# Directional derivative.

**Definition:** Let $f : D \to \mathbb{R}$, $D \subset \mathbb{R}^n$ open, $\mathbf{x}^0 \in D$, and $\mathbf{v} \in \mathbb{R}^n \setminus \{0\}$ a vector. Then

scalar

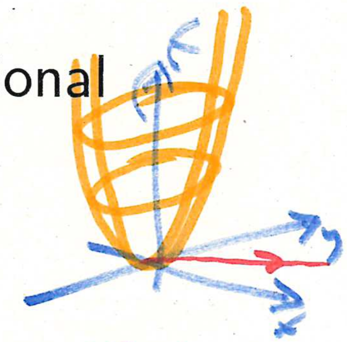
$$D_{\mathbf{v}} f(\mathbf{x}^0) := \lim_{t \to 0} \frac{f(\mathbf{x}^0 + t\mathbf{v}) - f(\mathbf{x}^0)}{t}$$

special case: $v = e_i$

$D_v f(x^0) = \frac{\partial f}{\partial x_i}(x^0)$

is called the directional derivative (Gateaux–derivative) of $f(\mathbf{x})$ in the direction of $\mathbf{v}$.

**Example:** Let $f(x, y) = x^2 + y^2$ and $\mathbf{v} = (1, 1)^T$. Then the directional derivative in the direction of $\mathbf{v}$ is given by:



$$\begin{aligned} D_{\mathbf{v}} f(x, y) &= \lim_{t \to 0} \frac{(x + 1 \cdot t)^2 + (y + 1 \cdot t)^2 - x^2 - y^2}{t} \\ &= \lim_{t \to 0} \frac{2xt + t^2 + 2yt + t^2}{t} \\ &= 2(x + y) \end{aligned}$$

$\|v\|_2 = \sqrt{2} \neq 1$

$D_{\frac{v}{\sqrt{2}}} f(x^0) = \frac{2}{\sqrt{2}}(x+y) = \sqrt{2}(x+y)$

# Remarks.

- For $\mathbf{v} = \mathbf{e}_i$ the directional derivative in the direction of $\mathbf{v}$ is given by the partial derivative with respect to $x_i$: $e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ith

$$D_{\mathbf{v}} f(\mathbf{x}^0) = \frac{\partial f}{\partial x_i}(\mathbf{x}^0)$$

- If $\mathbf{v}$ is a unit vector, i.e. $\|\mathbf{v}\| = 1$, then the directional derivative $D_{\mathbf{v}} f(\mathbf{x}^0)$ describes the slope of $f(\mathbf{x})$ in the direction of $\mathbf{v}$.

- If $f(\mathbf{x})$ is differentiable in $\mathbf{x}^0$, then all directional derivatives of $f(\mathbf{x})$ in $\mathbf{x}^0$ exist. With $\mathbf{h}(t) = \mathbf{x}^0 + t\mathbf{v}$ we have

$h'(t) = v \quad h(0) = x^0$

$$D_{\mathbf{v}} f(\mathbf{x}^0) \underset{(*)}{=} \frac{d}{dt}(f \circ \mathbf{h})|_{t=0} = \operatorname{grad} f(\mathbf{x}^0) \cdot \mathbf{v}$$

This follows directely applying the chain rule.

$$(*) = \lim_{h \to 0} \frac{f(x^0 + hv) - f(x^0)}{h} = \lim_{h \to 0} \frac{f(x^0 + tv + hv) - f(x^0 + tv)}{h}\Big|_{t=0} = \frac{d}{dt}(f \circ h)\Big|_{t=0}$$

# Properties of the gradient.

**Theorem:** Let $D \subset \mathbb{R}^n$ open, $f : D \to \mathbb{R}$ differentiable in $\mathbf{x}^0 \in D$. Then we have

a) The gradient vector $\operatorname{grad} f(\mathbf{x}^0) \in \mathbb{R}^n$ is orthogonal in the level set

$$N_{\mathbf{x}^0} := \{\mathbf{x} \in D \,|\, f(\mathbf{x}) = f(\mathbf{x}^0)\}$$

In the case of $n = 2$ we call the level sets contour lines, in $n = 3$ we call the level sets equipotential surfaces.

2) The gradient $\operatorname{grad} f(\mathbf{x}^0)$ gives the direction of the steepest slope of $f(\mathbf{x})$ in $\mathbf{x}^0$.

**Idea of the proof:** a) curve $h = h(t)$, $h(t) \in N_{x_0}$ $\forall t_s \in U$

$h(t_s) = x_s \in N_{x_0}$ & $h'(t_s)$ is tangential to $N_{x_0}$ in $x_s$

we know $D_v f(x_s) = 0 = \operatorname{grad} f(x_s) \cdot v$

$\perp$

a) application of the chain rule.

b) for an arbitrary direction $\mathbf{v}$ we conclude with the Cauchy–Schwarz inequality

$\|v\|_2 = 1$

$$|D_{\mathbf{v}} f(\mathbf{x}^0)| = |(\operatorname{grad} f(\mathbf{x}^0), \mathbf{v})| \;\leq\; \|\operatorname{grad} f(\mathbf{x}^0)\|_2 \underbrace{\|v\|_2}_{=1}$$

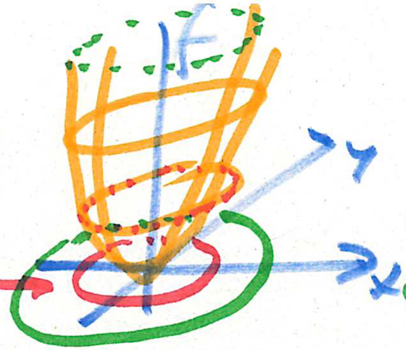
Equality is obtained for $\mathbf{v} = \operatorname{grad} f(\mathbf{x}^0)/\|\operatorname{grad} f(\mathbf{x}^0)\|_2$. $\|v\| = 1$

Level set

$n=2 \qquad f(x,y) = x^2 + y^2$

$N_{(1,0)} = \{ x \mid f(x,y) = f(1,0) = 1 \}$

$N_{(2,2)} = \{ x \mid f(x,y) = f(2,2) = 8 \}$

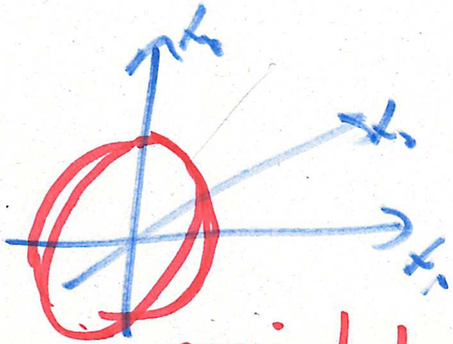


contour line

contour line

$n=2 \qquad f(x_1,x_2,x_3) = r(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$N_{(1,0,0)} = \{ x \mid f(x_1,x_2,x_3) = f(1,0,0) = 1 \}$



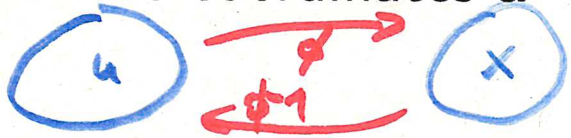
equipotential surface

# Curvilinear coordinates.

**Definition:** Let $U, V \subset \mathbb{R}^n$ be open and $\Phi : U \to V$ be a $\mathcal{C}^1$-map, for which the Jacobimatrix $\mathbf{J}\Phi(\mathbf{u}^0)$ is regular (invertible) at every $\mathbf{u}^0 \in U$.

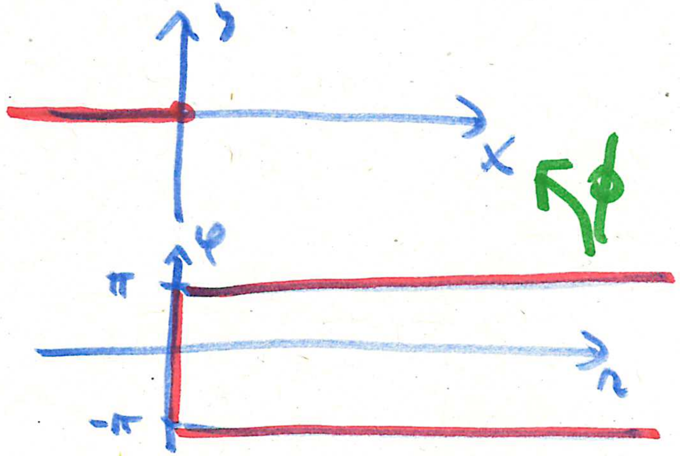
In addition there exists the inverse map $\Phi^{-1} : V \to U$ and the inverse map is also a $\mathcal{C}^1$–map.

Then $\mathbf{x} = \Phi(\mathbf{u})$ defines a coodinate transformation from the coordinates $\mathbf{u}$ to $\mathbf{x}$.




**Example:** Consider for $n = 2$ the polar coordinates $\mathbf{u} = (r, \varphi)$ with $r > 0$ and $-\pi < \varphi < \pi$ and set

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}$$




with the cartesian coordinates $\mathbf{x} = (x, y)$.

# Calculation of the partial derivatives.

For all $\mathbf{u} \in U$ with $\mathbf{x} = \mathbf{\Phi}(\mathbf{u})$ the following relations hold

$\frac{\partial}{\partial u}$

$$\Phi^{-1}(\Phi(\mathbf{u})) = \mathbf{u}$$

$$\mathbf{J}\,\Phi^{-1}(\mathbf{x}) \cdot \mathbf{J}\,\Phi(\mathbf{u}) = \mathbf{I}_n \qquad \text{(chain rule)}$$

$$\mathbf{J}\,\Phi^{-1}(\mathbf{x}) = (\mathbf{J}\,\Phi(\mathbf{u}))^{-1}$$

Let $\tilde{f} : V \to \mathbb{R}$ be a given function. Set

$$f(\Phi^{-1}(x)) = f(\mathbf{u}) := \tilde{f}(\Phi(\mathbf{u})) = \tilde{f}(x)$$

the by using the chain rule we obtain

$$\frac{\partial f}{\partial u_i} = \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial x_j} \frac{\partial \Phi_j}{\partial u_i} =: \sum_{j=1}^{n} g^{ij} \frac{\partial \tilde{f}}{\partial x_j}$$

with

$$g^{ij} := \frac{\partial \Phi_j}{\partial u_i}, \qquad \mathbf{G}(\mathbf{u}) := (g^{ij}) = (\mathbf{J}\,\Phi(\mathbf{u}))^T$$

# Notations.

We use the short notation

$$\frac{\partial}{\partial u_i} = \sum_{j=1}^{n} g^{ij} \frac{\partial}{\partial x_j}$$

Analogously we can express the partial derivatives with respect to $x_i$ by the partial derivatives with respect to $u_j$

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{n} g_{ij} \frac{\partial}{\partial u_j}$$

where

$$(g_{ij}) := (g^{ij})^{-1} = (\mathbf{J}\,\Phi)^{-T} = (\mathbf{J}\,\Phi^{-1})^{T}$$

We obtain these relations by applying the chain rule on $\Phi^{-1}$.

# Example: polar coordinates.

We consider polar coordinates

$$\mathbf{x} = \Phi(\mathbf{u}) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

We calculate

$$\mathbf{J}\,\Phi(\mathbf{u}) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

and thus

$(J\phi(u))^T$ $\quad$ $(J\phi(u))^{-T}$

$$(g^{ij}) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ & \\ -r\sin\varphi & r\cos\varphi \end{pmatrix} \qquad (g_{ij}) = \begin{pmatrix} \cos\varphi & -\dfrac{1}{r}\sin\varphi \\ & \\ \sin\varphi & \dfrac{1}{r}\cos\varphi \end{pmatrix}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\cos\varphi \frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial}{\partial \varphi}\right)\left(\cos\varphi \frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial}{\partial \varphi}\right) =$$

$$= \cos^2\varphi \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\varphi\sin\varphi \frac{\partial}{\partial\varphi} - \frac{1}{r}\cos\varphi\sin\varphi \frac{\partial^2}{\partial r\,\partial\varphi}$$

$$+ \frac{1}{r}\sin^2\varphi \frac{\partial}{\partial r} + \frac{1}{r^2}\sin\varphi\cos\varphi \frac{\partial}{\partial\varphi} + \frac{1}{r^2}\sin^2\varphi \frac{\partial^2}{\partial\varphi^2}$$

# Partial derivatives for polar coordinates.

The calculation of the partial derivatives gives

$$\frac{\partial}{\partial x} = \cos\varphi\,\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\,\frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \sin\varphi\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\,\frac{\partial}{\partial \varphi}$$

$$\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix} = (J_\Phi^T)^{-1}\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial \varphi}\end{pmatrix}$$

**Example:** Calculation of the Laplacian–operator in polar coordinates

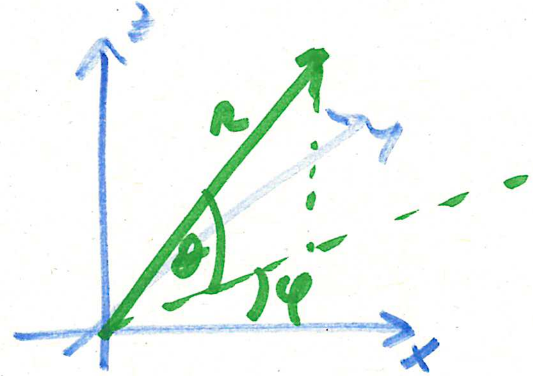
$$\frac{\partial^2}{\partial x^2} = \cos^2\varphi\,\frac{\partial^2}{\partial r^2} - \cancel{\frac{\sin(2\varphi)}{2r}\,\frac{\partial^2}{\partial r\partial\varphi}} + \frac{\sin^2\varphi}{r^2}\,\frac{\partial^2}{\partial\varphi^2} + \cancel{\frac{\sin(2\varphi)}{r^2}\,\frac{\partial}{\partial\varphi}} + \frac{\sin^2\varphi}{r}\,\frac{\partial}{\partial r}$$

$$\frac{\partial^2}{\partial y^2} = \sin^2\varphi\,\frac{\partial^2}{\partial r^2} + \cancel{\frac{\sin(2\varphi)}{2r}\,\frac{\partial^2}{\partial r\partial\varphi}} + \frac{\cos^2\varphi}{r^2}\,\frac{\partial^2}{\partial\varphi^2} - \cancel{\frac{\sin(2\varphi)}{r^2}\,\frac{\partial}{\partial\varphi}} + \frac{\cos^2\varphi}{r}\,\frac{\partial}{\partial r}$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r}\,\frac{\partial}{\partial r}$$

# Example: spherical coordinates.

We consider spherical coordinates




$$\mathbf{x} = \Phi(\mathbf{u}) = \begin{pmatrix} r\cos\varphi\cos\theta \\ r\sin\varphi\cos\theta \\ r\sin\theta \end{pmatrix}$$

The Jacobian–matrix is given by:

$$\mathbf{J}\,\Phi(\mathbf{u}) = \begin{pmatrix} \cos\varphi\cos\theta & -r\sin\varphi\cos\theta & -r\cos\varphi\sin\theta \\ \sin\varphi\cos\theta & r\cos\varphi\cos\theta & -r\sin\varphi\sin\theta \\ \sin\theta & 0 & r\cos\theta \end{pmatrix}$$

$(J\,\Phi(u))^T$, $\quad (J\,\Phi(u))^{-T}$

# Partial derivatives for spherical coordinates.

Calculating the partial derivatives gives

$$\frac{\partial}{\partial x} = \cos\varphi\,\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r\cos\theta}\,\frac{\partial}{\partial\varphi} - \frac{1}{r}\cos\varphi\,\sin\theta\,\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \sin\varphi\,\cos\theta\,\frac{\partial}{\partial r} + \frac{\cos\varphi}{r\cos\theta}\,\frac{\partial}{\partial\varphi} - \frac{1}{r}\sin\varphi\,\sin\theta\,\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial z} = \sin\theta\,\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\,\frac{\partial}{\partial\theta}$$

**Example:** calculation of the Laplace–operator in spherical coordinates

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2\cos^2\theta}\,\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2} + \frac{2}{r}\,\frac{\partial}{\partial r} - \frac{\tan\theta}{r^2}\,\frac{\partial}{\partial\theta}$$