

BSS RAM's with \aleph_1 -Oracle and the Axiom of Choice

Christine Gaßner

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BSS RAM's with ν -Oracle and the Axiom of Choice

(History and Outline)

↓ Stephen C. Kleene

Recursion Theory based on recursion and μ -operator

↓ Yiannis N. Moschovakis

Generalized Recursion Theory based on recursion and ν -operator

↓ Gaßner

ν -Operators for BSS RAM's over arbitrary mathematical structures

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Computable multi-valued correspondences

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Question:

Are there computable choice functions for these correspondences?

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Outline:

- BSS RAM's
- A characterization of [non-]deterministic semi-decidability
- $AC^{n,m}$ (in HPL) and effective $AC^{n,m}$ and AC^∞

Computation by BSS RAM's over Algebraic Structures

(The Machines and the Allowed Instructions)

Computation over $\mathcal{A} = (\underbrace{U_{\mathcal{A}}}_{\text{universe}}; \underbrace{C_{\mathcal{A}}}_{\text{constants}}; \underbrace{f_1, \dots, f_{n_1}}_{\text{operations}}; \underbrace{R_1, \dots, R_{n_2}}_{\text{relations}}, =)$.

Z_1	Z_2	Z_3	Z_4	Z_5	...
-------	-------	-------	-------	-------	-----

Registers for elements in $U_{\mathcal{A}}$

I_1	I_2	I_3	I_4	...	$I_{k_{\mathcal{M}}}$
-------	-------	-------	-------	-----	-----------------------

Registers for indices in \mathbb{N}

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Registers for indices in \mathbb{N}

- **Computation** instructions:

$$\ell: Z_j := f_k(Z_{j_1}, \dots, Z_{j_{m_k}})$$

$$\text{(e.g. } \ell: Z_j := Z_{j_1} + Z_{j_2} \text{)}$$

$$\ell: Z_j := d_k$$

$$(d_k \in C_{\mathcal{A}} \subseteq U_{\mathcal{A}})$$

- **Branching** instructions:

$$\ell: \text{if } Z_i = Z_j \text{ then goto } \ell_1 \text{ else goto } \ell_2$$

$$\ell: \text{if } R_k(Z_{j_1}, \dots, Z_{j_{n_k}}) \text{ then goto } \ell_1 \text{ else goto } \ell_2$$

- **Copy** instructions:

$$\ell: Z_{I_j} := Z_{I_k}$$

- **Index** instructions:

$$\ell: I_j := 1$$

$$\ell: I_j := I_j + 1$$

$$\ell: \text{if } I_j = I_k \text{ then goto } \ell_1 \text{ else goto } \ell_2$$

Uniform Computation over Algebraic Structures

(Input and Output Procedures of Machines in $M_{\mathcal{A}}$)

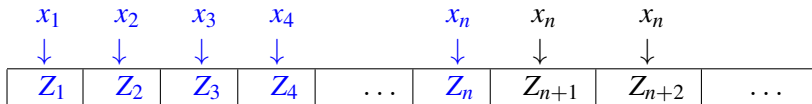
- $U_{\mathcal{A}}$ is the universe of \mathcal{A}
- Input and output space: $U_{\mathcal{A}}^{\infty} =_{\text{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^i$
- **Input** of $\vec{x} = (x_1, \dots, x_n) \in U_{\mathcal{A}}^{\infty}$:

x_1 x_2 x_3 x_4 x_n

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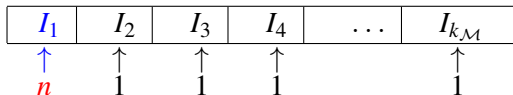
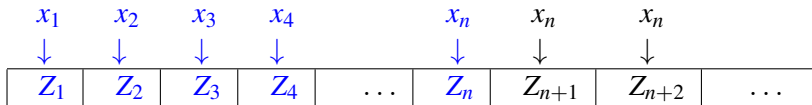
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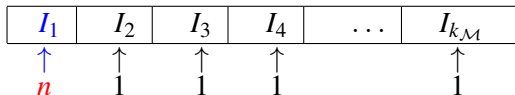
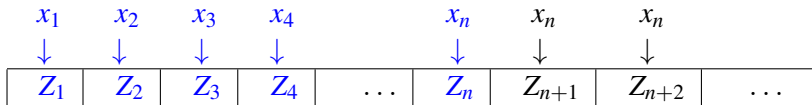
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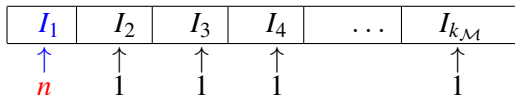
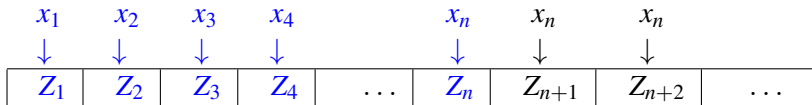


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- $U_{\mathcal{A}}$ is the universe of \mathcal{A}
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- **Input** of $\vec{x} = (x_1, \dots, x_n) \in U_{\mathcal{A}}^{\infty}$:



!!!

- **Output** of Z_1, \dots, Z_{I_1} .

$[\nu\text{-}]$ Semi-Decidability

(The Definitions)

$P \subseteq U_{\mathcal{A}}^{\infty}$ is a *decision problem*.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *semi-decidable*

if there is a BSS RAM \mathcal{M} such that $\vec{x} \in P \Leftrightarrow \underbrace{\mathcal{M}(\vec{x}) \text{ halts on } \vec{x}}_{\mathcal{M}(\vec{x})\downarrow}$.

We will also use:

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *nondeterministically semi-decidable*

if there is a nondeterministic BSS RAM \mathcal{M} such that $\vec{x} \in P$
 $\Leftrightarrow \underbrace{\mathcal{M} \text{ halts on } \vec{x} \text{ for some guesses.}}_{\mathcal{M}(\vec{x})\downarrow}$.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *ν -semi-decidable*

if there is a ν -oracle BSS RAM semi-deciding P .

...

ν -oracle BSS RAM = BSS RAM being able to use operator ν

...

μ -Oracle BSS RAM's with μ -Operators for $\mathbb{N} \subseteq U_{\mathcal{A}}$

(Kleene's Operator μ)

\mathcal{A} fixed, $\mathbb{N} \subseteq U_{\mathcal{A}}$ effectively enumerable over \mathcal{A} ,
 $f : U_{\mathcal{A}}^{\infty} \rightarrow \underbrace{\{a, b\}}_{\{1, 0\}}$ partial function, computable over \mathcal{A} .

Definition (Kleene's operator for \mathcal{A})

$$\begin{aligned} & \mu[f](x_1, \dots, x_n) \\ & =_{\text{df}} \min\{k \in \mathbb{N} \mid f(x_1, \dots, x_n, k) = 1 \ \& \ f(x_1, \dots, x_n, l) \downarrow \ \text{for } l < k, l \in \mathbb{N}\} \end{aligned}$$

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Example

$\mathcal{A} = (\mathbb{N}; 0; +, -; \leq, =)$

$$f_0(a_1, \dots, a_n, x) := \begin{cases} 1 & \text{if } \underbrace{x^n + a_n x^{n-1} + \dots + a_1 x^0}_{p(x)} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow \mu[f_0](a_1, \dots, a_n) = \text{the smallest zero of } p$

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Definition (Oracle Instruction with Kleene's operator)

$$\begin{array}{ccc} z_1 & \cdots & z_n \\ \downarrow & & \downarrow \\ \ell : Z_j := \mu[f](Z_1, \dots, Z_{I_1}), & & \text{if } I_1 = n \end{array}$$

no minimum \Rightarrow the machine loops forever

Properties

Any μ -semi-decidable problem is semi-decidable over \mathcal{A} .

ν -Oracle BSS RAM's for Structures with a and b

(Moschovakis' Operator ν)

\mathcal{A} is fixed. a, b are constants of \mathcal{A} .

$f : U_{\mathcal{A}}^{\infty} \rightarrow \{a, b\}$ partial function, computable over \mathcal{A} .

Definition (Moschovakis' operator for \mathcal{A})

$$\begin{aligned} & \nu[f](x_1, \dots, x_n) \\ =_{\text{df}} & \{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \dots, y_m) \in U_{\mathcal{A}}^{\infty}) (f(x_1, \dots, x_n, \underbrace{y_1, y_2, \dots, y_m}_{\vec{y} \in U_{\mathcal{A}}^{\infty}}) = a)\} \end{aligned}$$

Definition (Oracle instruction with Moschovakis' operator)

$$\begin{array}{ccc} & z_1 & \cdots & z_n \\ & \downarrow & & \downarrow \\ \text{NONDETERMINISTIC!} & \ell : Z_j := \nu[f](Z_1, \dots, Z_{l_1}) & & \end{array}$$

$\nu[f](z_1, \dots, z_n) \neq \emptyset \Rightarrow Z_j$ contains some $z \in \nu[f](z_1, \dots, z_n)$.

$\nu[f](z_1, \dots, z_n) = \emptyset \Rightarrow$ no stop (the machine loops forever).

Nondeterministic Machines versus ν -oracle Machines

(Guessing Solutions and Nondeterministic Semi-Decidability)

$f : U_{\mathcal{A}}^{\infty} \rightarrow \{a, b\}$ partial function, computable by \mathcal{M}_f over \mathcal{A} .

Properties (By ν -operator of a ν -oracle machine)

$$\begin{array}{ccccccc} & x_1 & \cdots & x_n & & x_1 & \cdots & x_n & y_1 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow \\ Z_j := \nu[f](Z_1, \dots, Z_{I_1}); & \dots & & Z_j := \nu[f](Z_1, \dots, Z_{I_1-1}, Z_{I_1}) & \dots & & & & \\ \downarrow & & & \downarrow & & & & & \\ y_1 & & & y_2 & & & & \Rightarrow f(x_1, \dots, x_n, y_1, \dots, y_m) = a & \end{array}$$

Properties (By input-guessing procedure of nondeterm. machine)

x_1	x_2		x_n	y_1	y_2		y_m	x_n	
\downarrow	\downarrow		\downarrow	\downarrow	\downarrow		\downarrow	\downarrow	
Z_1	Z_2	\dots	Z_n	Z_{n+1}	Z_{n+2}	\dots	Z_{n+m}	Z_{n+m+1}	\dots

Then, simulate \mathcal{M}_f .

Proposition

$A \subseteq U_{\mathcal{A}}^{\infty}$ is ν -semi-decidable iff A is nondeterm. semi-decidable.

ν_n -Oracle BSS RAM's versus ν -Oracle BSS RAM's

(For Motivation: Computable Choice Functions?)

$$\mathcal{A} = (\mathbb{N}; \mathbb{N}; ; =).$$

$$R(x_1, \dots, x_n) := \begin{cases} 1 & \text{if } x_i \neq x_j \text{ for all } i, j \text{ with } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Example (ν_n for fixed arity n)

$$\begin{array}{c} (z_1, \dots, z_n) \in \mathbb{N}^n \\ \downarrow \quad \downarrow \\ \ell : Z_j := \nu_n[R](Z_1, \dots, Z_n) \end{array}$$

Example (ν for any arity)

$$\begin{array}{c} (z_1, \dots, z_n) \in \mathbb{N}^\infty \\ \downarrow \quad \downarrow \\ \ell : Z_j := \nu[R](Z_1, \dots, Z_{I_1}) \end{array}$$

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Properties

In both cases, we get a $z \in \mathbb{N} \setminus \{z_1, \dots, z_n\}$ if $z_i \neq z_{i+k}$. ($k \geq 1$)

For the ν_n -computable correspondence $\mathbb{N}^n \ni (z_1, \dots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \dots, z_n\}$ we have a choice function computable by means of $n + 1$ constants.

For the ν -computable correspondence $\mathbb{N}^\infty \ni (z_1, \dots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \dots, z_n\}$ we do not have a computable choice function.

The Axiom of Choice in the Second-Order Logic

(Some Definitions and Relationships between Statements Related to AC in HPL)

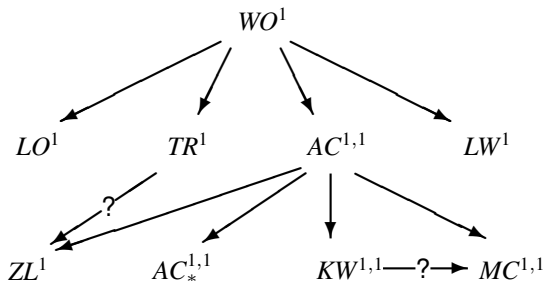
$$AC^{n,m} =_{\text{df}} \forall A \forall R \exists S (\text{cor}(R, A) \rightarrow \forall \vec{X} (A\vec{X} \rightarrow \exists !! \vec{W} (R(\vec{X} . \vec{W}) \wedge S(\vec{X} . \vec{W}))))$$

$$WO^n =_{\text{df}} \forall A \exists T (\text{wo}(T, A))$$

$$LO^n =_{\text{df}} \forall A \exists T (\text{lo}(T, A))$$

...

For Henkin-structures (satisfying the axioms of comprehension):



If we find neither $H_1 \text{---} ? \rightarrow H_2$ nor, for a statement H_3 , $H_1 \rightarrow H_3$ and $H_3 \rightarrow H_2$, then $H_1 \rightarrow H_2$ is not deducible from $^h ax.$ (Gaßner 1994)

Effective Second Order Logic and a Generalization

(The Axiom of Choice)

Definition (An effective form of the axiom of choice over \mathcal{A})

A semi-decidable

R semi-decidable correspondence with domain A

\Rightarrow There is a semi-decidable mapping S such that $(*)$ is satisfied.

Details (An effective $AC^{n,m}$)

$$\begin{aligned} A &\subseteq U_{\mathcal{A}}^n \\ R &\subseteq U_{\mathcal{A}}^{n+m} \\ S &\subseteq U_{\mathcal{A}}^{n+m} \end{aligned}$$

$(*)$

$$\forall \vec{X} (A\vec{X} \rightarrow \exists !! \vec{Y} (R\langle \vec{X}, \vec{Y} \rangle \wedge S\langle \vec{X}, \vec{Y} \rangle))$$

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(for tuples in $U_{\mathcal{A}}^n$ and $U_{\mathcal{A}}^m$)

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$R\langle \vec{X}, \vec{Y} \rangle$ means $(X_1, \dots, X_n, Y_1, \dots, Y_m) \in R$ (for tuples in $U_{\mathcal{A}}^n$ and $U_{\mathcal{A}}^m$)

$R\langle \vec{X}, \vec{Y} \rangle$ means $(X_1, a, \dots, X_{n-1}, a, X_n, b, Y_1, a, \dots, Y_{m-1}, a, Y_m) \in R$

(for tuples in $U_{\mathcal{A}}^{\infty}$)

Structures with AC^∞ and without AC^∞

(Some Examples)

Example (Structures with effective AC^∞)

- $(\{0, 1\}; 0, 1; ; =)$
- $(\mathbb{N}; 0; s; =), \quad s(n) = n + 1$
- $(\mathbb{Q}; \mathbb{Q}; +, -; \leq, =)$
- $(\mathbb{R}; \mathbb{R}; +, -; \leq, =)$
- ...

Example (Structures without effective AC^∞)

- ?

A Characterization of Semi-Decidability

(Transfer of a Method from the Second-Order Logic)

Properties (Representation of \mathcal{A} by predicates)

$$REL_{\mathcal{A}} =_{\text{df}} \{ \underbrace{R_1, \dots, R_{n_2}}_{R_j \subseteq U_{\mathcal{A}}^{n_j}}, \underbrace{F_1, \dots, F_{n_1}}_{F_j = \{(x_1, \dots, x_{m_j}, y) \mid y = f_j(x_1, \dots, x_{m_j})\}} \subseteq U_{\mathcal{A}}^{m_j+1} \}$$

$$R_j \subseteq U_{\mathcal{A}}^{n_j} \quad F_j = \{(x_1, \dots, x_{m_j}, y) \mid y = f_j(x_1, \dots, x_{m_j})\} \subseteq U_{\mathcal{A}}^{m_j+1}$$

π permutation of $U_{\mathcal{A}}$

$$\pi(A) =_{\text{df}} \bigcup_n \{ (\pi(x_1), \dots, \pi(x_n)) \mid (x_1, \dots, x_n) \in A \} \quad (A \subseteq U_{\mathcal{A}}^{\infty})$$

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\mathcal{G} is the group of relations-preserving automorphisms π of $U_{\mathcal{A}}$
with

$$\pi(R_j) = R_j,$$

$$\pi(F_j) = F_j.$$

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$$R_j \subseteq U_{\mathcal{A}}^{n_j} \quad F_j = \{(x_1, \dots, x_{m_j}, y) \mid y = f_j(x_1, \dots, x_{m_j})\} \subseteq U_{\mathcal{A}}^{m_j+1}$$

π permutation of $U_{\mathcal{A}}$

$$\pi(A) =_{\text{df}} \bigcup_n \{ (\pi(x_1), \dots, \pi(x_n)) \mid (x_1, \dots, x_n) \in A \} \quad (A \subseteq U_{\mathcal{A}}^{\infty})$$

\mathcal{G} is the group of relations-preserving automorphisms π of $U_{\mathcal{A}}$
with

$$\pi(R_j) = R_j,$$

$$\pi(F_j) = F_j.$$

Definition (Some subgroups of \mathcal{G})

$$\mathcal{G}(P) =_{\text{df}} \{ \pi \in \mathcal{G} \mid (\forall x \in P)(\pi(x) = x) \} \quad (P \subseteq U_{\mathcal{A}})$$

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A Characterization of Semi-Decidability

(Transfer of a Method from the Second-Order Logic)

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Theorem (A property of [non-deterministic] semi-decidability)

For any $A \subseteq U_{\mathcal{A}}^{\infty}$ that is [non-deterministically] semi-dec. over \mathcal{A} , there is a **finite** $P \subseteq U_{\mathcal{A}}$ such that $\mathcal{G}(P) \subseteq \text{sym}_{\mathcal{G}}(A)$.

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More general:

\mathcal{G} is the group of permutations of $U_{\mathcal{A}}$.

Let $\mathcal{I}_{\mathcal{A}} \subseteq \mathcal{P}(U_{\mathcal{A}} \cup \text{REL}_{\mathcal{A}})$ be a normal ideal in $U_{\mathcal{A}}$ with respect to \mathcal{G} .

Theorem (Normal ideals and [non-deterministic] semi-decidability)

For any $A \subseteq U_{\mathcal{A}}^{\infty}$ that is [non-deterministically] semi-dec. over \mathcal{A} , there is a $P \in \mathcal{I}_{\mathcal{A}}$ such that $\mathcal{G}(P) \subseteq \text{sym}_{\mathcal{G}}(A)$.

Structures with AC^∞ and without AC^∞

(Some Examples)

Example (Structures with effective AC^∞)

- $(\{0, 1\}; 0, 1; ; =)$
- $(\mathbb{N}; 0; s; =), \quad s(n) = n + 1$
- $(\mathbb{Q}; \mathbb{Q}; +, -; \leq, =)$
- $(\mathbb{R}; \mathbb{R}; +, -; \leq, =)$
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Example (Structures without effective AC^∞)

- $(\mathbb{N}; \mathbb{N}; ; =)$
- $(\mathbb{N} \times \mathbb{N}; \mathbb{N} \times \{0\}; f; \leq_{\text{lexi}}, =), \quad f(n, m) = (n, 0)$
(Note: \leq_{lexi} is a decidable well-ordering on $\mathbb{N} \times \mathbb{N}$.)
- $(\mathbb{Q}; \mathbb{N}; s; \leq, =), \quad s(n) = n + 1$
- ...

Thank you very much for your attention!

References

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