A tutorial in Generalized Baire Spaces: *Games, trees and models*

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Background

- Classical Baire space is the space of irrational numbers, it arises from analysis.
- Mostowski and others started the study of countable models of first order theories using analytic (and topological) methods.
- Stability theory, infinitary logic, and generalized quantifiers led to **uncountable** structures.
- **Generalized Baire spaces** are suitable for topological study of uncountable models of theories in first order logic and its extensions.

Models i.e. structures

- Relational structure (M,R,...).
- A set with relations, functions and constants.
- Partial orders, trees, linear orders, lattices, groups, semigroups, fields, monoids, graphs, hypergraphs, directed graphs.

Models and topology

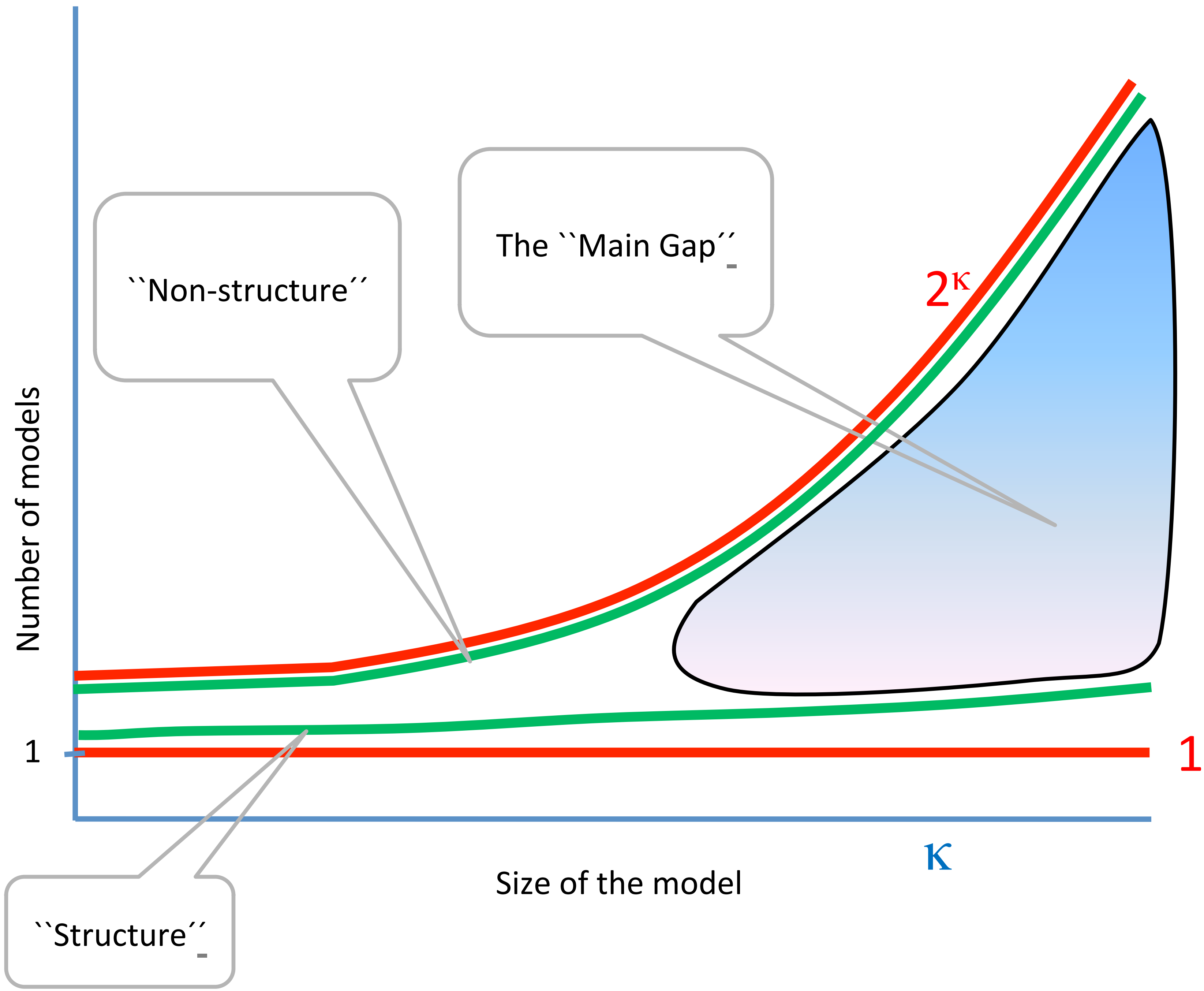
- A countable model is a point in 2^ω (mod ≅).
- A model of size κ is a point in 2^κ (mod ≅).
- Properties of models ~ subsets of 2^κ.
- Isomorphism of models:``analytic” subset of 2^κx2^κ.

The basic question

- How to identify a structure?
- Relevant even for finite structures.
- Can infinite structures be classified by invariants?

Shelah's Main Gap

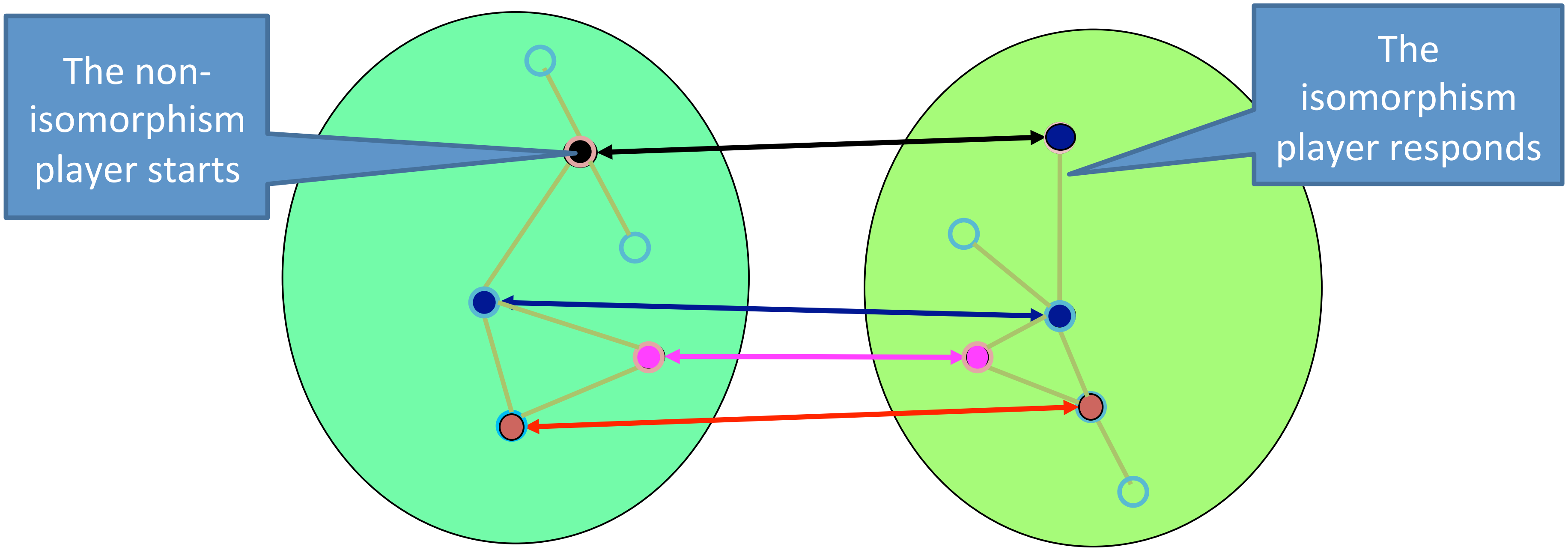
- M any structure.
- The first order theory of M is either of the two types:
  - Structure Case: All uncountable models can be characterized in terms of dimension-like invariants.
  - Non-structure case: In every uncountable cardinality *there are* non-isomorphic models that are ``extremely" difficult to distinguish from each other by means of invariants (but some other models of the theory may be easy to distinguish from each other).


``Non-structure´´
The ``Main Gap´´
2^{κ}
Number of models
1
1
``Structure´´
Size of the model
κ

The program

- To analyze further the non-structure case.
  - We replace isomorphism by a game.
  - We develop the topology of 2^{κ}.

Ehrenfeucht-Fraïssé game



Two players: The non-isomorphism player and the isomorphism player.

Approximating isomorphism

- M,N countable (graphs, posets,...)
- M $\ncong$ N
- The non-isomorphism player wins the EF game of length ω with the enumeration strategy τ.
- T(M,N)=the countable tree of plays against τ, where the isomorphism player has not lost yet.
- T(M,N) has no infinite branches, well-founded.

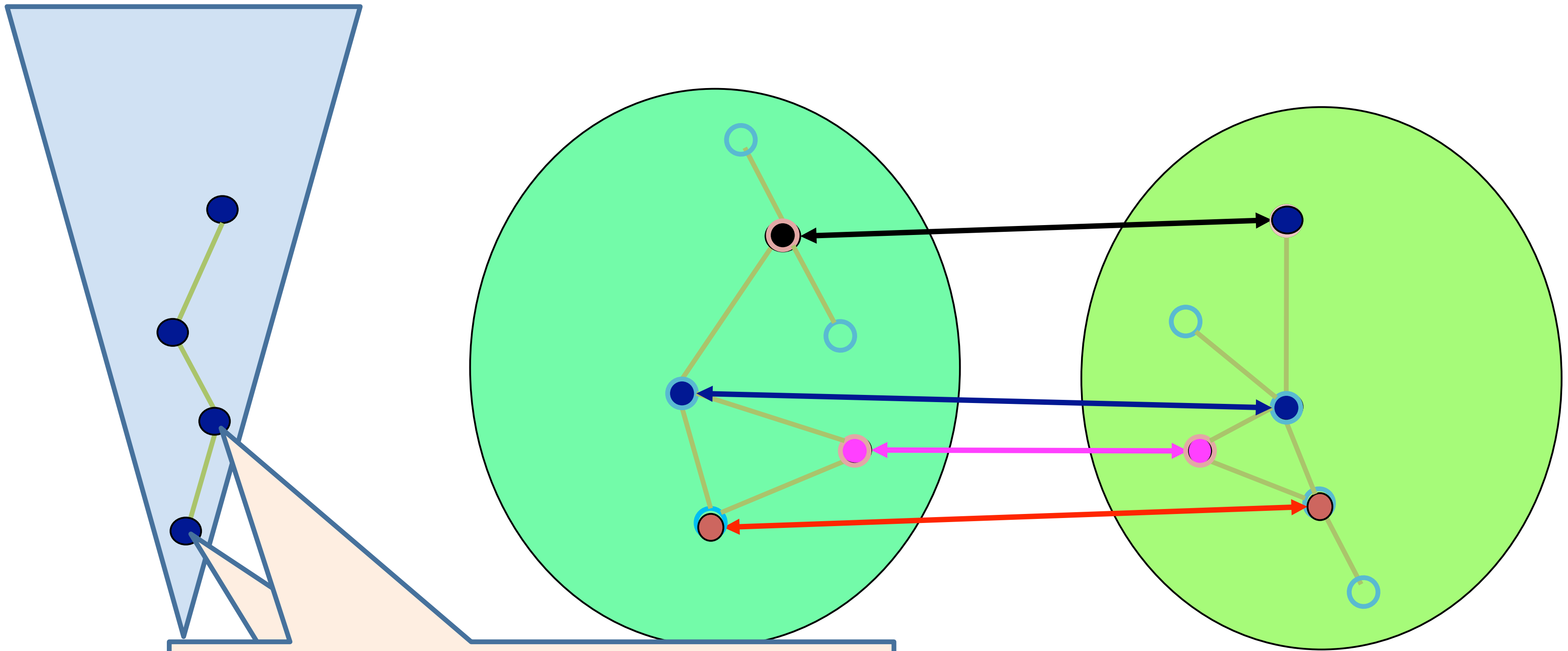
Approximating isomorphism (contd.)

- T(M,N) has a rank $\alpha<\omega_1$, which we can minimize.
- $\sigma_M=\sup_{\mathbf{a},\mathbf{b}}\{\mathrm{rank}(T((M,\mathbf{a}),(M,\mathbf{b}))) : (M,\mathbf{a}) \not\cong M,\mathbf{b})\}$.
- Scott rank of M.
- Scott ranks put countable models into a hierarchy, calibrated by countable ordinals.
- The orbit of M is a Borel subset of 2^ω.
- 60's and 70's: Scott, Vaught: invariant topology.
- 90's and 00's: Kechris, Hjorth, Louveau: Borel equivalence relations.

Game with a clock

- The isomorphism player loses the EF game of length ω, but maybe she can win if the non-isomorphism player is forced to obey a clock.

Ehrenfeucht-Fraïssé game with a clock



The non-isomorphism player goes up the clock-tree

The clock gives a chance

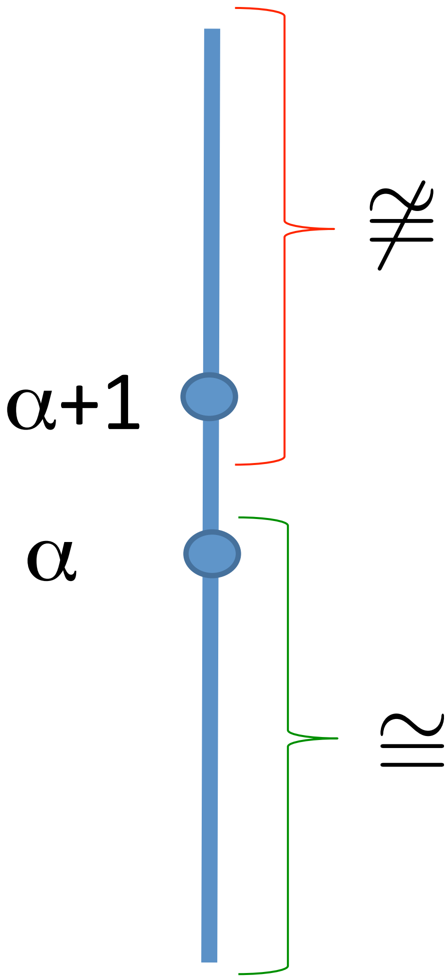
- Although the isomorphism player loses the EF-game of length ω, she wins the game if T(M,N) is the clock.
  - Recall: T(M,N)=the tree of plays against τ, where the isomorphism player has not lost yet.

A well-founded clock

- The tree B_α of descending sequences of elements of α is the canonical well-founded tree of rank α.

For countable M and N:

- TFAE:
  - M ≅ N
  - The isomorphism player wins the EF game clocked by B_α for some $\alpha<\omega_1$ such that the non-isomorphism player wins with clock $B_{\alpha+1}$.



An ordering of trees

- T≤T' if there is f:T→T' such that

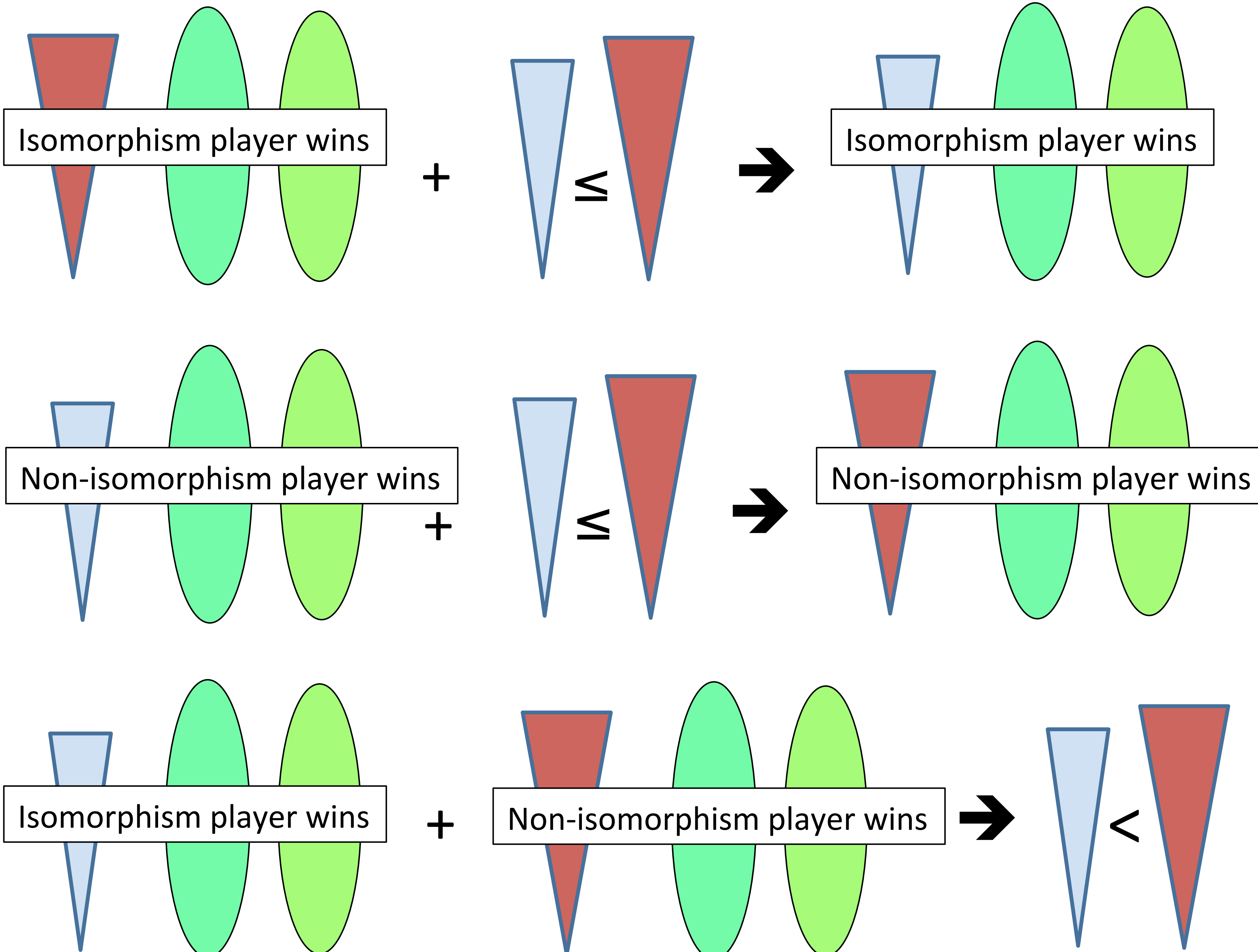
$$x <_T y \rightarrow f(x) <_T f(y).$$

- If T and T' do not have infinite branches, then T≤T' iff rank(T)≤rank(T').
- Fact: T≤T' iff II wins a comparison game on T and T'.

T≤T' ranks game clocks

- If T≤T' then a game clocked by T is
  - easier for the isomorphism player
  - harder for the non-isomorphism player

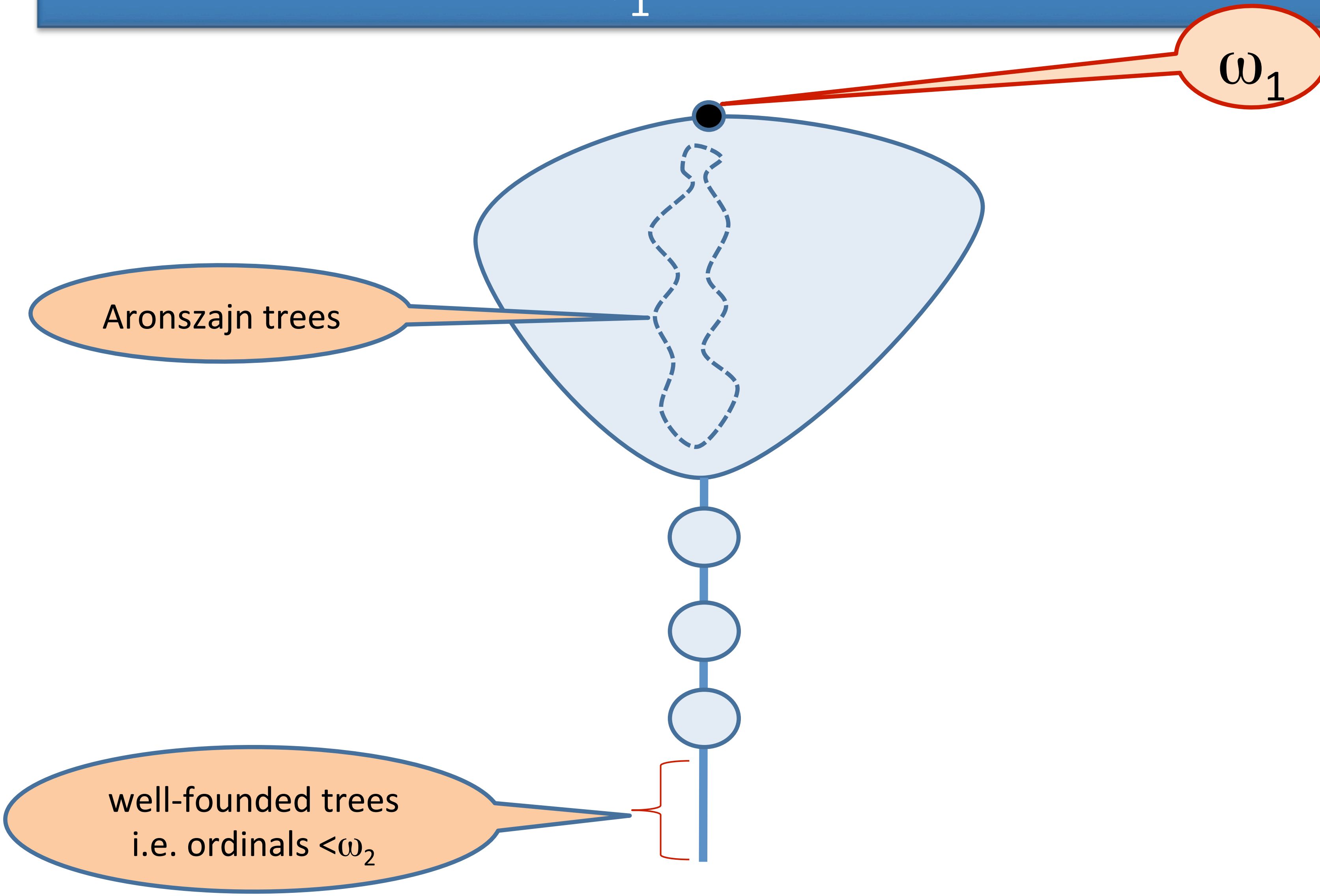
  than the same game clocked by T'.


Isomorphism player wins
+
≤
Isomorphism player wins
Non-isomorphism player wins
+
≤
Non-isomorphism player wins
Isomorphism player wins
+
Non-isomorphism player wins
<

There are incomparable trees

- (Todorčević) There are incomparable Aronszajn trees.
- A tree is a bottleneck if it is comparable with every other tree.
- (Mekler-V., Todorčević-V.) It is consistent that there are no non-trivial bottlenecks.
- (Todorčević) PFA→coherent Aronszajn trees are all comparable, and there is a canonical family of coherent Aronszajn trees that are bottlenecks in the class of trees of size $\aleph_1$.

The structure of trees of size and height $\aleph_1$ under $\leq$



A ``successor" operator on trees

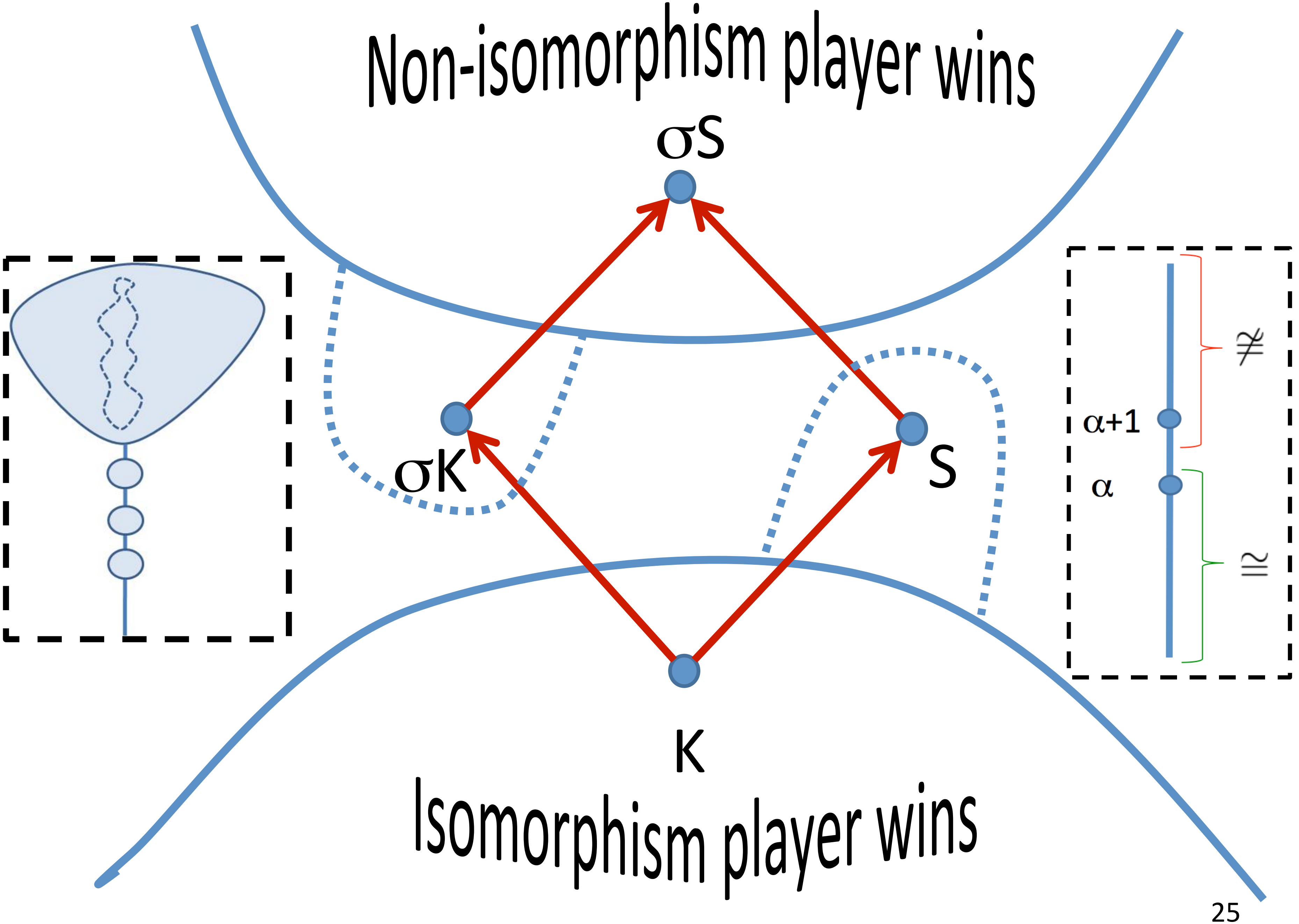
- T a tree
- σT = the tree of ascending chains in T
- T< σT
- $\sigma B_\alpha \equiv B_{\alpha+1}$
- Definition: T << T' iff σT ≤ T'.
- T << T' implies T<T'
- << is well-founded

The uncountable case

- M,N of size κ.
- M $\ncong$ N.
- The non-isomorphism player wins the EF game of length κ with the enumeration strategy τ.
- T(M,N)=the tree of plays against τ, where the isomorphism player has not lost yet.
- T(M,N) has no branches of length κ.
- The cardinality of T(M,N) is $\kappa^{<\kappa}$.

Watershed

- For M and N of cardinality κ TFAE:
  - $M \not\cong N$
  - The isomorphism player wins the EF game clocked by K for some tree K w/o κ-branches, $|K| \leq 2^{\kappa^{<\kappa}}$, but does not win the game clocked by σK. **(K=the tree of winning strategies of isomorphism player in short games).**
  - The non-isomorphism player does not win the EF game clocked by S for some tree S w/o κ-branches, $|S| \leq \kappa^{<\kappa}$, but wins if clock is σS. **(S is of the form T(M,N) for an enumeration strategy τ which renders S minimal in <<.)**


Non-isomorphism player wins
σS
σK
S
K
α+1
α
≇
≅
Isomorphism player wins

Non-determinacy of the EF game

- Determinacy of the EF game of length ω_1 in the class of models of size $\aleph_2$ is equiconsistent with the existence of a weakly compact cardinal. (Hyttinen-Shelah-V.)

Generalized Baire space

- $\omega_1^{\omega_1}$, models of size $\aleph_1$
  - G_δ-topology.
  - ω_1-metrizable, ω_1-additive.
  - meager ($\bigcup_{\alpha<\omega_1} A_\alpha$, A_α nowhere dense), Baire Category Theorem holds: B_α dense open ➔ $\bigcap_{\alpha<\omega_1} B_\alpha \neq \emptyset$.
  - dense set of continuum size.
  - A a topological space: Sikorski 50s, Juhasz &Weiss 70s, Todorčević 80s,
  - As **descriptive set theory "higher up":** Halko, Mekler, Shelah, Todorčević, V. 90s
- κ^κ, models of size κ
- λ^κ, κ=cof(λ), models of size λ, which are unions of chains of length κ of smaller models. (Dzamonja-V. 2011)

A Cantor-Bendixson Theorem

- Assume I(ω): There is a normal ideal on ω_2 such that the complement contains a dense σ-closed set.
- Every closed subset of $\omega_1^{\omega_1}$ is ω_1-perfect after removing up to ω_1 elements.
- V. 1991

Another application of I(ω)

- Assume I*(ω): The complement of the non-stationary ideal on ω_1-cofinal elements of ω_2 has a dense σ-closed set.
- Follows: The determinacy of the EF game of length ω_1 in the class of models of size $\aleph_2$. (Mekler-Shelah-V. 1993)

Descriptive Set Theory in $\omega_1^{\omega_1}$

- A set $A \subseteq \omega_1^{\omega_1}$ is analytic if it is the projection of a closed set $\subseteq \omega_1^{\omega_1} \times \omega_1^{\omega_1}$.
- Equivalently, there is a tree $T \subseteq \omega_1^{<\omega_1} \times \omega_1^{<\omega_1}$ such that for all f:

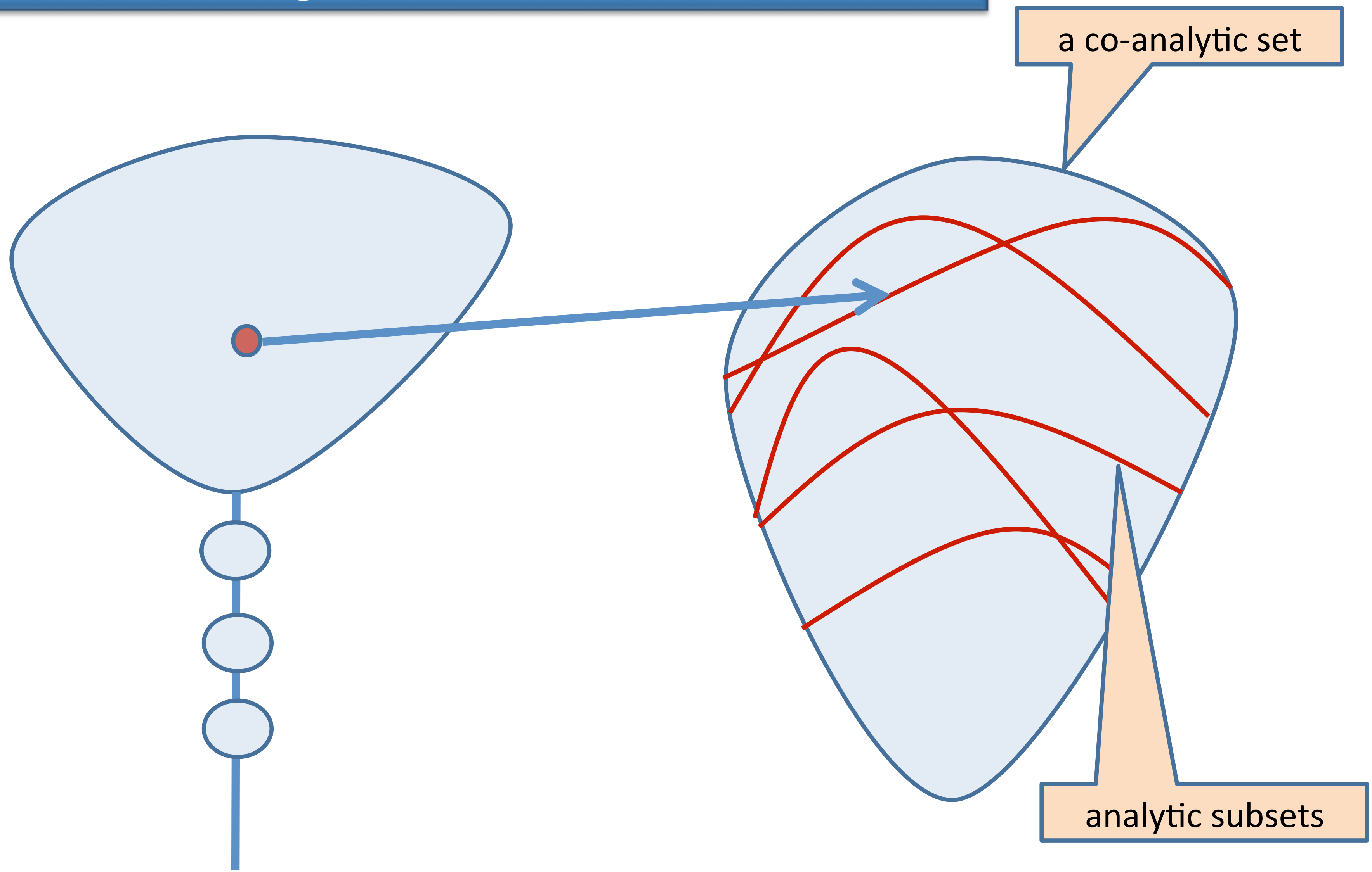
  $f \in A$ iff $T(f)$ has an uncountable branch,

  where $T(f)=\{\mathbf{g}(\alpha) : (\mathbf{g}(\alpha),\mathbf{f}(\alpha))\in T\}$ and $\mathbf{g}(\alpha)=(g(\beta))_{\beta<\alpha}$.

A Covering Theorem

- Every co-analytic subset A of ${\omega_1}^{\omega_1}$ is covered by canonical sets A_T, T a tree w/o uncountable branches, such that every analytic subset of A is covered by some A_T.
- CH implies the sets A_T are analytic and the trees T are of size $\aleph_1$.

Covering Theorem under CH



Proof

- Suppose A is co-analytic and $B \subseteq A$ is analytic.
- $f \in A$ iff $T(f)$ has no uncountable branches.
- $f \in B$ iff $S(f)$ has an uncountable branches.
- Let T' be the tree of $(\mathbf{f}(\alpha),\mathbf{g}(\alpha),\mathbf{h}(\alpha))$ where $\mathbf{g}(\alpha) \in T(f)$ and $\mathbf{h}(\alpha) \in S(f)$.
- If $f \in B$, there is an uncountable branch h in $S(f)$.
- Let $F(\mathbf{g}(\alpha))= (\mathbf{f}(\alpha),\mathbf{g}(\alpha),\mathbf{h}(\alpha))$.
- This is an order preserving mapping $T(f) \rightarrow T'$

Proof contd.

- So $T(f) \leq T'$
- Let $A_{T'} = \{f \in A : T(f) \leq T'\}$.
- Then $B \subseteq A_{T'}$.
- We have proved the Covering Theorem: If A is co-analytic, then A is the union of sets A_T such that if B is any analytic set $\subseteq A$, then there is a tree T w/o uncountable branches such that $B \subseteq A_T$.
- CH implies each A_T is analytic.

Souslin-Kleene, separation

- Souslin-Kleene: If A is analytic co-analytic, then $A=A_T$ for some T w/o uncountable branches.
- Separation: If A and B are disjoint analytic sets, then there is a set $C=(-B)_T$ which separates A and B.

Luzin Separation Theorem?

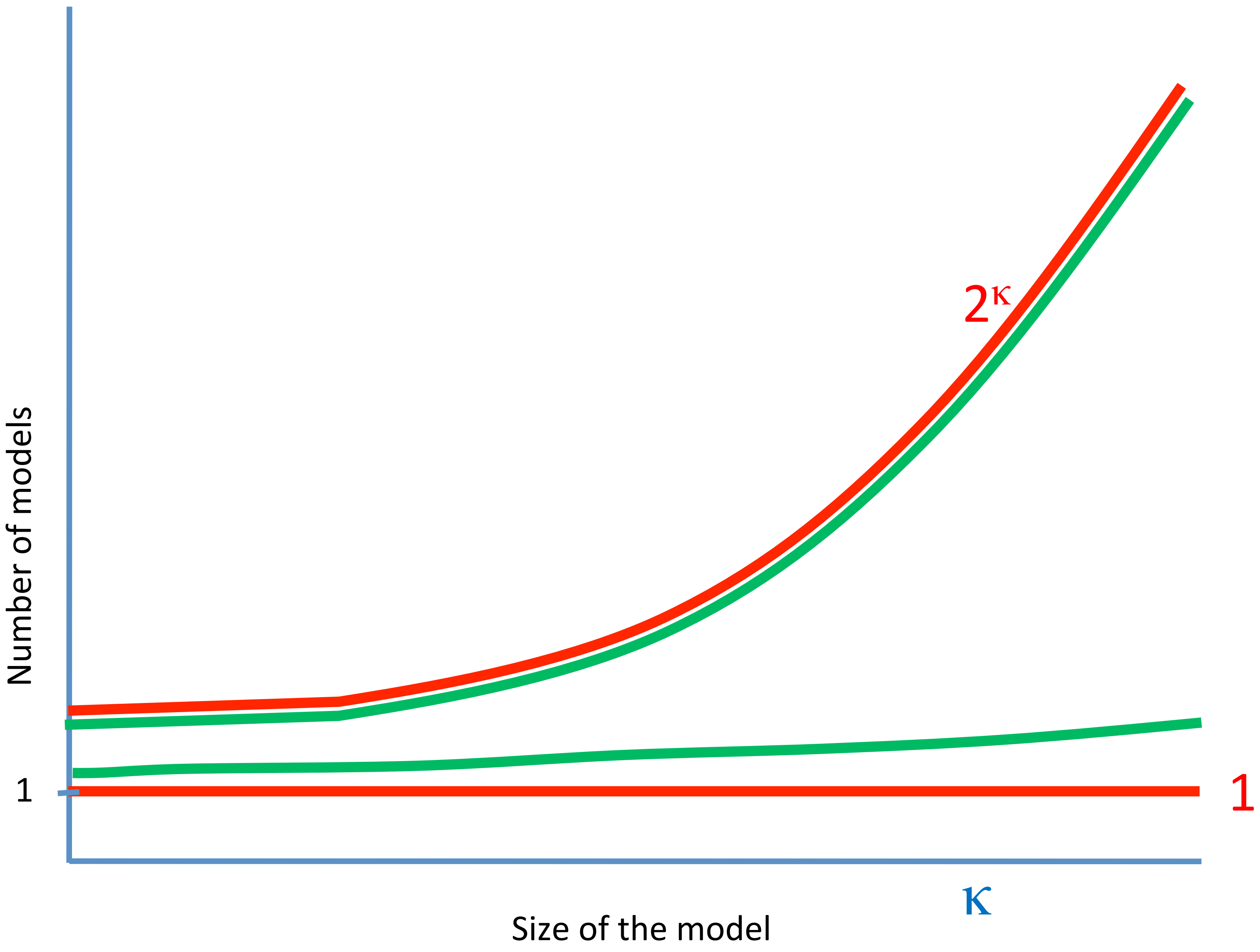
- Borel means closure of open under complements and unions of length ω_1.
- (Shelah-V. 2000)
  - Assume CH. There are disjoint analytic sets which cannot be separated by a Borel set.
  - Assume $\neg$CH+MA. Any two disjoint analytic sets of expansions of $(\omega_1,<)$ can be separated by a Borel set.
- (Mekler-V. 1993, Halko-Shelah 2001,)
  - CUB is not Borel, and ``CUB is analytic co-analytic" is independent of ZFC+CH, as is ``the orbit of the free group of $\aleph_1$ generators is analytic co-analytic".

The analogy

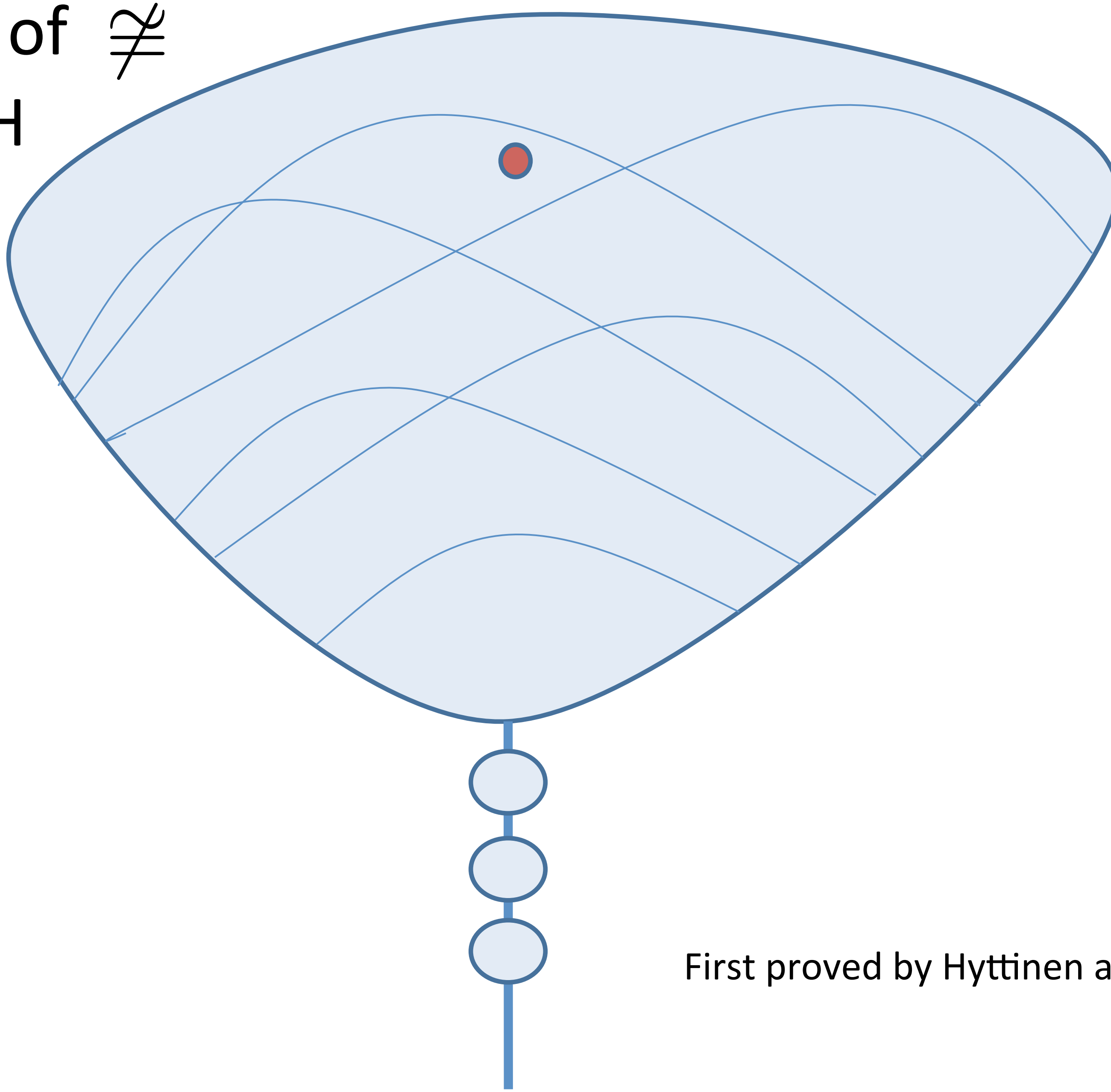
Ordinals	Trees
No descending chains	No uncountable branches
Finite	Countable
Successor ordinal	The tree of all chains of a tree
Ranked game	Clock tree
Comparison of ordinals	Order-preseving mappings
Undefinability of well-order	Undefinability of having an uncountable branch
Baire space ω^ω	Generalized Baire space ${\omega_1}^{\omega_1}$
Union of an analytic set of countable ordinals is countable	Union of an analytic set of trees with no uncountable branches is a tree with no uncountable branches

Definable trees and/or models?

- (J. Steel) Assuming large cardinals,
  - If $T \subseteq R^{<\omega_1}$ is in L(R), then ``T has an uncountable branch" is forcing absolute.
  - If M and N are in L(R) and their universe is ω_1, then $M \cong N$ is absolute with respect to forcing that preserves ω_1.


Number of models
2^{κ}
1
1
κ
Size of the model

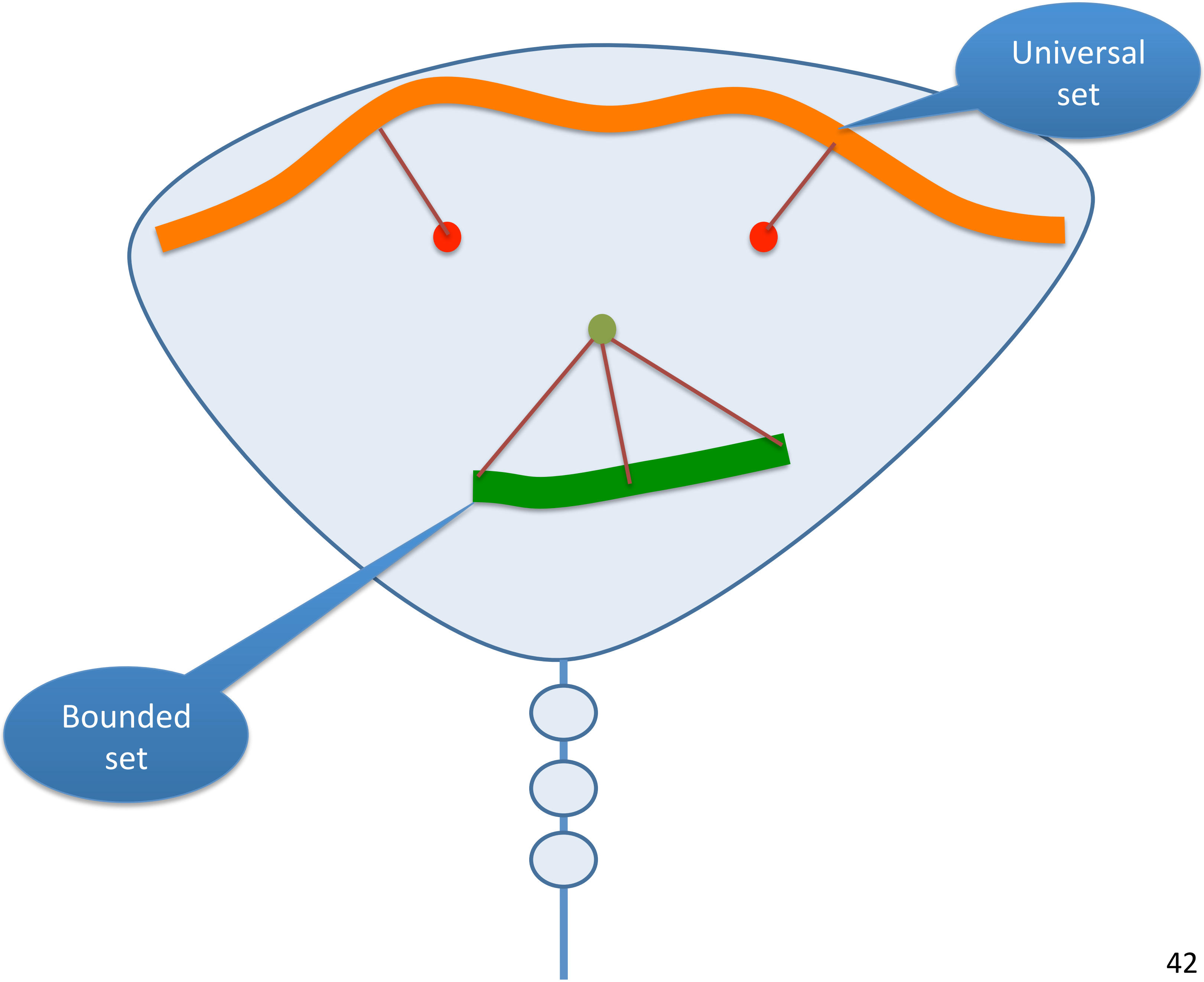
Degrees of $\ncong$ under CH



First proved by Hyttinen and Tuuri

Cardinal invariants about trees

- U(κ) **Universality Property**: There is a family of size κ of trees of size and height $\aleph_1$ w/o branches of length ω_1 such that every such tree is ≤ one in the family.
- B(κ) **Boundedness Property**: Every family of size < κ of trees of size and height $\aleph_1$ w/o branches of length ω_1 has a tree which is ≥ each one in the family.
- C(κ) **Covering Property**: Every co-analytic subset A of ${\omega_1}^{\omega_1}$ is covered by κ analytic sets, such that every analytic subset of A is covered by one of them.


Universal set
Bounded set

Cardinal invariants about trees

- $U(\kappa)$ **Universality Property**
- $B(\kappa)$ **Boundedness Property**
- $C(\kappa)$ **Covering Property**
- Assuming CH: $(U(\kappa)\&B(\lambda))\rightarrow C(\kappa)\&\lambda\leq\kappa$ and $(B(\kappa)\&\lambda<\kappa)\rightarrow\neg C(\lambda)$.
- $U(\kappa)$ & $B(\kappa)$ is consistent with κ anything between $\aleph_2$ and $2^{\aleph_1}$. (Mekler-V. 1993)
- $U(\lambda^+)$ & $B(\lambda^+)$ if $\aleph_1$ replaced by a singular strong limit λ, of cof ω. (Džamonja-V. 2008)

A more recent result of Shelah

- There are structures M and N such that
  - The cardinality of M and N is $\aleph_1$.
  - For all $\alpha<\omega_1$, the isomorphism player wins the EF game of length α.
  - M and N are non-isomorphic.
- The point: CH not assumed.

Kangas-Hyttinen-V. 2013

Theorem

Suppose that κ is a regular cardinal such that $\kappa = \aleph_\alpha$, $\beth_{\omega_1}(|\alpha| + \omega) \leq \kappa$ and $2^\lambda < 2^\kappa$ for all $\lambda < \kappa$. Let T be a countable complete first order theory. Then every model of T of size κ is $L^2_{\kappa\omega}$-characterizable if and only if T is a shallow, superstable theory without DOP or OTOP.

Summary

- In the non-structure case we can get models that are very close to being isomorphic in the sense that
  - the non-isomorphism player does not win even if he is given a large clock tree.
  - the isomorphism player wins in large clock trees.
- Structure of trees under $\leq$: an approach to infinite EF (and other!) games.

Thank you!