partial f}{\partial \mathbf{n}} \right) do$$

We denote by

$$\langle \nabla f, n \rangle = \frac{\partial f}{\partial \mathbf{n}}(\mathbf{x}) = D_{\mathbf{n}} f(\mathbf{x}) \qquad \text{for } \mathbf{x} \in \partial G$$

the directional derivative of $f(\mathbf{x})$ in the direction of the outer unit normal vector $\mathbf{n}(\mathbf{x})$.

Proof of the Green formulas.

We set

$$\mathbf{F}(\mathbf{x}) = f(\mathbf{x}) \cdot \nabla g(\mathbf{x})$$

Then we have

$$\begin{aligned} \boxed{\operatorname{div} \mathbf{F}(\mathbf{x})} &= \frac{\partial}{\partial x_1}\left(f \cdot \frac{\partial g}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(f \cdot \frac{\partial g}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(f \cdot \frac{\partial g}{\partial x_3}\right) \\ &= \boxed{f \cdot \Delta g + \langle \nabla f, \nabla g \rangle} \end{aligned}$$

Now we apply the Gauß theorem:

Gauß

$$\begin{aligned} \int_G (f \Delta g + \langle \nabla f, \nabla g \rangle)\, d\mathbf{x} &= \int_G \operatorname{div} \mathbf{F}(\mathbf{x})\, d\mathbf{x} = \oint_{\partial G} \langle \mathbf{F}, \mathbf{n} \rangle \, do \\ &= \oint_{\partial G} f \langle \nabla g, \mathbf{n} \rangle \, do = \oint_{\partial G} f \frac{\partial g}{\partial \mathbf{n}} \, do \end{aligned}$$

$\langle f \cdot \nabla g, n \rangle = f \langle \nabla g, n \rangle$

The second formula follows directely by exchanging f and g.