Games for Baire classes and partition classes

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Outline

Infinite games in descriptive set theory

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The games we will focus on today are those for characterizing classes of functions.

The tree game

Given $f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$, we can think of the task of finding a value $f(x)$ as the *goal* of a player in an infinite two-player game called the tree game (due to Semmes).

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At each round $n \in \mathbb{N}$,

1. player $\mathbf{I}$ picks a natural number x_n, and
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Letting $x := \langle x_0, x_1, x_2, \dots\rangle \in \mathbb{N}^\mathbb{N}$ and $(T, \phi) := \bigcup_n (T_n, \phi_n)$, the rules are:

- for all $n \in \mathbb{N}$ we have $T_n \subseteq T_{n+1}$ and $\phi_n \subseteq \phi_{n+1}$; and
- if $x \in \mathrm{dom}(f)$ then T has a unique infinite branch.

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Player $\mathbf{II}$ wins the run iff she follows the rules and we have that if $x \in \mathrm{dom}(f)$ then the sequence of labels along the unique infinite branch of T is exactly $f(x)$.

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$$\mathbf{\Lambda}_{\alpha,\beta} := \{f :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} ; \forall X \in \mathbf{\Sigma}^0_\alpha .\, f^{-1}[X] \in \mathbf{\Sigma}^0_\beta\}.$$

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7. $R_{1,\alpha+1} :=$? (Louveau-Semmes, unpublished).

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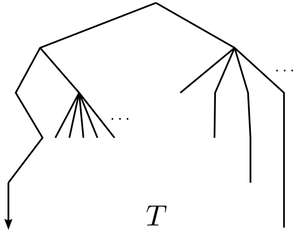

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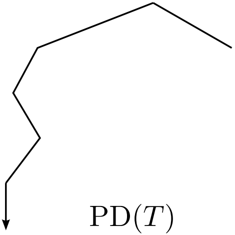

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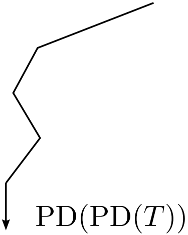

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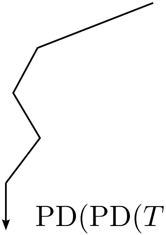

$\mathrm{PD}(\mathrm{PD}(T))$

This can be iterated transfinitely as usual

$$\begin{aligned}
\mathrm{PD}(T,0) &:= T\\
\mathrm{PD}(T,\alpha+1) &:= \mathrm{PD}(\mathrm{PD}(T,\alpha))\\
\mathrm{PD}(T,\lambda) &:= \bigcap_{\alpha<\lambda} \mathrm{PD}(T,\alpha) \quad \text{for limit } \lambda
\end{aligned}$$

Bisimulations and bisimilarity

Definition

Let $\mathcal{T} = (T, \phi_T)$ and $\mathcal{S} = (S, \phi_S)$ be labeled trees. A relation $Z \subseteq T \times S$ is a bisimulation between $\mathcal{T}$ and $\mathcal{S}$ if whenever $\sigma\ Z\ \tau$:

1. $|\sigma| = |\tau|$ and $\phi_T(\sigma) = \phi_S(\tau)$
2. $\sigma \neq \langle\rangle \Rightarrow \sigma\restriction(|\sigma|-1)\ Z\ \tau\restriction(|\tau|-1)$
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4. vice versa.

The union of all bisimulations between $\mathcal{T}$ and $\mathcal{S}$ is itself a bisimulation between $\mathcal{T}$ and $\mathcal{S}$, and the trees are called bisimilar if this relation is non-empty.

Tree game, revisited

In the tree game, the rule

"if $x \in \mathrm{dom}(f)$ then T has a unique infinite branch"

can be rewritten as

"if $x \in \mathrm{dom}(f)$ then $\mathrm{PD}(T, \omega_1)$ is non-empty and linear."

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We call the resulting game the relaxed tree game.

Main theorem: characterization of each Baire class

Recall that every ordinal α can be uniquely written as $\lambda + n$ for some limit λ and natural n. We then define

$$\begin{aligned} \alpha{\downarrow} &:= \lambda + \left\lceil \tfrac{n}{2} \right\rceil \\ \alpha{\downdownarrows} &:= \lambda + \left\lfloor \tfrac{n}{2} \right\rfloor \end{aligned}$$

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Theorem

Adding the rule

$$R_{1,\alpha+1} := \begin{array}{l} \mathrm{PD}(T, \alpha{\downarrow}) \text{ is bisimilar to a linear tree and} \\ \mathrm{PD}(T, \alpha{\downarrow\!\!\downarrow}) \text{ is bisimilar to a fin. branching tree} \end{array}$$

to the relaxed tree game characterizes $\mathbf{\Lambda}_{1,\alpha+1}$, i.e., the Baire class α functions.

Idea of the proof, hard direction

Assuming $f \in \mathbf{\Lambda}_{1,\alpha+1}$, for each $\sigma \in \mathbb{N}^{<\mathbb{N}}$ we have

$$f^{-1}[\sigma] \quad \in \quad \mathbf{\Sigma}^0_{\alpha+1}$$

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- every descendant τ' of τ makes a guess for the "$\exists k$" quantifier of (1) when the first $\forall$ is instantiated with $|\tau'| - |\tau| - 1$, and so on.

  (If no more quantifiers, then only add τ' to T in case the corresponding formula is true (open)/still possibly true (closed).)

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3. If the "first guess" $\phi(\tau), n$ that τ makes is bigger than the least correct pair, then $\tau \notin \mathrm{PD}(T, \alpha\downarrow)$.
   ($\Rightarrow \mathrm{PD}(T, \alpha\downarrow)$ is bisimilar to a f.b. tree)

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Partition classes

Let "$\mathrm{BC}\ \alpha$ on $\mathbf{\Pi}^0_\beta$" be the class of functions f for which there exists a partition of $\mathrm{dom}(f)$ into countably many $\mathbf{\Pi}^0_\beta$ parts, such that the restriction of f to each part is Baire class α.

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2. $\mathbf{\Lambda}_{2,3} =$ "$\mathrm{BC}\ 1$ on $\mathbf{\Pi}^0_2$" (Semmes);
3. $\mathbf{\Lambda}_{3,3} =$ "$\mathrm{BC}\ 0$ on $\mathbf{\Pi}^0_2$" (Andretta-Semmes);

Two definitions

The product $T \times (S, \phi_S)$ of a tree T and a labeled tree (S, ϕ_S) is the labeled tree with underlying set $T \times S$ and labeling function inherited from (S, ϕ_S):

$$\phi(\sigma, \tau) = \phi_S(\tau)$$

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Given a tree T and $\sigma \in \mathbb{N}^{<\mathbb{N}}$, we denote

$$T_\sigma := \{\tau \in T\,;\, \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}.$$

The game

Theorem

For $\alpha < \beta$, "$\mathrm{BC}\ \alpha$ on $\mathbf{\Pi}^0_\beta$" is characterized by the tree game with additional rules

1. $\mathrm{PD}(T, \beta{\downarrow})$ is linear;
2. $\mathrm{PD}(T, \beta{\twoheaddownarrow})$ is fin. branching; and
3. for each $n \in \mathbb{N}$ there exist a tree S and a labeled tree (U, ϕ_U) such that
    1. $(T_{\langle n \rangle}, \phi) \simeq S \times (U, \phi_U)$;
    2. $\mathrm{PD}(S, \beta{\downarrow} - 1)$ is linear; and
    3. $\mathrm{PD}(U, \alpha{\downarrow})$ is linear; and
    4. $\mathrm{PD}(U, \alpha{\twoheaddownarrow})$ is fin. branching.

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The games by Andretta, Semmes, and Andretta-Semmes are particular cases.

The game

Theorem

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The games by Andretta, Semmes, and Andretta-Semmes are particular cases.

Thanks for your attention!