Orbit Equivalence Relations and Borel Reducibility on the Generalized Baire Space

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joint work with Sy Friedman and Tapani Hyttinen

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Basic Notions

- $|\kappa^{<\kappa}| = \kappa > \omega$ fixed throughout the presentation; basic open set $[p] = \{\eta \in \kappa^\kappa \mid \eta \supset p\}$, $p \in \kappa^{<\kappa}$.
- *Borel sets*: close basic open sets under unions and intersections of length κ.
- *Standard Borel space*, a space homeomorphic to a Borel subset of κ^κ.
- Σ_1^1 a projection of a Borel set.
- A function is *Borel* if inverse image of every Borel set is Borel.
- For equivalence relations E and E' on standard Borel spaces B and B' respectively, E is *Borel reducible* E', if there is a Borel map $f\colon B \to B'$ which induces a one-to-one map from B/E to B'/E'.
- If a set contains an intersection of length $\leqslant \kappa$ of open dense sets, it is *co-meager*. A complement of such a set is *meager*.
- If G is a topological group which is a standard Borel space and acts in a Borel way on some standard Borel space X, denote by E_G^X the orbit equivalence relation.

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Basic Notions

Fact

- *(Halko-Shelah [HS01])* κ^κ *is not meager and Borel sets have the property of Baire.*
- *A Borel function is continuous on a co-meager set.*

Basic Notions

- E_0 e.r. on 2^κ: $\exists\alpha\forall\beta > \alpha(\eta(\beta) = \xi(\beta))$,
- E_1 e.r. on $(2^\kappa)^\kappa$: $\exists\alpha\forall\beta > \alpha(\eta_\beta = \xi_\beta)$.

Results

We will prove this in the end of the talk if there is time left:

Theorem

If G is a discrete group of size at most κ and acts in a Borel way on a standard Borel space X. Then $E_G^X \leqslant_B E_0$.

The converse is not true:

Theorem

There is E with $E \leqslant_B \mathrm{id}$ which is not induced by a Borel action of such a group. In fact the equivalence classes of E have size 2.

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Compare

Compare to the classical results for $\kappa = \omega$:

Theorem (Dougherty-Jackson-Kechris)

The following are equivalent:

1. $E \leqslant_B E_0$,
2. E *is hyperfinite.*
3. E *is realizable be a Borel action of* $\mathbb{Z}$.

Theorem (Feldman-Moore)

If E *is countable e.r., then it can be realized by a Borel action of a countable group.*

Theorem (Thm 7.4.10 in [Gao09])

$E_0 < E_\infty$, *where the latter is the universal countable e.r.*

Results

Theorem

$E_1 \leqslant_B E_0$.

Compare to the classical $\kappa = \omega$ result:

Theorem

(Kechris-Louveau) E_1 is not reducible to any equivalence relation induced by a Borel action of a Polish group.

Proof of $E_1 \leqslant_B E_0$

Think of E_0 on κ^κ.
For all limit α let E_1^α be the e.r. on $(2^\alpha)^\alpha$ defined analogously to E_1. Let $f\colon \bigcup_{\alpha\in\lim(\kappa)}(2^\alpha)^\alpha \to \kappa$ be such that

- if $p \in (2^\alpha)^\alpha$ and $q \in (2^\beta)^\beta$ with $\alpha \neq \beta$, then $f(\alpha) \neq f(\beta)$,
- if $p, q \in (2^\alpha)^\alpha$, then $f(p) = f(q) \iff (p,q) \in E_1^\alpha$.

For every $(\eta_\alpha)_{\alpha<\kappa} \in (2^\kappa)^\kappa$ let $\xi = F((\eta_\alpha)_{\alpha<\kappa})$ be defined by $\xi(\beta) = 0$ for successor β and $\xi(\beta) = f((\eta_\alpha \restriction \beta)_{\alpha<\beta})$ for limit β.

If $(\eta_\alpha)_{\alpha<\kappa}$ and $(\xi_\alpha)_{\alpha<\kappa}$ are E_1-equivalent, then $(\eta_\alpha \restriction \beta)_{\alpha<\beta}$ and $(\xi_\alpha \restriction \beta)_{\alpha<\beta}$ are E_1^α-equivalent for all $\beta > \gamma$ for the γ which witnesses the E_1-equivalence.

If $(\eta_\alpha)_{\alpha<\kappa}$ and $(\xi_\alpha)_{\alpha<\kappa}$ are not E_1-equivalent, then there is a cub-set of β for which $(\eta_\alpha \restriction \beta)_{\alpha<\beta}$ and $(\xi_\alpha \restriction \beta)_{\alpha<\beta}$ are not E_1^α-equivalent.

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Iterated Jump Operation

Definition

If E is an equivalence relation on 2^κ, its *jump* is e.r. on $(2^\kappa)^\kappa$: Two sequences $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are E^+-equivalent, if

$$\{[x_\alpha]_E \mid \alpha < \kappa\} = \{[y_\alpha]_E \mid \alpha < \kappa\}.$$

Suppose α is a limit and $E^{\beta+}$ is defined to be an equivalence relation on 2^κ for $\beta < \alpha$. Then $E^{+\alpha} = \bigoplus_{\beta<\alpha} E^{\beta+}$.

Results

Theorem

$E_0 <_B \mathrm{id}^+$ *(strict inequality).*

This implies $E_1 \leqslant_B \mathrm{id}^+$ based on which (together with the fact that an isomorphism relation can be Σ^1_1-complete in L [HK14]) one expects Hjorth's turbulence theory to be non-generalizable.

Results

Definition

For a regular cardinal $\mu < \kappa$ and $\lambda \in \{2, \kappa\}$ let $E^{\lambda}_{\mu-cub}$ be the equivalence relation on λ^{κ} such that η and ξ are $E^{\lambda}_{\mu-cub}$-equivalent if the set $\{\alpha \mid \eta(\alpha) = \xi(\alpha)\}$ contains a μ-cub,

Theorem

Every jump of identity $\mathrm{id}^{+\alpha}$, $\alpha < \kappa^{+}$, *is reducible to* E^{κ}_{μ}.

Corollary

If $\mathcal{M}$ *is a Borel set of structures with domain* κ *(in particular the models of a countable complete first-order classifiable shallow theory [FHK14]) and* $\cong_{\mathcal{M}}$ *the isomorphism relation on* $\mathcal{M}$*, then* $\cong_{\mathcal{M}} \leqslant_B E^{\kappa}_{\mu-cub}$.

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Proof of $\mathrm{id}^{+\alpha} \leqslant_B E^{\kappa}_{\mu-\mathrm{cub}}$

The previous proof was based on the idea that E_1 can be approximated by relations E_1^{α} such that both, the E_1-non-equivalence and E_1-equivalence reflect in a cub-set, the latter being in fact a final segment. This idea can be generalized. Borel sets can be coded by pairs (t, h) where t is a well-founded subtree of $\kappa^{<\omega}$ and h is a function on the leafs. Define $(t, h)\restriction\alpha$ to be $(t \cap \alpha^{<\omega}, h\restriction\alpha^{<\omega})$ for suitable ("good") $\alpha < \kappa$.

Proof of $\mathrm{id}^{+\alpha} \leqslant_B E^{\kappa}_{\mu-\mathrm{cub}}$

Lemma

Suppose (t, h) codes a Borel subset $B_{(t,h)}$ of $2^\kappa \times 2^\kappa$. Then

$$(\eta, \xi) \in B \iff (\eta \restriction \alpha, \xi \restriction \alpha) \in B_{(t,h)\restriction\alpha}$$

for cub-many α and $(\eta, \xi) \notin B \iff (\eta \restriction \alpha, \xi \restriction \alpha) \notin B_{(t,h)\restriction\alpha}$ for cub-many α.

Lemma

Let S be the set of Borel equivalence relations E such that for some Borel code (t, h), $E = B_{(t,h)}$ and $B_{(t,h)\restriction\alpha}$ is an equivalence relation for cub-many good $\alpha < \kappa$. Then S contains id *and is closed under jump and the join operation $\bigoplus$.*

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Question

It has been shown that $E^{\kappa}_{\mu-\text{cub}}$ is Σ^1_1-complete in L [HK14]. On the other hand it has been shown in [FHK14] that under some cardinality assumprions, T is classifiable if and only if $E^2_{\mu-cub} \not\leq_B \cong^{\kappa}_T$ for all regular $\mu < \kappa$. Thus a set of questions would be answered if the following question is answered positively:

Question

Is $E^{\kappa}_{\mu-cub}$ reducible to $E^2_{\mu-cub}$?

For example the following would follow: Suppose T_1 and T_2 are complete first-order theories with T_1 classifiable and shallow and T_2 non-classifiable. Also suppose that $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Then $\cong^{\kappa}_{T_1}$ is Borel reducible to $\cong^{\kappa}_{T_2}$.

Proof of $E_G^X \leqslant_B E_0$ for $|G| \leqslant \kappa$

Steps:

1. $E_G^X \leqslant_B E_G^{\mathcal{P}(G)^\kappa}$.
2. $E_G^{\mathcal{P}(G)^\kappa} \leqslant_B E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa}$.
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Step 1: $E_G^X \leqslant_B E_G^{\mathcal{P}(G)^\kappa}$.

Assume without loss of generality that X is a Borel subset of 2^κ. Let $\pi\colon \kappa \to 2^{<\kappa}$ be a bijection. Let $x \in X$ and for each $\alpha < \kappa$ let

$$Z_\alpha(x) = \{g \in G \mid gx \in [\pi(\alpha)]\}.$$

This defines a reduction: an element $x \in X$ is mapped to $(Z_\alpha(x))_{\alpha<\kappa}$.

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Step 2: $E_G^{\mathcal{P}(G)^\kappa} \leqslant_B E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa}$.

There is a normal subgroup $N \subseteq F_\kappa$ such that $G \cong F_\kappa/N$. Assume without loss of generality that $G = F_\kappa/N$. Let pr be the canonical projection map $F_\kappa \to F_\kappa/N$. For $(A_\alpha)_{\alpha<\kappa} \in \mathcal{P}(G)^\kappa$, let

$$F((A_\alpha)_{\alpha<\kappa}) = (\mathrm{pr}^{-1}[A_\alpha])_{\alpha<\kappa}.$$

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Step 3: $E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa} \leqslant_B E_0$.

The action of F_κ on $\mathcal{P}(F_\kappa)^\kappa$ induces an action of F_α on $\mathcal{P}(F_\alpha)^\alpha$. Denote $X_\alpha = \mathcal{P}(F_\alpha)^\alpha$ for all $\alpha \leqslant \kappa$. Let $f\colon \bigcup_{\alpha<\kappa} X_\alpha \to \kappa$ be a function such that if $x, y \in X_\alpha$ and are $E_{F_\alpha}^{\mathcal{P}(F_\alpha)^\alpha}$-equivalent, then $f(x) = f(y)$ and $f(x) \neq f(y)$ otherwise. For $x \in X_\kappa$ let $x(\alpha) = f(x \restriction \alpha)$. Then $x \mapsto (x(\alpha))_{\alpha<\kappa}$ is the reduction.

Note: the basic idea is that $|F_\alpha| < |F_\kappa| = \kappa$ for $\alpha < \kappa$ unlike in the case $\kappa = \omega$.

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