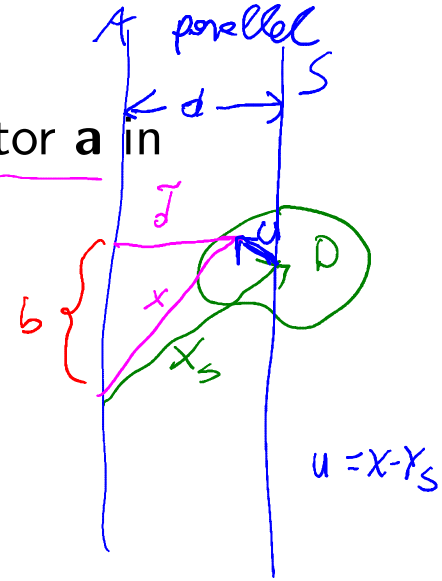
# The Theorem of Steiner.

**Theorem:** (Theorem of Steiner) For the moment of inertia of a homogeneous solid $K$ with total mass $m$ with respect to a given axis of rotation $A$ we have

$$\Theta_A = md^2 + \Theta_S$$

$S$ is the axis through to center of mass of the solid $K$ parallel to the axis $A$ and $d$ the distance of the center of mass $\mathbf{x}_s$ from the axis $A$.

**Idea of the proof:** Set $\mathbf{x} := \Phi(\mathbf{u}) = \mathbf{x}_s + \mathbf{u}$. Then with the unit vector $\mathbf{a}$ in direction of the axis $A$




$$\begin{aligned} \rightarrow \quad \Theta_A &= \rho \int_K \underbrace{(\langle \mathbf{x}, \mathbf{x} \rangle - \underbrace{\langle \mathbf{x}, \mathbf{a} \rangle^2}_{b^2})}_{\tilde{d}^2} d\mathbf{x} = (*) \\ &= \rho \int_D (\langle \mathbf{x}_s + \mathbf{u}, \mathbf{x}_s + \mathbf{u} \rangle - \langle \mathbf{x}_s + \mathbf{u}, \mathbf{a} \rangle^2) d\mathbf{x} \end{aligned}$$

$(*) = \rho \int_K (\langle x - x_s + x_s, x - x_s + x_s \rangle - \langle x - x_s + x_s, a \rangle^2) dx = \ldots = \Theta_S + \rho \underbrace{\int \langle x_s, x - x_s \rangle}_{\text{linear in } x - x_s} + \ldots + m d^2$

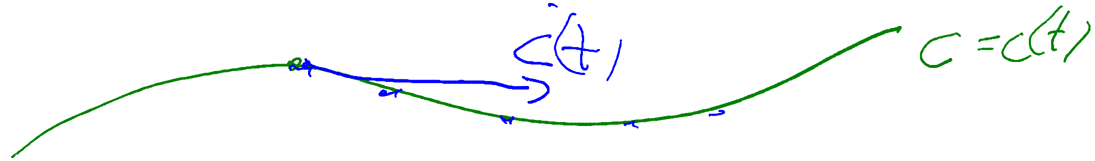
where

$$D := \{\mathbf{x} - \mathbf{x}_s \,|\, \mathbf{x} \in K\}$$

$\int (x - x_s) dx = 0$

$\Theta_S = \rho \int_K (\langle x - x_s, x - x_s \rangle - \langle x - x_s, a \rangle^2) dx$

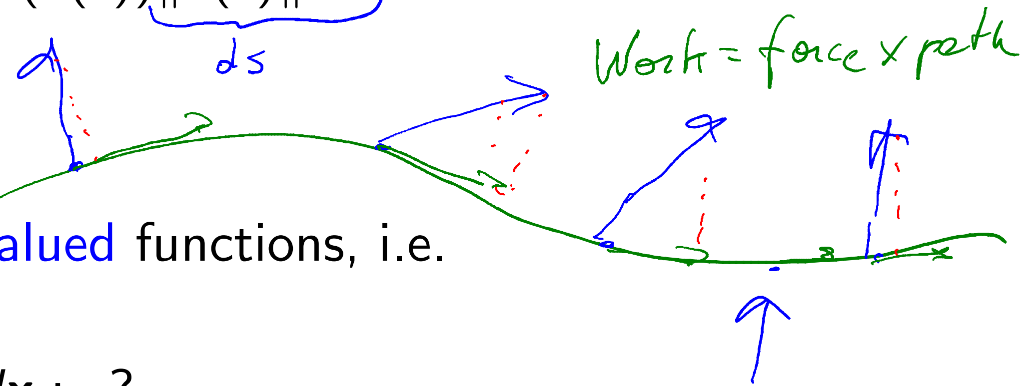
## 3.2 Line integrals




We already had a defintion of a line integral of a scalar field for a piecewise $\mathcal{C}^1$–curve $\mathbf{c} : [a, b] \to D$, $D \subset \mathbb{R}^n$, and a continuous scalar function $f : D \to \mathbb{R}$

$$\int_{\mathbf{c}} f(\mathbf{x})\, ds := \int_a^b f(\mathbf{c}(t)) \underbrace{\|\dot{\mathbf{c}}(t)\|\, dt}_{ds}$$

where $\| \cdot \|$ denotes the Euklidian norm.




**Generalisation:** Line integrals of vector valued functions, i.e.

$$\int_{\mathbf{c}} \mathbf{f}(\mathbf{x}) d\mathbf{x} := ?$$

**Application:** A point mass is moving along $\mathbf{c}(t)$ in a force field $\mathbf{f}(\mathbf{x})$.

**Question:** How much physical work has to be done along the curve?

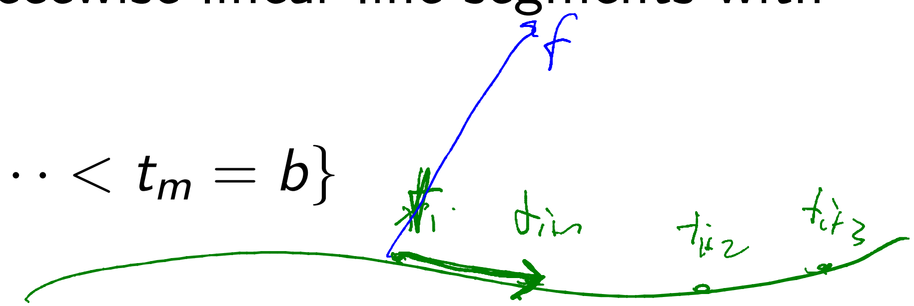
# Line integral on vector fields.

**Definition:** For a continuous vector field $\mathbf{f} : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^n$ open, and a piecewise $\mathcal{C}^1$–curve $\mathbf{c} : [a, b] \to D$ we define the line integral on vector fields by

$$\int_c \mathbf{f}(\mathbf{x}) d\mathbf{x} := \int_a^b \overbrace{\langle \mathbf{f}(\mathbf{c}(t)), \dot{\mathbf{c}}(t) \rangle}^{\text{scalar}} \, dt$$

**Derivation:** Approximate the curve by piecewise linear line segments with corners $\mathbf{c}(t_i)$, where

$$Z = \{a = t_0 < t_1 < \cdots < t_m = b\}$$




is a partition of the interval $[a, b]$.

Then the workload along the curve $\mathbf{c}(t)$ in the force field $\mathbf{f}(\mathbf{x})$ is approximately given by :

subdivisions

$$A \approx \sum_{i=0}^{m-1} \langle \mathbf{f}(\mathbf{c}(t_i)), \mathbf{c}(t_{i+1}) - \mathbf{c}(t_i) \rangle$$

Thus:

components segments

$$A \approx \sum_{j=1}^{n} \sum_{i=0}^{m-1} f_j(\mathbf{c}(t_i))\underbrace{(c_j(t_{i+1}) - c_j(t_i))}_{t_{i+1} - t_i} \quad \underbrace{(t_{i+1} - t_i)}$$

$\tau_{ij} \in [t_i, t_{i+1}]$

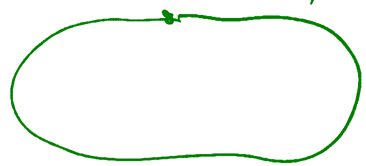
$$= \sum_{j=1}^{n} \sum_{i=0}^{m-1} f_j(\mathbf{c}(t_i))\dot{c}_j(\tau_{ij})\underbrace{(t_{i+1} - t_i)}_{dt}$$

For a sequence of partitions $Z$ with $\|Z\| \to 0$ the left side converges to the above defined line integral on vector fields.

**Remarks:** For a closed curve $\mathbf{c}(t)$, i.e. $\mathbf{c}(a) = \mathbf{c}(b)$, we use the notation

$$\oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x}$$

$c(a) = c(b)$



## Properties of the line integral on vector fields.

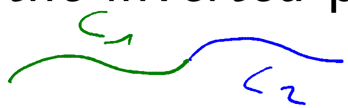
- Linearity:

$$\int_c (\alpha \mathbf{f}(\mathbf{x}) + \beta \mathbf{g}(\mathbf{x}))\, d\mathbf{x} = \alpha \int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} + \beta \int_c \mathbf{g}(\mathbf{x})\, d\mathbf{x}$$

- It is:

$$\int_{-c} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = -\int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x},$$

where $(-\mathbf{c})(t) := c(b + a - t)$, $a \leq t \leq b$, denotes the inverted path.



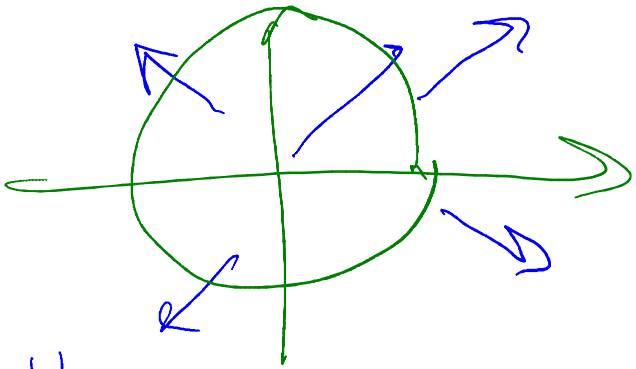

- It is

$$\int_{c_1 + c_2} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_{c_1} \mathbf{f}(\mathbf{x})\, d\mathbf{x} + \int_{c_2} \mathbf{f}(\mathbf{x})\, d\mathbf{x}$$

where $\mathbf{c}_1 + \mathbf{c}_2$ denotes the path composed by $\mathbf{c}_1$ and $\mathbf{c}_2$ such that the end point of $\mathbf{c}_1$ coincides with the starting point of $\mathbf{c}_2$.

$= (-1$

$$f(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} x \\ y \end{pmatrix}$$



$$\int f(x)\,dx = \int_0^{2\pi} \left\langle \frac{1}{R^2}\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, R\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle dt = 0$$

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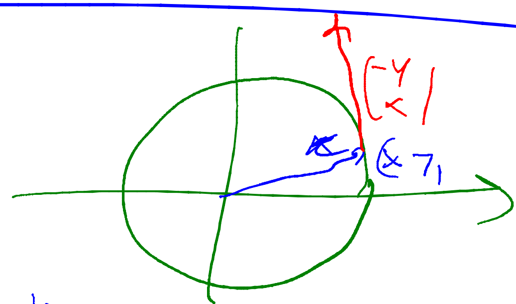
$$f(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$$



$$\int f(x)\,dx = \int_0^{2\pi} \left\langle \begin{pmatrix} -R\sin t \\ R\cos t \end{pmatrix}, R\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle dt = R^2\, 2\pi$$

# Further properties of the line integral on vector fields.

- The line integral on vector fields is invariant under paramterisation.
- It is

$$\int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_a^b \langle \mathbf{f}(\mathbf{c}(t)), \mathbf{T}(t)\rangle\, \|\dot{\mathbf{c}}(t)\|\, dt = \int_c \langle \mathbf{f}, \mathbf{T}\rangle\, ds$$

with the tangent unit vector $\mathbf{T}(t) := \dfrac{\dot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|}$.

- Formal notation:

$$\int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_c \sum_{i=1}^n f_i(\mathbf{x})\, dx_i = \sum_{i=1}^n \int_c f_i(\mathbf{x})\, dx_i$$

with

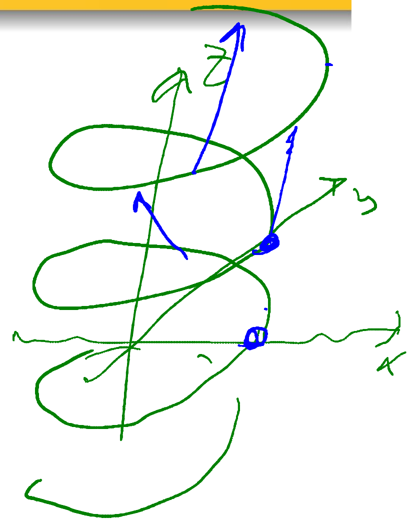
$$\int_c f_i(\mathbf{x})\, dx_i := \int_a^b f_i(\mathbf{c}(t))\dot{c}_i(t)\, dt$$

## Example.

Let $\mathbf{x} \in \mathbb{R}^3$ and

$$\mathbf{f}(\mathbf{x}) := (-y, x, z^2)^T$$

$$\mathbf{c}(t) := (\cos t, \sin t, at)^T \qquad \text{with } 0 \le t \le 2\pi$$



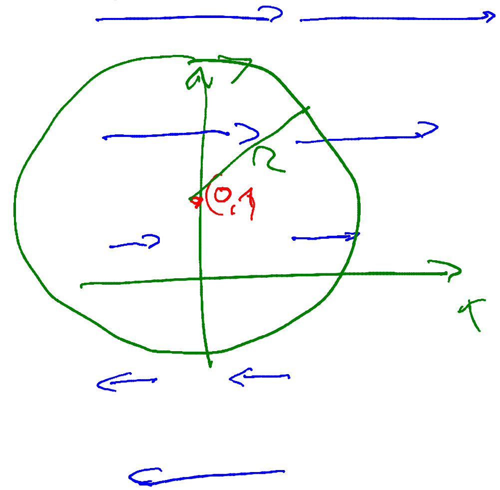

We calculate

$$\begin{aligned}
\int_c \mathbf{f}(\mathbf{x})\,d\mathbf{x} &= \int_c (\overset{f_1}{-y}\,dx + \overset{f_2}{x}\,dy + \overset{f_3}{z^2}\,dz) = \int_c \sum_i f_i\, dx_i = \\
&= \int_0^{2\pi} (\overset{f_1}{-\sin t})(\underset{\dot{c}_1}{-\sin t}) + \overset{f_2}{\cos t}\,\underset{\dot{c}_2}{\cos t} + \overset{f_3}{a^2 t^2}\,\underset{\dot{c}_3}{a})\,dt \\
&= \int_0^{2\pi} (1 + a^3 t^2)\,dt \\
&= 2\pi + \frac{a^3}{3}(2\pi)^3
\end{aligned}$$

**Definition:** Let $\mathbf{u}(\mathbf{x})$ be the velocity field of a moving fluid. We call the line integral $\oint_c \mathbf{u}(\mathbf{x})d\mathbf{x}$ along a closed curve the circulation of the field $\mathbf{u}(\mathbf{x})$.

depends on C

**Example:** For the field $\mathbf{u}(x,y) = (y,0)^T \in \mathbb{R}^2$ we obtain along the curve $\mathbf{c}(t) = (r\cos t, 1 + r\sin t)^T$, $0 \le t \le 2\pi$ the circulation




$\int_0^{2\pi} \left\langle \begin{pmatrix} 1+r\sin t \\ 0 \end{pmatrix}, \begin{pmatrix} -r\sin t \\ \dots \end{pmatrix} \right\rangle dt =$

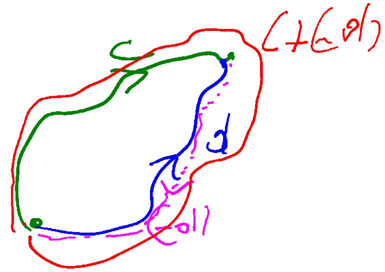
$$\oint_c \mathbf{u}(\mathbf{x})\,d\mathbf{x} = \int_0^{2\pi} (1 + r\sin t)(-r\sin t)dt$$

$$= \int_0^{2\pi} (-r\sin t - r^2\sin^2 t)dt$$

$$= \left[ r\cos t - \frac{r^2}{2}(t - \sin t\cos t) \right]_0^{2\pi} = -\pi r^2$$

# Curl free vector fields.

**Definition:** A continuous vector field $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in D \subset \mathbb{R}^n$, is called curl free, if the line integral along **all** closed and piecewise $\mathcal{C}^1$–curves $\mathbf{c}(t)$ in $D$ vanishes, i.e.




$$\oint_c \mathbf{f}(\mathbf{x})\,d\mathbf{x} = 0 \qquad \text{for all closed } \mathbf{c}.$$

**Remark:** A vector field is curl free if an only if the value of the line integral $\int_c \mathbf{f}(\mathbf{x})d\mathbf{x}$ depends only from the starting and the end point of the path, but not on the specific path $\mathbf{c}$. In this case we call the line integral path independent.

$$\int_c f(x)dx - \int_d f(x)dx = \int_c f(x)dx - (-1)\int_{(-d)} f(x)dx = \int_{c+(-d)} f(x)dx \overset{\text{curl free}}{=} 0$$

**Question:** Which criteria on the vector field $\mathbf{f}(\mathbf{x})$ **guarantee** the path independency of the line integral?

# Connected sets.

**Definition:** A subset $D \subset \mathbb{R}^n$ is called connected, if any two points in $D$ can be connected by a piecewise $\mathcal{C}^1$–curve:

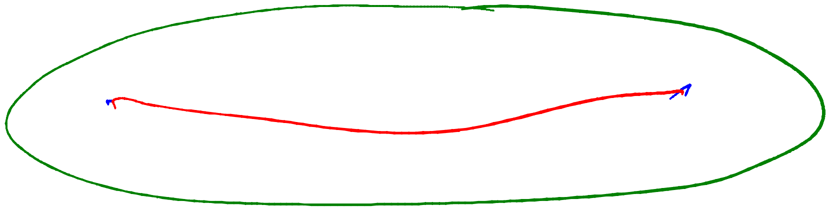
$$\forall \mathbf{x}^0, \mathbf{y}^0 \in D \ : \ \exists \mathbf{c} : [a, b] \to D \quad : \quad \mathbf{c}(a) = \mathbf{x}^0 \ \wedge \ \mathbf{c}(b) = \mathbf{y}^0$$

An open and connected set $D \subset \mathbb{R}^n$ is called domain in $\mathbb{R}^n$.

**Remark:** An **open** set $D \subset \mathbb{R}^n$ is **not** connected if and only if there exist **disjoint** and open sets $U_1, U_2 \subset \mathbb{R}^n$ with

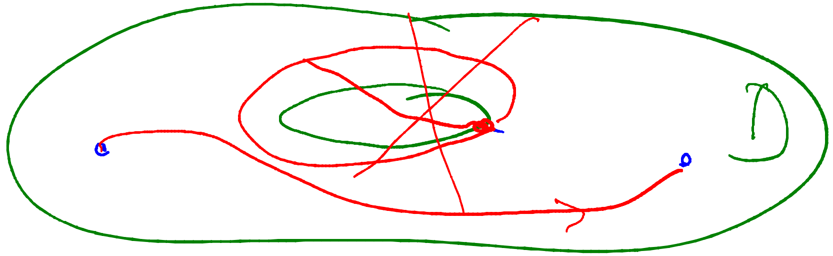
$$U_1 \cap D \neq \emptyset, \quad U_2 \cap D \neq \emptyset, \quad D \subset U_1 \cup U_2$$

Not connected sets are – in contrary to connected sets – a separable in at least two disjoint open sets.

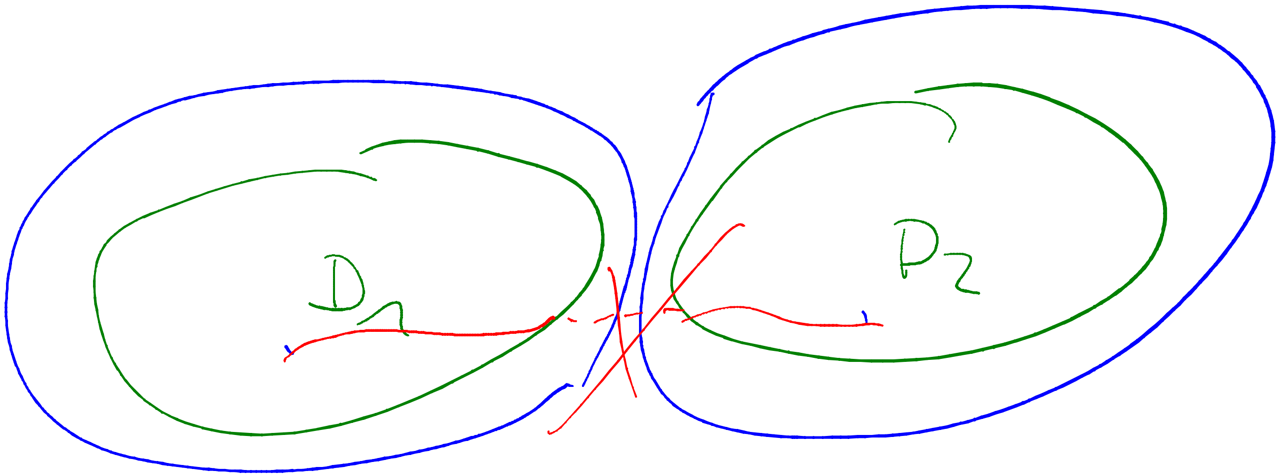


simply connected

= connected + every closed curve can be shrinked to a point in $D$




connected




not connected

**Definition:** Let $\mathbf{f} : D \to \mathbb{R}^n$ be a vector field on a domain $D \subset \mathbb{R}^n$. The vector field is called gradient field, if there is a scalar $\mathcal{C}^1$–function $\varphi : D \to \mathbb{R}$ with

$$\boxed{\mathbf{f}(\mathbf{x}) = \nabla\varphi(\mathbf{x})}$$

The function $\varphi(\mathbf{x})$ is called antiderivative or potential of $\mathbf{f}(\mathbf{x})$, and the vector field $\mathbf{f}(\mathbf{x})$ is called conservativ.

**Remark:** Suppose a mass point is moving in a conservative force field $\mathbf{K}(\mathbf{x})$, i.e. $\mathbf{K}$ has a potential $\varphi(\mathbf{x})$ such that $\mathbf{K}(\mathbf{x}) = \nabla\varphi(\mathbf{x})$. The the function $U(\mathbf{x}) = -\varphi(\mathbf{x})$ gives the potential energy:

Newton $\qquad K(x) = -\nabla U(x)$

$$\mathbf{K}(\mathbf{x}) = m\ddot{\mathbf{x}} = -\nabla U(\mathbf{x}) \qquad \langle K, \dot{x}\rangle$$

Multiplying this relation with $\dot{\mathbf{x}}$ we obtain

$$m\langle\ddot{\mathbf{x}}, \dot{\mathbf{x}}\rangle + \overbrace{\langle\nabla U(\mathbf{x}), \dot{\mathbf{x}}\rangle}^{\frac{d}{dt}U(x)} = \frac{d}{dt}\underbrace{\left(\frac{1}{2}m\|\dot{\mathbf{x}}\|^2 + U(\mathbf{x})\right)}_{\text{total energy}} = 0 \qquad \text{conserved}$$

$$\frac{1}{2}\frac{d}{dt}\|\dot{x}\|^2 = \frac{1}{2}\frac{d}{dt}\langle\dot{x},\dot{x}\rangle = \frac{1}{2}2\langle\dot{x},\ddot{x}\rangle$$

**Theorem:** (Fundamental theorem on line integrals)

Let $D \subset \mathbb{R}^n$ be a domain and $\mathbf{f}(\mathbf{x})$ a continuous vector field on $D$.

1) If $\mathbf{f}(\mathbf{x})$ has a potential $\varphi(\mathbf{x})$, then for all piecewise $\mathcal{C}^1$–curves $\mathbf{c} : [a, b] \to D$ we have:

$$\boxed{\int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \varphi(\mathbf{c}(b)) - \varphi(\mathbf{c}(a))}$$

In particular the line integral is path independent and $\mathbf{f}(\mathbf{x})$ is curl free.

+ simply connected $D$

2) In the opposite direction we have: If $\mathbf{f}(\mathbf{x})$ is curl free, then $\mathbf{f}(\mathbf{x})$ has a potential $\varphi(\mathbf{x})$.
Let $\mathbf{x}^0 \in D$ be a fixed point and $\mathbf{c}_\mathbf{x}$ (for $\mathbf{x} \in D$) denotes an arbitrary piecewise $\mathcal{C}^1$–curve in $D$ connecting the points $\mathbf{x}^0$ and $\mathbf{x}$, then $\varphi(\mathbf{x})$ is given by:

$$\varphi(\mathbf{x}) = \int_{c_\mathbf{x}} \mathbf{f}(\mathbf{x})\, d\mathbf{x} + \text{const.}$$

# Example I.

The central force field

$$\mathbf{K}(\mathbf{x}) := \frac{\mathbf{x}}{\|\mathbf{x}\|^3} = \underline{\underline{\frac{1}{|R|^2}\frac{x}{\|x\|}}}$$

has the potential

$$U(\mathbf{x}) = -\frac{1}{\|\mathbf{x}\|} = -(x_1^2 + x_2^2 + x_3^2)^{-1/2} \qquad \nabla U(x) = K(x)$$

since

$$\nabla U(\mathbf{x}) = (x_1^2 + x_2^2 + x_3^2)^{-3/2}(x, y, z)^T = \frac{\mathbf{x}}{\|\mathbf{x}\|^3}$$

The workload along a piecewise $\mathcal{C}^1$–curve $\mathbf{c} : [a, b] \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is given by

$$A = \underline{\int_c \mathbf{K}(\mathbf{x})\, d\mathbf{x}} = \left(\frac{1}{\|\mathbf{c}(a)\|} - \frac{1}{\|\mathbf{c}(b)\|}\right)$$

$$U(c(a)) - U(c(b))$$

# Example II.

The vector field

$$\mathbf{f}(\mathbf{x}) := \begin{pmatrix} 2xy + z^3 \\ x^2 + 3z \\ 3xz^2 + 3y \end{pmatrix}$$

has the potential

$$\varphi(\mathbf{x}) = x^2y + xz^3 + 3yz \qquad \nabla\varphi = f$$

For an arbitrary $\mathcal{C}^1$–curve $\mathbf{c}(t)$ from $P = (1, 1, 2)$ to $Q = (3, 5, -2)$ we have

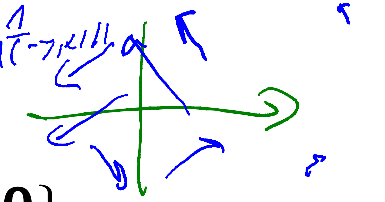
$$\underline{\int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x}} = \underline{\varphi(Q) - \varphi(P)} = -9 - 15 = -24$$

If we interpret $\mathbf{f}(\mathbf{x})$ as electrical field, then the line integral on vector fields represents the electrical voltage between the two points $P$ and $Q$.

# Example III.

Consider the vector field

$\frac{1}{x^2+y^2}\sqrt{x^2+y^2}\begin{pmatrix}-y\\x\end{pmatrix}\frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}\begin{pmatrix}-y\\x\end{pmatrix}\frac{1}{\|(-y,x)\|}$



$$\mathbf{f}(x,y) = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{mit } (x,y)^T \in D = \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

Not simply connected

For the unit sphere $\mathbf{c}(t) := (\cos t, \sin t)^T$, $0 \le t \le 2\pi$, we obtain

$$\begin{aligned}
\int_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} &= \int_0^{2\pi} \langle \mathbf{f}(\mathbf{c}(t)), \dot{\mathbf{c}}(t) \rangle \, dt \\
&= \int_0^{2\pi} \left\langle \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\rangle dt \\
&= \int_0^{2\pi} 1 \, dt = 2\pi
\end{aligned}$$

$\mathbf{f}(x,y)$ is therefore not curl free and ~~has no potential on $D$.~~

# Requirements for potentials.

**Remark:** If $\mathbf{f}(\mathbf{x})$, $\mathbf{x} \in D \subset \mathbb{R}^3$ is a $\mathcal{C}^1$–vector field with potential $\varphi(\mathbf{x})$, then

$$\text{curl}\, \mathbf{f}(\mathbf{x}) = \text{curl}\, (\nabla\varphi(\mathbf{x})) = 0 \qquad \text{für alle } \mathbf{x} \in D$$

Thus curl $\mathbf{f}(\mathbf{x}) = 0$ is a necessary condition for the existence of a potential.

If we define for a vector field $\mathbf{f} : D \to \mathbb{R}^2$, $D \subset \mathbb{R}^2$, the **scalar** curl

$$\text{curl}\, \mathbf{f}(x, y) := \frac{\partial f_2}{\partial x}(x, y) - \frac{\partial f_1}{\partial y}(x, y)$$

then curl $\mathbf{f}(x, y) = 0$ is a necessary condition even in 2 dimensions.

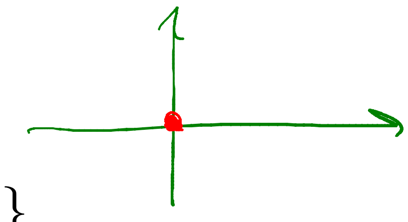
The condition

$$\text{curl}\, \mathbf{f}(\mathbf{x}) = 0$$

is a sufficient condition, if the domain $D$ is **simply connected**, i.e. if $D$ has no "holes".

# Example.

We consider the vector field



$$\mathbf{f}(x, y) = \frac{1}{x^2 + y^2} \begin{pmatrix} -y \\ x \end{pmatrix} \qquad \text{with } (x, y)^T \in D = \mathbb{R}^2 \setminus \{\mathbf{0}\}$$

not simply connected

Calculating the curl gives

$$\begin{aligned} \operatorname{curl} \left[ \frac{1}{r^2} \begin{pmatrix} -y \\ x \end{pmatrix} \right] &= \frac{\partial}{\partial x} \overbrace{\left( \frac{x}{x^2 + y^2} \right)}^{f_2} + \frac{\partial}{\partial x} \overbrace{\left( \frac{y}{x^2 + y^2} \right)}^{-f_1} \\ &= \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} + \frac{1}{x^2 + y^2} - \frac{2y^2}{(x^2 + y^2)^2} \\ &= 0 \end{aligned}$$

The curl of $\mathbf{f}(x, y)$ vanishes.

But $\mathbf{f}(x, y)$ has on the set $D = \mathbb{R}^2 \setminus \{\mathbf{0}\}$ no potential.

The domain is **not** simply connected.

# The integral theorem of Green for vector fields in $\mathbb{R}^2$.

**Theorem:** (Integral theorem of Green)

Let $\mathbf{f}(\mathbf{x})$ be a $\mathcal{C}^1$–vector field on a domain $D \subset \mathbb{R}^2$. Let $K \subset D$ be compact and projectable with respect to both coordinates, such that $K$ is bounded by a closed and piecewise $\mathcal{C}^1$–curve $\mathbf{c}(t)$.

The parameterisation of $\mathbf{c}(t)$ is chosen such that $K$ is always on the left when going along the curve with increasing parameter (positive circulation). Then:

$$\oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \int_K \text{curl}\, \mathbf{f}(\mathbf{x})\, d\mathbf{x}$$

**Remark:**

The integral theorem is also valid for domains which can be splittet in *finite* many domains which all are projectable with respect to both coordinate directions, so called Green domains.

# Alternative formulation of the integral theorem of Green I.

We have seen that the relation

$$\oint_c \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \oint_c \langle \mathbf{f}, \mathbf{T} \rangle\, ds$$

holds, where $\mathbf{T}(t) = \frac{\dot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|}$ denotes the tangent unit vector.

With the intergral thoerem of Green we obtain

$$\int_K \operatorname{curl} \mathbf{f}(\mathbf{x})\, d\mathbf{x} = \oint_{\partial K} \langle \mathbf{f}, \mathbf{T} \rangle\, ds$$

Is $\mathbf{f}(\mathbf{x})$ a velocity field, then the fluid motion described by $\mathbf{f}$ is curl free if $\operatorname{curl} \mathbf{f}(\mathbf{x}) = 0$, since

$$\oint_c \mathbf{f}(\mathbf{x}) d\mathbf{x}$$

is the circulation of $\mathbf{f}(\mathbf{x})$.