Adaptive FEM for PDE Constrained Optimization with Pointwise State Constraints

Winnifried Wollner

Workshop
PDE Constrained Optimization - recent challenges and future developments
Hamburg, March 27 - 29, 2008

Overview

1. Problem Formulation
2. A Posteriori Estimates
3. Numerical Example

Optimization Problem

$$\text{Minimize } J(q,u) = \frac{1}{2}\|u - u^d\|^2 + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)} \quad q \in Q_{\text{ad}}$$

$$\text{s.t.} \begin{cases} -\Delta u = q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\text{and } u \leq \psi \text{ on } \overline{\Omega}$$

With nice $\Omega \subset \mathbb{R}^n$, $n \leq 3$, $\alpha \geq 0$, $\psi > 0$ and at least one of the following holds:

- $\alpha > 0$,
- Q_{ad} is bounded in $L^2(\Omega)$.

Then there exists a (unique) solution $(\overline{q}, \overline{u}) \in L^2(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega) = Q \times W$ to this problem.

Necessary Conditions

Let $(\overline{q}, \overline{u})$ be a solution to the optimization problem, then there exist

- $\overline{z} \in L^2(\Omega) = Q$
- $\overline{\mu} \in M(\overline{\Omega}) = C^0(\overline{\Omega})^*$

such that:

$$
\begin{aligned}
(-\Delta \overline{u}, \varphi) &= (\overline{q}, \varphi) && \forall \varphi \in Q \\
(-\Delta \varphi, \overline{z}) &= (\overline{u} - u^d, \varphi) + \langle \varphi, \overline{\mu} \rangle && \forall \varphi \in W \\
\langle \overline{u}, \overline{\mu} \rangle &\geq \langle \varphi, \overline{\mu} \rangle && \forall \varphi \leq \psi \\
(\overline{z}, q - \overline{q}) &\geq (-\alpha \overline{q}, q - \overline{q}) && \forall q \in Q_{\text{ad}}
\end{aligned}
$$

For simplicity assume inactive control constraints.

Necessary Conditions

Let $(\overline{q}, \overline{u})$ be a solution to the optimization problem, then there exist

- $\overline{z} \in L^2(\Omega) = Q$
- $\overline{\mu} \in M(\overline{\Omega}) = C^0(\overline{\Omega})^*$

such that:

$$
\begin{aligned}
(-\Delta \overline{u}, \varphi) &= (\overline{q}, \varphi) && \forall \varphi \in Q \\
(-\Delta \varphi, \overline{z}) &= (\overline{u} - u^d, \varphi) + \langle \varphi, \overline{\mu} \rangle && \forall \varphi \in W \\
\langle \overline{u}, \overline{\mu} \rangle &\geq \langle \varphi, \overline{\mu} \rangle && \forall \varphi \leq \psi \\
(\overline{z}, q) &= (-\alpha \overline{q}, q) && \forall q \in Q
\end{aligned}
$$

For simplicity assume inactive control constraints.

Barrier Problem

Consider the barrier problem:

$$\text{Minimize } J_\gamma(q,u) = J(q,u) + b(u;\gamma) \quad q \in Q_{\mathrm{ad}}$$
$$\text{s.t.} \begin{cases} -\Delta u = q & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with the barrier functional:

$$b(v;\gamma) = \int_\Omega \frac{\gamma^m}{(m-1)(\psi - v)^{m-1}}\, dx$$

with order $m > 1$ which we assume to be sufficiently large.
Then there exists a (unique) solution $(\overline{q}_\gamma, \overline{u}_\gamma) \in Q_{\mathrm{ad}} \times W$.

Necessary Conditions

Let $(\overline{q}_\gamma, \overline{u}_\gamma)$ be a solution to the barrier problem then there exist

- $\overline{z}_\gamma \in H_0^1(\Omega) = V$
- $\overline{\mu}_\gamma = \frac{\psi\gamma^m}{(\psi - \overline{u}_\gamma)^m} \in L^2(\Omega)$

such that:

$$
\begin{aligned}
(\nabla \overline{u}_\gamma, \nabla\varphi) &= (\overline{q}_\gamma, \varphi) && \forall \varphi \in V \\
(\nabla\varphi, \nabla \overline{z}_\gamma) &= (\overline{u}_\gamma - u^d, \varphi) + (\varphi, \overline{\mu}_\gamma) && \forall \varphi \in V \\
(\overline{z}_\gamma, q - \overline{q}_\gamma) &\geq (-\alpha \overline{q}_\gamma, q - \overline{q}_\gamma) && \forall q \in Q_{\text{ad}}
\end{aligned}
$$

For simplicity assume inactive control constraints. In addition we get:

$$
\begin{aligned}
(-\Delta \overline{u}_\gamma, \varphi) &= (\overline{q}_\gamma, \varphi) && \forall \varphi \in Q \\
(-\Delta\varphi, \overline{z}_\gamma) &= (\overline{u}_\gamma - u^d, \varphi) + \langle \varphi, \overline{\mu}_\gamma \rangle && \forall \varphi \in W \\
(\overline{z}_\gamma, q) &= (-\alpha \overline{q}_\gamma, q) && \forall q \in Q
\end{aligned}
$$

Necessary Conditions

Let $(\overline{q}_\gamma, \overline{u}_\gamma)$ be a solution to the barrier problem then there exist

- $\overline{z}_\gamma \in H_0^1(\Omega) = V$
- $\overline{\mu}_\gamma = \frac{\psi\gamma^m}{(\psi - \overline{u}_\gamma)^m} \in L^2(\Omega)$

such that:

$$
\begin{aligned}
(\nabla \overline{u}_\gamma, \nabla \varphi) &= (\overline{q}_\gamma, \varphi) && \forall \varphi \in V \\
(\nabla \varphi, \nabla \overline{z}_\gamma) &= (\overline{u}_\gamma - u^d, \varphi) + (\varphi, \overline{\mu}_\gamma) && \forall \varphi \in V \\
(\overline{z}_\gamma, q) &= (-\alpha \overline{q}_\gamma, q) && \forall q \in Q
\end{aligned}
$$

For simplicity assume inactive control constraints. In addition we get:

$$
\begin{aligned}
(-\Delta \overline{u}_\gamma, \varphi) &= (\overline{q}_\gamma, \varphi) && \forall \varphi \in Q \\
(-\Delta \varphi, \overline{z}_\gamma) &= (\overline{u}_\gamma - u^d, \varphi) + \langle \varphi, \overline{\mu}_\gamma \rangle && \forall \varphi \in W \\
(\overline{z}_\gamma, q) &= (-\alpha \overline{q}_\gamma, q) && \forall q \in Q
\end{aligned}
$$

Necessary Conditions

Let $(\overline{q}_\gamma, \overline{u}_\gamma)$ be a solution to the barrier problem then there exist

- $\overline{z}_\gamma \in H_0^1(\Omega) = V$
- $\overline{\mu}_\gamma = \frac{\psi\gamma^m}{(\psi - \overline{u}_\gamma)^m} \in L^2(\Omega)$

such that:

$$
\begin{aligned}
(\nabla \overline{u}_\gamma, \nabla \varphi) &= (\overline{q}_\gamma, \varphi) && \forall \varphi \in V \\
(\nabla \varphi, \nabla \overline{z}_\gamma) &= (\overline{u}_\gamma - u^d, \varphi) + (\varphi, \overline{\mu}_\gamma) && \forall \varphi \in V \\
(\overline{z}_\gamma, q) &= (-\alpha \overline{q}_\gamma, q) && \forall q \in Q
\end{aligned}
$$

For simplicity assume inactive control constraints. In addition we get:

$$
\begin{aligned}
(-\Delta \overline{u}_\gamma, \varphi) &= (\overline{q}_\gamma, \varphi) && \forall \varphi \in Q \\
(-\Delta \varphi, \overline{z}_\gamma) &= (\overline{u}_\gamma - u^d, \varphi) + \langle \varphi, \overline{\mu}_\gamma \rangle && \forall \varphi \in W \\
(\overline{z}_\gamma, q) &= (-\alpha \overline{q}_\gamma, q) && \forall q \in Q
\end{aligned}
$$

Use standard Q_1 elements for discretization.
Then there exists a (unique) solution $(\overline{q}_\gamma^h, \overline{u}_\gamma^h) \in Q_{\mathrm{ad}}^h \times V^h$ and:

- $\overline{z}_\gamma^h \in V^h$
- $\overline{\mu}_\gamma^h = \frac{\psi\gamma^o}{(\psi - \overline{u}_\gamma^h)^o} \in L^2(\Omega)$

such that:

$$
\begin{aligned}
(\nabla \overline{u}_\gamma^h, \nabla\varphi) &= (\overline{q}_\gamma^h, \varphi) && \forall \varphi \in V^h \\
(\nabla\varphi, \nabla \overline{z}_\gamma^h) &= (\overline{u}_\gamma^h - u^d, \varphi) + (\varphi, \overline{\mu}_\gamma^h) && \forall \varphi \in V^h \\
(\overline{z}_\gamma^h, q - \overline{q}_\gamma^h) &\geq (-\alpha \overline{q}_\gamma^h, q - \overline{q}_\gamma^h) && \forall q \in Q_{\mathrm{ad}}^h
\end{aligned}
$$

Again for inactive control constraints.

Discretization

Use standard Q_1 elements for discretization.
Then there exists a (unique) solution $(\overline{q}_\gamma^h, \overline{u}_\gamma^h) \in Q_{\text{ad}}^h \times V^h$ and:

- $\overline{z}_\gamma^h \in V^h$
- $\overline{\mu}_\gamma^h = \frac{\psi\gamma^o}{(\psi - \overline{u}_\gamma^h)^o} \in L^2(\Omega)$

such that:

$$
\begin{aligned}
(\nabla \overline{u}_\gamma^h, \nabla \varphi) &= (\overline{q}_\gamma^h, \varphi) && \forall \varphi \in V^h \\
(\nabla \varphi, \nabla \overline{z}_\gamma^h) &= (\overline{u}_\gamma^h - u^d, \varphi) + (\varphi, \overline{\mu}_\gamma^h) && \forall \varphi \in V^h \\
(\overline{z}_\gamma^h, q) &= (-\alpha \overline{q}_\gamma^h, q) && \forall q \in Q_{\text{ad}}^h
\end{aligned}
$$

Again for inactive control constraints.

Overview

Error in the Cost Functional

Splitting the error:

$$J(\overline{q},\overline{u}) - J_\gamma(\overline{q}_\gamma^h,\overline{u}_\gamma^h) = \Big(\underbrace{J(\overline{q},\overline{u}) - J_\gamma(\overline{q}_\gamma,\overline{u}_\gamma)}_{\approx \eta_{\mathrm{hom}}}\Big) + \Big(\underbrace{J(\overline{q}_\gamma,\overline{u}_\gamma) - J_\gamma(\overline{q}_\gamma^h,\overline{u}_\gamma^h)}_{\approx \eta_{\mathrm{disc}}}\Big)$$

Let

$$\begin{aligned} \xi &= (\overline{q},\overline{u},\overline{z},\overline{\mu}) \\ \xi_\gamma &= (\overline{q}_\gamma,\overline{u}_\gamma,\overline{z}_\gamma,\overline{\mu}_\gamma) \\ \xi_\gamma^h &= (\overline{q}_\gamma^h,\overline{u}_\gamma^h,\overline{z}_\gamma^h,\overline{\mu}_\gamma^h) \end{aligned}$$

be the solutions to the three systems of necessary conditions.

Error in the Cost Functional

Splitting the error:

$$J(\overline{q},\overline{u}) - J_\gamma(\overline{q}^h_\gamma,\overline{u}^h_\gamma) = \underbrace{\Big(J(\overline{q},\overline{u}) - J_\gamma(\overline{q}_\gamma,\overline{u}_\gamma)\Big)}_{\approx \eta_{\mathrm{hom}}} + \underbrace{\Big(J(\overline{q}_\gamma,\overline{u}_\gamma) - J_\gamma(\overline{q}^h_\gamma,\overline{u}^h_\gamma)\Big)}_{\approx \eta_{\mathrm{disc}}}$$

Let

$$\begin{aligned} \xi &= (\overline{q},\overline{u},\overline{z},\overline{\mu}) \\ \xi_\gamma &= (\overline{q}_\gamma,\overline{u}_\gamma,\overline{z}_\gamma,\overline{\mu}_\gamma) \\ \xi^h_\gamma &= (\overline{q}^h_\gamma,\overline{u}^h_\gamma,\overline{z}^h_\gamma,\overline{\mu}^h_\gamma) \end{aligned}$$

be the solutions to the three systems of necessary conditions.

Estimating the Homotopy Error

Introduce the Lagrangian:

$$\mathcal{M}(q,u,z,\mu) = J(q,u) + (q,z) + (\Delta u, z) - \langle \psi - u, \mu \rangle$$

Using the NCs we obtain:

$$\begin{aligned} J(\overline{q},\overline{u}) - J_\gamma(\overline{q}_\gamma,\overline{u}_\gamma) &= \mathcal{M}(\xi) - \mathcal{M}(\xi_\gamma) - b(\overline{u}_\gamma;\gamma) \\ &= \frac{1}{2}\langle \overline{u}_\gamma - \overline{u}, \overline{\mu}_\gamma + \overline{\mu} \rangle - b(\overline{u}_\gamma;\gamma) \end{aligned}$$

Finally:

$$J(\overline{q},\overline{u}) - J_\gamma(\overline{q}_\gamma,\overline{u}_\gamma) \approx \frac{1}{2}\langle \overline{u}^h_\gamma - \psi, \overline{\mu}^h_\gamma \rangle - b(\overline{u}^h_\gamma;\gamma) = \eta_{\text{hom}}$$

Estimating the Discretization Error

Introduce the Lagrangian:

$$\mathcal{L}(q,u,z) = J_\gamma(q,u) + (q,z) - (\nabla u, \nabla z)$$

Using the NCs we obtain:

$$J_\gamma(\overline{q}_\gamma, \overline{u}_\gamma) - J_\gamma(\overline{q}_\gamma^h, \overline{u}_\gamma^h) = \frac{1}{2}\mathcal{L}'(\xi_\gamma^h)(\xi_\gamma - \tilde{\xi}^h)$$

with arbitrary $\tilde{\xi}^h \in Q^h \times V^h \times V^h$.
Use (with a higher order interpolant on a coarser mesh $I_{2h}^{(2)}$):

$$J_\gamma(\overline{q}_\gamma, \overline{u}_\gamma) - J_\gamma(\overline{q}_\gamma^h, \overline{u}_\gamma^h) \approx \frac{1}{2}\mathcal{L}'(\xi_\gamma^h)(I_{2h}^{(2)}\xi_\gamma^h - \xi_\gamma^h) = \eta_{\text{disc}}$$

Overview

Model Problem

From: Günther, Hinze

$$\min J(q,u) = \frac{1}{2}\|u-0.5\|^2 + \frac{1}{2}\|q-60\|^2,$$
$$(\nabla u, \nabla\varphi) + (u,\varphi) = (q,\varphi) \quad \forall\varphi \in H^1(\Omega),$$
$$0.45 \le u(x) \le \psi(x) \quad \text{in } \overline{\Omega},$$

on the domain $\Omega = (0,1)^2$ with data:

$$\psi(x) = \min\big(1, \max(0.5, 50\|x-(0.3,0.3)^T\|^2)\big)$$

And a reference value J^* is given as:

$$J^* = 1759.04733$$

Typical Mesh

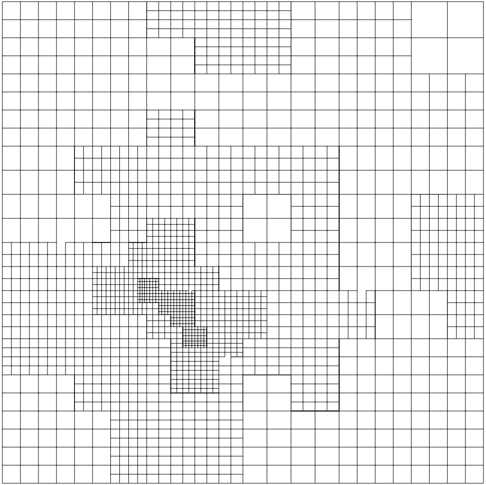


Figure: Mesh

State

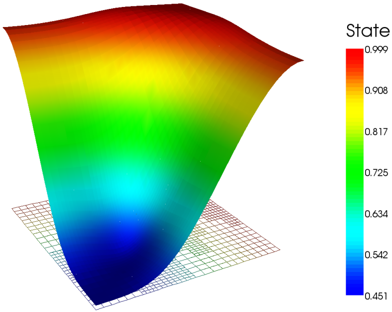


Figure: State

Control

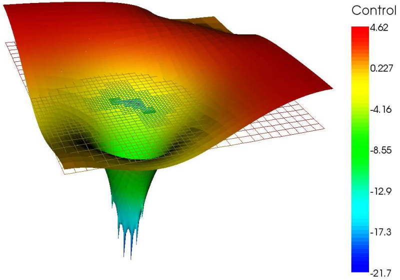


Figure: Control

Lagrange Multiplier for State Constraints

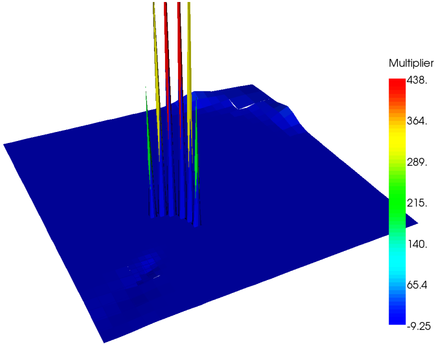


Figure: Lagrange Multiplier

Error in the Costfunctional

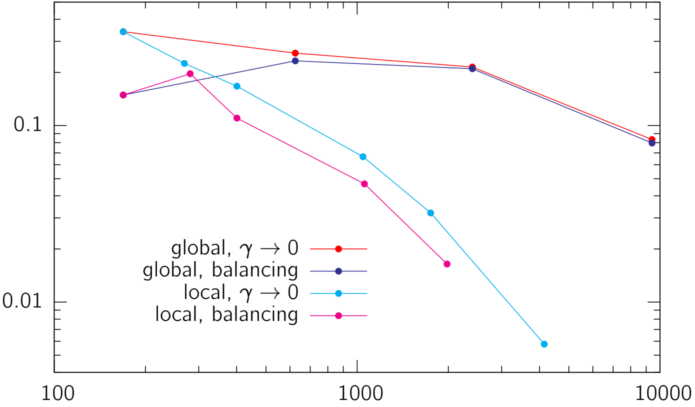


Figure: Error in the cost functional for different strategies

Effectivities

Table: Effectivity indices on globally refined meshes

γ	N		
	625	2401	9409
$3 \cdot 10^{-1}$	0.48	0.47	0.48
$1 \cdot 10^{-1}$	0.87	0.83	0.87
$3 \cdot 10^{-2}$	0.15	0.25	0.85
$1 \cdot 10^{-2}$	2.36	2.35	0.38
$3 \cdot 10^{-3}$	6.74	8.74	4.19
$1 \cdot 10^{-3}$	11.1	18.3	12.0

Transformed Problem

$$\min J(q,u) = \frac{1}{2}\|(0.45-\psi)u + \psi - 0.5\|^2 + \frac{1}{2}\|q-60\|^2,$$
$$(\nabla((0,45-\psi)u-\psi), \nabla\varphi) + ((0,45-\psi)u-\psi, \varphi) = (q,\varphi) \quad \forall \varphi \in H^1,$$
$$0 \le u(x) \le 1 \quad \text{in } \overline{\Omega},$$

on the domain $\Omega = (0,1)^2$.

Effectivities

Table: Effectivity indices on globally refined meshes

γ	N = 625	2401	9409
$3 \cdot 10^{-1}$	0.36	0.67	0.94
$1 \cdot 10^{-1}$	0.00	0.31	0.72
$3 \cdot 10^{-2}$	1.29	1.60	0.12
$1 \cdot 10^{-2}$	1.52	2.30	1.03
$3 \cdot 10^{-3}$	1.58	2.58	1.54
$1 \cdot 10^{-3}$	1.60	2.66	1.71

Effectivities

Table: Effectivity indices on locally refined meshes

(a) Local refinement balanced with η_{hom}

N	I_{eff}	γ	$\lvert J^* - J_\gamma \rvert$
169	0.48	$2 \cdot 10^{-2}$	$1.5 \cdot 10^{-1}$
281	3.83	$4 \cdot 10^{-3}$	$1.9 \cdot 10^{-1}$
401	1.27	$8 \cdot 10^{-3}$	$1.1 \cdot 10^{-1}$
1057	1.56	$4 \cdot 10^{-3}$	$4.7 \cdot 10^{-2}$
1981	0.65	$3 \cdot 10^{-3}$	$1.6 \cdot 10^{-2}$

(b) Local refinement for $\gamma \to 0$

N	I_{eff}	$\lvert J^* - J_\gamma \rvert$
169	1.98	$3.4 \cdot 10^{-1}$
269	7.80	$2.2 \cdot 10^{-1}$
401	3.45	$1.7 \cdot 10^{-1}$
1045	3.90	$6.7 \cdot 10^{-2}$
1749	2.09	$3.2 \cdot 10^{-2}$

Thank You for Your Attention!

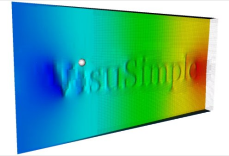
Optimization Library RoDoBo

http://rodobo.uni-hd.de/

FEM Toolbox Gascoigne3D

http://gascoigne.uni-hd.de/

Visualization Tool VisuSimple



http://visusimple.uni-hd.de/