Some Practical Considerations for Closed Loop Control of Distributed Parameter Systems

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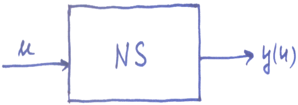
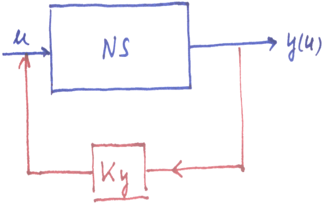

u
NS
y(u)


u
NS
y(u)
Ky

$$J(y,u) = \int_0^\infty \Big(\frac{1}{2}\int_\Omega |y(t,x)|^2\,\mathrm{d}x + \frac{\beta}{2}\,|u(t)|^2\Big)e^{-\mu t}\,\mathrm{d}t,$$

$$\begin{aligned}
y_t - \nu y_{xx} + y y_x &= 0 && \text{in } (0,\infty)\times(0,1),\\
\nu y_x(\cdot,0) + \sigma_0 y(\cdot,0) &= u && \text{in } (0,\infty)\\
\nu y_x(\cdot,1) + \sigma_1 y(\cdot,1) &= g && \text{in } (0,\infty)\\
y(0,\cdot) &= y_\circ && \text{in } (0,1),
\end{aligned}$$

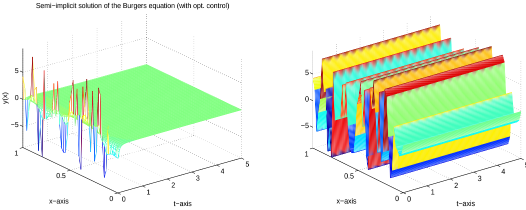


FIGURE 1. Optimal state with random noise (9.0) in the initial condition: feedback design (left) and open-loop design (right).

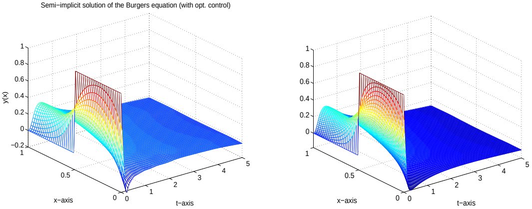


FIGURE 2. Optimal state with random noise (0.25) in the RHS: feedback design (left) and open-loop design (right).

$$\begin{cases} \min E\{\int_0^T ((y, Dy) + (u, Cu))\, dt + (y(T), Gy(T))\} \\ y_t = Ay + Bu + w \\ y(0) = y_0 \end{cases}$$

w white noise with intensity V, $Q_0 = E(y_0, \cdot)y_0$

$$u^* = -\mathcal{F}(t)y(t), \quad \mathcal{F}(t) = C^{-1}B^*\Pi(t)$$

$$\begin{cases} -\Pi_t(t) = A^*\Pi(t) + \Pi(t)A^* - \Pi(t)BC^{-1}B^*\Pi(t) + D \\ \Pi(T) = G \end{cases}$$

$$E_{\min} = tr\left(\Pi(0)Q_0 + \int_0^T \Pi(t)V(t)\, dt\right)$$

close the loop: $\quad y_t = Ay(t) - BC^{-1}B^*\Pi(t)\, y(t)$

Dynamic Programming Principle

$$\begin{cases} \min J(y_0, u) = \int_0^\infty L(y(t), u(t)) e^{-\mu t} dt \\ \dot{y}(t) = f(y(t), u(t)), \quad y(0) = y_0, \; u \in \mathcal{U}_{ad}. \end{cases}$$

value function $v(y_0) = \min_{u \in \mathcal{U}_{ad}} J(y_0, u)$.

$(DPP) \quad v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{ \int_0^T L(y(t; y_0, u), u(t) e^{-\mu t} dt + v(y(T; y_0, u)) e^{-\mu T} \}$

$$(HJB) \quad \mu v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{ (\nabla v(y_0), f(y_0, u)) + L(y_0, u) \}$$

replace y_0 by $y^*(t)$.

$$u^*(t) = \mathcal{F}(y^*(t)) \qquad \text{where}$$

$\mathcal{F}(y^*(t)) \in \operatorname{argmin}_{u \in \mathcal{U}_{ad}} \{ (\nabla v(y^*(t)), f(y^*(t), u)) + L(y^*(t), u) \}$

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(*DPP*) $v(y_0) = \min_{u \in \mathcal{U}_{ad}} \{ \int_0^T L(y(t; \, y_0, u), u(t) e^{-\mu t} dt + v(y(T; y_0, u)) e^{-\mu T} \}$

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$$(P)\begin{cases} \min_{u\in U_{ad}} \int_0^T e^{-\mu t}\, h(y) + \frac{\alpha}{2}\int_0^T |u|^2 \\ \dot{y} = f(y) + Bu(t)\,, \quad y(0) = y_0 \end{cases}$$

$v = v(t,y)$ ... value functional $\qquad \nabla = \nabla_y$

$$\begin{cases} v_t + \mu v = \min_{u\in U_{ad}} \left((f(t,y) + B^T u, \nabla v) + h(y) + \frac{\alpha}{2}|u|^2\right) \\ v(T,\cdot) = 0 \end{cases}$$

$$u = P_{U_{ad}}\left(-\frac{1}{\alpha}B^T \nabla v\right)$$

t-discretisation, infinite time horizon

$$t_j = jh$$

$$\begin{cases} y_{j+1} = y_j + h\,f(y_j, u_j) \\ y_0 \ \text{given} \end{cases}$$

$$J_h(y_0, u_h) = \frac{h}{2}[L(y_0, u_0) + \sum_{j=1}^{\infty} e^{-\mu hj}(L(y_j, u_{j-1}) + L(y_j, u_j))].$$

$$\upsilon_h(y_0) = \inf_{u_h \in \mathcal{U}^h_{ad}} J_h(y_0, u_h)$$

$$(HJB_h) \quad \upsilon_h(y_0) = \inf_{u \in \mathcal{U}^h_{ad}} \{\frac{h}{2}[L(y_0, u) + e^{-\mu h} L(y_0 + hf(y_0, u), u)] + e^{-\mu h} \upsilon_h(y_0 + hf(y_0, u))\}$$

$$y^*_{j+1} = y^*_j + hf(y^*_j, \mathcal{F}_h(y^*_j)),\ j \geq 0$$

numerically still infeasible, joint with L. Xie

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$$v_h(y_0) = \inf_{u_h \in \mathcal{U}_{ad}^h} J_h(y_0, u_h)$$

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discretization, finite time horizon

in time: TVD Runge-Kutta scheme

in space: local Lax-Friedrich scheme

POD-Galerkin Ansatz (for Navier Stokes):

$$y^l(t,x) = \sum_{i=1}^{l} \alpha_i(t)\varphi_i(x), \qquad \mathbf{u}(t,x) = \sum u_i(t)\,\hat{\varphi}_i\,(x),$$

$\{\varphi_i\}$ POD basis for velocities, $\{\hat{\varphi}_i\}$ POD basis for controls

$$\dot{\alpha} = -A\alpha - \alpha^T H\alpha + G(u)$$

$$A_{ij} = \frac{1}{Re}\langle \nabla\varphi_i,\ \nabla\varphi_j\rangle, \quad H_{ijk} = \langle \varphi_j \cdot \nabla\varphi_k,\ \varphi_i\rangle,$$

$$G_i = \langle \mathbf{u}, \varphi_i\rangle$$

Implementation Aspects

for NS: finite difference upwind scheme on staggered nonuniform grid, fractional ϑ-scheme for time-integration.

$$(HJB_h)\ v_h(\alpha) = \inf_{u \in \mathcal{U}_{ad}^h} \{ \frac{h}{2} \Big(L(\alpha, u) + e^{-\mu h} L(\alpha + hf(\alpha, u), u) \Big) + e^{-\mu h} v(\alpha + hf(\alpha, u)) \}$$

hypercube Γ_h in $\mathbb{R}^n$ such that $\alpha^*(t) \in \Gamma_h$ for all t,
α_j grid points in Γ_h, $\alpha = \sum \lambda_j \alpha_j \quad , \quad \sum \lambda_j = 1$

$$v_h(\alpha) = \sum \lambda_j v_h(\alpha_j)$$

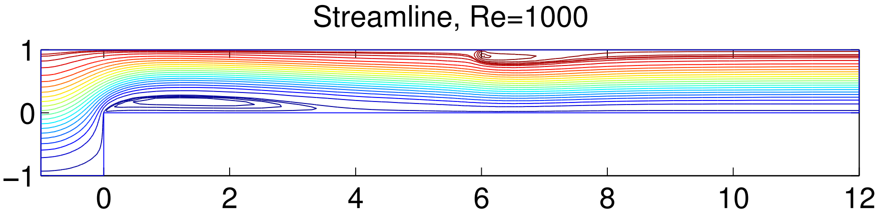
Solve (HJB_h) as fixed point equation,
(Bardi, Capuzzo-Dolcetta; Gonzales-Rofman)
nonuniform grid $\sim$ decay of POD eigenvalues.

Constrained nonlinear programming problem at each grid point of Γ_h : in parallel.

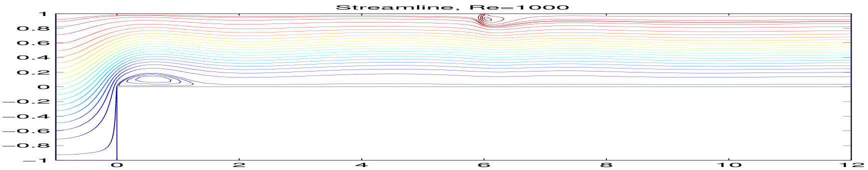
Validation

Reynolds number, *Re*	1000
Time horizon, T	10
No. of basis functions	6
No. of controls	2
Bounds on controls	$[-0.5, 0.5]$
Grid system	$12 \times 6 \times 6 \times 4 \times 3 \times 3$
Discount rate, μ	2
Observe region, Ω_o	$[-1, 12] \times [0, 1]$

Table: Parameter Settings



Streamline, Re=1000
1
0
−1
0
2
4
6
8
10
12



Streamline, Re=1000
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
2
4
6
8
10
12

2-point BVP-solutions

$\mu = 0$, no control constraints

$$(OS)\begin{cases} \dot{y} = f(y) + Bu\,, & y(0) = y_0 \\ \dot{\lambda} + f'(y)\lambda = -h'(y)\,, & \lambda(T) = 0 \\ u = \frac{1}{\alpha}\,\lambda \end{cases}$$

$$\begin{aligned} u(t) &= -\tfrac{1}{\alpha}\,B^T\nabla_y v(t,y(t)) = \mathcal{F}_T(t,y(t)) = \tfrac{1}{\alpha}\lambda(t;y_0) \\ &= \mathcal{F}_{T-t}(0,y(t)) \sim \mathcal{F}_T(0,y(t)) = \tfrac{1}{\alpha}\,\lambda\,(0;y(t)) \end{aligned}$$

finite dimensional realization

(i) choose grid $\sum \subset \mathbb{R}^n$ containing optimal trajectory

(ii) calculate $(y(\cdot,\bar{y}_0),\ \lambda(\cdot,\bar{y}_0))$ for all $\bar{y}_0 \in \sum$, thus $\mathcal{F}_T(0,\bar{y}_0)$ available

(iii) use interpolation for $\mathcal{F}_T(0,y),\ y \in \mathbb{R}^n$ arbitrary.

Remark: (OS) must be solved often ! (gridpoints in one direction)$^{\# \text{basis}}$ $\rightarrow$ POD use for open loop optimal control to estimate values in $\sum$.

Example

(Control-to-zero of Burgers equation).

	FE	POD	closed loop
cost	0.1694	0.1704	0.1733

Optimal Vortex Reduction by Non-Stationary Flows

Cost functionals:

- Tracking functional: $J_1(y) = \int_0^T \int_\Omega |y(t,x) - y_{des}(t,x)|^2 \, dx \, dt$
- Curl functional: $J_2(y) = \int_0^T \int_\Omega |\operatorname{curl} y(t,x)|^2 \, dx \, dt$
- Galilean functional: $J_3(y) = \int_0^T \int_\Omega |\det \nabla y| \, dx \, dt$
- Objective functional: $J_4(y,p) = \int_0^T \int_\Omega (|r(y,p)| - 1)^+ \, dx \, dt$,

$$r(y,p) = \frac{\omega}{\sigma} - \frac{\sigma_s(p_{x_1 x_1} - p_{x_2 x_2}) - 2\sigma_n p_{x_1 x_2}}{\sigma^{\frac{3}{2}}},$$

$$\omega = (y_2)_{x_1} - (y_1)_{x_2}, \; \sigma_s = (y_2)_{x_1} + (y_1)_{x_2}, \; \sigma_n = (y_1)_{x_1} - (y_2)_{x_2}.$$

jointly with B. Vexler

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jointly with B. Vexler

$$\begin{aligned} y &= 0 && \text{on } \partial\Omega \setminus (\Gamma_{in} \cup \Gamma_{out}), \\ y &= g(u)\,\hat{y}_{in} && \text{on } \Gamma_{in}, \end{aligned}$$

$\hat{y}_{in}$ parabolic profile

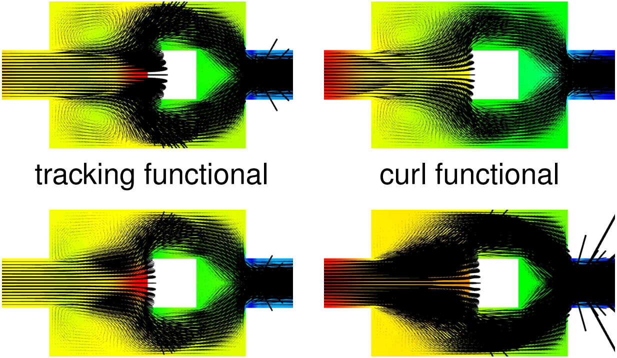
$$g(u)(t) = \frac{c_0}{T} + \sum_{k=1}^{n} \left(u_{2k-1} \sin(2\pi kt/T) + u_{2k} \cos(2\pi kt/T) \right).$$

$U = \mathbb{R}^{2n}$.

$$\int_0^T \int_{\Gamma_{in}} y \cdot n \, ds \, dt = c_0 \int_{\Gamma_{in}} \hat{y}_{in} \cdot n \, ds$$

independently of u.

Optimal Vortex Reduction by Non-Stationary Flows



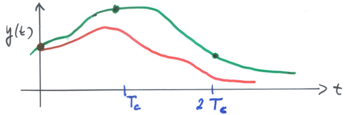
tracking functional

curl functional

Galilean functional

Objective functional

Receding horizon control



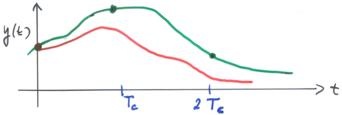
$T_c \cdots$ control-interval (stabilization, $y = 0$ steady state).

$$(P) \begin{cases} \min \int_0^\infty h^\circ(y(t), u(t))dt \quad \sim \int_0^{T_c} h^\circ(y, u)dt + \int_{T_c}^\infty h^\circ(y, u)dt \\ \frac{d}{dt} y(t) = f(y(t), u(t)), \qquad y(0) = y_0, \quad \text{in } Y. \end{cases}$$

$f(0, 0) = h^\circ(0, 0) = 0$

end-point restriction: $y(T_c) = 0$ or $y(T_c) \in S$.

Receding horizon control



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Alternative: replace $\int_T^\infty h^\circ(y(t), u(t)dt$ by $G(y(T))$

$$(P_k)\begin{cases} \inf_u \int_{(k-1)T}^{kT} h^\circ(y(t), u(t))dt + G(y(kT)) \\ \dot{y}(t) = f(y(t), u(t)), \\ y((k-1)T) = \bar{y}((k-1)T) \end{cases}$$

global solution by concatenation: $(\bar{y}_k, \bar{u}_k)$.

Definition

$G \in C(Y, \mathbb{R})$, $G(0) = 0$, is a control - Ljapunov- function, if

$\forall\, y_0 \in Y,\ T > 0\ \exists\, u = u(\cdot, y_0, T)$:

$$\int_0^T h^\circ(y(t), u(t))dt + G(y(T)) \leq G(y_0)$$

$\dot{y}(t) = f(y(t), u(t)), \quad y(0) = y_0.$

Alternative: replace $\int_T^\infty h^\circ(y(t), u(t)dt$ by $G(y(T))$

$$
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\dot{y}(t) = f(y(t), u(t)), \\
y((k-1)T) = \bar{y}((k-1)T)
\end{cases}
$$

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\int_0^T h^\circ(y(t), u(t))dt + G(y(T)) \leq G(y_0)
$$

$\dot{y}(t) = f(y(t), u(t)), \quad y(0) = y_0.$

JUSTIFY

$$V_T(y_0) = \inf_u \int_0^T h^\circ(y(t), u(t))dt + G(y(T)).$$

Theorem (H)

- $T \to V_T(y_0)$ monotonically decreasing.
- There exists $\rho_T < 1$: $G(\bar{y}(kT)) \leq \rho_T^k G(y_0)$ for all k.

CONSTRUCT

$G(y) = \frac{\alpha}{2}|y|^2 \cdots$ CLF ?

Definition

(P) is called closed loop dissipative, if $\exists$ feedback law $u = \Phi(y) \in U$ and $\alpha > 0$:

$$\alpha(f(y,u),y)_Y + h^\circ(y,u) \le 0,$$

CLD $\Rightarrow \frac{\alpha}{2}|y|^2$ is CLF. (domains)

Examples

Navier–Stokes equations, Boussinesq equations, geophysical flows, etc.

more generally: G requires to solve a Liapunov or Riccati equation.
K.Ito-K.

Applications

(i) dissipative Systems:

$$\frac{d}{dt}y(t) = A_0 y(t) + f(y(t)) + Bu(t)$$

$$A_0 \text{ dissipative}, \ f: \text{locally Lipschitz}, (f(y), y)_Y = 0$$

e.g. Navier–Stokes equations, Boussinesq equation, geophysical flows, Lorenz systems, etc.

$$h^\circ(y,u) = \frac{1}{2}|y|^2 + \frac{\beta}{2}|u|^2, \ u = -B^\star y,$$

$G(y) = \frac{\alpha}{2}|y|^2$ ist CLF if α sufficiently large.

(ii) wave equation withdamping

$$(\tfrac{d}{dt})^2 z(t) + \partial\psi(\tfrac{d}{dt}z(t)) + \Lambda z(t) \ni Bu(t),$$

$$\psi : \mathbb{R}^n \to \mathbb{R} \text{ convex }, \partial\psi(0) \ni 0, \Lambda \in \mathbb{R}^{n\times n}.$$

Coulomb friction: $\psi(z) = \sum_{i=1}^n \gamma_i |z_i|, \gamma_i \geq 0.$

CLF–function G is solution of algebraic Riccati equation.

(iii) Semilinear wave equation

$$\begin{cases} y_{tt} + \varphi(y_t) - y_{xx} = u\,\chi_I, & x \in (0,1),\ t > 0 \\ y(t,0) = 0,\ y_x(t,1) = 0 \end{cases}$$

$I = (x_1, x_2) \subset (0,1)$, φ Lipschitz, $\varphi(s)s \geq 0$,

$$\int_0^T h^\circ(y,u)dt = \frac{\gamma}{2}\int_0^T \left(|y_x|^2_{L^2} + |y_t|^2_{L^2} + |u|^2_{L^2(I)}\right) dt$$

$$G(y(T)) = \int_0^1 \left(|y_x(T)|^2 \ + \ |y_t(T)|^2\right) dx +$$

$$\int_0^1 a(x) y_x(T) y_t(T)\, dx \ + b\int_0^1 \ y(T) y_t(T)\, dx$$

Feedback Control of Schrödinger Equation

$$i\,\frac{\partial}{\partial t}\Psi(x,t) = (\mathcal{H}_0 + \epsilon(t)\mu)\Psi(x,t), \quad \Psi(x,0) = \Psi_0(x).$$

$\mathcal{H}_0$ positive, closed, self-adjoint operator in X,
μ self-adjoint dipole operator, $\epsilon(t) \in \mathbb{R}$ amplitude

$$i\,\frac{d}{dt}\mathcal{O}(t) = \mathcal{H}_0\mathcal{O}(t),$$

$$\mathcal{O}(t) = e^{-i(\lambda t - \theta)}\psi_{k_0},$$

$$V_1(\Psi(t),\mathcal{O}(t)) = \tfrac{1}{2}|\Psi(t) - \mathcal{O}(t)|_X^2$$

$$V_2(\Psi(t),\mathcal{O}(t)) = \tfrac{1}{2}\left(1 - |(\mathcal{O}(t),\Psi(t))_X|^2\right)$$

$$V_2(\Psi,\mathcal{O}) = 0 \quad \text{if and only if} \quad \Psi = e^{i\theta}\mathcal{O},$$

observe $\Psi(t) = \Psi_0 = 1$ for all t.

$$\frac{d}{dt} V_1(\Psi(t), \mathcal{O}(t)) = \epsilon(t)\, Im\, (\mathcal{O}(t), \mu\Psi(t))_X.$$

$$\epsilon(t) = -\frac{1}{\beta} Im\, (\mathcal{O}(t), \mu\Psi(t))_X = F_1(\Psi(t), \mathcal{O}(t)),$$

then

$$\frac{d}{dt} V_1(\Psi(t), \mathcal{O}(t)) = -\beta\, |\epsilon(t)|^2.$$

Similarly

$$\frac{d}{dt} V_2(\Psi(t), \mathcal{O}) = \epsilon(t)\, Im \left(\overline{(\mathcal{O}, \Psi(t))_X}\, (\mathcal{O}, \mu\Psi(t))_X \right).$$

$$\epsilon(t) = -\frac{1}{\beta} Im \left(\overline{(\mathcal{O}, \Psi(t))_X}\, (\mathcal{O}, \mu\Psi(t))_X \right) = F_2(\Psi(t), \mathcal{O}),$$

then

$$\frac{d}{dt} V_2(\Psi(t), \mathcal{O}) = -\beta\, |\epsilon(t)|^2.$$

Assume that

- $$\{e^{i(\lambda_k-\lambda_{k_0})\tau}\}\cup\{e^{-i(\lambda_k-\lambda_{k_0})\tau}\}_{k\neq k_0} \quad \text{is } \omega\text{-independent in } L^2(0,T),$$

  for some $T>0$

- $$\mu_{k_0}^k=(\psi_{k_0},\mu\psi_k)_X\neq 0 \text{ for all } k=1,2,\ldots.$$

- $$\{S(t)\Psi_0,\ t\geq 0\} \text{ compact in } X.$$

Theorem

$\lim_{t\to\infty} V_1(\Psi(t),\mathcal{O}(t))=0$, *for the feedback law* F_1*.*
If, in addition, $(\Psi_0,\psi_{k_0})_X\neq 0$*, then* $\lim_{t\to\infty} V_2(\Psi(t),\mathcal{O}(t))=0$*,*
for the feedback law F_2*.*

If $|\lambda_k+\lambda_\ell-2\lambda_{k_0}|>\delta$ is violated: $\tilde{\mu}(t)=\sum_{j=1}^m \epsilon_j(t)\,\mu_j$,

$$V_1(t,\Psi) = 1 - (\mathcal{O}(t),\Psi)_{H\times H}$$

satisfies the Hamilton Jacobi equation

$$\frac{\partial V_1}{\partial t} + \min_{\epsilon}[\frac{\beta}{2}|\epsilon|^2 + (V_1)_\Psi(A_0\Psi + \epsilon B\Psi)] + \frac{1}{2\beta}|(\mathcal{O}(t), B\Psi)_{H\times H}|^2 = 0,$$

where $A_0 \begin{pmatrix} 0 & \mathcal{H}_0 \\ -\mathcal{H}_0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & \mu \\ -\mu & 0 \end{pmatrix}$

$$\epsilon^* = \frac{1}{\beta}(\mathcal{O}(t), B\Psi)_{H\times H} = F_1(\Psi, \mathcal{O}(t))$$

minimizes

$$J(\epsilon) = \int_0^T (\,\frac{\beta}{2}|\epsilon(t)|^2 + \frac{1}{2\beta}|(\mathcal{O}(t), B\Psi(t))_{H\times H}|^2\,)\,dt + V_1(\Psi(T), \mathcal{O}(T)),$$

over $\epsilon \in L^2(0,T)$.