State Constrained Optimal Control with Discontinuous States

Anton Schiela

Zuse Institute Berlin

DFG Research Center MATHEON

Uni Hamburg, 28.3.2008

Outline

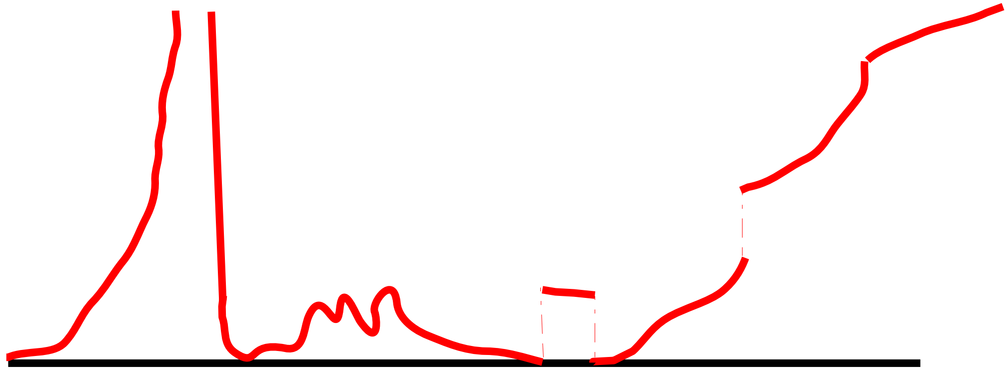
$$\min \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad Ay - Bu = 0$$

$$y \geq 0$$

State constrained optimal control is well understood for continous states

This talk: states are possibly **discontinuous, unbounded**

Main result: Existence of **measure valued** Lagrange multipliers



A simple model problem

MATHEON

$$\min \frac{1}{2}\|y - y_d\|^2_{L_2} + \frac{\alpha}{2}\|u\|^2_U \qquad Ay - Bu = 0$$
$$y \geq 0$$

Basic theory:

existence of minimizers

Slater condition

measure valued Lagrange multipliers

regularity of minimizers

$$y_* - y_d + A^* p + m = 0$$
$$\alpha u_* - B^* p = 0$$
$$m \leq 0$$
$$\langle m, y_* \rangle = 0$$

Control-to-State Mapping:

$$y = Su$$

notorious assumption for state constraints:

$S : U \to C(\overline{\Omega})$ is continuous

„Classic“ References:
Casas, Alibert/Raymond,...

Why must all states be continuous?

MATHEON

Hahn Banach separation theorem:

fundamental existence principle for dual spaces

separation of two convex sets, one of them must have **non-empty interior**

Optimal Control:

$$\mathbb{R}_+ \times \{(y,u) : Ay - Bu = 0\}$$ **empty interior**

$$-\mathbb{R}_+ \times \{(y,u) : y \geq 0\}$$ **non-empty interior in** $\mathrm{C}(\overline{\mathbf{\Omega}})$

Lagrange-Multiplier Theorems:

Robinson, Zowe-Kurcyusz

sum-rules of convex and non-smooth analysis

} **apply Hahn-Banach and need regularity conditions**

Continuity seems essential, because

Lagrange-Multipliers „are“ measures in this case

Control problems with discontinuous states

$$\min \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad Ay - Bu = 0$$

$$y \geq 0$$

Control-to-State mapping:

$S : U \to Y \qquad Y \not\hookrightarrow C(\overline{\Omega})$

Space of states is **too large and irregular** to have a non-empty positive cone

Important classes:

elliptic boundary control in 3d, most of parabolic control

related: constraints on derivatives of the state

Common practice:

usually there is a subspace U_∞ with $S : U_\infty \to C(\overline{\Omega})$

impose additional control constraints, such that $u \in U_\infty$

A first idea

MATHEON

Problem: no choice of space of states is correct

either: no continuous control-to-state mapping

or: no application of Hahn-Banach possible

Eliminate state

$$u \in U: \quad \min \frac{1}{2}\|Su - y_d\|^2_{L_2} + \frac{\alpha}{2}\|u\|^2_U \qquad Su \geq 0$$

Yields existence of Lagrange multipliers in U^*

If S is continously invertible, then

$$\langle m, u \rangle = \langle m, S^{-1}y \rangle = \langle S^{-*}m, y \rangle$$

The catch:

this works also for $Su = 0$, which is ill-posed

does not capture the structure of the problem

poor regularity results, no new information

Extending the space

MATHEON ZIB

Problems of the latter approach:

primal space was too small, Slater condition could not be exploited

reason for poor regularity of dual variables

Use larger space (but still sufficiently small)

$$u \in U \qquad \min \frac{1}{2}\|Su + w - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad w = 0$$

$$Su + w \geq 0$$

Extending the space

MATHEON ZIB

Problems of the first approach:

primal space was too small, Slater condition could not be exploited

reason for poor regularity of dual variables

Use larger space (but still sufficiently small)

$$\begin{array}{lcr} u \in U & \min \frac{1}{2}\|Su + w - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 & w = 0 \\ w \in C(\overline{\Omega}) & & Su + w \geq 0 \end{array}$$

This does not change the problem, but the space

Extending the space

Problems of the first approach:

primal space was too small, Slater condition could not be exploited

reason for poor regularity of dual variables

Use larger space (but still sufficiently small)

$$u \in U \qquad \min \frac{1}{2}\|Su + w - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad w = 0$$

$$w \in C(\overline{\Omega}) \qquad Su + w \geq 0$$

This does not change the problem, but the space

First step of proof:

existence of Lagrange multipliers for this „transformed" setting in $U \times W$

Slater condition can now be exploited

Remaining proof:

back-transformation to a space $U \times Y$ via the relation $y = Su + w$

density arguments

Main theorem (control-to-state mapping version)

$$\min \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad y = Su$$

$$y \geq 0$$

Assumptions:

$$\left.\begin{array}{l} S : U \to L_2(\Omega) \text{ is continuous} \\ S : U_\infty \to C(\overline{\Omega}) \text{ is continuous} \end{array}\right\} U_\infty \text{ is } \textbf{dense} \text{ in } U$$

Slater condition: $\exists \breve{u} : S\breve{u} \geq \delta > 0$

Conclusions:

existence of a Lagrange multiplier m

m is a measure, satisfying additionally $|\langle m, Su \rangle| \leq C\|u\|_U$

Main theorem (control-to-state mapping version)

MATHEON

$$\min \frac{1}{2}\|y-y_d\|_{L_2}^2+\frac{\alpha}{2}\|u\|_U^2 \qquad y=Su$$

$$y \geq 0$$

Conclusions in detail:

$$Y := \{y \in L_2(\Omega) : \exists u \in U, w \in C(\overline{\Omega}) : y = Su + w\}$$

$$\|y\|_Y = \inf_{Su+w=y} \|w\|_\infty + \|u\|_U$$

$$\exists m, v \in Y^* \subset M(\overline{\Omega}) :$$

$$y_* - y_d + v + m = 0 \qquad m \leq 0$$

$$\alpha u_* - S^* v = 0 \qquad \langle m, y_* \rangle = 0$$

Note: by our assumptions $C(\overline{\Omega})$ is dense in Y, so "$Y^* \subset M(\overline{\Omega})$" is justified

Intermezzo: some reflection

MATHEON

$$Y := \{y \in L_2(\Omega) : \exists u \in U, w \in C(\overline{\Omega}) : y = Su + w\}$$

$$\|y\|_Y = \inf_{Su+w=y} \|w\|_\infty + \|u\|_U$$

The space Y:

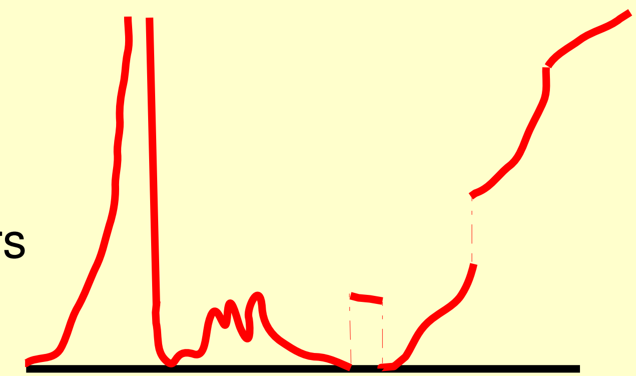
most regular case: $S : U \to W \hookrightarrow C(\overline{\Omega}) \quad \Rightarrow \quad Y \equiv C(\overline{\Omega})$

most irregular case $S = Id : L_2(\Omega) \to L_2(\Omega) \quad \Rightarrow \quad Y \equiv L_2(\Omega)$

The dual space Y*:

can be identified with a subspace of $M(\overline{\Omega})$

less regular states imply more regular Lagrange multipliers



Lagrange multipliers as sensitivities:

regularity of Lagrange multipliers characterizes meaningful perturbations

Application: boundary control in 3d

$$\min \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad y \geq 0$$

$$\int_\Omega \langle \nabla y, \nabla \varphi \rangle + y\varphi \, dx - \int_\Gamma u\gamma(\varphi)\, dx = 0 \quad \forall \varphi \in H^1(\Omega)$$

$$\left.\begin{aligned} S &: L_2(\Gamma) \to L_p(\Omega)\, \forall p < \infty \\ S &: L_q(\Gamma) \to C(\overline{\Omega})\, \forall q > 2 \end{aligned}\right\} L_q(\Gamma) \text{ **dense in** } L_2(\Gamma)$$

$$\exists m, v \in Y^* \subset M(\overline{\Omega}):$$

$$y_* - y_d + v + m = 0 \qquad m \leq 0$$

$$\alpha u_* - S^* v = 0 \qquad \langle m, y_* \rangle = 0$$

Towards an adjoint equation

MATHEON ZIB

$$\exists m, v \in Y^* :$$

$$y_* - y_d + v + m = 0 \qquad m \le 0$$

$$\alpha u_* - S^* v = 0 \qquad \langle m, y_* \rangle = 0$$

adjoint state p

$$\int_\Omega (y - y_d)\varphi \, dx + \int_\Omega \langle \nabla p, \nabla \varphi \rangle + p\varphi \, dx + \int_{\overline{\Omega}} \varphi \, dm = 0 \quad \forall \varphi \in Y$$

$$\alpha u_* - \gamma(p) = 0 \quad a.e.\ in\ \Gamma$$

Traditional way:

analysis of PDEs with right hand sides in Y^*

relate S^* to such a PDE

Closed operators

MATHEON

Aim:

theory in terms of **differential** operators, not control-to-state mappings

direct derivation of the desired optimality system

$$\langle Ay, \varphi \rangle = \int_\Omega \langle \nabla y, \nabla \varphi \rangle + y\varphi \, dx$$

Variants:

$A : H^1(\Omega) \to H^1(\Omega)^*$ — isomorphism, variational setting

$A : W^{1,p}(\Omega) \to W^{1,p'}(\Omega)^*$ $\quad \frac{1}{p} + \frac{1}{p'} = 1$ — isomorphism, cf. Amann '94

Sobolev embedding:

$$p > d : \quad W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$$

A closed, densely defined, bijective operator:

$$A : C(\overline{\Omega}) \supset \mathrm{dom} A \to W^{1,p'}(\Omega)^* \qquad \mathrm{dom} A = W^{1,p}(\Omega)$$

The adjoint of a densely defined operator

$$A : C(\overline{\Omega}) \supset \mathrm{dom} A \to W^{1,p'}(\Omega)^* \qquad \mathrm{dom} A = W^{1,p}(\Omega)$$

$$\langle Ay, \varphi \rangle = \int_\Omega \langle \nabla y, \nabla \varphi \rangle + y\varphi \, dx$$

Adjoint:

$$A^* : W^{1,p'}(\Omega) \supset \mathrm{dom} A^* \to M(\overline{\Omega})$$

$$\langle Ay, \varphi \rangle = \langle y, A^* \varphi \rangle \quad \forall y \in \mathrm{dom} A$$

$$\mathrm{dom} A^* = \varphi : \langle Ay, \varphi \rangle \;\; \text{is continuous on} \;\; \mathrm{dom} A$$

$$\langle \varphi, A^* p \rangle = \int_\Omega \langle \nabla \varphi, \nabla p \rangle + \varphi p \, dx \qquad \text{"}\forall \, \varphi \in C(\overline{\Omega})\text{"}$$

This is defined elementarily for $\varphi \in W^{1,p}(\Omega)$
and has a unique continuous extension to $C(\overline{\Omega})$, since $p \in \mathrm{dom} A^*$

Main theorem (differential operator version)

$$\min \frac{1}{2}\|y - y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad Ay - Bu = 0$$
$$y \geq 0$$

Assumptions:

- $A : L_2(\Omega) \supset \mathrm{dom} A \to P^*$ / $A : C(\overline{\Omega}) \supset \mathrm{dom}_\infty A \to P^*_\infty$ closed, densely defined, bijective
- $B : U \to P^*$ / $B : U_\infty \to P^*_\infty$ continuous
- U_∞ dense in U
- Slater condition

Main theorem (differential operator version)

$$\min \frac{1}{2}\|y-y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad Ay - Bu = 0$$
$$y \geq 0$$

Conclusions:

$A : Y \supset \mathrm{dom} A \to P_\infty^*$ is densely defined

$\exists m \in Y^*, p \in \mathrm{dom} A^* \subset P_\infty$

$y_* - y_d + A^* p + m = 0$ measure valued Lagrange multiplier $m \in Y^*$

$\alpha u_* - B^* p = 0$ regular adjoint state $p \in P_\infty$

$m \leq 0$ well defined complementarity, positivity

$\langle m, y_* \rangle = 0$

$$Y := \{y \in L_2(\Omega) : \exists u \in U, w \in C(\overline{\Omega}) : y = Su + w\}$$
$$\|y\|_Y = \inf_{Su+w=y} \|w\|_\infty + \|u\|_U$$

Boundary control in 3d revisited

$$\min \frac{1}{2}\|y-y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad y \geq 0$$

$$\int_\Omega \langle \nabla y, \nabla \varphi \rangle + y\varphi \, dx - \int_\Gamma u\gamma(\varphi)\, dx = 0 \quad \forall \varphi \in H^1(\Omega)$$

$$\int_\Omega (y-y_d)\varphi \, dx + \int_\Omega \langle \nabla p, \nabla \varphi \rangle + p\varphi \, dx + \int_{\overline{\Omega}} \varphi \, dm = 0 \quad \forall \varphi \in Y$$

$$\alpha u_* - \gamma(p) = 0 \quad a.e.\ in\ \Gamma$$

$$m \leq 0$$

$$\langle m, y_* \rangle = 0$$

regular adjoint state: $p \in W^{1,p'}(\Omega)$

Lagrange multiplier: $m \in Y^* \subset M(\overline{\Omega})$

regularity of control: $u \in L_2(\Gamma) \cap W^{1-p',p}(\Gamma)$

$$p' < \frac{d}{d-1}$$

Boundary control in 3d revisited

MATHEON

$$\min \frac{1}{2}\|y-y_d\|_{L_2}^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad y \ge 0$$

$$\int_\Omega \langle \nabla y, \nabla \varphi\rangle + y\varphi\, dx - \int_\Gamma u\gamma(\varphi)\, dx = 0 \quad \forall \varphi \in H^1(\Omega)$$

$$\int_\Omega (y-y_d)\varphi\, dx + \underbrace{\int_\Omega \langle \nabla p, \nabla \varphi\rangle + p\varphi\, dx}_{\langle \varphi, A^* p\rangle} + \int_{\overline{\Omega}} \varphi\, dm = 0 \quad \forall \varphi \in Y$$

$$\alpha u_* - \gamma(p) = 0 \quad a.e.\ in\ \Gamma$$

$$m \le 0$$

$$\langle m, y_*\rangle = 0$$

regular adjoint state: $p \in W^{1,p'}(\Omega)$

Lagrange multiplier: $m \in Y^* \subset M(\overline{\Omega})$

regularity of control: $u \in L_2(\Gamma) \cap W^{1-p',p}(\Gamma)$

$p' < \frac{d}{d-1}$

Conclusions

MATHEON ZIB

Optimality Conditions for States Constraints with Discontinuous States

- new results about optimality conditions
- exploit higher regularity of states for higher regularity of controls
- crucial density relation
- measure valued Lagrange multipliers with additional regularity
- direct approach in terms of differential operators

Reference: ZIB-Report 07-35

Conclusions

MATHEON

Optimality Conditions for States Constraints with Discontinuous States

- new results about optimality conditions
- exploit higher regularity of states for higher regularity of controls
- crucial density relation
- measure valued Lagrange multipliers with additional regularity
- direct approach in terms of differential operators

Thank you!

Reference: ZIB-Report 07-35