3.1 Area integrals

scalar

Given a function $f : D \to \mathbb{R}$ with domain of defintion $D \subset \mathbb{R}^n$.

Aim: Calculate the volume under the graph of $f(\mathbf{x})$:

$$\boxed{V = \int_D f(\mathbf{x}) d\mathbf{x}}$$

Remember (Analysis II): Riemann–Integral of a function f on the interval $[a, b]$:

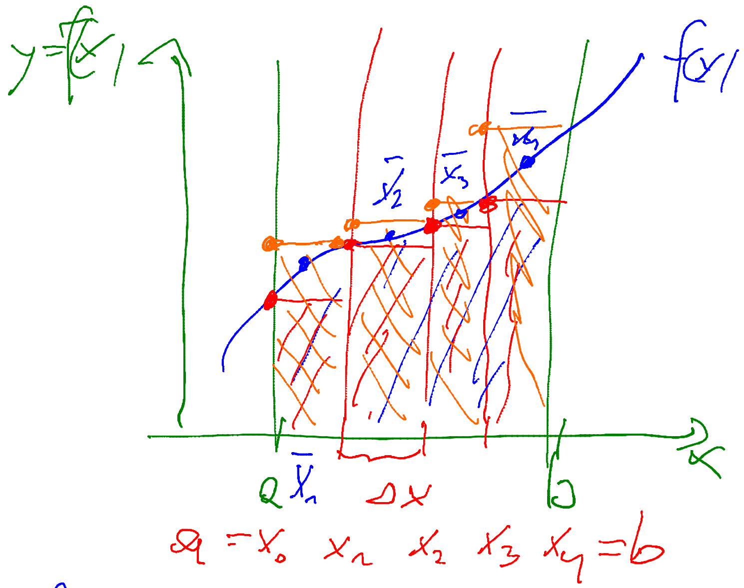
$$I = \int_a^b f(x) dx$$

The integral I is defined as limit of Riemann upper– and lower-sums, if the limits exist and coincide.

$$U_f = \sum_{i=1}^{4} f(x_{i-1}) \Delta x$$

$$O_f = \sum_{i=1}^{4} f(x_i) \Delta x$$

$$U_f \le O_f$$



$$R_f = \sum_{i=1}^{4} f(\bar{x}_i) \Delta x \qquad U_f < R_f < O_f$$

if $\Delta x \to 0$

if $U_f \to I \leftarrow O_f$ then f integrable

$$I = \int_a^b f(x)\,dx$$

1 Dimension:

f monoton $\longrightarrow$ f integrable

f continuous $\longrightarrow$ f integrable

In practice:

given $f = f(x)$, you look for a primitive function F

$F = F(x)$, $F'(x) = f(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

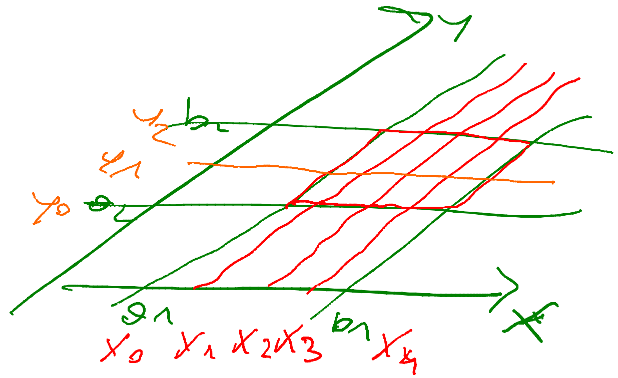
Construction of area integrals.

Procedure: Same as in the one dimensional case.

But: the domain of definition D is more complex.

Starting point: consider the case of two variables $n = 2$ and a domain of definition $D \subset \mathbb{R}^2$ of the form

$$D = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$$



i.e. D is compact cuboid (rectangle).

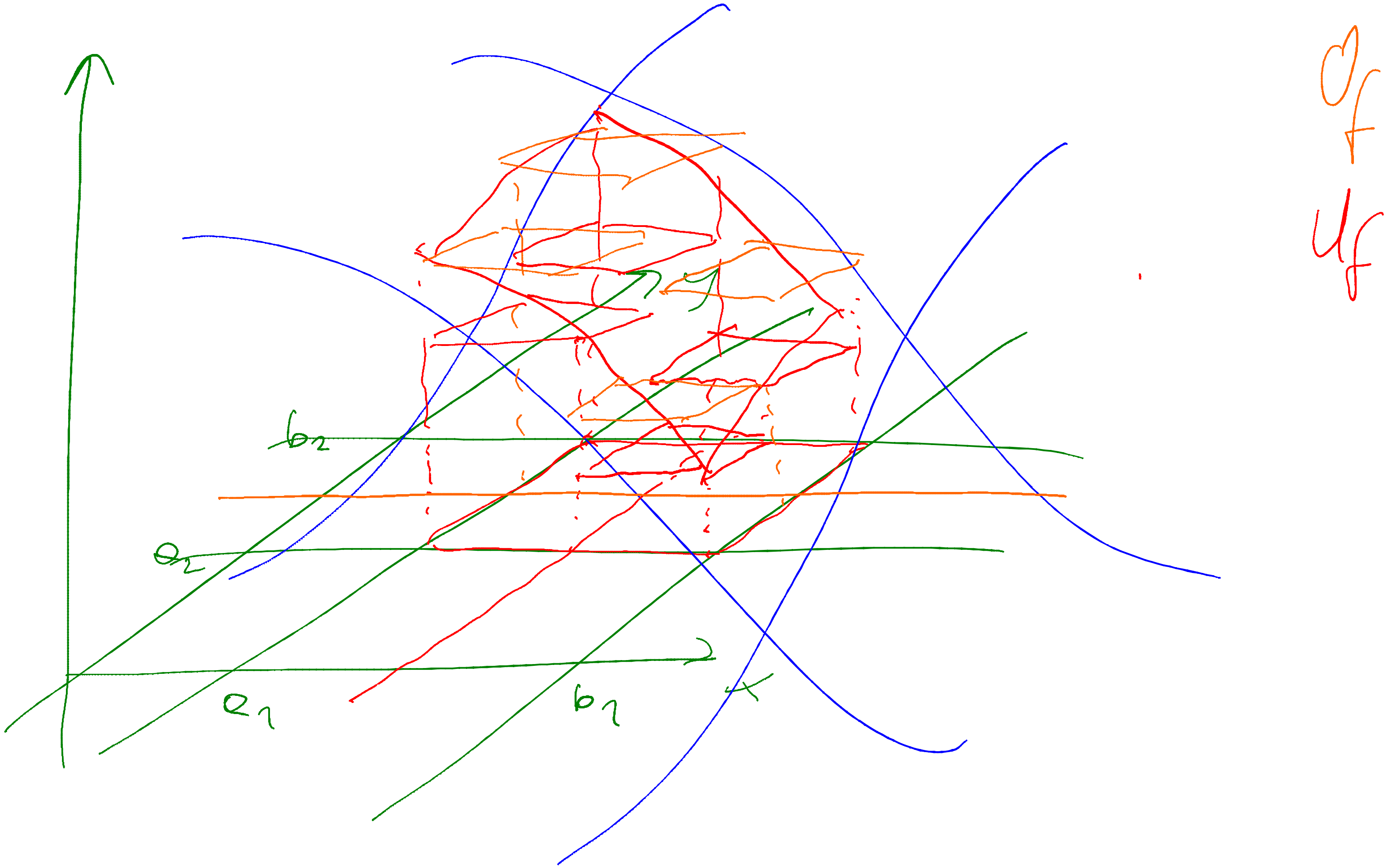
Let $f : D \to \mathbb{R}$ be a bounded function.

Definition: We call $Z = \{(x_0, x_1, \ldots, x_n), (y_0, y_1, \ldots, y_m)\}$ a partition of the cuboid $D = [a_1, b_1] \times [a_2, b_2]$ if it holds

$$a_1 = x_0 < x_1 < \cdots < x_n = b_1$$

$$a_2 = y_0 < y_1 < \cdots < y_m = b_2$$

$\mathbf{Z}(D)$ denotes the set of partitions of D.


b_2
a_2
a_1
b_1
x
y
σ_f
u_f

Partitions and Riemann sums.

Definition:

- The fineness of a partition $Z \in \mathbf{Z}(D)$ is given by

$$\|Z\| \;:=\; \max_{i,j} \{|x_{i+1} - x_i|, |y_{j+1} - y_j|\}$$

Δx_i Δy_j

- For a given partition Z the sets

$$\boxed{Q_{ij} \;:=\; [x_i, x_{i+1}] \times [y_j, y_{j+1}]}$$

are called the subcuboid of the partition Z. The volume of the subcuboid Q_{ij} is given by

$$\boxed{\text{vol}(Q_{ij}) \;:=\; (x_{i+1} - x_i) \cdot (y_{j+1} - y_j)}$$

- For arbitrary points $x_{ij} \in Q_{ij}$ of the subcuboids we call

$$\boxed{R_f(Z)} := \sum_{i,j} f(\mathbf{x}_{ij}) \cdot \text{vol}(Q_{ij})$$

a Riemann sum of the partition Z.

Riemann upper and lower sums.

Definition:

In analogy to the integral for the univariate case we call for a partition Z

lower Sum:

$$U_f(Z) := \sum_{i,j} \inf_{\mathbf{x} \in Q_{ij}} f(\mathbf{x}) \cdot \mathrm{vol}(Q_{ij})$$

Upper Sum:

$$O_f(Z) := \sum_{i,j} \sup_{\mathbf{x} \in Q_{ij}} f(\mathbf{x}) \cdot \mathrm{vol}(Q_{ij})$$

the Riemann lower sum and the Riemann upper sum of $f(\mathbf{x})$, respectively.

Remark:

A Riemann sum for the partition Z lies always between the lower and the upper sum of that partition i.e.

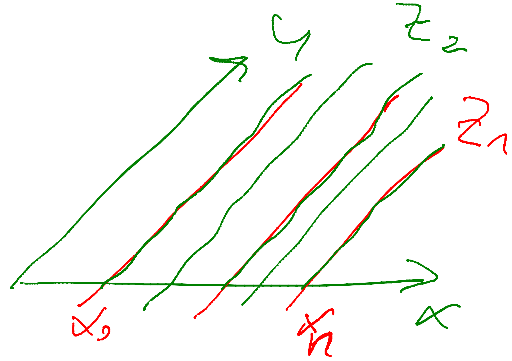
$$\boxed{U_f(Z) \leq R_f(Z) \leq O_f(Z)}$$

!

Remark.

$Z_0 = \{(x_0, x_1) \times (y_0, y_1)\}$

If a partition Z_2 is obtained from a partition Z_1 by adding additional intermediate points x_i and/or y_j, then



monoton increasing — monoton decreasing

$$U_f(Z_2) \geq U_f(Z_1) \quad \text{and} \quad O_f(Z_2) \leq O_f(Z_1)$$

finer — finer

For arbitrary two partitions Z_1 and Z_2 we always have:

$U_f(Z_0) \leq U_f(Z) \leq O_f(Z_0)$

$$U_f(Z_1) \leq O_f(Z_2)$$

$U_f(Z_0) \leq O_f(Z) \leq O(Z_0)$

U_f bounded from below — O_f bounded from above

Question: what happens to the lower and upper sums in the limit $\|Z\| \to 0$:

$$\boxed{\begin{aligned} U_f &:= \sup\{U_f(Z) \,:\, Z \in \mathbf{Z}(D)\} \\ O_f &:= \inf\{O_f(Z) \,:\, Z \in \mathbf{Z}(D)\} \end{aligned}}$$

Observation: Both values U_f and O_f exist since lower and upper sum are monoton and bounded.

U_f, O_f exist, but in general $U_f \neq O_f$

Definition:

1. The Riemann lower and upper integral of a function $f(\mathbf{x})$ on D is given by

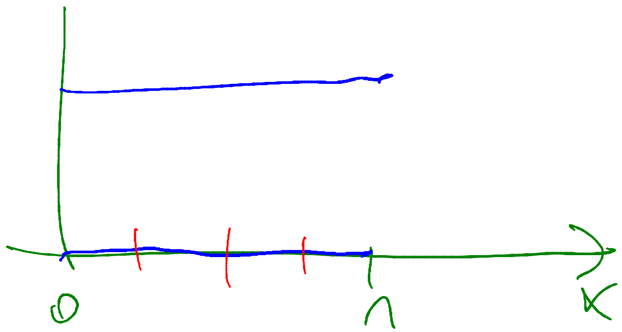
$$\int_{\underline{D}} f(\mathbf{x}) d\mathbf{x} \quad := \quad \sup\{U_f(Z) \,:\, Z \in \mathbf{Z}(D)\} \quad = U_f$$

$$\int_{\overline{D}} f(\mathbf{x}) d\mathbf{x} \quad := \quad \inf\{O_f(Z) \,:\, Z \in \mathbf{Z}(D)\} \quad = O_f$$

Definition

2. The function $f(\mathbf{x})$ is called Riemann–integrable on D, if lower and upper intergral conincide. The Riemann–integral of $f(\mathbf{x})$ on D is then given by

$$\int_D f(\mathbf{x}) d\mathbf{x} := \int_{\underline{D}} f(\mathbf{x}) d\mathbf{x} = \int_{\overline{D}} f(\mathbf{x}) d\mathbf{x}$$



$$f(x) = \begin{cases} 0 & x \text{ rational } x \in \mathbb{Q} \\ 1 & x \text{ irrational } x \in \mathbb{R}/\mathbb{Q} \end{cases}$$

$U_f(Z) = 0 \qquad O_f(Z) = 1 \qquad$ for any partition Z

$\Rightarrow U_f = 0 \qquad O_f = 1$

$$\boxed{f \text{ not Riemann integrable}}$$

Remark.

Up to now we habe "only" considered the case of **two** variables:

$$f : D \to \mathbb{R}, \qquad D \in \mathbb{R}^2$$

In higher dimensions, $n > 2$, the procdeure is the same.

Notation: for $n = 2$ and $n = 3$

$$\int_D f(x, y)dxdy \quad \text{bzw.} \quad \int_D f(x, y, z)dxdydz$$

or

$$\iint_D f(x, y)dxdy \quad \text{bzw.} \quad \iiint_D f(x, y, z)dxdydz$$

respectively.

Theorem:

a) Linearity

$$\longrightarrow \quad \int_D (\alpha f(\mathbf{x}) + \beta g(\mathbf{x}))d\mathbf{x} = \alpha \int_D f(\mathbf{x})d\mathbf{x} + \beta \int_D g(\mathbf{x})d\mathbf{x}$$

b) Monotonicity

$U_f(Z) \le U_g(Z) \quad \forall Z$

$O_f(Z) \le O_g(Z) \quad \forall Z$

If $f(\mathbf{x}) \le g(\mathbf{x})$ for all $\mathbf{x} \in D$, then:

$$\boxed{\int_D f(\mathbf{x})d\mathbf{x} \le \int_D g(\mathbf{x})d\mathbf{x}}$$

if $f = 0$

$0 \le g(x) \Rightarrow 0 \le \int_D g(x)\,dx$

c) Positivity

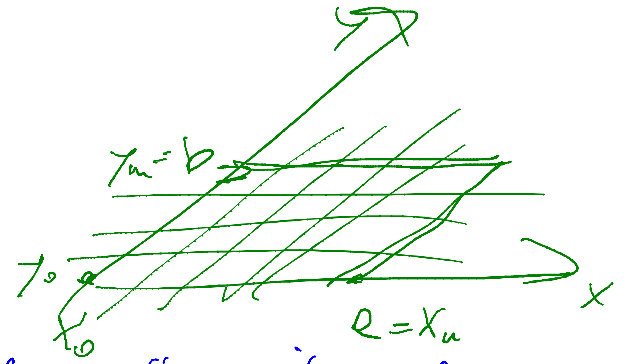
If for all $\mathbf{x} \in D$ the relation $f(\mathbf{x}) \ge 0$ holds, i.e. $f(\mathbf{x})$ is non–negativ, then

$$\int_D f(\mathbf{x})d\mathbf{x} \ge 0$$

$$f(x,y) = x^2 + y \qquad (x,y) \in [0,a] \times [0,b]$$

equidistant $\quad x_i = \frac{a}{n} i \qquad i = 0, \dots, n \qquad \Delta x_i = \frac{a}{n}$

$$y_j = \frac{b}{m} j \qquad j = 0, \dots, m \qquad \Delta y_j = \frac{b}{m}$$



$$O_f = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_i^2 + y_j) \underbrace{\frac{a}{n} \frac{b}{m}}_{v(Q_{ij})}$$

= is O_f because function increasing in x and y, we take always the right value

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\frac{a^2}{n^2} i^2 + \frac{b}{m} j \right) \frac{a}{n} \frac{b}{m} = \frac{a^3}{n^3} \frac{b}{m} \Big(\underbrace{\sum_{i=1}^{n} i^2}_{\frac{n(n+1)(2n+1)}{6}} \Big) \Big(\underbrace{\sum_{j=1}^{m} 1}_{m} \Big) + \frac{a}{n} \frac{b^2}{m^2} \Big(\underbrace{\sum_{j=1}^{n} 1}_{n} \Big) \Big(\underbrace{\sum_{j=1}^{m} j}_{\frac{m(m+1)}{2}} \Big) =$$

$$= a^3 b \, \frac{1 (1 + \frac{1}{n})(2 + \frac{1}{n})}{6} + a b^2 \, \frac{1 (1 + \frac{1}{m})}{2} \xrightarrow[m \to \infty]{n \to \infty} \frac{a^3}{3} b + \frac{a b^2}{2}$$

$$U_f = \sum_{i=1}^{n} \sum_{j=1}^{m} (x_{i-1}^2 + y_{j-1}) \frac{a}{n} \frac{b}{m} = \dots = a^3 b \, \frac{(1 - \frac{1}{n}) 1 (2 - 2\frac{1}{n} + \frac{1}{n})}{6} + a b^2 \, \frac{(1 - \frac{1}{m}) 1}{2}$$

$$\longrightarrow \frac{a^3}{3} b + \frac{a b^2}{2} \qquad U_f = O_f \quad f \text{ integrable} \quad \boxed{\int_0^a \int_0^b (x^2 + y) \, dy \, dx = \frac{a^3}{3} b + \frac{a b^2}{2}}$$

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \dots$$

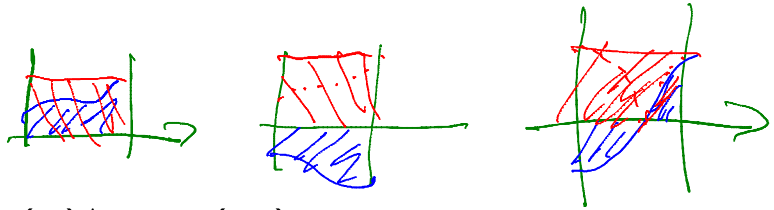
Additional properties of the integral.

Theorem:

a) Let D_1, D_2 and D be cuboids, $D = D_1 \cup D_2$ and $\mathrm{vol}(D_1 \cap D_2) = 0$, then $f(\mathbf{x})$ is on D integrable if and only if $f(\mathbf{x})$ is integrable on D_1 and D_2. And we have

$$\int_D f(\mathbf{x})d\mathbf{x} = \int_{D_1} f(\mathbf{x})d\mathbf{x} + \int_{D_2} f(\mathbf{x})d\mathbf{x}$$

b) The following estimate holds for the integral



$$\left| \int_D f(\mathbf{x})d\mathbf{x} \right| \quad \leq \quad \sup_{\mathbf{x}\in D} |f(\mathbf{x})| \cdot \mathrm{vol}(D)$$

c) Riemann criterion

$f(\mathbf{x})$ is integrable on D if and only if :

$$\forall \varepsilon > 0 \quad \exists Z \in \mathbf{Z}(D) \quad : \quad O_f(Z) - U_f(Z) < \varepsilon$$

Fubini's theorem.

Theorem: (Fubini's theorem) Let $f : D \to \mathbb{R}$ be integrable, $D = [a_1, b_1] \times [a_2, b_2]$ be a cuboid. If the integrals

$$\boxed{F(x) = \int_{a_2}^{b_2} f(x, y)dy} \quad \text{und} \quad \boxed{G(y) = \int_{a_1}^{b_1} f(x, y)dx}$$

exist for all $x \in [a_1, b_1]$ and $y \in [a_2, b_2]$, respectively, then

$$\int_D f(\mathbf{x})d\mathbf{x} = \int_{a_1}^{b_1} \underbrace{\int_{a_2}^{b_2} f(x, y)dy}_{F(x)}dx = \int_{a_1}^{b_1} F(x)\, dx$$

$$\int_D f(\mathbf{x})d\mathbf{x} = \int_{a_2}^{b_2} \underbrace{\int_{a_1}^{b_1} f(x, y)dx}_{G(y)}dy = \int_{a_2}^{b_2} G(y)\, dy$$

holds true.

Importance:

Fubini's theorem allows to reduce higher-dimensional integrals to one-dimensional integrals.

$$\int_0^b \int_0^a f(x,y)\,dx\,dy = \int_0^b \int_0^a (x^2+y)\,dx\,dy = \int_0^b \left[\frac{x^3}{3}\Big|_0^a + y\cdot 1\Big|_0^a \right] dx$$

$$= \int_0^b \left(\frac{a^3}{3} + ya \right) dy = \frac{a^3}{3} b + a \frac{b^2}{2}$$

Example.

Given the cuboid $\boxed{D = [0,1] \times [0,2]}$ and the function

$$\boxed{f(x,y) = 2 - xy}$$

We will show that continuous functions are integrable on cuboids. Thus we can apply Fubini's theorem:

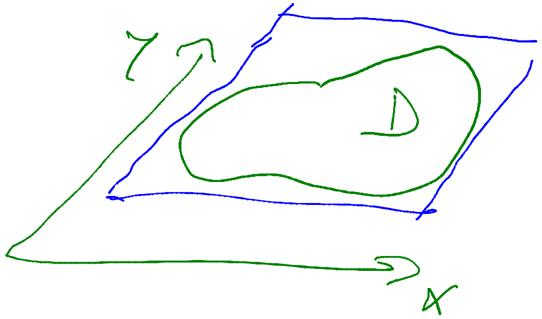
$$\int_D f(\mathbf{x}) d\mathbf{x} = \int_0^2 \underbrace{\int_0^1 f(x,y) dx}_{G(y)} dy = \int_0^2 \left[2x - \frac{x^2 y}{2} \right]_{x=0}^{x=1} dy$$

$$= \int_0^2 \left(2 - \frac{y}{2} \right) dy = \left[2y - \frac{y^2}{4} \right]_{y=0}^{y=2} = 3$$

Remark: Fubini's theorem requires the integrability of $f(\mathbf{x})$. The existence of the two integrals $F(x)$ and $G(y)$ does **not** guarantee the integrability of $f(\mathbf{x})$!

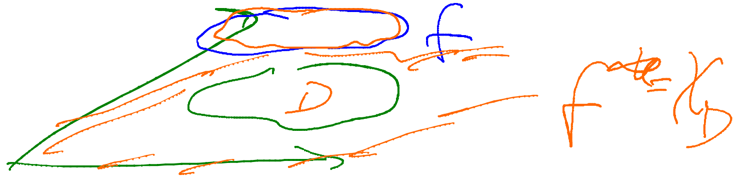
The characteristic function.

Definition: Let $D \subset \mathbb{R}^n$ compact and $f : D \to \mathbb{R}$ bounded. We set

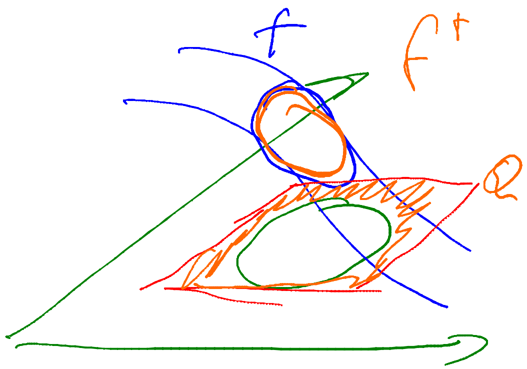


$$f^*(\mathbf{x}) := \begin{cases} f(\mathbf{x}) & : \quad \text{if } \mathbf{x} \in D \\ 0 & : \quad \text{if } \mathbf{x} \in \mathbb{R}^n \setminus D \end{cases}$$

In particular for $f(\mathbf{x}) = 1$ we call $f^*(\mathbf{x})$ the characteristic function of D.
The characteristic function of D is called $\mathcal{X}_D(\mathbf{x})$.



Let Q be the smallest cuboid with $D \subset Q$. The function $f(\mathbf{x})$ is called integrable on D, if $f^*(\mathbf{x})$ is integrable on Q. We set



$$\int_D f(\mathbf{x}) d\mathbf{x} := \int_Q f^*(\mathbf{x}) d\mathbf{x}$$

Measurability and null sets.

Definition: The compact set $D \subset \mathbb{R}^n$ is called measurable, if the integral

closed and bounded

$$\mathrm{vol}(D) := \int_D 1 d\mathbf{x} = \int_Q \mathcal{X}_D(\mathbf{x}) d\mathbf{x}$$

exists. We call $\mathrm{vol}(D)$ the volume of D in $\mathbb{R}^n$.

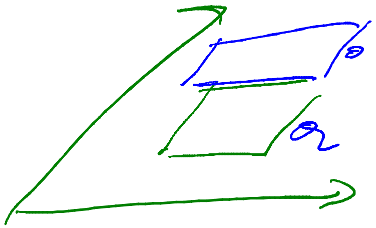
The compact set D is called null set, if D is measurable and if $\mathrm{vol}(D) = 0$ holds.

Remark:

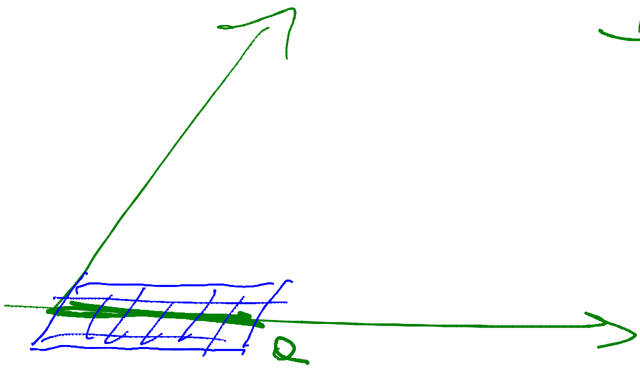
- If D a cuboid, then $Q = D$ and thus

$$\boxed{\int_D f(\mathbf{x}) d\mathbf{x} = \int_Q f^*(\mathbf{x}) d\mathbf{x} = \int_Q f(\mathbf{x}) d\mathbf{x}}$$

$\int \mathcal{X}_Q = V(Q) = (b_2 - a_2)(b_1 - a_1)$



  i.e. the introduced concepts of integrability coincide.
- Cuboids are measurable sets.
- $\mathrm{vol}(D)$ is the volume of the cuboid on $\mathbb{R}^n$.



$$D = [0, a] \times [0, 0]$$

$$\int \chi_D(x)\, dx = 0$$

$$\left.\begin{aligned} O_f &= 0 \\ U_f &\geq 0 \end{aligned}\right\} \Rightarrow$$

Three more properties of integration.

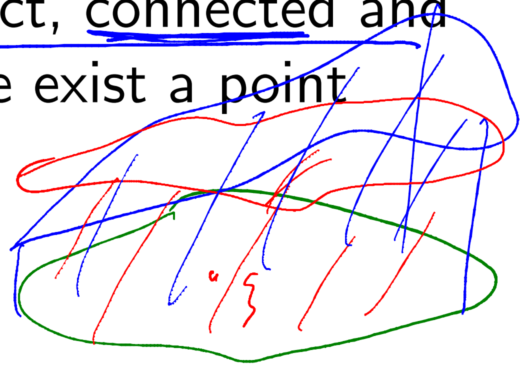
We have the following theorems for integrals in higher dimensions.

Theorem: Let $D \subset \mathbb{R}^n$ be compact. D is measurable if and only if the boundary ∂D of D is a null set.

Theorem: Let $D \subset \mathbb{R}^n$ be compact and measurable. Let $f : D \to \mathbb{R}$ be continuous. Then $f(\mathbf{x})$ is integrable on D.

Theorem: (Mean value theorem) Let $D \subset \mathbb{R}^n$ be compact, connected and measurable, and let $f : D \to \mathbb{R}$ be continuous, then there exist a point $\xi \in D$ with

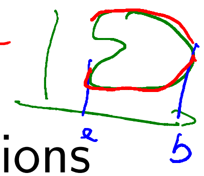
$$\int_D f(\mathbf{x})d\mathbf{x} = f(\xi) \cdot \mathrm{vol}(D)$$



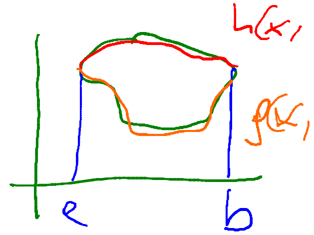
"Normal" areas.

Definition:

not normal

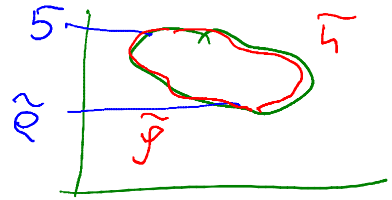


- A subset $D \subset \mathbb{R}^2$ is called "normal" area, there exist continuous functions g, h and $\tilde{g}, \tilde{h}$ with



$$D = \{(x, y) \mid a \leq x \leq b \text{ und } g(x) \leq y \leq h(x)\}$$

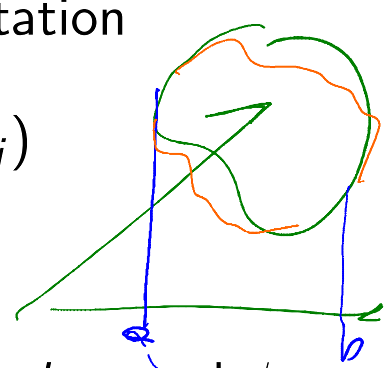
~~and~~ or



$$D = \{(x, y) \mid \tilde{a} \leq y \leq \tilde{b} \text{ und } \tilde{g}(y) \leq x \leq \tilde{h}(y)\}$$

respectively.

- A subset $D \subset \mathbb{R}^3$ is called "normal" area , if there is a representation



$$D = \{ (x_1, x_2, x_3) \mid a \leq x_i \leq b, \ g(x_i) \leq x_j \leq h(x_i) \text{ and } \varphi(x_i, x_j) \leq x_k \leq \psi(x_i, x_j) \}$$

with a permutation (i, j, k) of $(1, 2, 3)$ and continuos functions g, h, φ and ψ.