TREE-FORCING

In ZFC, we can always build a model for a finite fragment of ZFC. Moreover, by LS, we can consider countable transitive model of ZFC^* (← This denotes the finite fragment).

DEF: $(P, \leq)$ is a partial order. We say that G is a GENERIC FILTER for $\{D : D \in \mathcal{D}\}$ iff:

① $\forall p, q \in G$, $\exists r \leq p, q$ s.t. $r \in G$

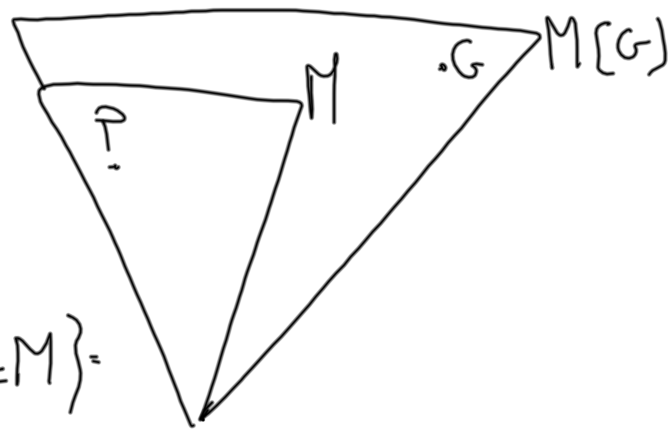
② If $q \in G$ and $p \geq q$ then $p \in G$

$\Big\}$ G is a FILTER

③ $\forall D \in \mathcal{D}$, $G \cap D \neq \emptyset$.

What is $\{D : D \in \mathcal{D}\}$? It is a family of DENSE subsets of P, ie,

$D \subseteq P$ is DENSE iff $\forall p \in P \, \exists q \in D$ s.t. $q \leq p$.

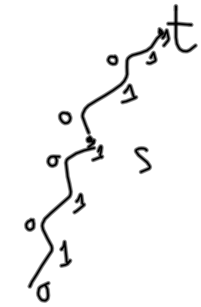


- $P \in M$
- since M transitive, $P \subseteq M$
- $\{D \subseteq P : D \text{ dense} \wedge D \in M\} =$
  $= \{D_n : n \in \omega\}$

Build the generic as follows:

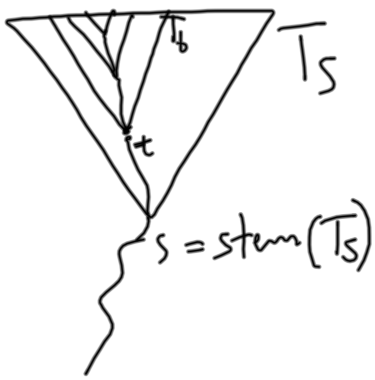
$$\begin{array}{ccccc} p_0 & \geq & p_1 & \geq & p_2 \cdots\cdots \\ \cap & & \cap & & \cap \\ D_0 & & D_1 & & D_2 \end{array}$$

COHEN FORCING: $\mathbb{P}$ is simply the set of finite sequences of 0's and 1's.



$t \leq s \iff t$ extends s

REMARK 1 $D_m := \{ t \in 2^{<\omega} : |t| \geq m \}$ is DENSE



$$\mathbb{P} := \{ T_s : s \in 2^{<\omega} \}$$

ordered by

$$T \leq S \iff T \subseteq S$$

Let $G \subseteq \mathbb{P}$ be generic over M.

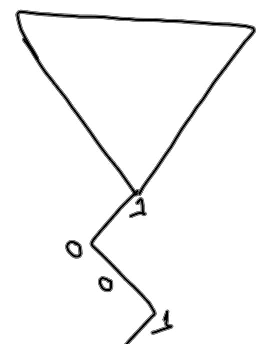
Let $x := \bigcup \{ \mathrm{stem}(T) : T \in G \}$.

$T_s \Vdash \varphi \iff \forall G$ generic s.t. $T_s \in G$, we have $M[G] \models \varphi$.

<u>REMARK:</u>

$T_t \leq T_s$ and $T_s \Vdash \varphi$, then $T_t \Vdash \varphi$

φ is $\bigwedge \cdot X(0)=1$
$\cdot\ X(1)=0$
$\cdot\ X(2)=0$
$\cdot\ X(3)=1$



$T_S \Vdash \varphi$

φ_4 is $X(4)=0$ Q: $T_S \Vdash \varphi_4$? $T_S \Vdash \neg\varphi_4$?

NO   NO

$T_{S^\frown 0} \Vdash \varphi_4$ and $T_{S^\frown 1} \Vdash \neg\varphi_4$

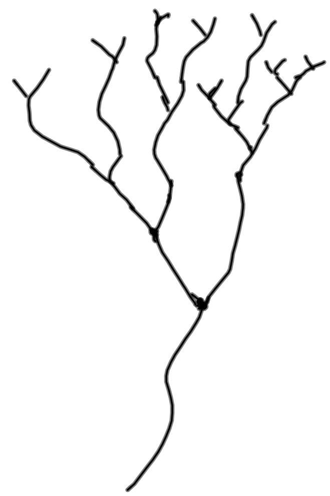
$$T_s \Vdash (1,0,0,1) \subseteq X$$

$$T_{s^\frown 0} \Vdash (1,0,0,1,0) \subseteq X$$

$$T_{s^\frown 1} \Vdash (1,0,0,1,1) \subseteq X$$

<u>DEF</u>: Let $\mathbb{S}$ be the partial order consisting of perfect trees ordered by inclusion, i.e.,

$$T \leq S \Leftrightarrow T \subseteq S.$$



LEMMA: $\mathbb{S}$ preserves $\aleph_1$.

PROOF: AIM: Given $A \in M[G]$ countable seq. of ordinals $\exists B \in M$ countable seq. of ordinals s.t. $B \supseteq A$.

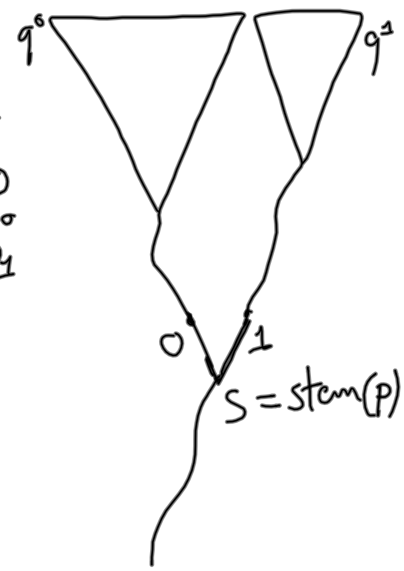
$F : \omega \to On$ and $\mathrm{ran}(F) = A$.

Fix $p \in \mathbb{S} \xrightarrow{\text{AIM}}$ Build $q \in \mathbb{S}$, $q \leq p$ and $B \in M$ s.t. $q \Vdash \mathrm{ran}(F) \subseteq B$.

- pick $p_{s^\frown 0}$, $p_{s^\frown 1}$

  and take $q^0 \leq p_{s^\frown 0}$, $b_0^0 \in On$ s.t. $q^0 \Vdash \dot{F}(0) = b_0^0$

  $q^1 \leq p_{s^\frown 1}$, $b_1^0 \in On$ s.t. $q^1 \Vdash \dot{F}(0) = b_1^0$



$p^0 := q^0 \cup q^1 \in \mathbb{S}$ and $stem(p_0) = s$

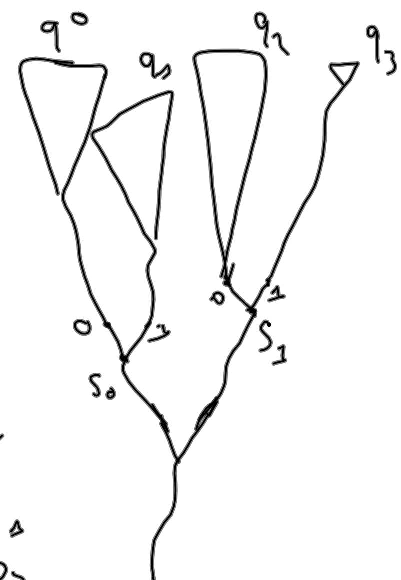
$B_0 := \{b_0, b_1\}$

$p^0_{s_0{}^\frown 0} \geq q^0 , b^1_0 \quad q^0 \Vdash F(1)=b^1_0$

$p^0_{s_0{}^\frown 1} \geq q^1 , b^1_1 \quad q^1 \Vdash F(1)=b^1_1$

$p^0_{s_1{}^\frown 0} \geq q^2 , b^1_2 \quad q^2 \Vdash F(1)=b^1_2$

$p^0_{s_1{}^\frown 1} \geq q^3 , b^1_3 \quad q^3 \Vdash F(1)=b^1_3$



$p^1 := q^0 \cup q^1 \cup q^2 \cup q^3$ and $B^1 := \{ b^1_0, b^1_1, b^1_2, b^1_3 \}$

By induction, we proceed analogously, and we get

$\{p^n : n \in \omega\}$ of trees in $\mathbb{S}$ s.t.

- $p^{n+1} \leq p^n$
- p^{n+1} and p^n have the same kth-splitnodes, for $k \leq n+1$

And so $\bigcap_{n \in \omega} p_n =: q \in \mathbb{S}$

Further let $B = \bigcup_{n \in \omega} B_n$

CLAIM:

$$q \Vdash \operatorname{ran}(\dot{F}) \subseteq B$$

$$q \Vdash \dot{F}(0) = b_0^0 \vee \dot{F}(1) = b_1^0$$

$$q \Vdash \bigvee_{j \leq 3} \dot{F}(1) = b_j^1$$

$\vdots$