Another example.

$f: \mathbb{R}^2 \to \mathbb{R}$?

Consider the equation $g(x,y) = e^{y-x} + 3y + x^2 - 1 = 0$.

It is

$$\frac{\partial g}{\partial y}(x,y) = e^{y-x} + 3 > 0 \qquad \text{for all } x \in \mathbb{R}.$$

Therefore the equation con be solved fpr every $x \in \mathbb{R}$ with respect to $\boxed{y =: f(x)}$ and $f(x)$ is a continuous differentiable function. Implicit differentiation ives

$\frac{d}{dx}$ / $g(x, f(x)) = e^{f(x)-x} + 3f(x) + x^2 - 1 = 0$

$$e^{y-x}(\underset{f'}{y'} - 1) + 3y' + 2x = 0 \quad \Longrightarrow \quad y' = \frac{e^{y-x} - 2x}{e^{y-x} + 3}$$

Differentiating again gives

$\frac{d}{dx}\left(e^{y(x)-x}(y'-1) + 3y' + 2x\right) = 0$

$$e^{y-x}y'' + e^{y-x}(y'-1)^2 + 3y'' + 2 = 0 \quad \Longrightarrow \quad y'' = -\frac{2 + e^{y-x}(y'-1)^2}{e^{y-x} + 3}$$

But: Solving the equation with respect to y (in terms of elementary functions) is not possible in this case!

$$0 = f(x,y) \qquad x \in \mathbb{R}^{n-m}, \; y \in \mathbb{R}^{m}$$

$$\boxed{f(x_0,y_0) = 0}$$

Taylor

$$\boxed{0 = f(x,y)} = \underbrace{f(x_0,y_0)}_{=0} + \boxed{\underbrace{\frac{\partial f}{\partial y}(x_0,y_0)}_{m \times m \text{ Matrix Jacobian}}(y-y_0) + \underbrace{\frac{\partial f}{\partial x}(x_0,y_0)}_{m \times n-m \text{ Jacobian}}(x-x_0)} + \dots$$

if $\frac{\partial f}{\partial y}(x_0,y_0)$ is regular $\Longrightarrow$

$$\left(\frac{\partial f}{\partial y}(x_0,y_0)\right)^{-1}\left(\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)\right) + y - y_0 = 0$$

$$y = y_0 - \left(\frac{\partial f}{\partial y}\right)^{-1}\left(\frac{\partial f}{\partial x}\right)(x-x_0) \qquad \text{linearised version}$$

general remark.

Implicit differentiation of a implicitely defined function

$$g(x,y) = 0, \quad \frac{\partial g}{\partial y} \neq 0$$

$y = f(x)$, with $x, y \in \mathbb{R}$, gives

$0 = \frac{d}{dx}\left(g(x, y(x))\right) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' = 0$

$$f'(x) = -\frac{g_x}{g_y}$$

$$f''(x) = -\frac{g_{xx}g_y^2 - 2g_{xy}g_xg_y + g_{yy}g_x^2}{g_y^3}$$

$f'(x_0) = -\frac{g_x(x_0,y_0)}{g_y(x_0,y_0)} = 0$

Therefore the opint x^0 is a $\boxed{\text{stationary point of}}$ $f(x)$ if

$$g(x^0,y^0) = \boxed{g_x(x^0,y^0) = 0} \quad \text{and} \quad g_y(x^0,y^0) \neq 0$$

$y = f(x)$

And x^0 is a local maximum (minimum) if

$-f'' = \frac{g_{xx}g_y^2 - 0 + 0}{g_y^3}$

$$= \frac{g_{xx}(x^0,y^0)}{g_y(x^0,y^0)} > 0 \qquad \left(\text{bzw. } \frac{g_{xx}(x^0,y^0)}{g_y(x^0,y^0)} < 0\right)$$

$$g(x, y(x)) = 0$$

$$0 = \frac{d}{dx} g(x, y(x)) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\begin{pmatrix}\frac{dx}{dx} \\ \frac{dy}{dx}\end{pmatrix} = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\begin{pmatrix}1 \\ y'\end{pmatrix} =$$

$$= \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} y' = 0$$

$$g_y = \frac{\partial g}{\partial y}$$

$$g_x = \frac{\partial g}{\partial x}$$

$$g_x + g_y f' = 0 \Rightarrow f' = -\frac{g_x}{g_y}$$

$$0 = \frac{d}{dx}\left(g_x + g_y f'\right) = g_{xx} + g_{xy} f' + \left(g_{yx} + g_{yy} f'\right) f' + g_y f'' = 0$$

$$0 = g_{xx} - 2 g_{xy} \frac{g_x}{g_y} + g_{yy} \frac{g_x^2}{g_y^2} + g_y f'' = 0$$

$$\Rightarrow f'' = -\frac{g_{xx} g_y^2 - 2 g_{xy} g_y g_x + g_{yy} g_x^2}{g_y^3}$$

Implicit representation of curves.

Consider the set of solutions of a scalar equation

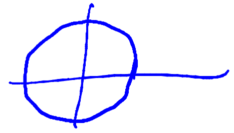
$$g(x,y) = 0$$

$g: \mathbb{R}^2 \to \mathbb{R}^1$

$g(x,y) = x^2+y^2+1 = 0$ no sol.

$g(x,y) = x^2+y^2 = 0$ $(x,y)=(0,0)$

$g(x,y) = x^2+y^2-1 = 0$



If

$$\operatorname{grad} g = (g_x, g_y) \neq 0$$

then $g(x,y)$ defines locally a function $y = f(x)$ or $x = \bar{f}(y)$.

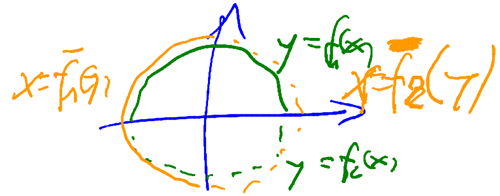
Definition: A solution point (x^0, y^0) of the equation $g(x,y) = 0$ with

- $\operatorname{grad} g(x^0, y^0) \neq 0$ is called regular point,
- $\operatorname{grad} g(x^0, y^0) = 0$ is called singular point.

Example: Consider (again) the equation for a circle

$\operatorname{grad} g = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \neq 0$ on $x^2+y^2-R^2=0$

$$g(x,y) = x^2 + y^2 - r^2 = 0 \quad \text{mit } r > 0.$$



on the circle there are no singular points!

Horizontal and vertical tangents.

Remarks:

a) If for a regular point (x^0, y^0) we have

$y = y(x)$ $\qquad y' = -\frac{g_x}{g_y}$

$$g_x(\mathbf{x}^0) = 0 \quad \text{und} \quad g_y(\mathbf{x}^0) \neq 0$$

$$\boxed{y'(x_0) = -\frac{g_x(x_0,y_0)}{g_y(x_0,y_0)} = 0}$$

then the set of solutions contains a horizontal tangent in $\mathbf{x}^0$.

b) If for a regular point (x^0, y^0) we have

$x = f(y)$ $\qquad x' = -\frac{g_y}{g_x}$

$$g_x(\mathbf{x}^0) \neq 0 \quad \text{und} \quad g_y(\mathbf{x}^0) = 0$$

$$\boxed{x'(y_0) = -\frac{g_y(x_0,y_0)}{g_x(x_0,y_0)} = 0}$$

then the set of solutions contains a vertical tangent in $\mathbf{x}^0$.

c) If $\mathbf{x}^0$ is a singular point, then the set of solutions is approximated at $\mathbf{x}^0$ "in second order" by the following quadratic equation

$$g_{xx}(\mathbf{x}^0)(x - x^0)^2 + 2g_{xy}(\mathbf{x}^0)(x - x^0)(y - y^0) + g_{yy}(\mathbf{x}^0)(y - y^0)^2 = 0$$

$$0 = g(x,y) = \underbrace{g(x_0,y_0)}_{=0} + \underbrace{\operatorname{grad} g(x_0,y_0)}_{=0} \cdot \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \frac{1}{2}(x-x_0, y-y_0) \begin{pmatrix} g_{xx} & g_{yx} \\ g_{xy} & g_{yy} \end{pmatrix}(x_0,y_0) \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} + \dots$$

Remarks.

Due to c) for $g_{xx}, g_{xy}, g_{yy} \neq 0$ we obtain:

$\det \mathbf{H}g(\mathbf{x}^0) > 0$: $\mathbf{x}^0$ is an isolated point of the set of solutions

$\det \mathbf{H}g(\mathbf{x}^0) < 0$: $\mathbf{x}^0$ is a double point

$\det \mathbf{H}g(\mathbf{x}^0) = 0$: $\mathbf{x}^0$ is a return point or a cusp

Geometric interpretation:

a) If $\det \mathbf{H}g(\mathbf{x}^0) > 0$, then both Eigenvalues of $\mathbf{H}g(\mathbf{x}^0)$ are or strictly positiv or strictly negativ, i.e. $\mathbf{x}^0$ is a strict local minimum or maximum of $g(\mathbf{x})$.

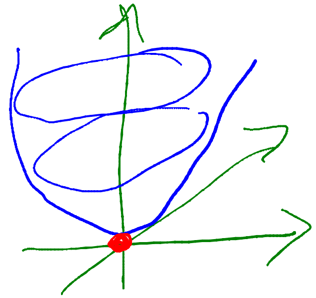
b) If $\det \mathbf{H}g(\mathbf{x}^0) < 0$, then both Eigenvalues of $\mathbf{H}g(\mathbf{x}^0)$ have opposite sign, i.e. $\mathbf{x}^0$ is a saddel point of $g(\mathbf{x})$.

c) If $\det \mathbf{H}g(\mathbf{x}^0) = 0$, then the stationary point $\mathbf{x}^0$ of $g(\mathbf{x})$ is degenerate.

$f(x,y) = x^2 + y^2 = 0$ $\qquad (x_0, y_0) = (0,0)$

$\operatorname{grad} f = 2\begin{pmatrix} x \\ y \end{pmatrix} = 0$ at $(0,0)$ $\Rightarrow$ singular point

$H_f = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\qquad \det H_f(0,0) > 0$ $\quad$ local min.

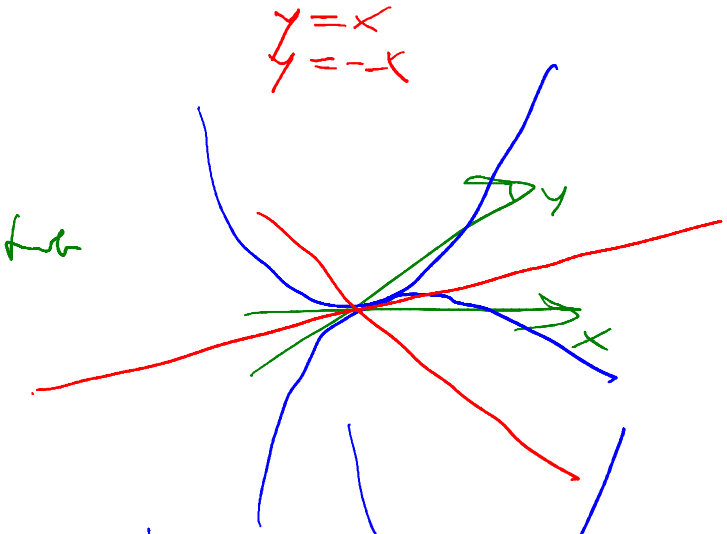


$f(x,y) = x^2 - y^2 = 0$ $\qquad (x_0, y_0) = (0,0)$

$\operatorname{grad} f = 2\begin{pmatrix} x \\ -y \end{pmatrix} = 0$ at $(0,0)$ $\quad$ singular point

$H_f = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $\qquad \det H_f(0,0) < 0$ $\quad$ double point

$\operatorname{grad} f(1,1) = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \neq 0$ $\quad$ regular point

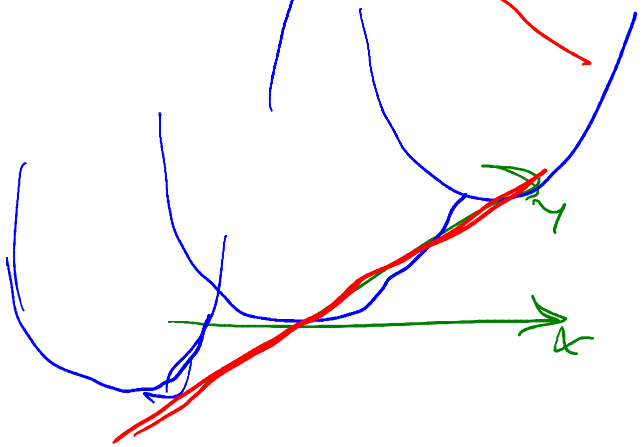


$f(x,y) = x^2 = 0$ $\qquad (x_0, y_0) = (0,0)$

$\operatorname{grad} f = 2\begin{pmatrix} x \\ 0 \end{pmatrix} = 0$ at $(0,0)$

$H_f = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ $\qquad \det H = 0$ $\quad$ degenerate

$\operatorname{grad} f = 2\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$ at $(0,y)$ $\quad$ all degenerate points



Example 1.

Consider the singular point $\mathbf{x}^0 = \mathbf{0}$ of the implicit equation

$$g(x, y) = y^2(x-1) + x^2(x-2) = 0$$

$g(0,0) = 0$

Calculate the partial derivatives up to order 2:

grad g

$$\begin{cases} g_x = y^2 + 3x^2 - 4x \\ g_y = 2y(x-1) \end{cases}$$

grad $g(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

Hg

$$\begin{cases} g_{xx} = 6x - 4 \\ g_{xy} = 2y \\ g_{yy} = 2(x-1) \end{cases}$$

$$\mathbf{H}g(\mathbf{0}) = \begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}$$

det $H > 0$

Therefore $\mathbf{x}^0 = \mathbf{0}$ is an isolated point.

$\varphi(x,y) = y^2(x-1) + x^2(x-2) = 0$

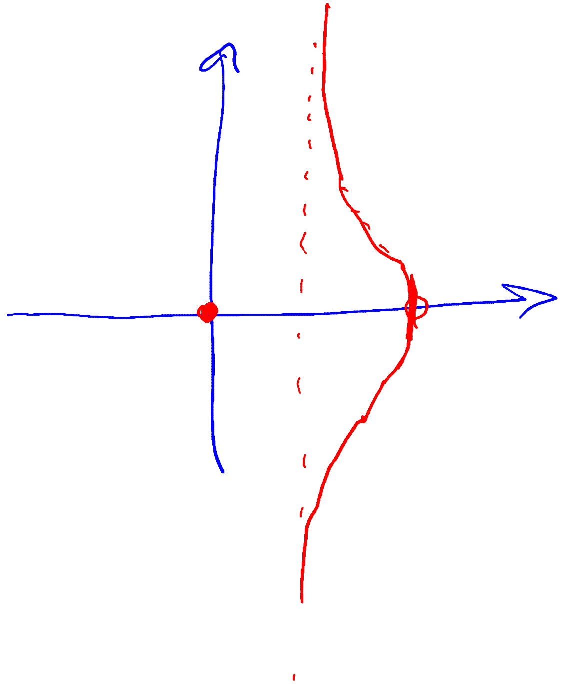
$x=0 \Rightarrow y=0$

$x=2 \Rightarrow y=0$

$y^2 = -\frac{x^2(x-2)}{x-1} > 0$ if $1 < x < 2$

$(x_0,y_0) = (2,0)$ $\quad \varphi(2,0) = 0$

$\operatorname{grad} \varphi = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \neq 0$ regular $\quad \frac{\partial \varphi}{\partial y} -$



Example 2.

Consider the singular point $\mathbf{x}^0 = \mathbf{0}$ of the implicit equation

$g(0,0)=0$

$$g(x,y) = y^2(x-1) + x^2(x+q^2) = 0$$

Calculate the partial derivatives up to order 2:

$\operatorname{grad} g(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ singular

$$\begin{aligned}
g_x &= y^2 + 3x^2 + 2xq^2 \\
g_y &= 2y(x-1) \\
g_{xx} &= 6x + 2q^2 \\
g_{xy} &= 2y \\
g_{yy} &= 2(x-1) \\
\mathbf{H}g(\mathbf{0}) &= \begin{pmatrix} 2q^2 & 0 \\ 0 & -2 \end{pmatrix}
\end{aligned}$$

$\det H < 0$ double point

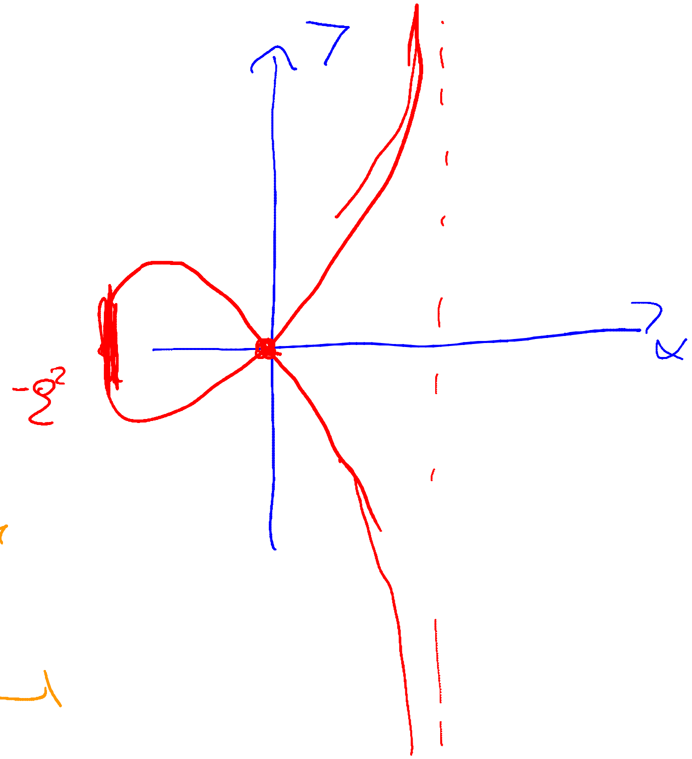
Therefore $\mathbf{x}^0 = \mathbf{0}$ is an double point.

$$f(x,y) = y^2(x-1) + x^2(x+\varepsilon^2) = 0$$

$x=0 \Rightarrow y=0$

$x=-\varepsilon^2 \Rightarrow y=0$

$$y^2 = -\frac{x^2(x+\varepsilon^2)}{x-1} \qquad -\varepsilon^2 \le x < 1$$



$(-\varepsilon^2, 0)$ $\quad \operatorname{grad} f(-\varepsilon^2, 0) = \begin{pmatrix} \varepsilon^4 \\ 0 \end{pmatrix} \neq 0$ regular

$\frac{\partial f}{\partial y} = 0$ at $(-\varepsilon^2, 0)$ vertical tangent

Example 3.

Consider the singular point $\mathbf{x}^0 = \mathbf{0}$ of the implicit equation

$$g(x, y) = y^2(x-1) + x^3 = 0$$

$g(0,0) = 0$

Calculate the partial derivatives up to order 2:

$$\begin{aligned}
g_x &= y^2 + 3x^2 \\
g_y &= 2y(x-1) \\
g_{xx} &= 6x \\
g_{xy} &= 2y \\
g_{yy} &= 2(x-1) \\
\mathbf{H}g(\mathbf{0}) &= \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}
\end{aligned}$$

$\operatorname{grad} g(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ singular.

$\det H = 0$

Therefore $\mathbf{x}^0 = \mathbf{0}$ is a cusp (or a return point).

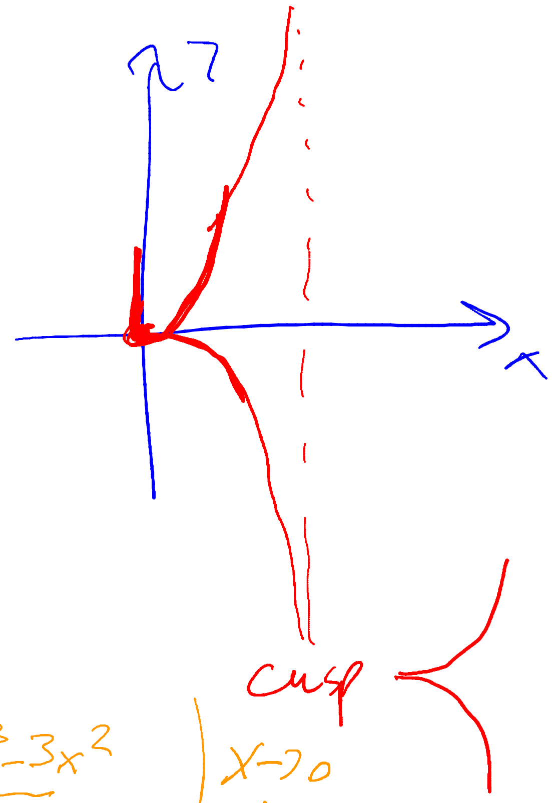
$$f(x,y) = y^2(x-1) + x^3 = 0$$

$x=0 \to y=0$

$$y^2 = -\frac{x^3}{x-1} \qquad 0 \le x < 1$$

$y^2 < 0 \quad x < 0$

$y^2 < 0 \quad x > 1$



$$\frac{x^3}{1-x} = y^2 \quad \text{monotone increasing}$$

$$\operatorname{grad} f = \begin{pmatrix} y^2 + 3x^2 \\ 2y(x-1) \end{pmatrix} = \begin{pmatrix} -\frac{x^3}{x-1} + 3x^2 \\ 2\sqrt{-\frac{x^3}{x-1}}\, x-1 \end{pmatrix} = \begin{pmatrix} \frac{-x^3+3x^3-3x^2}{x-1} \\ 2\sqrt{-x^4+x^3} \end{pmatrix} \underset{\approx}{x \to 0}$$

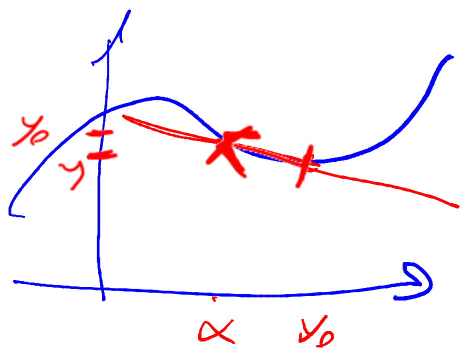
$$\begin{pmatrix} -3x^2 \\ 2\sqrt{x^3} \end{pmatrix} = \begin{pmatrix} -3\sqrt{x} \\ 2 \end{pmatrix} \xrightarrow{x \to 0} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$f(x,y) = 0 \qquad f(x_0,y_0) = 0$

$$0 = f(x,y) = \underbrace{f(x_0,y_0)}_{=0} + \underbrace{\operatorname{grad} f(x_0,y_0) \cdot \overbrace{\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}}_{=0} + \dots$$

if $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ $\overbrace{\text{become } \underline{\text{tangent vector}}}^{\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}$

$\operatorname{grad} f(x_0,y_0) \perp \text{tangent}$



Implicit representation of surfaces.

- The set of solutions of a scalar equation $g(x, y, z) = 0$ for $\operatorname{grad} g \neq \mathbf{0}$ is *locally* a surface in $\mathbb{R}^3$.
- For the tangential in $\mathbf{x}^0 = (x^0, y^0, z^0)^T$ with $g(\mathbf{x}^0) = 0$ and $\operatorname{grad} g(\mathbf{x}^0) \neq \mathbf{0}^T$ we obtain by Taylor expanding (denoting $\Delta\mathbf{x}^0 = \mathbf{x} - \mathbf{x}^0$)

$$0 = g(x) = \underbrace{g(x_0)}_{=0} + \operatorname{grad} g \cdot \Delta\mathbf{x}^0 = g_x(\mathbf{x}^0)(x - x^0) + g_y(\mathbf{x}^0)(y - y^0) + g_z(\mathbf{x}^0)(z - z_0) = 0$$

  i.e. the gradient is vertical to the surface $g(x, y, z) = 0$.
- If for example $g_z(\mathbf{x}^0) \neq 0$, then locally there exists a a representation at $\mathbf{x}^0$ of the form

$$z = f(x, y)$$

  and for the partial derivatives of $f(x, y)$ we obtain

$$\operatorname{grad} f(x, y) = (f_x, f_y) = -\frac{1}{g_z}(g_x, g_y) = \left(-\frac{g_x}{g_z}, \frac{g_y}{g_z}\right)$$

  using the implicit function theorem.

$$0 = f(x,y,z(x,y))$$

$$0 = \operatorname{grad}_{(x,y)} f = \begin{pmatrix} f_x + f_z z_x \\ f_y + f_z z_y \end{pmatrix} = \frac{1}{f_z} \begin{pmatrix} \frac{f_x}{f_z} + z_x \\ \frac{f_y}{f_z} + z_y \end{pmatrix}$$

$$\begin{pmatrix} z_x \\ z_y \end{pmatrix} = 0 - \begin{pmatrix} \frac{f_x}{f_z} \\ \frac{f_y}{f_z} \end{pmatrix}$$