

Building κ -complete filters for supercompact κ

Andrew Brooke-Taylor

University of Bristol

Generalised Baire Space workshop
Amsterdam, 4 November 2014

Two aspects of generalised Baire space:

Two aspects of generalised Baire space:

- ▶ generalised descriptive set theory

Two aspects of generalised Baire space:

- ▶ generalised descriptive set theory
- ▶ generalised cardinal characteristics of the continuum

Two aspects of generalised Baire space:

- ▶ generalised descriptive set theory
- ▶ generalised cardinal characteristics of the continuum

I'll be focusing on the latter.

Two aspects of generalised Baire space:

- ▶ generalised descriptive set theory
- ▶ generalised cardinal characteristics of the continuum

I'll be focusing on the latter.

First order of business:

Which proofs lift from the classical ω case to the generalised setting?

Assumptions on κ

Some results lift for any infinite cardinal (e.g., $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$).

Assumptions on κ

Some results lift for any infinite cardinal (e.g., $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$).

Almost always we assume κ is regular satisfying $\kappa^{<\kappa} = \kappa$.

Assumptions on κ

Some results lift for any infinite cardinal (e.g., $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$).

Almost always we assume κ is regular satisfying $\kappa^{<\kappa} = \kappa$.

Most of the time we want to assume κ is inaccessible.

Assumptions on κ

Some results lift for any infinite cardinal (e.g., $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$).

Almost always we assume κ is regular satisfying $\kappa^{<\kappa} = \kappa$.

Most of the time we want to assume κ is inaccessible.

Frequently it is convenient to assume κ is weakly compact (e.g., Suzuki: $\mathfrak{s}(\kappa) \geq \kappa^+$ iff κ is weakly compact).

Assumptions on κ

Some results lift for any infinite cardinal (e.g., $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$).

Almost always we assume κ is regular satisfying $\kappa^{<\kappa} = \kappa$.

Most of the time we want to assume κ is inaccessible.

Frequently it is convenient to assume κ is weakly compact (e.g., Suzuki: $\mathfrak{s}(\kappa) \geq \kappa^+$ iff κ is weakly compact).

Sometimes even stronger assumptions are provably necessary (e.g., Zapletal: $\text{Con}(\mathfrak{s}(\kappa) = \kappa^{++}) \rightarrow \text{Con}(o(\kappa) = \kappa^{++})$).

Assumptions on κ

Some results lift for any infinite cardinal (e.g., $\mathfrak{b}(\kappa) \leq \mathfrak{d}(\kappa)$).

Almost always we assume κ is regular satisfying $\kappa^{<\kappa} = \kappa$.

Most of the time we want to assume κ is inaccessible.

Frequently it is convenient to assume κ is weakly compact (e.g., Suzuki: $\mathfrak{s}(\kappa) \geq \kappa^+$ iff κ is weakly compact).

Sometimes even stronger assumptions are provably necessary (e.g., Zapletal: $\text{Con}(\mathfrak{s}(\kappa) = \kappa^{++}) \rightarrow \text{Con}(o(\kappa) = \kappa^{++})$).

I'll be talking about a case where we assume κ is supercompact.

The problem

In a number of classical arguments for the ω case, one builds up a filter with nice properties recursively.

In the κ analogue, we generally want to build up a κ -*complete* filter.

The problem

In a number of classical arguments for the ω case, one builds up a filter with nice properties recursively.

In the κ analogue, we generally want to build up a κ -complete filter.

Question

How can we preserve κ -completeness at limit stages of the construction?

A simple example

$$u(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a filter base for a uniform ultrafilter on } \kappa\}$$

(An ultrafilter $\mathcal{U}$ on κ is *uniform* if every $X \in \mathcal{U}$ has cardinality κ .)

A simple example

$$u(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a filter base for a uniform ultrafilter on } \kappa\}$$

(An ultrafilter $\mathcal{U}$ on κ is *uniform* if every $X \in \mathcal{U}$ has cardinality κ .)

Mathias forcing $\mathbb{M}_{\mathcal{F}}$

Conditions of the form (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$;

$(s, A) \leq (t, B)$ iff $A \subseteq B$, s end-extends t , and $t^{-1}(1) \setminus s^{-1}(1) \subset B$.

A simple example

$u(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a filter base for a uniform ultrafilter on } \kappa\}$

(An ultrafilter $\mathcal{U}$ on κ is *uniform* if every $X \in \mathcal{U}$ has cardinality κ .)

Mathias forcing $\mathbb{M}_{\mathcal{F}}$

Conditions of the form (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$;

$(s, A) \leq (t, B)$ iff $A \subseteq B$, s end-extends t , and $t^{-1}(1) \setminus s^{-1}(1) \subset B$.

Making $u(\aleph_0) < 2^{\aleph_0}$

Start with a model of $2^{\aleph_0} = \aleph_2$.

Do an ω_1 -length finite support iteration of Mathias forcing with ultrafilters, where at each stage of the iteration the ultrafilter used contains the Mathias generic subsets from the previous stages.

A simple example

$$u(\kappa) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a filter base for a uniform ultrafilter on } \kappa\}$$

(An ultrafilter $\mathcal{U}$ on κ is *uniform* if every $X \in \mathcal{U}$ has cardinality κ .)

Mathias forcing $\mathbb{M}_{\mathcal{F}}$

Conditions of the form (s, A) with $s \in 2^{<\omega}$ and $A \in \mathcal{F}$;

$(s, A) \leq (t, B)$ iff $A \subseteq B$, s end-extends t , and $t^{-1}(1) \setminus s^{-1}(1) \subset B$.

Making $u(\aleph_0) < 2^{\aleph_0}$

Start with a model of $2^{\aleph_0} = \aleph_2$.

Do an ω_1 -length finite support iteration of Mathias forcing with ultrafilters, where at each stage of the iteration the ultrafilter used contains the Mathias generic subsets from the previous stages. This is c.c.c., so $2_0^{\aleph_0}$ remains large, but the set of Mathias reals added forms an ultrafilter filter base of size $\aleph_1$.

Generalising to κ

$\mathbb{M}_{\mathcal{U}}^{\kappa}$ is κ^+ -c.c., but to preserve cardinals we also want it to be κ -closed, which is equivalent to $\mathcal{U}$ being κ -complete.

For $\alpha < \kappa$, why should the intersection of the first α -many κ -Mathias reals be non-empty?

Generalising to κ

$\mathbb{M}_{\mathcal{U}}^{\kappa}$ is κ^+ -c.c., but to preserve cardinals we also want it to be κ -closed, which is equivalent to $\mathcal{U}$ being κ -complete.

For $\alpha < \kappa$, why should the intersection of the first α -many κ -Mathias reals be non-empty?

Garti & Shelah:

For the limit stages of the iteration, one has to employ the arguments in [Džamonja & Shelah].

The Džamonja-Shelah technique

We want the ultrafilters used along the way in the iteration to cohere (κ -completely), so their union is a final, all-encompassing ultrafilter.

The Džamonja-Shelah technique

We want the ultrafilters used along the way in the iteration to cohere (κ -completely), so their union is a final, all-encompassing ultrafilter.

Central idea

As usual with a large cardinal κ , we this final ultrafilter $\mathcal{U}$ should be given by

$$X \in \mathcal{U} \leftrightarrow X \subseteq \kappa \wedge j(X) \ni \kappa.$$

The Džamonja-Shelah technique

We want the ultrafilters used along the way in the iteration to cohere (κ -completely), so their union is a final, all-encompassing ultrafilter.

Central idea

As usual with a large cardinal κ , we this final ultrafilter $\mathcal{U}$ should be given by

$$X \in \mathcal{U} \leftrightarrow X \subseteq \kappa \wedge j(X) \ni \kappa.$$

In this case, the j in question will be the lift of a λ -supercompactness embedding for κ from the ground model. We can carefully control the behaviour of this lift, ensuring that reasonable sets that we want to be in the ultrafilter are in it, and κ -completeness comes for free.

See the whiteboard.

The main claim (1.18 of Džamonja & Shelah's paper)

In V^{S_κ} , there exist sequences

$$\begin{aligned}\bar{\alpha} &= \langle \alpha_i : i < \Upsilon^+ \rangle, \\ \bar{p}^* &= \langle p_i^* : i < \Upsilon^+ \rangle, \text{ and} \\ \bar{q}^* &= \langle q_i^* = ({}^1q_i, {}^2q_i) : i < \Upsilon^+ \rangle,\end{aligned}$$

such that the following hold.

- ▶ $\bar{\alpha}$ is a strictly increasing continuous sequence of ordinals less than Υ^+ .
- ▶ Each p_i^* is purely full in P_{α_i+1} .
- ▶ $\bar{p}^*$ is a decreasing sequence of conditions in P_{Υ^+} .
- ▶ $\bar{q}^* \in M^{S_\kappa}$, and in M^{S_κ} we have for each $i < \Upsilon^+$ that

$$(p_i^*, {}^1q_i) \in P_{\Upsilon^+} * \dot{S}^*$$

and

$$(p_i^*, {}^1q_i, {}^2q_i) \in P_{\Upsilon^+} * \dot{S}^* * \dot{P}'_{j(\alpha_i+1)}.$$

- ▶ In M^{S_κ} , $\langle (p_i^*, {}^1q_i, {}^2q_i) : i < \Upsilon^+ \rangle$ is a decreasing sequence of conditions in $P_{\Upsilon^+} * \dot{S}^* * \dot{P}'_{\sup_{i < \Upsilon^+} (j(\alpha_i+1))}$.
- ▶ In M^{S_κ} , $(p_{i+1}^*, {}^1q_{i+1})$ forces that ${}^2q_{i+1}$ is a common extension of

$$\{j(r) : r \in G_{P_{\alpha_i+1}}\}$$

- ▶ If $\dot{B}$ is an S_κ -name for a P_{α_i+1} -name for a subset of κ then there is an $S_\kappa * \dot{P}_{\Upsilon^+}$ -name $\tau_{\dot{B}}$ for an element of $\{0, 1\}$ such that:
 1. in V , $(\mathbb{1}_{S_\kappa}, \dot{p}_{i+1}^*)$ forces $\tau_{\dot{B}}$ to be a $P_{\alpha_i+1+1} \downarrow p_{i+1}^*$ -name, and
 2. $M \models [(\mathbb{1}_{S_\kappa}, \dot{p}_{i+1}^*, q_{i+1}^*) \Vdash \check{\kappa} \in j(\dot{B}) \leftrightarrow \tau_{\dot{B}} = \check{1}]$.
- ▶ If $\text{cf}(i) > \kappa$, then in $V^{S_\kappa * \dot{P}_{\alpha_i}}$ we have that

$$p_i^*(\alpha_i) = \left\{ \dot{B}[G_{P_{\alpha_i}}] : \begin{array}{l} \dot{B} \text{ is a } P_{\alpha_i} \downarrow (p_i^* \restriction \alpha_i)\text{-name for a subset of } \\ \kappa \text{ and } \tau_{\dot{B}}[G_{P_{\alpha_i}}] = 1 \end{array} \right\}.$$

In particular, this is a normal ultrafilter on κ .

Singular cardinals

Džamonja and Shelah originally used the technique to prove the following result:

Theorem

Suppose there is a supercompact cardinal κ . Then there is a forcing extension in which there is a singular strong limit cardinal μ of cofinality ω with $2^{\mu^+} > \mu^+$, and μ^{++} -many graphs on μ^+ that taken jointly embed every graph on μ^+ .

Proof.

Use the technique we've described to get a universal family of graphs at κ^+ . Then apply Prikry forcing at κ . □

Downsides

Key constraint:

The ultrafilters used are normal. This is not always desirable.

Downsides

Key constraint:

The ultrafilters used are normal. This is not always desirable.

E.g.

κ -Mathias forcing with a normal ultrafilter always adds a dominating κ -real. For $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$, we want an iterated forcing that blows up $\mathfrak{a}(\kappa)$ but keeps $\mathfrak{b}(\kappa)$ small, and for this we do *not* want to add dominating κ -reals.

Definition

An ultrafilter is Canjar if Mathias forcing with it does not add a dominating real.

Open questions

Open questions

- ▶ Exhibit a (κ -complete) Canjar ultrafilter on a measurable cardinal κ .

Open questions

- ▶ Exhibit a (κ -complete) Canjar ultrafilter on a measurable cardinal κ .
- ▶ Characterise Canjar ultrafilters on measurable cardinals.

Open questions

- ▶ Exhibit a (κ -complete) Canjar ultrafilter on a measurable cardinal κ .
- ▶ Characterise Canjar ultrafilters on measurable cardinals.
- ▶ More generally, look at κ -complete ultrafilters on κ satisfying any Boolean combination of the κ -analogues of rapidity, being a p-point, and being a q-point.

Open questions

- ▶ Exhibit a (κ -complete) Canjar ultrafilter on a measurable cardinal κ .
- ▶ Characterise Canjar ultrafilters on measurable cardinals.
- ▶ More generally, look at κ -complete ultrafilters on κ satisfying any Boolean combination of the κ -analogues of rapidity, being a p-point, and being a q-point.
- ▶ Show $\text{Con}(\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa))$ for $\kappa > \omega$, possibly from large cardinals.