Vector valued functions.

Definition: Let $D \subset \mathbb{R}^n$ be open and let $\mathbf{f} : D \to \mathbb{R}^m$ be a vector valued function.

The function $\mathbf{f}$ is called partial differentiable on $\mathbf{x}^0 \in D$, if for all $i = 1, \ldots, n$ the limits

$$\frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}^0) = \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{x}^0 + t\mathbf{e}_i) - \mathbf{f}(\mathbf{x}^0)}{t}$$

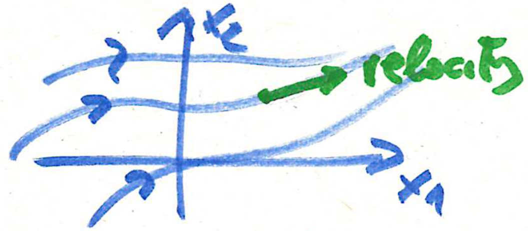
exist. The calculation is done componentwise

$$\frac{\partial \mathbf{f}}{\partial x_i}(\mathbf{x}^0) = \begin{pmatrix} \dfrac{\partial f_1}{\partial x_i} \\ \vdots \\ \dfrac{\partial f_m}{\partial x_i} \end{pmatrix} \qquad \text{for } i = 1, \ldots, n$$

Vectorfields.

Definition: If $m = n$ the function $\mathbf{f} : D \to \mathbb{R}^n$ is called a vectorfield on D. If every (coordinate-) function $f_i(\mathbf{x})$ of $\mathbf{f} = (f_1, \ldots, f_n)^T$ is a $\mathcal{C}^k$–function, then $\mathbf{f}$ is called $\mathcal{C}^k$–vectorfield.

$n=2$



Examples of vectorfields:

- velocity fields of liquids or gases;
- elektromagnetic fields;
- temperature gradients in solid states.

Definition: Let $\mathbf{f} : D \to \mathbb{R}^n$ be a partial differentiable vector field. The divergence on $\mathbf{x} \in D$ is defined as

$$\operatorname{div} \mathbf{f}(\mathbf{x}^0) := \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(\mathbf{x}^0)$$

or

$$\operatorname{div} \mathbf{f}(\mathbf{x}) = \nabla^T \mathbf{f}(\mathbf{x}) = (\nabla, \mathbf{f}(\mathbf{x})) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots \frac{\partial}{\partial x_n}\right) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Rules of computation and the rotation.

The following rules hold true:

$$\operatorname{div}(\alpha\,\mathbf{f} + \beta\,\mathbf{g}) = \alpha \operatorname{div}\mathbf{f} + \beta \operatorname{div}\mathbf{g} \quad \text{for } \mathbf{f}, \mathbf{g} : D \to \mathbb{R}^n$$

$$\operatorname{div}(\varphi \cdot \mathbf{f}) = (\nabla\varphi, \mathbf{f}) + \varphi \operatorname{div}\mathbf{f} \quad \text{for } \varphi : D \to \mathbb{R}, \mathbf{f} : D \to \mathbb{R}^n$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\varphi f_i) = \sum_{i=1}^{n} \varphi \frac{\partial}{\partial x_i} f_i + f_i \frac{\partial}{\partial x_i}\varphi = \varphi \operatorname{div} f + (f, \nabla\varphi)$$

Remark: Let $f : D \to \mathbb{R}$ be a C^2–function, then for the Laplacian we have

$$\Delta f = \operatorname{div}(\nabla f) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_i} f\right) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f$$

Definition: Let $D \subset \mathbb{R}^3$ open and $\mathbf{f} : D \to \mathbb{R}^3$ a partial differentiable vector field. We define the rotation as

$$\overset{\text{curl}}{\cancel{\operatorname{rot}}}\,\mathbf{f}(\mathbf{x}^0) := \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right)^T \Bigg|_{\mathbf{x}^0}$$

Alternative notations and additional rules.

$$\overset{\text{curl}}{\cancel{\mathrm{rot}}}\,\mathbf{f}(\mathbf{x}) = \nabla \times \mathbf{f}(\mathbf{x}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix} = \begin{pmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{pmatrix}$$

Remark: The following rules hold true:

$$\begin{aligned} \overset{\text{curl}}{\cancel{\mathrm{rot}}}\,(\alpha\,\mathbf{f} + \beta\,\mathbf{g}) &= \alpha\,\overset{\text{curl}}{\cancel{\mathrm{rot}}}\,\mathbf{f} + \beta\,\overset{\text{curl}}{\cancel{\mathrm{rot}}}\,\mathbf{g} \\ \text{curl}\ \cancel{\mathrm{rot}}\,(\varphi \cdot \mathbf{f}) &= (\nabla\varphi) \times \mathbf{f} + \varphi\,\overset{\text{curl}}{\cancel{\mathrm{rot}}}\,\mathbf{f} \end{aligned}$$

Remark: Let $D \subset \mathbb{R}^3$ and $\varphi : D \to \mathbb{R}$ be a $\underline{\mathcal{C}^2}$–function. Then

$$\overset{\text{curl}}{\cancel{\mathrm{rot}}}\,(\nabla\varphi) = 0\,,$$

using the exchangeability theorem of Schwarz. I.e. gradient fileds are rotation-free everywhere.

$$\operatorname{curl}(\varphi \cdot f) = \left(\frac{\partial(\varphi f_3)}{\partial x_2} - \frac{\partial(\varphi f_2)}{\partial x_3}, \frac{\partial(\varphi f_1)}{\partial x_3} - \frac{\partial(\varphi f_3)}{\partial x_1}, \frac{\partial(\varphi f_2)}{\partial x_1} - \frac{\partial(\varphi f_1)}{\partial x_2} \right) =$$

$$= \varphi \operatorname{curl} f + \underbrace{\left(f_3 \frac{\partial \varphi}{\partial x_2} - f_2 \frac{\partial \varphi}{\partial x_3}, f_1 \frac{\partial \varphi}{\partial x_3} - f_3 \frac{\partial \varphi}{\partial x_1}, f_2 \frac{\partial \varphi}{\partial x_1} - f_1 \frac{\partial \varphi}{\partial x_2} \right)}_{\nabla \varphi \times f}$$

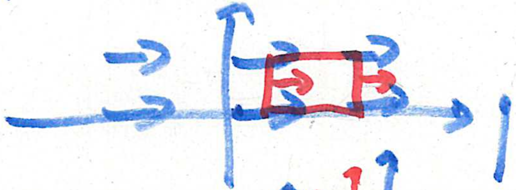
$$\operatorname{curl}(\nabla \varphi) = \left(\underbrace{\frac{\partial}{\partial x_2}\left(\frac{\partial}{\partial x_3}\varphi\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial}{\partial x_2}\varphi\right)}_{=0 \text{ if } \varphi \in C^2 \text{ (Schwarz)}}, \quad \dots \quad , \quad \right)$$

$=$

divergence is a measure of sources and sinks

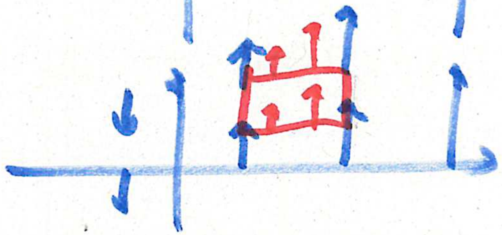
$$f(x_1,x_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\mathrm{div} f = 0$$



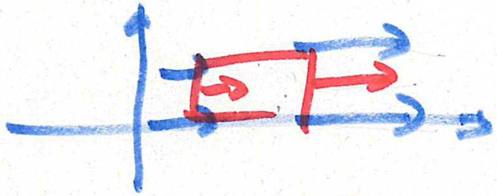
$$f(x_1,x_2) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$$

$$\mathrm{div} f = 0$$



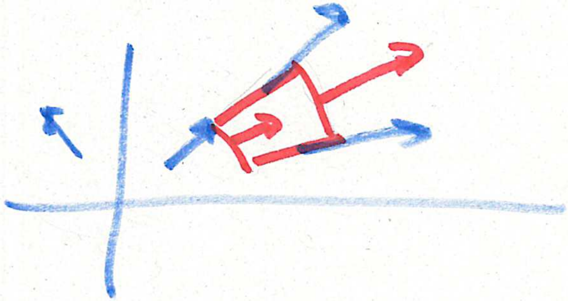
$$f(x_1,x_2) = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$$ source!

$$\mathrm{div} f = 1$$



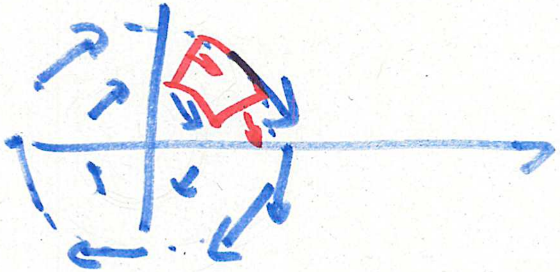
$$f(x_1,x_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mathrm{div} f = 1+1 = 2$$

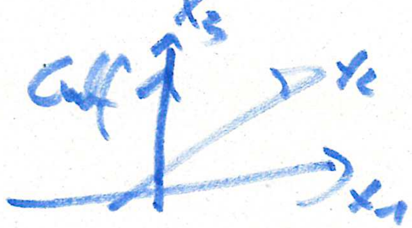


$$f(x_1,x_2) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$$

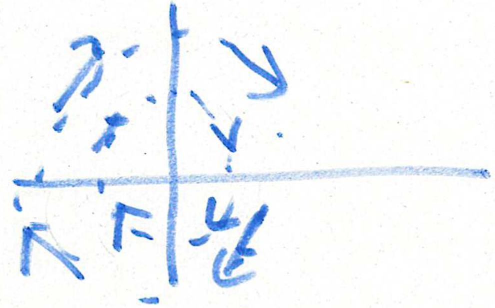
$$\mathrm{div} f = 0$$



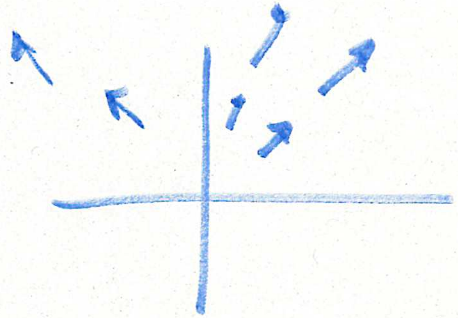
$$f = f(x_1, x_2, x_3) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \\ f_3 \end{pmatrix} \qquad \text{curl}\, f = \left(0, 0, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$



$$f = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \qquad \text{curl}\, f = -1 - 1 = -2$$



$$f = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \qquad \text{curl}\, f = 0$$



$$f = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{curl}\, f = 0$$