We know: given vector field $f = f(x)$

$\oint_C f(x)\,dx = 0$ $\forall$ curves c curl free

$\Rightarrow$ $\int_C f(x)\,dx$ is path independent

if $f = \nabla\phi$ gradient field $\Rightarrow$ $\int_C f(x)\,dx$ path independent

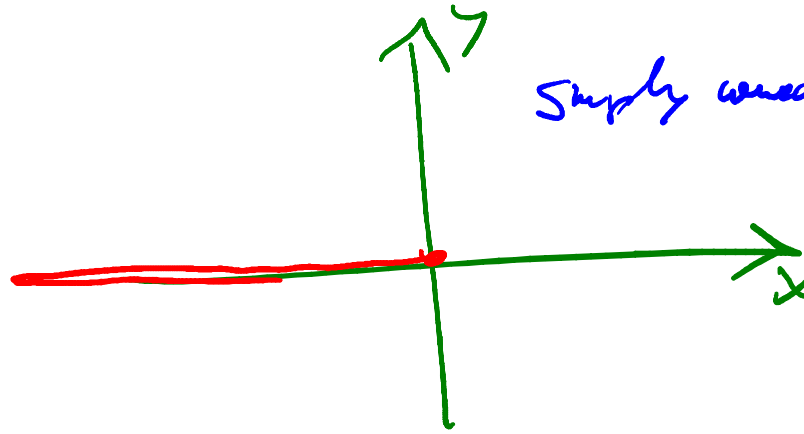
$= \phi(x(b)) - \phi(x(a))$

curl $f = 0$ is necessary for $f = \nabla\phi$

if curl $f = 0$ and D is simply connected $\Rightarrow$ $f = \nabla\phi$

$$f(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix}-y\\ x\end{pmatrix}$$

$$D = \mathbb{R}^2 \setminus \{(x,y) \mid y=0,\ x\le 0\}$$



simply connected

Example.

We consider the vector field

$$\mathbf{f}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{with } (x,y)^T \in D = \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

Calculating the curl gives

$$\begin{aligned} \operatorname{curl}\left[\frac{1}{r^2}\begin{pmatrix} -y \\ x \end{pmatrix}\right] &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) \\ &= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \\ &= 0 \end{aligned}$$

The curl of $\mathbf{f}(x,y)$ vanishes.

But $\mathbf{f}(x,y)$ has on the set $D = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ no potential.

The domain is **not** simply connected.

$\mathbb{R}^3$

$$\operatorname{curl} f = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{pmatrix} \partial_2 f_3 - \partial_3 f_2 \\ \partial_3 f_1 - \partial_1 f_3 \\ \partial_1 f_2 - \partial_2 f_3 \end{pmatrix}$$

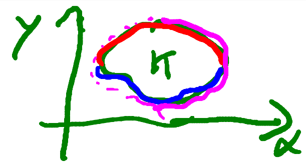
$\mathbb{R}^2$

$$f = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} \qquad \tilde{f}(x,y,z) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ 0 \end{pmatrix}$$

$$\operatorname{curl} \tilde{f} = \begin{pmatrix} 0 \\ 0 \\ \partial_1 f_2 - \partial_2 f_1 \end{pmatrix}$$

$$\boxed{\text{define} \quad \operatorname{curl} f = \partial_1 f_2 - \partial_2 f_1 \quad \text{in } \mathbb{R}^2}$$

The integral theorem of Green for vector fields in $\mathbb{R}^2$.

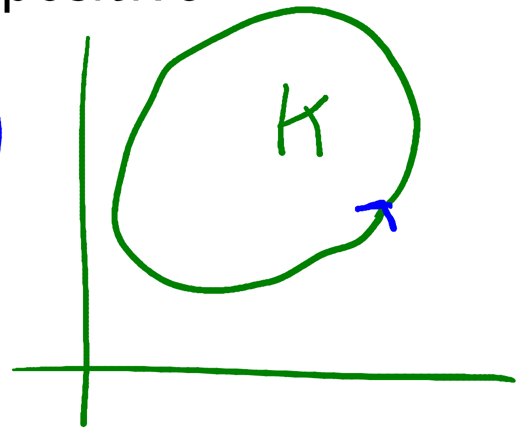


Theorem: (Integral theorem of Green)

Let $\mathbf{f}(\mathbf{x})$ be a $\mathcal{C}^1$–vector field on a domain $D \subset \mathbb{R}^2$. Let $K \subset D$ be compact and projectable with respect to both coordinates, such that K is bounded by a closed and piecewise $\mathcal{C}^1$–curve $\mathbf{c}(t)$.

The parameterisation of $\mathbf{c}(t)$ is chosen such that K is always on the left when going along the curve with increasing parameter (positive circulation). Then:

$$\boxed{\oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_K \underbrace{\operatorname{curl} \mathbf{f}(\mathbf{x})}_{\text{scalar}}\, d\mathbf{x}}$$



Remark:

The integral theorem is also valid for domains which can be splittet in *finite* many domains which all are projectable with respect to both coordinate directions, so called Green domains.

Alternative formulation of the integral theorem of Green I.

We have seen that the relation

$$\int_a^b \langle f(c(t)), \dot c(t)\rangle\, dt \overset{c=c(t)}{\underset{c(a)=c(b)}{=}} \oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \oint_c \langle \mathbf{f}, \mathbf{T}\rangle\, ds = \int_a^b \langle f(c(t)), \frac{\dot c(t)}{\|\dot c(t)\|}\rangle \underbrace{\|\dot c(t)\|\, dt}_{ds}$$

holds, where $\mathbf{T}(t) = \frac{\dot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|}$ denotes the tangent unit vector.

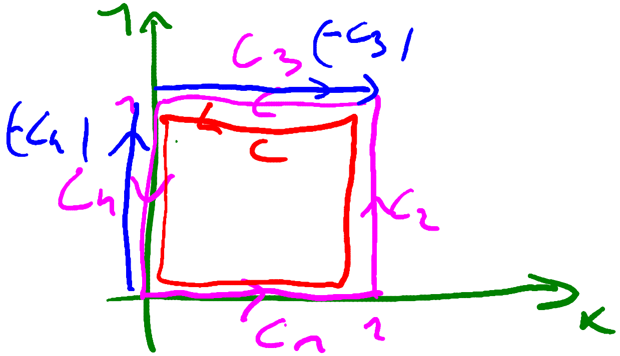
With the intergral thoerem of Green we obtain

$$\boxed{\int_K \operatorname{curl} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \oint_{\partial K} \langle \mathbf{f}, \mathbf{T}\rangle\, ds}$$

Is $\mathbf{f}(\mathbf{x})$ a velocity field, then the fluid motion described by $\mathbf{f}$ is curl free if $\operatorname{curl} \mathbf{f}(\mathbf{x}) = 0$, since

$$\oint_c \mathbf{f}(\mathbf{x}) d\mathbf{x}$$

is the circulation of $\mathbf{f}(\mathbf{x})$. on every curve c which is the boundary of a K where $\operatorname{curl} f = 0$



$$c_1(t)=\begin{pmatrix}t\\0\end{pmatrix}\qquad \dot c_1(t)=\begin{pmatrix}1\\0\end{pmatrix}$$

$$c_2(t)=\begin{pmatrix}1\\t\end{pmatrix}\qquad \dot c_2(t)=\begin{pmatrix}0\\1\end{pmatrix}$$

$$(-c_3)(t)=\begin{pmatrix}t\\1\end{pmatrix}\qquad (-c_3)^\cdot(t)=\begin{pmatrix}1\\0\end{pmatrix}$$

$$(-c_4)(t)=\begin{pmatrix}0\\t\end{pmatrix}\qquad (-c_4)^\cdot(t)=\begin{pmatrix}0\\1\end{pmatrix}$$

$$f(x,y)=\begin{pmatrix}f_1(x,y)\\f_2(x,y)\end{pmatrix}$$

$$\int_{c_1} f\,dx = \int_0^1 \left\langle \begin{pmatrix}f_1\\f_2\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\rangle dt = \int_0^1 f_1(t,0)\,dt \quad .$$

$$\int_{c_2} f\,dx = \int_0^1 \left\langle \begin{pmatrix}f_1\\f_2\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\rangle dt = \int_0^1 f_2(1,t)\,dt \quad \ldots$$

$$\int_{(-c_3)} f\,dx = \qquad = \int_0^1 f_1(t,1)\,dt \quad ..$$

$$\int_{(-c_4)} f\,dx = \qquad = \int_0^1 f_2(0,t)\,dt \quad \ldots.$$

$$\oint_c f(x)\,dx = \int_{c_1} + \int_{c_2} - \int_{(-c_3)} - \int_{(-c_4)} = \int_0^1 \big(f_1(t,0) - f_1(t,1)\big)dt + \int_0^1 \big(f_2(1,t) - f_2(0,t)\big)dt =$$

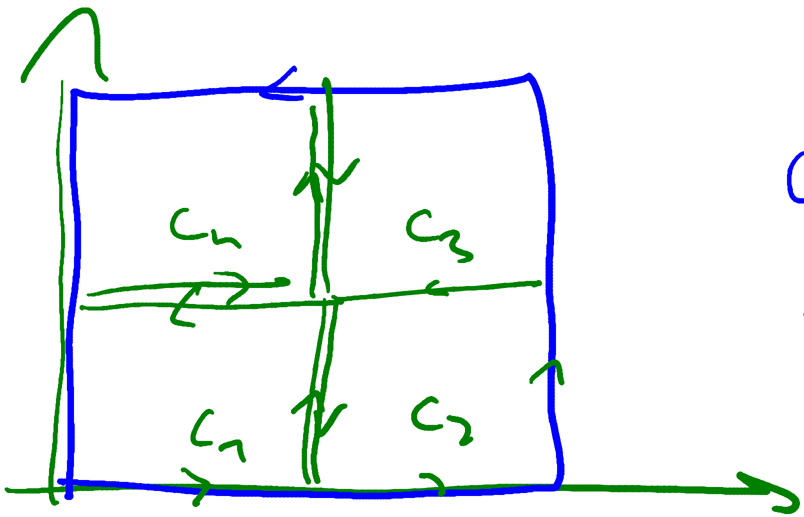
$$\quad . \qquad .. \qquad \ldots \qquad \ldots$$

$$= \int_0^1 \big(f_1(t,0) - f_1(t,1)\big)\,dt + \int_0^1 f_2(1,t) - f_2(0,t)\,dt$$

$$= \int_0^1 \Big(\cancel{f_1(t,0)} - \big(\cancel{f_1(t,0)} + f_{1y}(t,0)\cdot 1 + \dots\big)\Big)\,dt + \int_0^1 \Big(\cancel{f_2(0,t)} + f_{2x}(0,t)\cdot 1 + \dots - \cancel{f_2(0,t)}\Big)\,dt$$

$$= -\int_0^1 f_{1y}(t,0)\,dt + \int_0^1 f_{2x}(0,t)\,dt \approx f_{2x}(0,0) - f_{1y}(0,0)$$

$$= \operatorname{curl} f((0,0))$$



$$\oint_C f(x)\,dx = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} =$$

$$= \operatorname{curl} f(m_{C_1}) + \operatorname{curl} f(m_{C_2}) + \operatorname{curl} f(m_{C_3}) + \operatorname{curl} f(m_{C_4})$$

$$\approx \int_A \operatorname{curl} f \; da$$

If we substitute in the above equations the vector $\mathbf{T}$ by the outer normal vector $\mathbf{n} = (T_2, -T_1)^T$, we obtain

$$\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} T_2 \\ -T_1 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

$$\boxed{\oint_{\partial K} \langle \mathbf{f}, \mathbf{n} \rangle \, ds} = \oint_{\partial K} (f_1 T_2 - f_2 T_1) ds = \oint_{\partial K} \left\langle \underbrace{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}}_{f}, \mathbf{T} \right\rangle ds$$

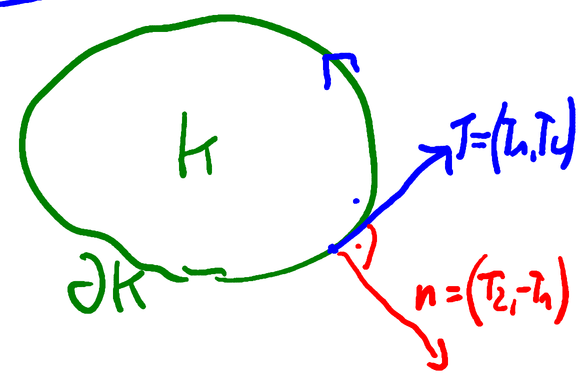
$$\overset{\text{Green}}{=} \int_K \overset{\text{curl}}{\cancel{\text{rot}}} \underbrace{\begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}}_{f} d\mathbf{x} = \boxed{\int_K \operatorname{div} \mathbf{f} \, d\mathbf{x}}$$

$$\text{curl} \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} = \partial_1 f_1 - \partial_2(-f_2) = \operatorname{div} f$$

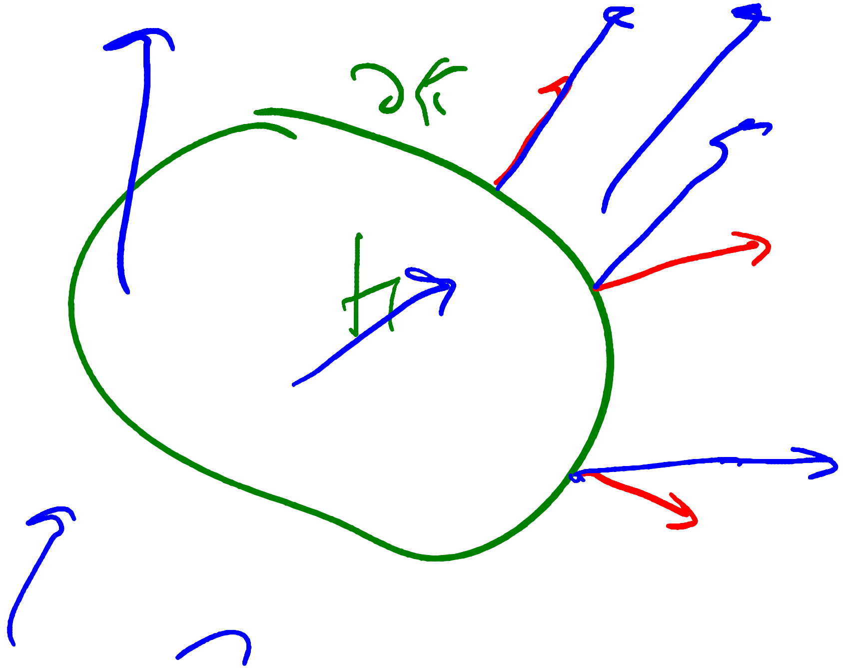
and thus the relation

Gauss

$$\boxed{\int_K \operatorname{div} \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = \oint_{\partial K} \langle \mathbf{f}, \mathbf{n} \rangle \, ds}$$



If $\mathbf{f}(\mathbf{x})$ is the velocity field of a fluid motion, then the right side describes describes the total flow of the fluid through the boundary of K. Therefore if $\operatorname{div} \mathbf{f}(\mathbf{x}) = 0$, then the fluid motion is is source and sink free (or divergence free).

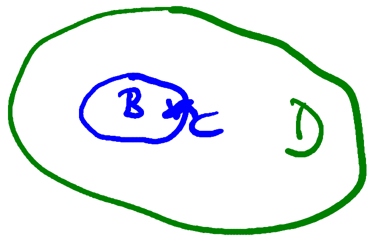


$$\oint_{\partial K} \langle f, n \rangle \, ds$$

Back again to the existence of potentials.

Conclusion: If $\operatorname{curl} \mathbf{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in D$, $D \subset \mathbb{R}^2$ a domain, then we have

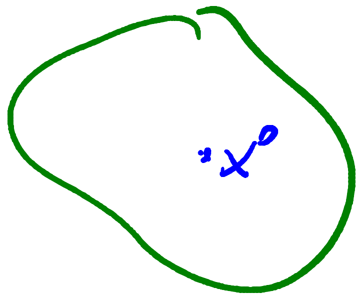
$$\oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = 0$$



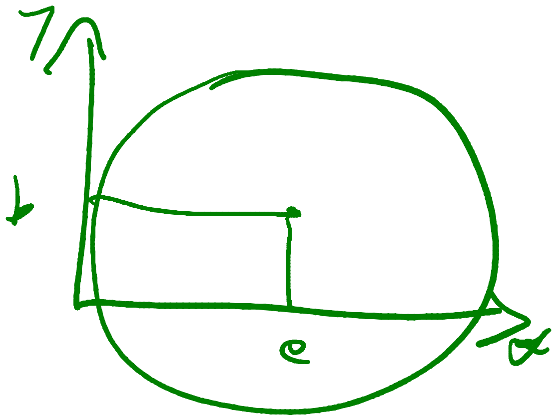
for every closed piecewise $\mathcal{C}^1$–curve, which surounds a Green domain $B \subset D$ completely.

Definition: A domain $D \subset \mathbb{R}^n$ is called simply connected, if any closed curve $\mathbf{c} : [a, b] \to D$ can be shrinked continuously in D to a point in D. More precise: There is a continuous map for $\mathbf{x}^0 \in D$

$$\Phi : \overbrace{[a, b]}^{\text{parametrisation}} \times [0, 1] \to D$$



with $\boxed{\Phi(t, 0) = \mathbf{c}(t)}$, for all $t \in [a, b]$ and $\boxed{\Phi(t, 1) = \mathbf{x}^0} \in D$, for all $t \in [a, b]$. The map $\Phi(t, s)$ is called a homotopy.



$$c(t) = \begin{pmatrix} a + r\cos t \\ b + r\sin t \end{pmatrix}$$

shrink it to (a, b)

$$\phi(t, s) = \begin{pmatrix} a + r(1-s)\cos t \\ b + r(1-s)\sin t \end{pmatrix}$$

$$\phi(t, 0) = c(t)$$

$$\phi(t, 1) = \begin{pmatrix} a \\ b \end{pmatrix}$$

continuous in s !

Criteria for integrability for potentials.

Theorem: Let $D \subset \mathbb{R}^n$ be a simply connected domain. A $\mathcal{C}^1$–vector field $\mathbf{f} : D \to \mathbb{R}^n$ has a potential on D if and only if the integrability criteria

$$\mathbf{J}\,\mathbf{f}(\mathbf{x}) = (\mathbf{J}\,\mathbf{f}(\mathbf{x}))^T \qquad \text{for all } \mathbf{x} \in D$$

Jacobian is symmetric

are satisfied, i.e. if

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k} \qquad \forall j, k$$

Remark: For $n = 2, 3$ the integrability criteria coincide with

$$\operatorname{rot} \mathbf{f}(\mathbf{x}) = 0$$

$n=2$: $Jf = \begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{pmatrix}$ is symmetric iff $\partial_2 f_1 - \partial_1 f_2 = 0 = \operatorname{rot} f$

$n = 3$

$$Jf = \begin{pmatrix} \partial_1 f_1 & \partial_1 f_2 & \partial_1 f_3 \\ \partial_1 f_2 & \partial_2 f_2 & \partial_3 f_2 \\ \partial_1 f_3 & \partial_2 f_3 & \partial_3 f_3 \end{pmatrix} \iff \operatorname{curl} f = 0$$

Symm

Example.

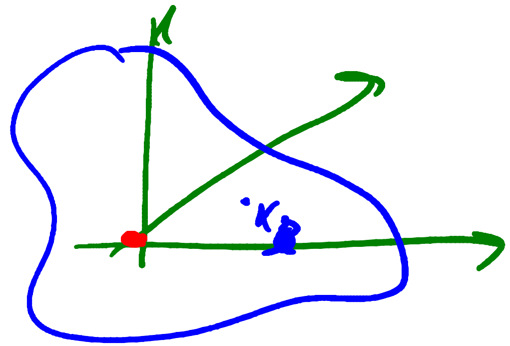
For $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ let the vector field be

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{2xy}{r^2} + \sin z \\ \ln r^2 + \frac{2y^2}{r^2} + ze^y \\ \frac{2yz}{r^2} + e^y + x\cos z \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \text{with } r^2 = x^2 + y^2 + z^2.$$

We would like to study the existence of a potential for $\mathbf{f}(\mathbf{x})$.

The set $D = \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is apparentely simply connected. In addition we have

$$\operatorname{curl} \mathbf{f}(\mathbf{x}) = 0$$



Thus $\mathbf{f}(\mathbf{x})$ has a potential.

Calculation of the potential.

We need to have: $\boxed{\mathbf{f}(\mathbf{x}) = \nabla\varphi(\mathbf{x})}$. Thus:

$r^2 = x^2 + y^2 + z^2$

$$\frac{\partial\varphi}{\partial x} = f_1(x,y,z) = \frac{2xy}{r^2} + \sin z$$

By integration with respect to the variable x we obtain

$\frac{\partial}{\partial x}(y \ln r^2) = y\,\frac{2x}{r^2}$

$$\varphi(\mathbf{x}) = y \ln r^2 + x \sin z + \underbrace{c(y,z)}_{?}$$

with an unknown function $c(y,z)$.

Pluging into the equation

$$= \frac{\partial\varphi}{\partial y} = f_2(x,y,z) = \ln r^2 + \frac{2y^2}{r^2} + ze^y$$

gives

$$= \ln r^2 + \frac{2y^2}{r^2} + \frac{\partial c}{\partial y} = \ln r^2 + \frac{2y^2}{r^2} + ze^y$$

Calculation of the potential (continuation).

From this we get the condition

$$\frac{\partial c}{\partial y} = ze^y$$

and therefore

$$c(y, z) = ze^y + d(z)$$

for an unknown function $d(z)$. So far we know:

$$\varphi(\mathbf{x}) = \underline{y \ln r^2 + x \sin z + ze^y} + d(z)$$

The last condition is

$$\frac{2yz}{r^2} + x\cos z + e^y + d'(z) = \frac{\partial \varphi}{\partial z} = f_3(x, y, z) = \frac{2yz}{r^2} + e^y + x \cos z$$

Therefore $d'(z) = 0$ and the potential is given by

$$\varphi(\mathbf{x}) = \underline{y \ln r^2 + x \sin z + ze^y} + c \quad \text{for } c \in \mathbb{R}$$

3.3 Surface integrals

Definition: Let $D \subset \mathbb{R}^2$ be a domain and $\mathbf{p} : D \to \mathbb{R}^3$ a $\mathcal{C}^1$–map

$$\boxed{\mathbf{x} = \mathbf{p}(\mathbf{u})} \quad \text{with } \boxed{\mathbf{x}} \in \mathbb{R}^3 \text{ and } \boxed{\mathbf{u} = (u_1, u_2)^T} \in D \subset \mathbb{R}^2$$

If for all $\mathbf{u} \in D$ the two vectors

$$\frac{\partial \mathbf{p}}{\partial u_1} \quad \text{and} \quad \frac{\partial \mathbf{p}}{\partial u_2}$$

are linear independent, we call

$$F := \{\mathbf{p}(\mathbf{u}) \mid \mathbf{u} \in D\}$$

a surface or a piece o surface. The map $\mathbf{x} = \mathbf{p}(\mathbf{u})$ is called a parameterisation or parameter representation of the surface F.

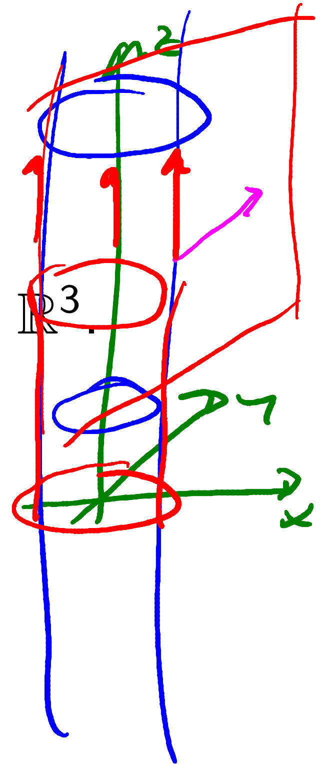
Example I.

We consider for a given $r > 0$ the map

r fixed

$$\mathbf{p}(\varphi, z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix} \qquad \boxed{\text{for } (\varphi, z) \in \mathbb{R}^2.}$$

$(\varphi, z) \in (0, 2\pi] \times \mathbb{R}$



The corresponding parameterized surface is an unbounded cylinder in $\mathbb{R}^3$.

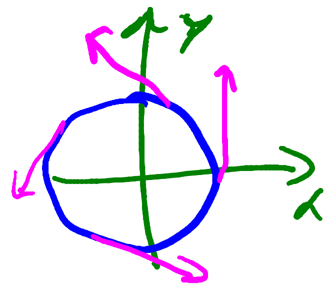
If we restrict the area of definition, e.g.

$$(\varphi, z) \in K := \boxed{[0, 2\pi] \times [0, H] \subset \mathbb{R}^2}$$

we obtain a bounded cylinder of height H.

The partial derivatives

$$\frac{\partial \mathbf{p}}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \\ 0 \end{pmatrix}, \qquad \frac{\partial \mathbf{p}}{\partial z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



of $\mathbf{p}(\varphi, z)$ are linearly independent on $\mathbb{R}^2$.