Topics in generalized descriptive set theory

Amsterdam workshop on generalized descriptive set theory

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Introduction

Descriptive set theory provides an powerful theory to study properties of definable sets of reals such as the complexity of sets, uniformization and regularity properties, and determinacy of infinite games with definable winning conditions.

The framework for descriptive set theory is Polish metric spaces. The spaces studied in many fields (topology, functional analysis, probability) are of this form. Descriptive set theory allows us to determine the complexity of sets and structures which occur in these fields.

Introduction

Generalized descriptive set theory tries to lift descriptive set theory to the context of larger spaces, for instance generalized Baire and Cantor spaces.

It connects various topics in set theory such as infinite combinatorics, forcing, large cardinals, topology, cardinal characteristics, and Ramsey theory.

There are many applications to the model theory of uncountable structures and classification theory (Friedman, Hyttinen, Kulikov, Väänänen).

We focus on the analogues to complexity classes of sets and the spaces studied in descriptive set theory and draw connections to various topics in set theory.

Generalized Baire spaces

Let κ be an infinite cardinal with $\kappa = \kappa^{<\kappa}$.

Given a cardinal μ, we equip the set ${}^{\kappa}\mu$ consisting of all functions $x : \kappa \longrightarrow \mu$ with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^{\kappa}\mu \mid s \subseteq x\},$$

where s is an element of the set ${}^{<\kappa}\mu$ of all functions $t : \alpha \longrightarrow \mu$ with $\alpha < \kappa$.

We call the space ${}^{\kappa}\kappa$ the *generalized Baire space* for κ.

$\mathbf{\Sigma}^1_1$ sets

A subset A of ${}^\kappa\kappa$ is a $\mathbf{\Sigma}^1_1$-*subset* if it is equal to the projection of a closed subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. We let $\mathbf{\Sigma}^1_1(\kappa)$ denote the class of all such subsets.

It is easy to see that a subset of ${}^\kappa\kappa$ is an element of $\mathbf{\Sigma}^1_1(\kappa)$ if and only if it is equal to a continuous image of a closed subset of ${}^\kappa\kappa$.

If κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$, then a subset of ${}^\kappa\kappa$ is contained in $\mathbf{\Sigma}^1_1(\kappa)$ if and only if it is definable over the structure $(\mathrm{H}_{\kappa^+}, \in)$ by a Σ_1-formula with parameters.

A subset A of ${}^\kappa\kappa$ is a $\mathbf{\Pi}^1_1$-*subset* if its complement is $\mathbf{\Sigma}^1_1$.

$\mathbf{\Pi}^1_1$ sets and trees

The structure of $\mathbf{\Pi}^1_1$ sets is connected with the combinatorics of trees.

Let $\mathcal{T}_\kappa$ be the set of subtrees of ${}^{<\kappa}\kappa$ without cofinal branches, ordered by $T \leq T'$ if there is a strict order preserving map $f\colon T \to T'$.

Lemma (Mekler-Väänänen)

1. *$\mathcal{T}_\kappa$ is $\mathbf{\Pi}^1_1$-complete.*
2. *Every $\mathbf{\Sigma}^1_1$-subset of $\mathcal{T}_\kappa$ is bounded by a tree in $\mathcal{T}_\kappa$.*

Theorem (Mekler-Väänänen)

There is a $<\kappa$-closed κ^+-c.c. forcing which changes the cofinality of $\mathcal{T}_\kappa$ to any regular cardinal μ with $\kappa < \mu \leq 2^\kappa$.

Embedding structure of trees

Suppose that $A \subseteq \kappa$ is bi-stationary. Let $T(A)$ denote the tree of continuous increasing sequences in A.

Lemma (Todorcevic)

Suppose that A, B are bi-stationary subsets of ω_1 and $A \setminus B$ is stationary. Then $T(A) \not\leq T(B)$.

Theorem (Mekler-Shelah-Hyttinen-Rautila)

It is consistent that the set of $T(A)$ for A bi-stationary in ω_1 is bounded (or unbounded) in $\mathcal{T}_{\omega_1}$.

This also implies that the complexity of the club filter on κ can vary between $\mathbf{\Sigma}^1_1$-complete and $\mathbf{\Delta}^1_1$.

Embedding structure of trees

Let $Stat_\kappa$ denote the set of bi-stationary subsets of κ, ordered by inclusion up to nonstationary sets.

Theorem (Thompson-S.)

1. *$Stat_{\omega_1}$ embeds into $\mathcal{T}_{\omega_1}$.*
2. *Every scattered linear order of size κ^+ embeds into $Stat_\kappa$.*

A long-standing open question:

Question

Is it provable that $\mathcal{T}_\kappa$ has no maximal element without the assumption $\kappa^{<\kappa} = \kappa$?

Cardinal characteristics

The method of Mekler-Väänänen is connected with other cardinal characteristics. we order ${}^\kappa\kappa$ by $f \leq^* g$ if $\{\alpha < \kappa \mid f(\alpha) \leq g(\alpha)\}$ is bounded in κ.

Theorem (Cummings-Shelah)

The dominating and bounding numbers of ${}^\kappa\kappa$ can be changed by nonlinear iterations of κ-Hechler forcing.

Theorem (Thompson-S.)

The dominating and bounding numbers of the following types of structures can be changed by nonlinear iterations.

1. *Trees in $\mathcal{T}_\kappa$.*
2. *Posets of size κ without increasing κ-chains.*
3. *Graphs of size κ without cliques of size κ.*

It can also be shown that it is consistent that the least number of $\mathbf{\Sigma}^1_1$ sets needed to cover a given proper $\mathbf{\Pi}^1_1$ set varies for different sets.

For some other characteristics than the bounding and dominating numbers, it is much more difficult to prove analogues to classical results (Brendle, Brooke-Taylor, Hyttinen, Zapletal).

Dichotomies and regularity properties

Definition

Suppose that $\kappa > \omega$ is regular, $\lambda > \kappa$ is inaccessible. Suppose that G is $Col(\kappa, < \lambda)$-generic over V.

1. $V[G]$ is a *κ-Silver model.*
2. $L(P(\kappa))^{V[G]}$ is a *κ-Solovay model.*

Theorem (S.)

In every κ-Solovay model

1. *Every subset of ${}^\kappa\kappa$ has the perfect set property.*
2. *The perfect set game for any subset of ${}^\kappa\kappa$ is determined.*

A similar result holds for the Banach-Mazur game.

Theorem (Laguzzi)

In every κ-Solovay model, every subset of ${}^\kappa\kappa$ is κ-Miller measurable.

Remark (Motto Ros-S.)

There is a clopen subset of $[\kappa]^\kappa$ without the Ramsey property.

Ramsey theory

Lemma (S.)

In every κ-Solovay model, for every function $f\colon [{}^{\kappa}2]^n \to {}^{\kappa}2$ definable from ordinals and subsets of κ, there is a perfect set C such that $f \restriction [C]^n$ is continuous.

Theorem (Blass)

If all sets of reals have the property of Baire, then for every coloring $f\colon [{}^{\omega}2]^m \to 2$, there is a perfect set C such that on $[C]^m$, the color depends only on the splitting type of the tuple.

Theorem (Lücke-Weinert-S.)

1. *In every κ-Solovay model, the analogue to Blass' theorem holds for $m = 2$.*
2. *If the analogue to Blass' theorem holds for $m = 3$, then κ is weakly compact.*

Is the analogue to Blass' theorem for $m = 3$ consistent?

Generalized Choquet spaces

We consider a generalization of the notion of Polish space to κ which includes ${}^{\kappa}\kappa$ and other interesting spaces (joint work with Sam Coskey).

Choquet games

Definition

Let X be a topological space. The *strong Choquet game* for X is played by two players, I (empty) and II (nonempty):

$$\begin{array}{cccccccc} \mathrm{I} & U_0, x_0 & & U_1, x_1 & & \cdots & \\ \mathrm{II} & & V_0 & & V_1 & & \cdots \end{array}$$

I plays U_n, x_n such that $x_n \in U_n$ and U_{n+1} is an open subset of V_n if $n > 0$. II responds with V_n such that $x_n \in V_n$ and V_n is an open subset of U_n.

II *wins* if $\bigcap_{n<\omega} U_n \neq \emptyset$.

In the (weak) Choquet game, player I omits the points x_n.

Theorem (Choquet)

A separable normal space is Polish if and only if player II has a winning strategy in the strong Choquet game.

Definition

Let X be a topological space. The *strong κ-Choquet game* for X is played by two players, I (empty) and II (nonempty):

$$\begin{array}{lccccccccc} \text{I} & U_0, x_0 & & U_1, x_1 & & \cdots & & U_\lambda, x_\lambda & & \cdots \\ \text{II} & & V_0 & & V_1 & & \cdots & & V_\lambda & & \cdots \end{array}$$

In the first half of each round, I plays U_α, x_α such that $x_\alpha \in U_\alpha$ and U_α is a nonempty relatively open subset of $\bigcap_{\beta<\alpha} U_\beta$.

In the second half of each round, II responds with V_α such that $x_\alpha \in V_\alpha$ and V_α is a relatively open subset of U_α.

II *wins* if $\bigcap_{\alpha<\gamma} U_\alpha \neq \emptyset$ for all $\gamma \leq \kappa$.

Definition

We say that X is a *strong κ-Choquet space* if X is normal of weight $\leq \kappa$ and II has a winning strategy in the strong κ-Choquet game for X.

Dynamic games

Definition

1. Suppose that $\lambda \leq \kappa$. The λ-*dynamic* strong κ-Choquet game is played as the strong κ-Choquet game, except that rather than open sets, the players may play the intersection of fewer than λ many open sets (intersected with the run up until that point).
2. A space X is said to be *λ-dynamic strong κ-Choquet* if it is normal of weight $\leq \kappa$ and player II has a winning strategy in the λ-dynamic strong κ-Choquet game on X.

Lemma

Suppose that λ, μ are cardinals with $\omega \leq \lambda < \mu \leq \kappa$.

1. *Every λ-dynamic strong κ-Choquet space is μ-dynamic strong κ-Choquet.*
2. *If λ is regular, then there is a μ-dynamic strong κ-Choquet space of weight $2^{<\lambda}$ which is not λ-dynamic strong κ-Choquet.*

Proof.

For 1, Player II can simulate the λ-dynamic game. For 2, let $\mathcal{C}_\lambda$ be the club filter on λ. Player I wins the λ-dynamic game by extending with 0 in limits. Player II wins the μ-dynamic game by playing a singleton. $\square$

Tactics

A tactic for player II is a strategy which depends only on the point played in the last move and on the intersection of the sequence of the played (open) sets.

Theorem

Suppose that X is κ-dynamic strong κ-Choquet. Then there is a winning tactic for player II (in the κ-dynamic strong κ-Choquet game on X).

Proof sketch.

Since the κ-topology on X is strong κ-Choquet, and X is κ-additive, it can be shown that there is a continuous open surjection $f\colon {}^{\kappa}\kappa \to X$. Note that player II has a winning tactic in the strong κ-Choquet game on ${}^{\kappa}\kappa$, in fact player II wins every run no matter what he or she plays.

The tactic on ${}^{\kappa}\kappa$ can be pushed forward to a winning tactic for player II in the κ-dynamic strong κ-Choquet game on X. □

Example

The following are (κ-dynamic) strong κ-Choquet spaces:

- Polish spaces
- Superclosed subsets of ${}^{\kappa}\kappa$, i.e. sets of the form $[T]$, where T is a $<\kappa$-closed subtree of ${}^{<\kappa}\kappa$ without end nodes
- ${}^{\kappa}\kappa$ with the linear order topology from the lexicographical order is a connected 1-dimensional (κ-dynamic) strong κ-Choquet space.
- Suppose that $\mathcal{L}$ is a relational language of cardinality at most κ and let $a(R)$ denote the number of arguments of $R \in \mathcal{L}$. Let

$$X_{\mathcal{L},\kappa} = \prod_{R \in \mathcal{L}} 2^{(\kappa^{a(R)})}$$

denote the space of $\mathcal{L}$-structures of cardinality κ with the $<\kappa$-support topology (the logic space).

Continuous images

Suppose that X is a strong κ-Choquet space. We call X *κ-perfect* if no singleton is the intersection of $< \kappa$ many open sets.

Proposition

1. *If X is strong κ-Choquet and $f\colon X \to Y$ is continuous, open, and surjective, then Y is strong κ-Choquet.*
2. *The $<\kappa$-supported product of κ many strong κ-Choquet spaces is strong κ-Choquet.*
3. *If X is a κ-perfect κ-Choquet space, then there is a continuous injection $f\colon 2^\kappa \to X$.*
4. *If X is a nonempty strong κ-Choquet space, then there is a continuous surjection $f\colon {}^\kappa\kappa \to X$.*

Borel isomorphisms

Theorem

1. *If X is a strong κ-Choquet space, then there is a subtree $T \subseteq {}^{<\kappa}\kappa$ without end nodes and a continuous bijection $f\colon [T] \to X$.*
2. *Any two κ-perfect, strong κ-Choquet spaces of size $> \kappa$ are κ-Borel isomorphic.*

Definition (Motto Ros)

A set X with a κ-algebra is *standard κ-Borel* if it is isomorphic to a κ-Borel subset of ${}^{\kappa}\kappa$ with its κ-Borel subsets.

Proposition (Motto Ros-S.)

For every κ-Borel subset B of a strong κ-Choquet space (X, τ), there is a topology $\bar{\tau}$ on X such that

1. *B is $\bar{\tau}$-clopen,*
2. *(X, τ) and $(X, \bar{\tau})$ are κ-Borel isomorphic, and*
3. *$(X, \bar{\tau})$ is homeomorphic to a superclosed subsets of ${}^{\kappa}\kappa$ (hence, in particular, (X, τ) is strong κ-Choquet.*

Standard κ-Borel spaces

Theorem (Motto Ros-S.)

The following are equivalent for a topological space (X, τ).

1. (X, τ) *s a standard κ-Borel space.*
2. (X, τ) *is κ-Borel isomorphic to a (super-)closed subset of ${}^{\kappa}\kappa$.*
3. (X, τ) *is κ-Borel isomorphic to a (zero-dimensional κ-additive) strong κ-Choquet space.*
4. *There is a (zero-dimensional κ-additive) strong κ-Choquet topology $\bar{\tau}$ on X such that (X, τ) and $(X, \bar{\tau})$ are κ-Borel isomorphic.*

Universality properties

Theorem (Motto Ros-S.)

For every strong κ-Choquet space (X, τ), the following are equivalent:

1. *(X, τ) is zero-dimensional and κ-additive.*
2. *(X, τ) is homeomorphic to a closed subset of ${}^{\kappa}\kappa$.*

Theorem (Motto Ros-S.)

Let κ be weakly compact. For every strong κ-Choquet space (X, τ) the following are equivalent:

1. *(X, τ) is κ-compact, zero-dimensional, and κ-additive.*
2. *(X, τ) is homeomorphic to a closed subset of ${}^{\kappa}2$.*

Using Fraisse constructions, we can construct Urysohn type κ-ultrametric spaces for various distance sets, for instance ${}^{\kappa}\kappa$ is the Urysohn κ-ultrametric space for the distance set κ.

Classes of continuous images

We consider several classes of continuous images of closed subsets of ${}^{\kappa}\kappa$, for instance the continuous images of ${}^{\kappa}\kappa$ (joint work with Philipp Lücke).

Classes of continuous images

Example

The club filter

$$\mathrm{Club}_\kappa \;=\; \{x \in {}^\kappa\kappa \mid \exists C \subseteq \kappa \text{ } club\ \forall \alpha \in C \ x(\alpha) = 1\}$$

is a continuous image of the space ${}^\kappa\kappa$.

Let T denote the tree consisting of all pairs (s, t) in ${}^\gamma 2 \times {}^\gamma 2$ such that $\gamma \in \mathrm{Lim} \cap \kappa$, $t(\alpha) \leq s(\alpha)$ for all $\alpha < \gamma$ and t is the characteristic function of a club subset of γ.

Then T is isomorphic to the tree ${}^{<\kappa}\kappa$, because it is closed under increasing sequences of length $<\kappa$ and every node has κ-many direct successors.

If we equip the set $[T]$ of all κ-branches through T with the topology whose basic open sets consists of all extensions of elements of T, then we obtain a topological space homeomorphic to ${}^\kappa\kappa$.

Since the projection $p : [T] \longrightarrow {}^\kappa\kappa$ onto the union of the first coordinate is continuous, we can conclude that the set Club_κ is equal to a continuous image of ${}^\kappa\kappa$.

Given an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, we study the following subclasses of $\mathbf{\Sigma}^1_1(\kappa)$ that arise by restricting the classes of used continuous functions and closed subsets.

- The class $\mathbf{\Sigma}^1_1(\kappa)$ of continuous images of closed subsets of ${}^\kappa\kappa$.
- The class $\mathbf{C}(\kappa)$ of continuous images of ${}^\kappa\kappa$.
- The class $\mathbf{I}_{cl}(\kappa)$ of continuous injective images of closed subsets of ${}^\kappa\kappa$.
- The class $\mathbf{I}(\kappa)$ of continuous injective images of ${}^\kappa\kappa$.

We will compare these classes with the following collections.

- The class $\mathbf{\Sigma}^0_1(\kappa)$ of open subsets of ${}^\kappa\kappa$.
- The class $\mathbf{B}(\kappa)$ of κ-Borel subsets of ${}^\kappa\kappa$, i.e. the subsets contained in the smallest algebra of sets on ${}^\kappa\kappa$ that contains all open subsets and is closed under κ-unions.

In the case $\kappa = \omega$, the relationship of the above classes is described by the following complete diagram.

$$\begin{tikzcd} \mathbf{\Sigma}^0_1(\omega) \arrow[r] & \mathbf{I}(\omega) \arrow[r] & \mathbf{I}_{cl}(\omega) \arrow[r, equal] & \mathbf{B}(\omega) \arrow[r] & \mathbf{\Sigma}^1_1(\omega) \arrow[r, equal] & \mathbf{C}(\omega) \end{tikzcd}$$

Our results will show that these classes behave in a very different way if κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. These results are summarized by the following complete diagram.

$$\begin{tikzcd} \mathbf{B}(\kappa) \arrow[r] & \mathbf{I}_{cl}(\kappa) \arrow[r, shift right] & \mathbf{\Sigma}^1_1(\kappa) \arrow[l, dashed, shift right, "\mathrm{V{=}L}"'] \\ \mathbf{\Sigma}^0_1(\kappa) \arrow[u] \arrow[r] & \mathbf{I}(\kappa) \arrow[u] \arrow[r] & \mathbf{C}(\kappa) \arrow[u] \end{tikzcd}$$

In particular, the following statements fails to generalize to higher cardinalities.

- Every closed subset of the space ${}^{\omega}\omega$ is equal to a continuous image of ${}^{\omega}\omega$.
- Every continuous injective image of the space ${}^{\omega}\omega$ is a Borel subset.

In the following, we construct the corresponding counterexamples.

Retracts

An important topological property of spaces of the form ${}^{\omega}\mu$ is the fact that non-empty closed subsets are retracts of the whole space, i.e. given an non-empty closed subset A of ${}^{\omega}\mu$ there is a continuous surjection $f : {}^{\omega}\mu \longrightarrow A$ such that $f \restriction A = \mathrm{id}_A$.

An easy argument shows that this property fails if κ is uncountable.

Proposition

Suppose that κ is an uncountable regular cardinal and $\mu > 1$ is a cardinal. Let A denote the set of all x in ${}^{\kappa}\mu$ such that $x(\alpha) = 1$ for only finitely many $\alpha < \kappa$. Then A is a closed subset of ${}^{\kappa}\mu$ that is not a retract of ${}^{\kappa}\mu$.

Proof.

Assume, towards a contradiction, that there is a continuous function $f : {}^{\kappa}\mu \longrightarrow A$ with $f \upharpoonright A = \mathrm{id}_A$. We construct a strictly increasing sequence $\langle \gamma_n < \kappa \mid n < \omega \rangle$ of ordinals such that $\gamma_0 = 1$ and

$$N_{x_n \upharpoonright \gamma_{n+1}} \;\subseteq\; f^{-1} \text{ " } N_{x_n \upharpoonright (\gamma_n + 1)}$$

holds for all $n < \omega$ and the unique $x_n \in {}^{\kappa}2$ with

$$x_n^{-1} \text{ " } \{1\} \;=\; \{\gamma_0, \ldots, \gamma_n\}.$$

Let $x \in {}^{\kappa}2$ be the unique function with

$$x^{-1} \text{ " } \{1\} \;=\; \{\gamma_n \mid n < \omega\}.$$

Then our construction yields

$$f(x) \upharpoonright \sup_{n<\omega} \gamma_n \;=\; x \upharpoonright \sup_{n<\omega} \gamma_n$$

and this implies that $f(x) \notin A$, a contradiction. $\square$

Continuous Images of ${}^{\kappa}\kappa$

Every closed subset of ${}^{\omega}\omega$ is a continuous image of ${}^{\omega}\omega$ and hence every $\mathbf{\Sigma}^1_1$-subset is equal to a continuous image of ${}^{\omega}\omega$.

The following result shows that this statement also does not generalize to uncountable regular cardinals.

Theorem

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Then there is a closed non-empty subset of ${}^{\kappa}\kappa$ that is not equal to a continuous image of ${}^{\kappa}\kappa$.

Proof.

It suffices to construct a closed subset A of ${}^{\kappa}\kappa$ with the property that A is not equal to the projection $p[T]$ of a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without terminal nodes.

Given $\lambda \leq \kappa$ closed under Gödel pairing and $x \in {}^{\lambda}2$, define a binary relation $\in_x$ on λ by setting

$$\alpha \in_x \beta \iff x(\prec\alpha, \beta\succ) = 1.$$

Define

$$W = \{x \in {}^{\kappa}2 \mid (\kappa, \in_x) \text{ is a well-order}\}.$$

Then W is a closed subset of ${}^{\kappa}\kappa$.

Assume, towards a contradiction, that there is a $<\kappa$-closed subtree T of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ without terminal nodes such that $W = p[T]$.

Proof (cont.).

Given $(s,t) \in T$ and $\alpha < \kappa$, define

$$r(s,t,\alpha) \ = \ \sup\{\mathrm{rnk}_{\in_x}(\alpha) \mid x \in p([T] \cap N_{(s,t)})\} \ \leq \ \kappa^+$$

Then $r(\emptyset,\emptyset,\alpha) = \kappa^+$ for every $\alpha < \kappa$.

Claim.

Let $(s,t) \in T$ and $\alpha < \kappa$ with $r(s,t,\alpha) = \kappa^+$. If $\gamma < \kappa^+$, then there is $(u,v) \in T$ extending (s,t) and $\alpha < \beta < \kappa$ such that $\mathrm{dom}(u)$ is closed under Gödel pairing, $\beta < \mathrm{lh}(u)$, $\beta \in_u \alpha$, and $r(u,v,\beta) \geq \gamma$.

Proof of the Claim.

There is a $(x,y) \in [T] \cap N_{(s,t)}$ with $\mathrm{rnk}_{\in_x}(\alpha) \geq \gamma + \kappa$. Hence we can find $\alpha < \beta < \kappa$ with $\gamma \leq \mathrm{rnk}_{\in_x}(\beta) < \gamma + \kappa$. Pick $\delta > \max\{\alpha, \beta, \mathrm{lh}(s)\}$ closed under Gödel pairing and define (u,v) to be the node $(x \restriction \delta, y \restriction \delta)$ extending (s,t). Since $\in_u$ is a well-ordering of $\mathrm{lh}(u)$, we have $\beta \in_u \alpha$. Finally, (x,y) witnesses that $r(u,v,\beta) \geq \gamma$. □

Proof (cont.).

Claim.

If $(s,t) \in T$ and $\alpha < \kappa$ with $r(s,t,\alpha) = \kappa^+$, then there is a node (u,v) in T extending (s,t) and $\alpha < \beta < \mathrm{lh}(u)$ such that $\mathrm{lh}(u)$ is closed under Gödel pairing, $\beta \in_u \alpha$ and $r(u,v,\beta) = \kappa^+$. □

This claim shows that there are strictly increasing sequences $\langle (s_n,t_n) \mid n < \omega \rangle$ of nodes in T and $\langle \beta_n \mid n < \omega \rangle$ of elements of κ with

- $\mathrm{lh}(s_n)$ *is closed under Gödel pairing,*
- $\beta_{n+1} \in_{s_{n+1}} \beta_n$.

Let $s = \bigcup_{n<\omega} s_n$ and $t = \bigcup_{n<\omega} t_n$. Then $(s,t) \in T$, since T is ω-closed. By our assumptions on T, there is a cofinal branch (x,y) in $[T]$ through (s,t). Then $\beta_{n+1} \in_x \beta_n$ for every $n < \omega$ and this shows that $x \notin W = p[T]$, a contradiction. □

Continuous injective images of ${}^{\kappa}\kappa$

Next we construct a continuous injective image of ${}^{\kappa}\kappa$ that is not a κ-Borel subset of ${}^{\kappa}\kappa$. In order to prove that certain sets are not κ-Borel, we need to introduce an important regularity property of subsets of ${}^{\kappa}\kappa$.

We say that a subset A of ${}^{\kappa}\kappa$ is *κ-Baire measurable* if there is an open subset U of ${}^{\kappa}\kappa$ and a sequence $\langle N_\alpha \mid \alpha < \kappa\rangle$ of nowhere dense subsets of ${}^{\kappa}\kappa$ such that the symmetric difference $A_\Delta U$ is a subset of $\bigcup_{\alpha<\kappa} N_\alpha$.

Every κ-Borel subset of ${}^{\kappa}\kappa$ is κ-Baire measurable. Moreover it is consistent that all $\mathbf{\Delta}^1_1$-subsets of ${}^{\kappa}\kappa$ are κ-Baire measurable.

Theorem

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Then there is a continuous injective image of ${}^{\kappa}\kappa$ that is not κ-Baire measurable.

To motivate this result, we first consider the case $\kappa = \aleph_1 = 2^{\aleph_0}$ and show that a well-known collection of combinatorial objects provides an example of a set with the above properties.

Definition

- Given $\gamma \in \mathrm{On}$, a sequence $\langle C_\alpha \mid \alpha < \gamma \rangle$ is a *coherent C-sequence* if the following statements hold for all $\alpha < \gamma$.
  - If α is a limit ordinal, then C_α is a closed unbounded subset of α.
  - If $\alpha = \bar{\alpha} + 1$, then $C_\alpha = \{\bar{\alpha}\}$.
  - If $\bar{\alpha} \in \mathrm{Lim}(C_\alpha)$, then $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$.
- A coherent C-sequence $\langle C_\alpha \mid \alpha < \gamma \rangle$ with $\gamma \in \mathrm{Lim}$ is *trivial* if there is a closed unbounded subset C_γ of γ that *threads* $\vec{C}$, i.e. the sequence $\langle C_\alpha \mid \alpha \leq \gamma \rangle$ is also a coherent C-sequence.

Let $\mathcal{C}oh(\omega_1)$ be the set of all coherent C-sequences of length ω_1 equipped with the topology whose basic open sets consist of all extensions of coherent C-sequences of limit length less than ω_1.

Claim.

The space $\mathcal{C}oh(\omega_1)$ is homeomorphic to ${}^{\omega_1}\omega_1$.

Proof of the Claim.

Let $\mathcal{T}$ denote the tree of all coherent C-sequences of limit length less than ω_1. Then $\mathcal{T}$ is isomorphic to ${}^{<\omega_1}\omega_1$, because $\mathcal{T}$ is σ-closed and every node in $\mathcal{T}$ has $\aleph_1$-many direct successors. This isomorphism gives us a homeomorphism of the above spaces. $\square$

Define $\mathcal{T}\!hr(\omega_1)$ to be set of all pairs $(\vec{C}, C)$ such that $\vec{C}$ is an element of $\mathcal{C}oh(\omega_1)$ and C is a thread through $\vec{C}$. We equip $\mathcal{T}\!hr(\omega_1)$ with the topology whose basic open sets consist of all component-wise extensions of pairs $(\vec{D}, D)$ such that $\vec{D}$ is a coherent C-sequence of length $\gamma \in \mathrm{Lim} \cap \omega_1$ and D is a thread through $\vec{D}$.

Claim.

The space $\mathcal{T}\!hr(\omega_1)$ is homeomorphic to ${}^{\kappa}\kappa$. □

Let $\mathcal{T}\!riv(\omega_1) = p[\mathcal{T}\!hr(\omega_1)]$ denote the set of all trivial coherent C-sequences of length ω_1.

Claim.

The set $\mathcal{T}\!riv(\omega_1)$ is a continuous injective image of ${}^{\kappa}\kappa$.

Proof of the Claim.

Since every coherent C-sequence of length ω_1 is threaded by at most one club subset of ω_1, the projection $p : \mathcal{T}\!hr(\kappa, \nu) \longrightarrow \mathcal{C}oh(\kappa, \nu)$ is injective. By the definition of the topologies, it is also continuous. □

We call a subset A of ${}^{\kappa}\kappa$ *super-dense* if $A \cap \bigcap_{\alpha<\kappa} U_\alpha \neq \emptyset$ whenever $\langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of dense open subsets of some non-empty open subset of ${}^{\kappa}\kappa$.

Proposition

Assume that A and B are disjoint super-dense subsets of ${}^{\kappa}\kappa$.
If $A \subseteq X \subseteq {}^{\kappa}\kappa \setminus B$, then X is not κ-Baire measurable. □

The club filter Club_κ is always a super-dense subset of ${}^{\kappa}\kappa$.

We will show that both $\mathit{Triv}(\omega_1)$ and its complement are super-dense. By the above claims, this shows that there is a continuous injective image of ${}^{\omega_1}\omega_1$ that is not $\aleph_1$-Baire measurable.

Let $\vec{C}_0$ be a coherent C-sequence of length $\gamma_0 < \omega_1$ and $\langle U_\alpha \mid \alpha < \kappa \rangle$ be a sequence of dense open subsets of $N_{\vec{C}_0}$.

We construct a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \omega_1 \rangle$ and a strictly increasing continuous sequence $\langle \gamma_\alpha \mid \alpha < \omega_1 \rangle$ of ordinals less than ω_1 such that the following statements hold for every $\alpha < \omega_1$.

- $\langle C_\beta \mid \beta < \gamma_\alpha \rangle$ is a coherent C-sequence extending $\vec{C}_0$.
- $N_{\langle C_\beta \mid \beta < \gamma_{\alpha+1} \rangle}$ is a subset of U_α.
- If $\alpha \in \mathrm{Lim}$, then $C_{\gamma_\alpha} = \{\gamma_{\bar{\alpha}} \mid \bar{\alpha} < \alpha\}$.

Then $\vec{C}$ is a coherent C-sequence that is contained in $\bigcap_{\alpha < \kappa} U_\alpha$ and the club $C = \{\gamma_\alpha \mid \alpha < \omega_1\}$ witnesses that $\vec{C}$ is trivial.

If we replace the third statement by

- $\operatorname{otp}(C_{\gamma_\alpha}) \leq \omega$,

then $\vec{C}$ is a non-trivial coherent C-sequence in $\bigcap_{\alpha < \kappa} U_\alpha$.

To prove the general theorem stated above, we pick an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, fix a bijection $f : {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa \longrightarrow \kappa$ and define A to be the set of all $x \in {}^{\kappa}\kappa$ such that the following statements hold for some $y \in {}^{\kappa}\kappa$ and a club subset C of κ.

- $C = \{\alpha < \kappa \mid x(\alpha) = y(\alpha)\}$.
- If $\alpha \in C$, then $x(\alpha) = f(x \restriction \alpha, \ y \restriction \alpha)$.

Given $x \in A$, it is easy to see that y and C with the above properties are uniquely determined.

A small modification of the above arguments shows that A is a continuous injective image of ${}^{\kappa}\kappa$ and a super-dense subset of ${}^{\kappa}\kappa$. Since A is disjoint from the club filter, it follows that A is not κ-Baire measurable.

The remaining implications

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

$$\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[r] & \mathbf{I}_{cl}(\kappa) \arrow[r, shift right] & \mathbf{\Sigma}^1_1(\kappa) \arrow[l, dashed, shift right, "V=L"'] \\
\mathbf{\Sigma}^0_1(\kappa) \arrow[u] \arrow[r] & \mathbf{I}(\kappa) \arrow[u] \arrow[r] & \mathbf{C}(\kappa) \arrow[u]
\end{tikzcd}$$

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

The following implications are trivial.

$$
\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[d, dashed, shift right, "\times" description] & \mathbf{I}_{cl}(\kappa) \arrow[r] \arrow[d, dashed, shift right, "\times" description] & \mathbf{\Sigma}^1_1(\kappa) \\
\mathbf{\Sigma}^0_1(\kappa) \arrow[u, shift right] \arrow[r, shift right] & \mathbf{I}(\kappa) \arrow[u, shift right] \arrow[l, dashed, shift right, "\times" description] \arrow[r, shift right] & \mathbf{C}(\kappa) \arrow[u] \arrow[l, dashed, shift right, "\times" description]
\end{tikzcd}
$$

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

The above constructions yields the following implications.

$$\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[drr, dashed, "\times" marking] & \mathbf{I}_{cl}(\kappa) & \mathbf{\Sigma}^1_1(\kappa) \\
\mathbf{\Sigma}^0_1(\kappa) & \mathbf{I}(\kappa) \arrow[ul, dashed, "\times" marking] & \mathbf{C}(\kappa)
\end{tikzcd}$$

As mentioned above, the following diagram completely describes the provable and consistent relations between the considered classes in the case where κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$.

We present results that yield the following implications.

$$
\begin{tikzcd}
\mathbf{B}(\kappa) \arrow[r, shift right] & \mathbf{I}_{cl}(\kappa) \arrow[l, dashed, shift right, "\times" description] & \mathbf{\Sigma}^1_1(\kappa) \arrow[l, dashed] \\
\mathbf{\Sigma}^0_1(\kappa) & \mathbf{I}(\kappa) & \mathbf{C}(\kappa) \arrow[ul, "\times" description]
\end{tikzcd}
$$

Theorem

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and A be a subset of ${}^{\kappa}\kappa$ such that

$$A = \{y \in {}^{\kappa}\kappa \mid \mathrm{L}[x, y] \models \varphi(x, y)\}$$

for some $x \in {}^{\kappa}\kappa$ and a Σ_1-formula $\varphi(u, v)$. Then A is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$.

Corollary

Every κ-Borel subset of ${}^{\kappa}\kappa$ is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$.

Corollary

There is a continuous injective image of a closed subset of ${}^{\kappa}\kappa$ that is not a κ-Borel subset of ${}^{\kappa}\kappa$.

Corollary

Assume that $\mathrm{V} = \mathrm{L}[x]$ *for some subset* x *of* κ*. Then every* $\mathbf{\Sigma}^1_1$*-subset of* ${}^\kappa\kappa$ *is a continuous injective image of a closed subset of* ${}^\kappa\kappa$*.*

Theorem

Let κ *be an uncountable regular cardinal,* $\delta > \kappa$ *be an inaccessible cardinal and* G *be* $\mathrm{Col}(\kappa, {<}\delta)$*-generic over* V*. In* $\mathrm{V}[G]$*, the club filter* Club_κ *is not equal to a continuous injective image of* ${}^\kappa\kappa$*.*

Corollary

It is consistent that there is a continuous image of ${}^\kappa\kappa$ *that is not equal to a continuous injective image of a closed subset of* ${}^\kappa\kappa$*.*

Trees of higher cardinalities

Let W be the closed set of all x in ${}^{\kappa}\kappa$ coding a well-order of κ. By the above results, W is not a continuous image of ${}^{\kappa}\kappa$. But it is easy to show that W equal to a continuous image of ${}^{\kappa}(\kappa^{+})$.

Therefore it is also interesting to investigate continuous images of closed subsets of ${}^{\mu}\kappa$ for some cardinal $\mu \geq \kappa$.

Since every subset of ${}^{\kappa}\kappa$ is a continuous image of ${}^{\kappa}(2^{\kappa})$, the above results show that

$$c(\kappa) \ = \ \min\{\mu \in \mathrm{On} \mid \textit{Closed subsets of } {}^{\kappa}\kappa \textit{ are cont. images of } {}^{\kappa}\mu\}$$

is a well-defined cardinal characteristic with

$$\kappa \ < \ c(\kappa) \ \leq \ 2^{\kappa}.$$

The following result shows that we can manipulate the value of $c(\kappa)$ by forcing.

Theorem

Assume that

- κ *is an uncountable cardinal with* $\kappa = \kappa^{<\kappa}$,
- $\mu \geq 2^\kappa$ *is a cardinal with* $\mu = \mu^\kappa$, *and*
- $\theta \geq \mu$ *is a cardinal with* $\theta = \theta^\kappa$.

Then the following statements hold in a cofinality preserving forcing extension $\mathrm{V}[G]$ *of the ground model* V.

- $2^\kappa = \theta$.
- *Every closed subset of* ${}^\kappa\mu$ *is an continuous image of* ${}^\kappa\mu$.
- *There is a closed subset* A *of* ${}^\kappa\kappa$ *that is not equal to an continuous image of* ${}^\kappa\bar{\mu}$ *for some* $\bar{\mu} < \mu$ *with* $\bar{\mu}^{<\kappa} < \mu$.

Kurepa trees as continuous images

The techniques developed in the proofs of the above results also allows us to discuss the question whether the set of all cofinal branches through a κ-Kurepa tree can be a continuous image of ${}^{\kappa}\kappa$.

Theorem

- *Let ν be an infinite cardinal and $\kappa = \nu^+ = \nu^{\aleph_0}$. If T is a κ-Kurepa subtree of ${}^{<\kappa}\kappa$, then $[T]$ is not a continuous image of ${}^{\kappa}\kappa$.*
- *Let ν be an uncountable regular cardinal, $\kappa > \nu$ be an inaccessible cardinal and $(G * H)$ be $(\mathrm{Add}(\omega, 1) * \mathrm{Col}(\nu, {<}\kappa))$-generic over V. In $\mathrm{V}[G, H]$, there is a κ-Kurepa subtree T of ${}^{<\kappa}\kappa$ such that $[T]$ is a retract of ${}^{\kappa}\kappa$.*
- *Let κ be an inaccessible cardinal and T be a slim κ-Kurepa subtree of ${}^{<\kappa}\kappa$. Then $[T]$ is not a continuous image of ${}^{\kappa}\kappa$.*

Questions

Do classes of continuous images in strong κ-Choquet spaces X of size $> \kappa$ have similar properties as for ${}^{\kappa}\kappa$, for example

Question

Is there a closed subset of X which is not a continuous image of ${}^{\kappa}\kappa$?

Which Ramsey-type theorems hold for uncountable cardinals, for example

Question

For which uncountable cardinals does the analogue of the Halpern-Läuchli theorem hold?

What are the optimal versions of dhichotomy theorems, for example

Question

Does the Hurewicz dichotomy hold for all subsets of ${}^{\kappa}\kappa$ in κ-Solovay models, and does it need an inaccessible cardinal?