The Hurewicz dichotomy for generalized Baire spaces

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A theorem of Hurewicz

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- notice that a subset of ${}^{\omega}\omega$ is compact (resp. $\boldsymbol{K}_\sigma$) iff it is bounded (resp. eventually bounded), or
- use the Baire category theorem and the fact that basic (cl)open sets of ${}^{\omega}\omega$ are trivially never compact.

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Let X be Polish. Then X contains a **closed** *subspace homeomorphic to ${}^{\omega}\omega$ iff X is not $\boldsymbol{K}_\sigma$.*

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Theorem (Kechris, Saint Raymond)

Every analytic subset of a Polish X satisfies the Hurewicz dichotomy.

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Question

Given a cardinal κ, is it possible to have that every κ-analytic set $A \subseteq {}^\kappa\kappa$ satisfies the Hurewicz dichotomy?

The generalized Hurewicz dichotomy at a given κ

Theorem (Lücke-M.-Schlicht)

Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$. Then there is a partial order $\mathbb{P}(\kappa)$ with the following properties:

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$\mathbb{P}(\kappa)$ is a $< \kappa$-support iteration of a forcing $\mathbb{P}_\kappa(A)$ which turns a given κ-analytic $A \subseteq {}^\kappa\kappa$ of the ground model into a $\boldsymbol{K}_\kappa$ set.

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The following characterization of κ-compact and $\boldsymbol{K}_\kappa$ sets $A \subseteq {}^\kappa\kappa$ is true only when κ is **weakly compact***:*

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(Thus in the non-weakly compact case we have just to inscribe a closed copy of ${}^\kappa 2$ inside any $A \subseteq {}^\kappa\kappa$ which is not covered by a $\boldsymbol{K}_\kappa$ set.)

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In this case we can design $\mathbb{P}_\kappa(A)$ *so that* A *becomes a* $\boldsymbol{K}_\kappa$ *set in the forcing extension (and this remains true if we further force with a* $< \kappa$*-closed poset).*

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Case B: κ is weakly compact (in the forcing extension)

Then A contains a closed copy of ${}^\kappa\kappa$ iff it contains the body $[T]$ of a κ-Miller tree $T \subseteq {}^{<\kappa}\kappa$, i.e. a tree closed under $<\kappa$-sequences and cofinally κ-splitting.

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Lemma (Halko)

$A \subseteq {}^\kappa\kappa$ is contained in a κ-compact set if and only if the canonical tree

$$T = \{\boldsymbol{N}^\kappa_{x \restriction \alpha} \mid x \in A, \alpha < \kappa\}$$

is a κ-tree ($=$ levels have size $< \kappa$) without κ-Aronzajn subtrees.

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When κ is not weakly compact we let $\mathbb{P}_{\kappa}(A) = \mathbb{K}(A)$ be the following variant of Todorcevic's forcing for adding a κ-Kurepa tree without κ-Aronzajn subtrees.

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2. Given $p, q \in \mathbb{K}(A)$, we set $p \leq_{\mathbb{K}(A)} q$ iff $\alpha_q \leq \alpha_p$, $c_q \subseteq c_p$ and $T_p(\gamma)$ is an end-extension of $T_q(\gamma)$ for every $\gamma \in \mathrm{ran}(c_q)$.

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- *There is an $\exists^x$-perfect embedding into T (a combinatorial condition which allows us to inscribe ${}^{\kappa}2$ in $p[T]$ as a closed subset).*
- *If $\mathbb{P}$ is a $< \kappa$-closed forcing adding a new subset of κ, then $\mathbb{P}$ adds a new element of $p[T]$.*

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We now let $\mathbb{P}(\kappa)$ be a $< \kappa$-support interation of length 2^κ of forcings of the form $\mathbb{K}(p[T])$ in which, using a book-keeping procedure, we deal with all trees $T \subseteq {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ appearing in any intermediate stage of the iteration itself.

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Let G be $\mathbb{P}(\kappa)$-generic over V and notice that κ is not weakly compact in $\mathrm{V}[G]$ as well (because $\mathbb{P}(\kappa)$ is $< \kappa$-closed). Let $T \subseteq {}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ be any tree in $\mathrm{V}[G]$ and set $A = p[T]^{\mathrm{V}[G]}$.

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This concludes the proof of the main theorem.

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This concludes the proof of the main theorem. Using a small variation of the above argument one can get

Theorem (Lücke-M.-Schlicht)

Let κ be an uncountable regular cardinal, $\lambda > \kappa$ be an inaccessible cardinal, and G be $\mathbf{Col}(\kappa, < \lambda)$-generic over V. Then in $\mathrm{V}[G]$ every κ-analytic set satisfies the Hurewicz dichotomy.

Hurewicz dichotomy everywhere

For $\kappa = \kappa^{<\kappa}$ we constructed a forcing $\mathbb{P}(\kappa)$ such that:

1. $\mathbb{P}(\kappa)$ is $< \kappa$-directed closed and κ^+-c.c.;
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Hurewicz dichotomy and the perfect set property

Definition

A set $A \subseteq {}^{\kappa}\kappa$ has the κ-**Perfect Set Property** (κ-PSP) if either $|A| \leq \kappa$, or A contains a closed copy of ${}^{\kappa}2$.

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Let $\kappa = \kappa^{<\kappa} > \omega$ be non-weakly compact, and assume that all closed sets have the κ-PSP. Then A is contained in a $\boldsymbol{K}_\kappa$ set iff $|A| \leq \kappa$. Hence if A has the κ-PSP as well, then it satisfies the **strong** *Hurewicz dichotomy.*

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Theorem (Schlicht)

After collapsing an inaccessible $\lambda > \kappa$ to κ^+, we get that all projective subsets of ${}^{\kappa}\kappa$ have the κ-PSP.

Hurewicz dichotomy and the perfect set property

Definition

A set $A \subseteq {}^\kappa\kappa$ has the κ-**Perfect Set Property** (κ-PSP) if either $|A| \leq \kappa$, or A contains a closed copy of ${}^\kappa 2$.

Proposition (Lücke-M.-Schlicht)

Let $\kappa = \kappa^{<\kappa} > \omega$ be non-weakly compact, and assume that all closed sets have the κ-PSP. Then A is contained in a $\boldsymbol{K}_\kappa$ set iff $|A| \leq \kappa$. Hence if A has the κ-PSP as well, then it satisfies the **strong** *Hurewicz dichotomy.*

Theorem (Schlicht)

After collapsing an inaccessible $\lambda > \kappa$ to κ^+, we get that all projective subsets of ${}^\kappa\kappa$ have the κ-PSP.

Thus it is possible to have that all κ-analytic subsets of ${}^\kappa\kappa$ have the κ-PSP **and** satisfy the Hurewicz dichotomy (independently of whether κ is weakly compact or not after the collapse).

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So the answer to the previous question is positive!

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- κ-**Sacks measurable** if for every perfect tree $T \subseteq {}^{<\kappa}\kappa$ there is a perfect tree $S \subseteq T$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$.

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Can we separate the κ-Miller measurability from the κ-PSP (and if yes, at which level)?

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Lemma

Assume that all κ-analytic subsets of ${}^\kappa\kappa$ satisfy the Hurewicz dichotomy. If $A \subseteq {}^\kappa\kappa$ is κ-analytic, then for every closed copy C of ${}^\kappa\kappa$ there is a closed copy $D \subseteq C$ of ${}^\kappa\kappa$ such that eihter $D \subseteq A$ or $D \cap A = \emptyset$.

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Proof.

If there is no such $D \subseteq A \cap C$, then by the Hurewicz dichotomy $A \cap C$ is a contained in a $\boldsymbol{K}_\kappa$ set K.

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Thus by forcing with $\mathbb{P}(\kappa)$ over L we can separate the κ-Sacks measurability and the κ-Miller measurability (for suitable κ's) from the κ-PSP.

Failures of the Hurewicz dichotomy

Theorem (Lücke-M.-Schlicht)

Assume $\mathsf{V} = \mathsf{L}$. *If* κ *is an uncountable regular cardinal, then there is a* **closed** *subset of* ${}^{\kappa}\kappa$ *which does not satisfy the Hurewicz dichotomy.*

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The (generalized) Hurewicz dichotomy is in fact extremely delicate.

Theorem (Lücke-M.-Schlicht)

Let κ *be an uncountable cardinal such that* $\kappa^{<\kappa} = \kappa$*, and suppose that* x *is a Cohen real over* V*. Then in* $\mathrm{V}[x]$ *there is a* **closed** *set* $A \subseteq {}^{\kappa}\kappa$ *which does not satisfy the Hurewicz dichotomy.*

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Proof.

Let $A = ({}^\kappa\kappa)^{\mathrm{V}}$.

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Let $A = ({}^{\kappa}\kappa)^{\mathrm{V}}$. Then A is the body (in $\mathrm{V}[x]$) of the tree $T = ({}^{<\kappa}\kappa)^{\mathrm{V}}$, it cannot contain a perfect subtree, but it is unbounded (hence not $\boldsymbol{K}_{\kappa}$) in $({}^{\kappa}\kappa)^{\mathrm{V}[x]}$. □

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Remark: The Cohen real destroys the Hurewicz dichotomy everywhere!

Open problems

If we collapse an inaccessible $\lambda > \kappa$ to κ^+ with κ non-weakly compact, we get that all projective sets have the κ-PSP, and hence they satisfy the Hurewicz dichotomy. Until now, this is the unique technique that we have to force the Hurewicz dichotomy for co-κ-analytic sets, and we do not know if it works also for weakly compact cardinals κ.

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1. If the Hurewicz dichotomy holds for co-κ-analytic sets, is there an inner model with an inaccessible cardinal?

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Question

1. If the Hurewicz dichotomy holds for co-κ-analytic sets, is there an inner model with an inaccessible cardinal?
2. If κ is weakly compact and we collapse an inaccessible $\lambda > \kappa$ to κ^+, do we still have that the Hurewicz dichotomy holds for all projective sets?

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Is it consistent that for a weakly compact κ, all κ-analytic sets have the κ-PSP but there is a closed set not satisfying the Hurewicz dichotomy?

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Question

Can we separate the κ-Miller measurability from the κ-PSP in the non-weakly compact case?

Thank you for your attention!