Analytic and Discrete Shape Hessian Approximations

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DFG SPP 1253 - Optimization with Partial Differential Equations

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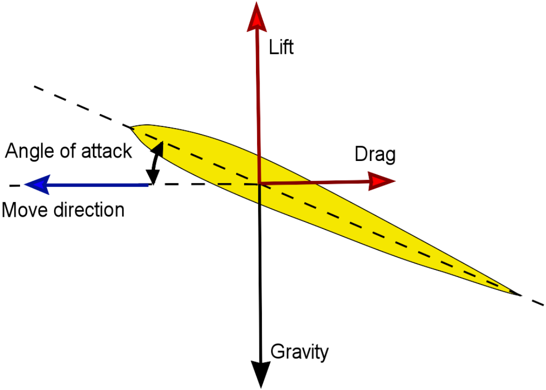
Universität Trier

Outline

Introduction



Literature

Discussed in:

- J. Sokolowski, J.-P. Zolésio: “Introduction to Shape Optimization (1992)”
- M.C. Delfour, J.-P. Zolésio: “Shapes and Geometries (2001)”
- E. Arian and S. Ta’asan: “Analysis of the Hessian for Aerodynamic Optimization (1996)”
- B. Mohammadi and O. Pironneau: “Applied Shape Optimization for Fluids (2001)”
- K. Eppler: Habilitation Thesis (2007)
- ...

Minimize Dissipation of Kinetic Energy into Heat

Objective

$$\min_{(u,p,\Omega)} \dot{E}(u,\Omega) := \frac{1}{2} \int_{\Omega} \nu \sum_{j,k=1}^{3} \left(\frac{\partial u_k}{\partial x_j} \right)^2 dA$$

Constraints

a) Stokes Equation

$$\begin{aligned} -\nu \Delta u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

b) Navier-Stokes Equation

$$\begin{aligned} -\nu \Delta u + \rho u \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}$$

Shape Derivatives

a) Stokes

$$d\dot{E}_S(u,\Omega)[V] = \int_\Gamma \langle V, n\rangle \left[-\nu \sum_{k=1}^{3} \left(\frac{\partial u_k}{\partial n}\right)^2\right] dS$$

b) Navier-Stokes

$$d\dot{E}_{NS}(u,\Omega)[V] = \int_\Gamma \langle V, n\rangle \left[\nu \sum_{k=1}^{3} \left(\frac{\partial u_k}{\partial n}\right)^2 - \frac{\partial u_k}{\partial n}\frac{\partial \lambda_k}{\partial n}\right] dS$$

$$\begin{aligned} -\nu\Delta\lambda + \rho\lambda\nabla u - \rho\left(\nabla\lambda\right)^T u + \nabla\lambda_p &= -2\Delta u && \text{in } \Omega \\ \operatorname{div} \lambda_p &= 0 && \text{in } \Omega \end{aligned}$$

Shape Derivatives

Advantages:

- No mesh sensitivities!
- Every surface mesh node is design variable
- Search space not limited by parameterization

Disadvantages:

- High frequency noise
- No regularity enforced by parameterization
- Small maximum step length

Solution

Smooth gradient via Hessian approximation

Symbol of an Operator

Suppose Fourier disturbance (oscillation) of design q

$$\tilde{q}(x) = e^{i\omega x}$$

First order differential operator $H := \frac{\partial}{\partial x}$

$$H\tilde{q} = i\omega\tilde{q}$$

Second order differential operator $H := \frac{\partial^2}{\partial x^2}$

$$H\tilde{q} = -\omega^2\tilde{q}$$

Dirichlet to Neumann Map:

$$H\tilde{q} = |\omega|\tilde{q}$$

Symbol of the Stokes Hessian

Idea: Track propagation of boundary oscillations into state variables

$$\Omega_\epsilon[\tilde{q}] := \{(x,y) : x \in \mathbb{R}, y \geq \epsilon\tilde{q}(x)\}$$

Perturbed gradient

$$\tilde{J}_S = -\,2\nu \sum_{k=1}^{2} \frac{\partial u_k}{\partial y} \frac{\partial u_k'[\tilde{q}]}{\partial y}$$

Linearized State Equation

$$\begin{aligned} -\nu\Delta u'[\tilde{q}] + \nabla p'[\tilde{q}] &= 0 \;\text{ in } \Omega \\ \operatorname{div} u'[\tilde{q}] &= 0 \\ u'[\tilde{q}] &= -\frac{\partial u_i}{\partial n}\tilde{q} \;\text{ on } \partial\Omega \end{aligned}$$

Symbol of the Stokes Hessian

Suppose state variables oscillate in both directions:

$$u'[\tilde{q}] = \hat{u}_k e^{i\omega_1 x} e^{\omega_2 y}$$

Boundary Condition gives

$$\hat{u}_k = \frac{\partial u_i}{\partial y}$$

State Equation in Fourier Space gives

$$\begin{bmatrix} -\nu(-\omega_1^2 + \omega_2^2) & 0 & i\omega_1 \\ 0 & -\nu(-\omega_1^2 + \omega_2^2) & \omega_2 \\ i\omega_1 & \omega_2 & 0 \end{bmatrix} \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{p} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Non-Contradiction

Only frequencies non-contradicting the above:

$$\omega_1 = |\omega_2|$$

Symbol of the Stokes Hessian

Symbol of the Stokes Hessian

Stokes Hessian H is pseudo-differential operator:

$$H\tilde{q} = |\omega_1|\tilde{q}$$

Properties:

- Boundary oscillation and gradient oscillation have same phase
- Frequency of boundary oscillation and amplitude of gradient oscillation scale linearly

Discrete Symbol Approximation

Idea

Observe oscillations discretely

Normal perturbation of boundary:

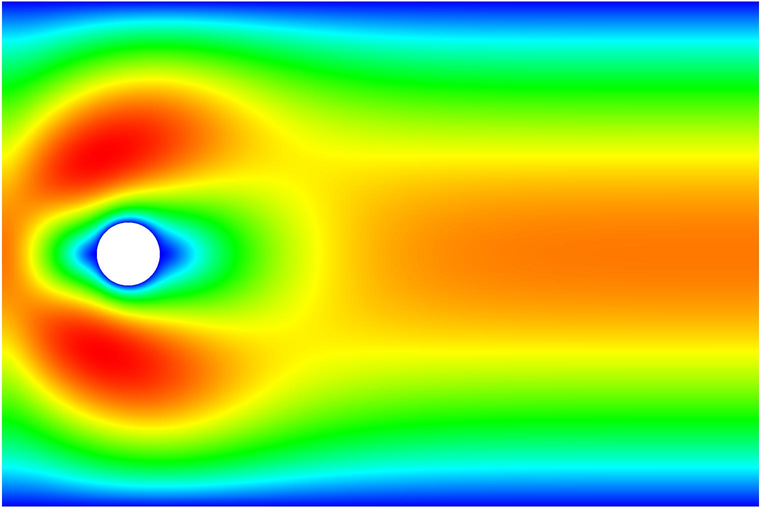
$$\Gamma_1^{\epsilon}(\tilde{q}_\omega) = \{x(\varphi) + \epsilon \tilde{q}_\omega(\varphi) n(\varphi) : \varphi \in [0, \ell]\}$$

Shape Hessian

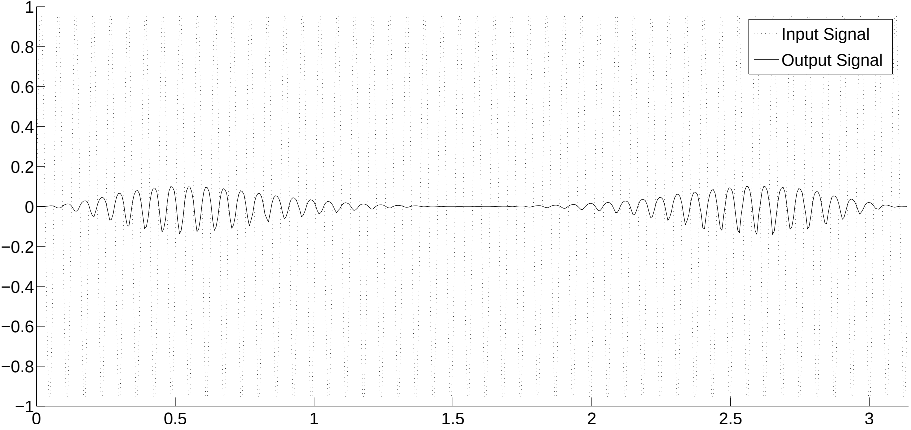
$$d^2 J(u, \Gamma_1)[V][\tilde{q}_\omega] = \lim_{\epsilon \to 0} \frac{dJ(u, \Gamma_1^{\epsilon})[V] - dJ(u, \Gamma_1)[V]}{\epsilon}$$

replaced by finite difference

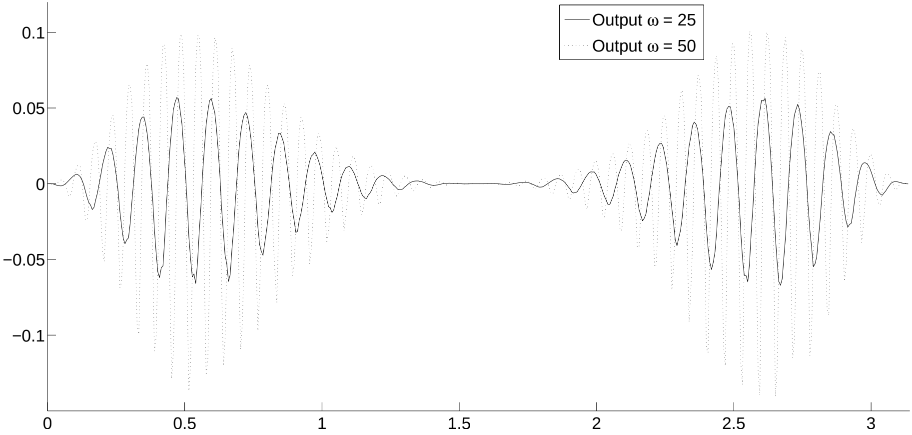
Stokes: Initial Domain



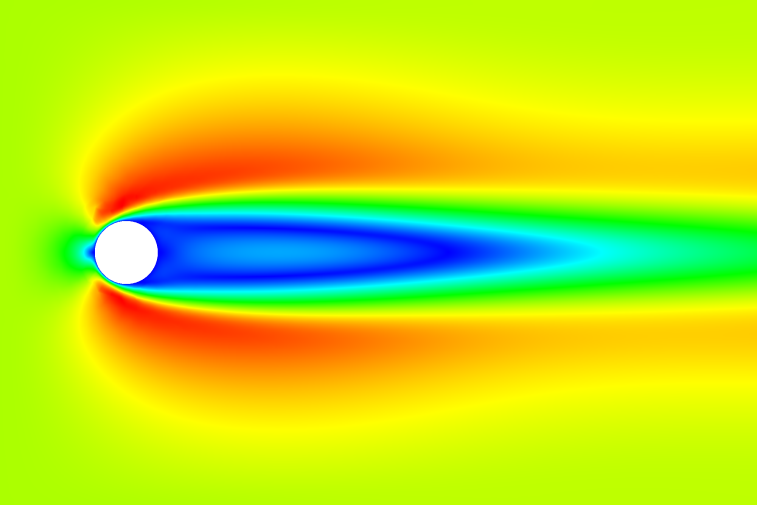
Stokes: Oscillations Stay in Phase



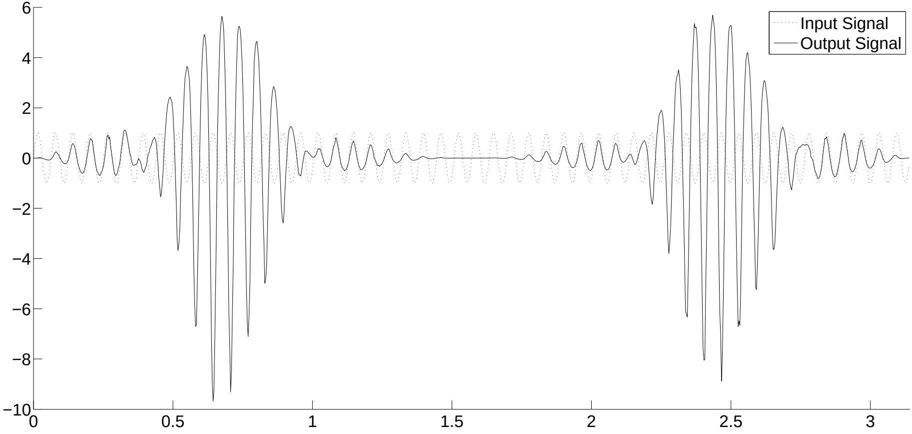
Stokes: Amplitude Scales Linearly With Frequency



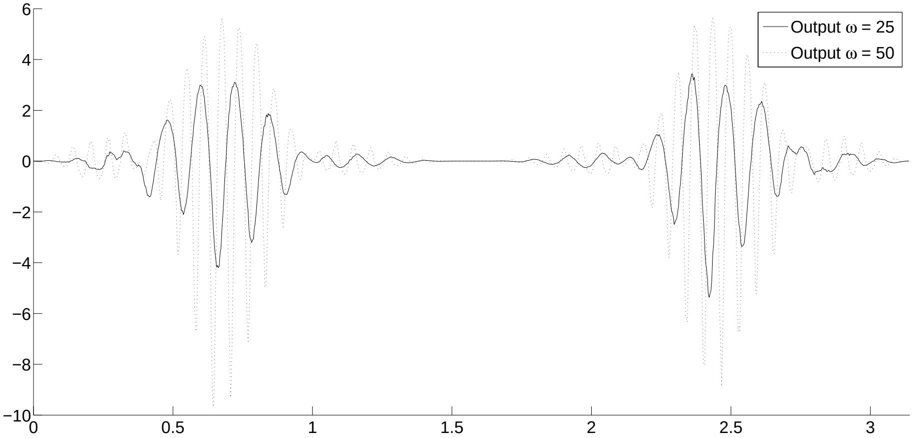
Navier-Stokes: Initial Domain



Navier-Stokes: Oscillations Stay in Phase



Navier-Stokes: Amplitude Scales Linearly With Frequency



Preconditioning

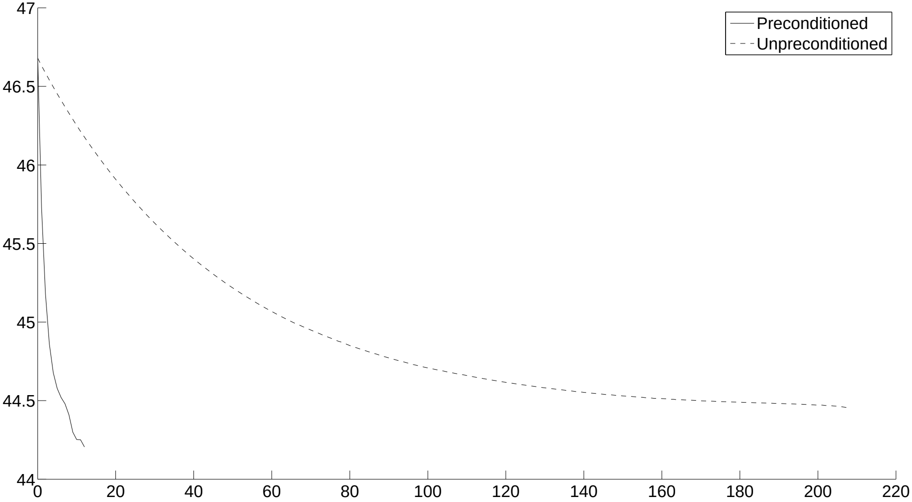
- Symbol $|\omega|$ requires construction of discrete Dirichlet To Neumann map
- Laplace-Beltrami Operator Δ_Γ numerically cheap to construct but oversmooths
- Determine maximum frequency ω_{max} possible in the mesh
- Preconditioner:

$$H_h \approx \alpha \Delta_\Gamma + \mathrm{Id}$$

Over- and undersmoothing cancel in L_1 sense:

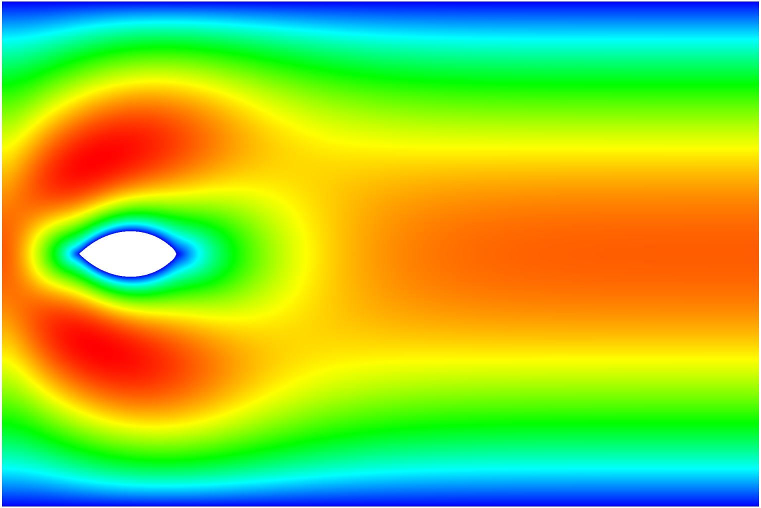
$$\alpha = \frac{\frac{3}{2}\omega_{max} - 3}{\omega_{max}^2}$$

Performance: Stokes

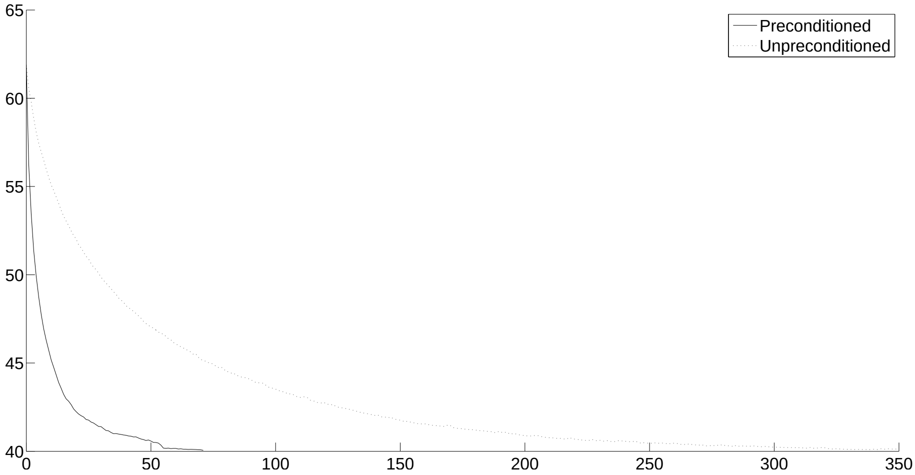


- Optimum in iteration 12 vs 200: 96% less iterations

Optimal Shape: Stokes

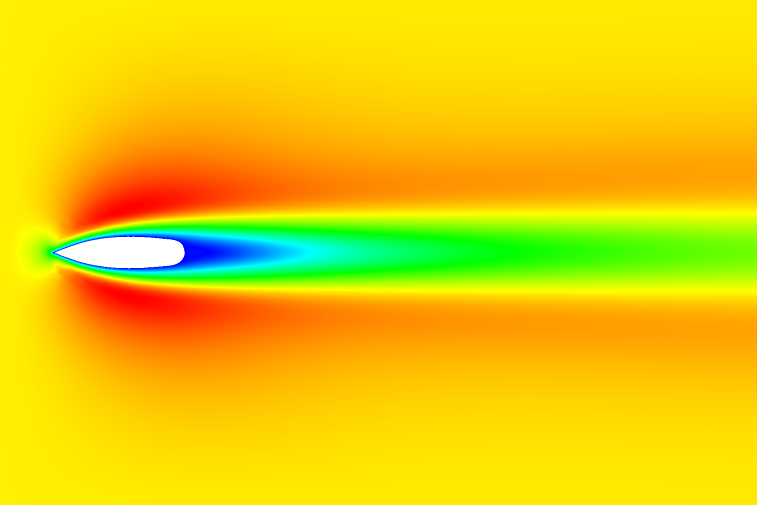


Performance: Navier-Stokes



- Optimum in iteration 71 vs 350: 80% less iterations

Optimal Shape: Navier-Stokes



Outlook and Future Work

- Preconditioned sensitivity-free drag reduction for a 3D aircraft using compressible RANS equation with turbulence modeling
- Extracting more information from the Shape Hessian
- Coupling to robust design
- Convergence acceleration with Multi-Level/MGOPT