A Liouville hyperbolic souvlaki

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Abstract

We construct a transient bounded-degree graph no transient subgraph of which embeds in any surface of finite genus.

Moreover, we construct a transient, Liouville, bounded-degree, Gromov–hyperbolic graph with trivial hyperbolic boundary that has no transient subtree. This answers a question of Benjamini. This graph also yields a (further) counterexample to a conjecture of Benjamini and Schramm.

1 Introduction

A well-known result of Benjamini & Schramm [3] states that every non-amenable graph contains a non-ambenable tree. This naturally motivates seeking for other properties that imply a subtree with the same property. However, there is a simple example of a transient graph that does not contain a transient tree [3] (such a graph had previously also been obtained by McGuinness [13]). We improve this by constructing —in Section 7— a transient bounded-degree graph no transient subgraph of which embeds in any surface of finite genus (even worse, every transient subgraph has the complete graph K^r as a minor for every r). This answers a question of I. Benjamini (private communication).

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Given these examples, it is natural to ask for conditions on a transient graph that would imply a transient subtree. In this spirit, Benjamini [4, Open Problem 1.62] asks whether hyperbolicity is such a condition. We answer this in the negative by constructing —in Section 6— a transient hyperbolic (bounded-degree) graph that has no transient subtree. While preparing this manuscript, T. Hutchcroft and A. Nachmias (private communication) provided a simpler example with these properties, which we sketch in Section 6.1.

A related result of Thomassen states that if a graph satisfies a certain isoperimetric inequality, then it must have a transient subtree [15].

The starting point for this paper was the following problem of Benjamini and Schramm

Conjecture 1.1 ([2, 1.11. Conjecture])**.** *Let M be a connected, transient, Gromov-hyperbolic, Riemannian manifold with bounded local geometry, with the property that the union of all bi-infinite geodesics meets every ball of sufficiently large radius. Then M admits non constant bounded harmonic functions. Similarly, a Gromov-hyperbolic bounded valence, transient graph, with C-dense bi-infinite geodesics has non constant bounded harmonic functions.*

We remark that in order to disprove —the second sentence of— this, it suffices to find a transient, Gromov-hyperbolic bounded valence (aka. degree) graph with the Liouville property; for given such a graph G, one can attach a disjoint 1-way infinite path to each vertex of G, to obtain a graph having 1-dense bi-infinite geodesics while preserving all other properties. As pointed out by I. Benjamini (private communication), it is not hard to prove that any 'lattice' in a horoball in 4-dimensional hyperbolic space has these properties. We prove that our example also has these properties, thus providing a further counterexample to Conjecture 1.1. A perhaps surpising aspect of our example is that all of its geodesics eventually coincide despite its transience; see Section 2.

Although we do not formally provide a counterexample to the first sentence of Conjecture 1.1, we believe it is easy to obtain one by blowing up the edges of our graph into tubes.

In Section 2.1 we provide a sketch of this construction, from which the expert reader might be able to deduce the details.

2 The hyperbolic Souvlaki

In this section we construct a bounded-degree graph Ψ with the following properties

1. it is hyperbolic, and its hyperbolic boundary consists of a single point;
2. for every vertex x of Ψ, there is a unique infinite geodesic starting at x, and any two 1-way infinite geodesics of Ψ eventually coincide;
3. it is transient;
4. every subtree of Ψ is recurrent;
5. it has the Liouville property.

This graph thus yields a counterexample to [4, Open Problem 1.62] and Conjecture 1.1 as mentioned in the Introduction.

2.1 Sketch of construction

Let us sketch the construction of this graph Ψ, and outline the reasons why it has the above properties. It consists of an 1-way infinite path $S = s_0 s_1 \ldots$, on which we glue a sequence M_i of finite increasing subgraphs of an infinite '3-dimensional' hyperbolic graph H_3. For example, H_3 could be the 1-skeleton of a regular tiling of 3-dimensional hyperbolic space, and the M_i could be taken to be copies of balls of increasing radii around some origin in H_3, although it was more convenient for our proofs to construct different H_3 and M_i.

In order to glue M_i on S, we identify the subpath $s_{2^i} \ldots s_{2^{i+2}-1}$ with a geodesic of the same length in M_i. Thus M_i intersects M_{i-1} and M_{i+1} but no other M_j, and this intersection is a subpath of S; see Figure 4. (Our graph can be quasi-isometrically embedded in $\mathbb{H}^5$, but probably not in $\mathbb{H}^4$.) We call this graph a *hyperbolic souvlaki*, with *skewer* S and *meatballs* M_i. We detail its construction in Section 2.

To prove that this graph is transient, we construct a flow of finite energy from s_0 to infinity (Section 4). This flow carries a current of strength 2^{-i} inside M_i out of each vertex in $s_{2^i} \ldots s_{2^{i+1}-1}$, and distributes it evenly to the vertices in $s_{2^{i+1}} \ldots s_{2^{i+2}}$ for every i. These currents can be thought of as flowing on spheres of varying radii inside M_i, avoiding each other, and it was important to have at least three dimensions for this to be possible while keeping the energy dissipated under control.

To prove that our graph has the Liouville property, we observe that random walk has to visit S infinitely often, and has enough time to 'mix' inside the M_i between subsequent visits to S (Section 5).

2.2 Formal construction

We now explain our precise construction, which is similar but not identical to the above sketch. We start by constructing a hyperbolic graph H_3 which we will use as a model for the 'meatballs' M_i; more precisely, the M_i will be chosen to be increasing subgraphs of H_3.

Let T_3 denote the infinite tree with one vertex r, which we call the *root*, of degree 3 and all other vertices of degree 4. For $n = 1, 2, \ldots,$ we put a cycle —of length 3^n— on the vertices of T_3 that are at distance n from r in such a way that the resulting graph is planar[1]; see Figure 1. We denote this graph by H_2. It is not hard to see that H_2 is hyperbolic.

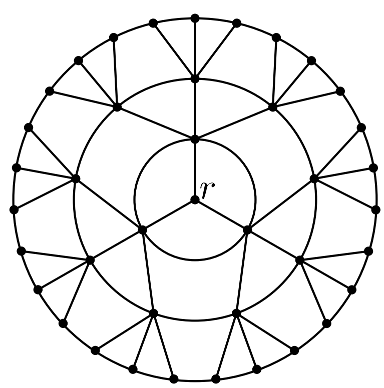

Figure 1: The ball of radius 3 around the root of H_2.

Recall that a *ray* is a 1-way infinite path. We will now turn H_2 into a '3-dimensional' hyperbolic graph H_3, in such a way that each ray inside T_3 (or H_2) starting at r gives rise to a subgraph of H_3 isomorphic to the graph W of Figure 2, which is a subgraph of the Cayley graph of the Baumslag-Solitar group $BS(1, 2)$. Formally, we construct W from infinitely many

[1]Formally, we pick a cyclic ordering on the neighbours of r and a linear ordering on the outer neighbours of every other vertex of T_3. Given a cyclic ordering on the vertices at level n of T_3, we get a cyclic ordering at level $n + 1$ by replacing each vertex by the linear ordering on its outer neighbours. Now we add edges between any two vertices that are adjacent in any of these cyclic orderings.

vertex disjoint double rays[2] $D_0, D_1, D_2, ..,$ where $D_i = ...r_i^{-2} r_i^{-1} r_i^0 r_i^1 r_i^2....$ Then we add all edges of the form $r_i^k r_{i+1}^{2k}$.

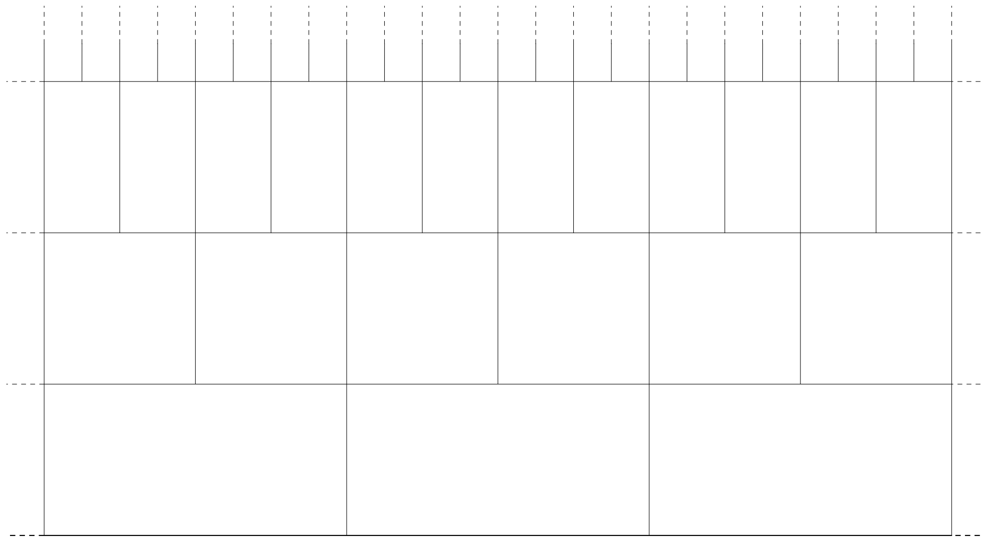

Figure 2: The graph W: a subgraph of the standard Cayley graph of the Baumslag-Solitar group $BS(1, 2)$. It is a plane hyperbolic graph.

To define H_3, we let the *height* $h(t)$ of a vertex $t \in V(H_2)$ be its distance $d(r, t)$ from the root r. For a vertex w of W, we say that its height $h(w)$ is n if w lies in D_n, the nth horizontal double ray in Figure 2.

We define the vertex set of H_3 to consist of all ordered pairs (t, w) where t is a vertex of H_2 and w is a vertex of W and $h(w) = h(t)$. The edge set of H_3 consists of all pairs of pairs $(t, w)(t', w')$ such that either

- $tt' \in E(H_2)$ and $ww' \in E(W)$, or
- $tt' \in E(H_2)$ and $w = w'$, or
- $t = t'$ and $ww' \in E(W)$.

Thus every vertex t of H_2 gives rise to a double ray in H_3, which consists of those vertices of H_3 that have t as their first coordinate. Similarly, every vertex w of W gives rise to a cycle in H_3, the length of which depends on $h(w)$. We call the vertices on any such cycle *cocircular*. Every ray of T_2

[2] A *double ray* is a 2-way infinite path.

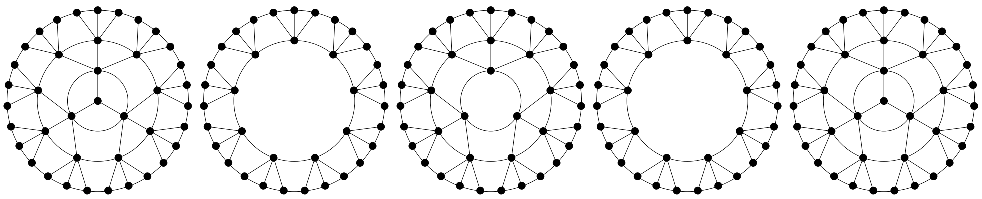

Figure 3: A subgraph of H_3. Edges of the form $(t, w)(t', w')$ with $t = t'$ and $ww' \in E(W)$ are missing from the figure: these are all the edges joining corresponding vertices in consecutive components of the figure.

starting at r gives rise to a copy of W, and if two such paths share their first k vertices, then the corresponding copies of W share their first k levels of h. It is not hard to prove that H_3 is a hyperbolic graph, but we will omit the proof as we will not use this fact.

We next construct Ψ by glueing a sequence of finite subgraphs M_n of H_3 along a ray S. We could choose the subgraph M_n to be a ball in H_3, but we found it more convenient to work with somewhat different subgraphs of H_3: we let M_n be the finite subgraph of H_3 spanned by those vertices (t, w) such that w lies in a certain *box* $B_n \subseteq W$ of W defined as follows. Consider a subpath P_n of the bottom double-ray of W of length $3 \cdot 2^n$, and let B_n consist of those vertices w that lie in or above P_n (as drawn in Figure 2) and satisfy $h(w) \leq n$.

This completes the definition of M_n. We let S_n denote the vertices of M_n corresponding to P_n, and we index the vertices of S_n as $\{r(x), 0 \leq x \leq 3 \cdot 2^n\}$. Note that S_n is a geodesic of M_n. We subdivide S_n into three parts: $L_n := \{r(x), 0 \leq x < 2^n\}, m_n := r(2^n)$ and $R_n := \{r(x), 2^n < x \leq 3 \cdot 2^n\}$. We define the *ceiling* F_n of M_n to be its vertices of maximum height, i.e. the vertices $(t, w) \in V(M_n)$ with $h(w) = n$.

Finally, it remains to describe how to glue the M_n together to form Ψ. We start with a ray S, the first vertex of which we denote by o and call the *root* of Ψ. We glue M_1 on S by identifying S_1 with the initial subpath of S

of length $|S_1|$. Then, for $n = 2, 3, \ldots$, we glue M_n on S in such a way that L_n is identified with R_{n-1} (where we used the fact that $|L_n| = |R_{n-1}| = 2^n$ by construction), m_n is identified with the following vertex of S, and R_n is identified with the subpath of S following that vertex and having length $|R_n| = 2^{n+1}$. Of course, we perform this identification in such a way that the linear orderings of L_n and R_n are given by the induced linear ordering of S. We let Ψ denote the resulting graph. We think of M_n as a subgraph of Ψ.

2.3 Properties of Ψ

By construction, for $j > i$ we have $M_i \cap M_j = \emptyset$ unless $j = i + 1$, in which case $M_i \cap M_j = R_i = L_j \subset S$. The following fact is easy to see.

For every n, R_n separates L_n (and o) from infinity. (1)

The following property will be important for the proof of Liouvilleness.

There is a uniform lower bound $p > 0$ for the probability $\mathbb{P}_v [\tau_{F_n} < \tau_{S_n}]$ that random walk from any vertex of L_n will visit the ceiling F_n before returning to S_n. (2)

Indeed, we can let p be the probability for random walk on H_2 starting at the root o to never visit o again; this is positive because H_2 is transient. Then (2) holds because in a random walk from S_n on M_n, any steps inside the copies of H_2 behave like random walk on H_2 until hitting F_n, and the steps 'parallel' to S_n do not have any influence.

3 Hyperbolicity

In this section we prove that Ψ is hyperbolic in the sense of Gromov [8].

Lemma 3.1. *The graph Ψ is hyperbolic, and has a one-point hyperbolic boundary.*

Proof. We claim that for every vertex $x \in V(\Psi)$, there is a unique 1-way infinite geodesic starting at x. Indeed, this geodesic $x_0 x_1 \ldots$, takes a step from x_i towards the root of T_3 inside the copy of H_2 corresponding x_i whenever such an edge exists in Ψ, and it takes a horizontal step in the direction of infinity whenever such an edge does not exist.

The hyperbolicity of Ψ now follows from a well-known fact saying that a space is hyperbolic if and only if any two geodesics with a common starting point are either at bounded distance or diverge exponentially in a certain sense; see [14] for details. We skip the details here as in our case the condition is trivially satisfied do to the above claim.

As all infinite geodesics eventually coincide with S, we also immediately have that the hyperbolic boundary of G consists of just one point. $\square$

4 Transience

In this section we prove that Ψ is transient. We do so by displaying a flow from o to infinity having finite Dirichlet energy; transience then follows from Lyons' criterion:

Theorem 4.1 (T. Lyons' criterion (see [11] or [10])). *A graph G is transient, if and only if G admits a flow of finite energy from a vertex to infinity.*

We refer the reader to [10] or [7] for the basics of electrical networks needed to understand this theorem.

To construct this flow f, we start with the flow t on the tree $T_3 \subset H_2$ which sends the amount 3^{-n} through each directed edge of T_3 from a vertex of distance $n-1$ from the root to a vertex of distance n from the root. Note that t has finite Dirichlet energy.

Our flow f will be as described in the introduction, that is, it is composed of flows $g(n)$ in M_n. These flows flow from L_n to R_n. The flow $g(n)$ in turn is composed of *'atomic'* flows, one for each $v \in L_n$. Roughly, these atomic flows imitate t from above for some levels, then use the edges parallel to S_n to bring it 'above' R_n, and then collect it back to (two vertices of) S_n imitating t in the inverse direction. A key idea here is that although the energy dissipated along the long paths parallel to S_n is proportional to their length, by going up enough levels with the t-part of these flows, we can ensure that the flow i carried by each such path is very small compared to its length ℓ. Thus its contribution $i^2\ell$ to the Dirichlet energy can be controlled: although going up one level doubles ℓ, and triples the number of long paths we have, each of them now carries $1/3$ of the flow, and so its contribution to the energy is multiplied by a factor of $1/9$. Thus all in all, we save a factor of $6/9$ by going up one more level – and we have made the M_i high enough that we can go up enough levels.

We now describe $g(n)$ precisely. For every $n \in \mathbb{N}$, let us first enumerate the vertices of L_n as $l^j = l^j_n$, with j ranging from 1 to $|L_n| = 2^n$, in the order they appear on S_n as we move from the midpoint m_n towards the root o.

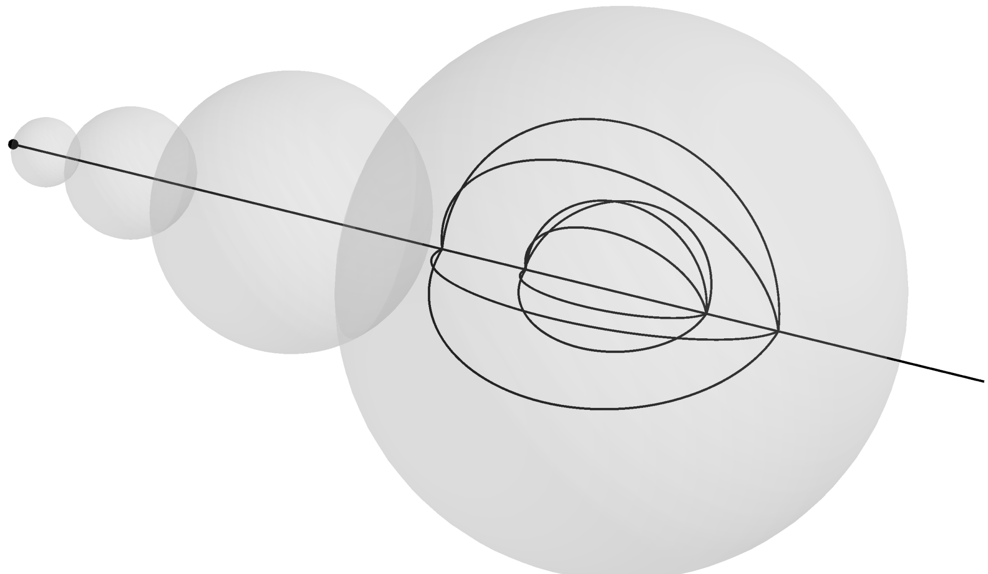

Figure 4: The structure of the graph Ψ, with the 'balls' intersecting along the ray and the flow inside the ball.

Likewise, we enumerate the vertices of R_n as $r^j = r^j_n$, with j ranging from 1 to $|R_n| = 2|L_n|$, in the order they appear on S_n as we move from the midpoint m_n towards infinity. Thus r^1, l^1 are the two neighbours of m_n on S. We will let $g(n)$ be the union of $|L_n|$ subflows $g^j = g^j_n$, where g^j flows from l^j into r^{2j} and r^{2j-1}. More precisely, g^j sends $1/|L_n| = 2^{-n}$ units of current out of l^j, and half as many units of current into each of r^{2j} and r^{2j-1}.

We define g^j as follows. In the copy of H_2 containing the source l^j of g^j, we multiply the flow t from above by the factor 2^{-n}, and truncate it after j layers; we call this the out-part of g^j. Then, from each endpoint x of that flow, we send the amount of flow that x receives from l^j, which equals $2^{-n}3^{-j}$, along the horizontal path P_x joining x to the copy C_1 of H_2 containing r^{2j-1}. We let half of that flow continue horizontally to reach the copy C_2 of H_2 containing r^{2j}; call this the middle-part of g^j. Finally, inside each of C_1, C_2, we put a copy of the out-part of g^j multiplied by 1/2 and with directions inverted; this is called the in-part of g^j. Note that the union of these three parts is a flow of intensity 2^{-n} from l^j to r^{2j} and r^{2j-1}, each of the latter receiving 2^{-n-1} units of current.

Let us calculate the energy $E(g^j)$. The contribution to $E(g^j)$ by its out-part is bounded above by $2^{-2n}E(t)$ because that part is contained in the flow $2^{-n}t$. Similarly, the contribution of the in-part is half of the contribution of the out-part. The contribution of the middle-part is $3^j \cdot (2j+1)2^j \cdot (2^{-n}3^{-j})^2$: the factor 3^j counts the number of horizontal paths used by the flow, each of which has length $(2j+1)2^j$, and carries $2^{-n}3^{-j}$ units of current (except for its last 2^j edges, from C_1 to C_2, which carry half as much, but we can afford to be generous). Note that this expression equals $2^{-2n}(2j+1)(6/9)^j$, which is upper bounded by $k2^{-2n}$ for some constant k.

Adding up these contributions, we see that $E(g^j) \leq K2^{-2n}$ for some constant K (which depends on neither n nor j).

Now let $g(n)$ be the union of the 2^n flows g^j. Note that g^j, g^i are disjoint for $i \neq j$, and therefore the energy $E(g(n))$ of g is just the sum $\sum_{j<2^n} E(g^j)$. By the above bound, this yields $E(g(n)) \leq K2^{-n}$.

Now let $f = \bigcup_{n\in\mathbb{N}} g(n)$ be the union of all the flows $g(n)$. Then $g(n), g(m)$ are disjoint for $n \neq m$, because they are in different $M_i's$. Thus $E(f) = \sum_n E(g(n)) \leq K$ is finite. Since $g(n)$ removes as much current from each vertex of L_n as $g(n-1)$ inputs, f is a flow from o to infinity. Hence Ψ is transient by Lyons' criterion (Theorem 4.1).

5 Liouville property

In this section we prove that Ψ is *Liouville*, i.e. it admits no bounded non-constant harmonic functions.

We remark that a well-known theorem of Ancona [1] states that in any non-amenable hyperbolic graph the hyperbolic boundary coincides with the Martin boundary. We cannot apply this fact to our case in order to deduce the Liouville property from the fact that our hyperbolic boundary is trivial, because our graph turns out to be amenable.

We will use some elementary facts about harmonic functions that can be found e.g. in [6].

Let h be a bounded non-constant harmonic functions on a graph G. We may assume that the range of h is contained in $[0, 1]$. Recall that, by the bounded martingale convergence theorem, if $(X_n)_{n\in\mathbb{N}}$ is a simple random walk on G, then $h(X_n)$ converges almost surely. We call such a function h *sharp*, if this limit $\lim_n h(X_n)$ is either 0 or 1 almost surely. It is well-known that if a graph admits a bounded non-constant harmonic function, then it admits a sharp harmonic function, see [6, Section 4].

So let us assume from now on that $h : V(\Psi) \to [0,1]$ is a sharp bounded harmonic function on Ψ.

We first recall some basic facts from [6, Section 7]; we repeat some of the proofs for the convenience of the reader.

Lemma 5.1. *If h is a sharp harmonic function, then $h(z) = \mathbb{P}_z [\lim h(Z_n) = 1]$ for every vertex z, where Z_n denotes a random walk from z.*

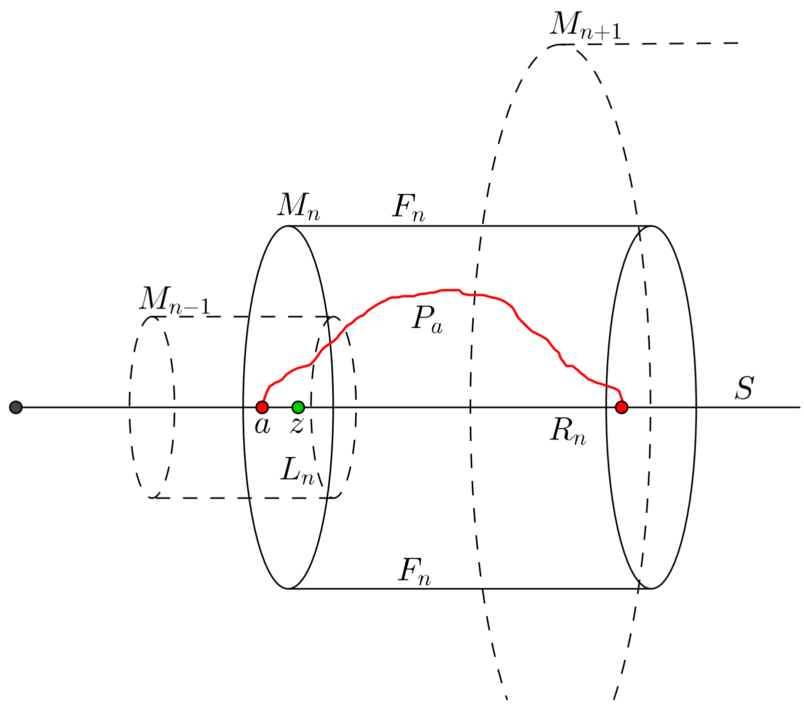

Figure 5: The path P_α in the proof of the Liouville property.

Lemma 5.2. *If h is a sharp harmonic function that is not constant, then for every $\epsilon > 0$ there are $a, z \in V$ with $h(a) < \epsilon$ and $h(z) > 1 - \epsilon$.*

Let $\mathcal{A}$ be a shift-invariant event of our random walk, i.e. an event not depending on the first n steps for every n. (The only kind of event we will later consider is the event 1^s that $s(Z_n)$ converges to 1, where s is our fixed sharp harmonic function.)

For $r \in (0, 1/2]$, let

$A_r := \{v \in V \mid \mathbb{P}[\mathcal{A}] > 1 - r\}$ and
$Z_r := \{v \in V \mid \mathbb{P}[\mathcal{A}] < r\}$.

Note that $A_r \cap Z_r = \emptyset$ for every such r.

By Lemma 5.1, if we let $\mathcal{A} := 1^s$ then we have $A_r = \{v \in V \mid s(v) > 1-r\}$ and $Z_r = \{v \in V \mid s(v) < r\}$.

Lemma 5.3. *For every $\epsilon, \delta \in (0, 1/2]$, and every $v \in A_\epsilon$, we have $\mathbb{P}_v[\text{visit } V \setminus A_\delta] < \epsilon/\delta$. Similarly, for every $v \in Z_\epsilon$, we have $\mathbb{P}_v[\text{visit } V \setminus Z_\delta] < \epsilon/\delta$.*

Proof. Start a random walk (Z_n) at v, and consider a stopping time τ at the first visit to $V \setminus A_\delta$. If τ is finite, let $z = Z_\tau$ be the first vertex of random walk outside A_δ. Since $z \notin A_\delta$, the probability that $s(X_n)$ converges to 1 for a random walk (X_n) starting from z is at least δ by the definition of A_δ. Thus, subject to visiting $V \setminus A_\delta$, the event $\mathcal{A}$ fails with probability at least δ since it is a shift-invariant event. But $\mathcal{A}$ fails with probability less than ϵ because $v \in A_\epsilon$, and so $\mathbb{P}_v[\text{visit } V \setminus A_\delta] < \epsilon/\delta$ as claimed.

The second assertion follows by the same arguments applied to the complement of $\mathcal{A}$. $\square$

Corollary 5.4. *If a random walk from $v \in A_\epsilon$ (respectively, $v \in Z_\epsilon$) visits a set $W \subset V$ with probability at least κ, then there is a v–W path all vertices of which lie in $A_{\epsilon/\kappa}$ (resp. $Z_{\epsilon/\kappa}$).*

Proof. Apply Lemma 5.3 with $\delta = \epsilon/\kappa$. Then the probability that random walk always stays within $A_{\epsilon/\kappa}$ is larger than $1 - \kappa$. Hence there is a nonzero probability that random walk meets W and along its trace only has vertices from $A_{\epsilon/\kappa}$. $\square$

Easily, h is uniquely determined by its values on the skewer S. Indeed, for every other vertex v, note that random walk X_n from v visits S almost surely, and so $h(v) = \mathbb{E}h(X_{\tau(R)})$, where $\tau(S)$ denotes the first hitting time of S by X_n. The same argument implies that

h is radially symmetric, i.e. for every two cocircular vertices v, w, we have $h(v) = h(w)$. (3)

Indeed, this follows from the fact that cocircular vertices have the same hitting distribution to S, which is easy to see (for any vertex on a circle, random walk has the same probability to move to some other circle).

We claim that, given an arbitrarily small $\epsilon > 0$, all but finitely many of the L_n contain a vertex in Z_ϵ.

Indeed if not, then since random walk from o has to visit all L_n by transience and (1) (where we use the fact that $L_n = R_{n-1}$), we would have $\mathbb{P}[\lim h(X_n) = 1] = 0$ for random walk X_n from o. But that probability is equal to $h(o)$ by Lemma 5.1, and if it is zero, then using Lemma 5.1 again easily implies that h is identically zero, contrary to our assumption that it is not constant.

Similarly, all but finitely many of the L_n contain a vertex in A_ϵ. Thus we can find a late enough M_n such that L_n contains a vertex $a \in A_\epsilon$ as well as a vertex $z \in Z_\epsilon$. We assume that a and z are the last vertices of L_n (in the ordering of L_n induced by the well-ordering of S) that are in A_ϵ and Z_ϵ, respectively. Assume without loss of generality that a appears before z in the ordering of L_n.

Note that, since R_n separates a from infinity (1), random walk from a visits R_n almost surely. Thus we can apply Corollary 5.4 with $W := R_n$ and $\kappa = 1$ to obtain an a–R_n path P_a with all its vertices in A_ϵ. We may assume that $P_a \subset M_n$ by taking a subpath contained in M_n if needed. Indeed, P_a can meet L_n only in vertices that are not past a in the linear ordering of L_n.

Let O_a denote the set of vertices $\{x = (t, w) \in M_n \mid \text{ there is } (t', w') \in V(P_a) \text{ with } w' = w\}$ obtained by 'rotating' P_a around S. By (3), we have $O_a \subset A_\epsilon$ since $P_a \subset A_\epsilon$. Note that O_a separates z from the ceiling F_n of M_n. But as random walk from $z \in Z_\epsilon$ visits F_n before returning to S with probability uniformly bounded below by (2), we obtain a contradiction to Lemma 5.3 with $\delta = 1/2$ for ϵ small enough compared to that bound.

6 A transient hyperbolic graph with no transient subtree

In this section we explain how our souvlaki construction can be slightly modified so that it does not contain any transient subtrees but remains transient and hyperbolic (and Liouville). This answers a question of I. Benjamini (private communication). The question is motivated by the fact that it is not too easy to come up with transient graphs that do not have transient subtrees [3].

We start with a very fast growing function $f : \mathbb{N} \to \mathbb{N}$, whose precise definition we reveal at the end of the proof. Roughly speaking, we will attach a sequence of finite graphs $(M_{f(n)})_{n\in\mathbb{N}}$ similar to the 'meatballs' from above to a ray S (the 'skewer') in such a way that most of the intersection of S with a fixed meatball is not contained in any other meatball. Formally, we let P_m be the 'bottom path' of M_m as defined in Section 2, and we tripartition $P_{f(n)}$ as follows: Let L_n consist of the first 2^n vertices on $P_{f(n)}$, and R_n consist of the last 2^{n+1}. The set of the remaining vertices of $P_{f(n)}$ we denote by Z_n, which by our choice of f will be much larger than R_n. As before, we glue the $M_{f(n)}$ on S by identifying P_n with a subpath of S. We start by glueing $M_{f(1)}$ on the initial segment of S of the appropriate length. Then we recursively glue the other $M_{f(n)}$ in such a way that L_n is identified with R_{n-1}. We call the resulting graph $\bar{\Psi}$.

Theorem 6.1. *$\bar{\Psi}$ is a bounded degree transient gromov-hyperbolic graph that does not contain a transient subtree.*

Proof. The hyperbolicity and the transience $\bar{\Psi}$ can be proved by the arguments we used for the original souvlaki Ψ. So it remains to show that $\bar{\Psi}$ does not have a transient subtree.

Let T be any subtree of $\bar{\Psi}$. We want to prove that T is not transient. Easily, we may assume that T does not have any degree 1 vertices. We will show that the following quotient Q of T is not transient: for each n, we identify all vertices in L_n to a new vertex v_n.

Note that the vertices v_n and v_{n+1} are cut-vertices of Q; let Q_n be the block of Q incident with both v_n and v_{n+1} containing these two vertices (a *block* is a maximally 2-connected subgraph). We will show that in Q_n the effective resistence from v_n to v_{n+1} is bounded away from 0, from which the recurrence of T will follow using Lyons' criterion.

Let $d = |L_{n+1}|$. We claim that there is some constant $c = c(d)$ only depending on d such that there are at most c vertices of Q_n with a degree greater than 2: indeed, $Q_n \setminus \{v_n, v_{n+1}\}$ is a forest with $d(v_n)+d(v_{n+1})$ leaves.

Next, we observe that Q_n has maximum degree at most d. Furthermore, the distance between v_n and v_{n+1} in Q_n is at least Z_n, which —by the choice of f— is huge compared to d and so also compared to c. Hence it remains to prove the following:

Lemma 6.2. *For every constant C and every m there is some $s = s(m, C)$, such that for every finite graph K with maximum degree at most C and at most C vertices of degree greater than 2, and any two vertices x, y of K at distance x and y at least s, the effective resistence between x and y in K is at least m.*

Proof. We start with a large $R \in \mathbb{N}$ the value of which we reveal later, and set $s = R \cdot C$.

Let K' be the graph obtained from K by contracting all vertices of degree 2. We colour an edge of K' black if it is subdivided at least R times in K. Note that K' has at most C vertices. Thus every x-y-path in K' has length at most C, but in K any such path has length at least s. Therefore each x-y-path in K' contains a black edge. Hence in K' there is an x-y-cut consisting of black edges only. This cut has at most C^2 edges. Thus the effective resistence in K between x and y is at least the one of that cut considered as a set of paths in K, which is as large as we want: indeed, we can pick R so large that the latter resistance exceeds m. $\square$

Now we reveal how large we have picked $f(n)$: recall that $d = 2^{n+1}$ and that $|Z_n| = f(n) - 3 \cdot 2^n$. We pick $f(n)$ large enough that $|Z_n| \geq s(1, max(c(d), d))$, where s is as given by the last lemma. With these choices the effective resistence between v_n and v_{n+1} in Q_n is at least 1. So Q cannot be transient by Lyons' criterion (Theorem 4.1). By Rayleigh's monotonicity law [10], T is recurrent too.

$\square$

6.1 Another transient hyperbolic graph with no transient subtree

We now sketch another construction of a transient hyperbolic graph with no transient subtree, provided by Tom Hutchcroft and Asaf Nachmias (private communication).

Let $[0,1]^3$ be the unit cube. For each $n \geq 0$, let D_n be the set of closed dyadic subcubes of length 2^{-n}. For each $n \geq 0$, let G_n be the graph with vertex set $\bigcup_{i=0}^{n} D_i$, and where two cubes x and y are adjacent if and only if

- $x \supset y$, $x \in D_i$ and $y \in D_{i+1}$ for some $i \in \{0, \dots n-1\}$,
- $y \supset x$, $y \in D_i$ and $x \in D_{i+1}$ for some $i \in \{0, \dots n-1\}$, or
- $x, y \in D_i$ for some $i \in \{0, \dots, n\}$ and $x \cap y$ is a square.

Then the graphs G_n are uniformly Gromov hyperbolic and, since the subgraph of G_n induced by D_n is a cube in $\mathbb{Z}^3$ (of size 4^n), the effective resistance between two corners this cube are bounded above uniformly in n. Moreover, the distance between these two points in G_n is at least n.

Let T be a binary tree, and let G be the graph formed by replacing each edge of T at height k from the root with a copy of G_{3^k}, so that the endpoints of each edge of T are identified with opposite corners in the corresponding copy of D_{3^k}. Since the graphs G_n are uniformly hyperbolic and T is a tree, it is easily verified that G is also hyperbolic. The effective resistance from the root to infinity in G is at most a constant multiple of the effective resistance to infinity of the root in T, so that G is transient. However, G does not contain a transient tree, since every tree contained in G is isomorphic to a binary tree in which each edge at height k from the root has been stretched by at least 3^k, plus some finite bushes.

7 A transient graph with no embeddable transient subgraph

We say that a graph H has a graph K as a *minor*, if K can be obtained from H by deleting vertices and edges and by contracting edges. Let K^r denote the complete graph on r vertices.

Proposition 7.1. *There is a transient bounded degree graph G such that every transient subgraph of G has a K^r minor for every $r \in \mathbb{N}$.*

In particular, G has no transient subgraph that embeds in any surface of finite genus.

We now construct this graph G. We will start with the infinite binary tree with root o, and replace each edge at distance r from o with a gadget D_{2^r} which we now define. Given n $(= 2^r)$, the vertices of D_n are organized in $2n+1$ levels numbered $-n, \dots, -1, 0, 1, \dots, n$. Each level i has $2^{n-|i|}$ vertices, and two levels i, j form a complete bipartite graph whenever $|i-j| = 1$; otherwise there is no edge between levels i, j. Any edge of D_n from level $i \geq 0$ to level $i+1$ or from level $-i$ to level $-(i+1)$ is given a resistance equal to $2^{n-|i|}$ (we will later subdivide such edges into paths of that many edges each having resistance 1). With this choice, the effective resistance R_i between levels i and $i+1$ of D_n is $2^{n-|i|}$ divided by the number of edges between those two levels, that is, $R_i = \frac{2^{n-|i|}}{2^{n-|i|}2^{n-|i|-1}} = 2^{-n+|i|+1}$, and so the effective resistance in D_n between its two vertices at levels n and $-n$ is $O(1)$

Let G' be the graph obtained from the infinite binary tree with root o by replacing each edge e at distance n from o with a disjoint copy of D_n, attaching the two vertices at levels n and $-n$ of D_n to the two end-vertices of e. We will later modify G' to obtain a bounded degree G with similar properties satisfying Proposition 7.1.

Note that as D_n has effective resistance $O(1)$, the graph G' is transient by Lyons' criterion.

We are claiming that if H is a transient subgraph of G', then H has a K^r minor for every $r \in \mathbb{N}$.

This will follow from the following basic fact of finite extremal graph theory [12, 9, 5]

Theorem 7.2. *For every $r \in \mathbb{N}$ there is a constant c_r such that every graph of average degree at least c_r has a K^r minor.*

Lemma 7.3. *If H is a transient subgraph of G', then H has a K^r minor for every $r \in \mathbb{N}$.*

Proof. Suppose that H has no K^r minor for some r, and fix any $m \in \mathbb{N}$. Consider for every copy C of the gadget D_n in G' where $n > m$, the bipartite subgraph $G_m = G_m(C)$ of H spanned by levels m and $m+1$ of $C \cap H$. By Theorem 7.2, the average degree of G_m is at most c_r. Thus, if we identify each of the partition classes of G_m into one vertex, we obtain a graph with 2 vertices and at most $\frac{3}{2}2^{n-m}c_r$ parallel edges, each of resistance 2^{n-m}, so that the effective resistance of the contracted graph is greater than $1/c_r =: C_r$.

Now repeating this argument for $m+1, m+2, \ldots$, we see that the effective resistance between the two partition classes of G_{m+k} (which is edge-disjoint to G_m) is also at least the same constant C_r. This easily implies that the effective resistance between the two endvertices of $C \cap H$ for any copy C of D_n is $\Omega(n)$. Since G' has 2^r copies of D_{2^r} at each 'level' r, we obtain that the effective resistance from o (which we may assume without loss of generality to be contained in H) to infinity in H is $\Omega(\sum_r 2^r/2^r) = \infty$.

Thus H can have no electrical flow from a vertex to infinity, and by Lyons' criterion (Theorem 4.1) it is not transient. $\square$

Recall that the edges of G' had resistances greater than 1. By replacing each edge of resistance k by a path of length k with edges having resistance 1, we do not affect the transience of G'. We now modify G' further into a graph G of bounded degree, which will retain the desired property.

Let x be a vertex of some copy C of D_n, at some level $j \neq n, -n$ of C. Then x sends edges to the two neighbouring levels $j \pm 1$. Each of those levels L, L', sends $2^{k \pm 1}$ edges to x for some k. Now disconnect all the edges from L to x, attach a binary tree T_L of depth $k \pm 1$ to x, and then reconnect those edges, one at each leaf of T_L. Do the same for the other level L', attaching a new tree $T_{L'}$ of appropriate depth to x. Note that after doing this for every such x, the graph G obtained has maximum degree 6 (we do not need to modify the vertices at levels $n, -n$ in C, as they already had degree 6.

Now let's check that G is still transient, by considering the obvious flow to infinity. Each new tree of the form T_L we attached has effective resistance from its root to the union of its leaves less that 1. Moreover, between any two levels of size about 2^k in some D_n we introduced as many such trees as there are vertices in the levels. An easy calculation yields that the extra effective resistance we introduced between two levels is about $2/2^k$; hence the total resistance we introduced to each D_n taking into account all its levels is bounded by a constant. Thus the effective resistance of each D_n remains bounded by a constant (independent of n), and so G is still transient.

Note that G' can be obtained from G by contracting edges. Thus any transient $H \subseteq G$ has a transient minor $H' \subseteq G'$, because contracting edges preserves transience by Lyons' criterion. As we have proved that H' has a K^r minor (Lemma 7.3), so does H as any minor of H' is a minor of H.

Despite Proposition 7.1, the following remains open

Question 7.4 (I. Benjamini (private communication)). *Does every bounded-degree transient graph have a transient subgraph which is sphere-packable in* R^3*?*

8 Problems

It is not hard to see that our hyperbolic souvlaki Ψ is *amenable*, that is, we have $\inf_{\emptyset \neq S \subset \Psi \text{ finite}} \frac{|\partial S|}{|S|} = 0$, where $\partial S = \{v \in V(\Psi) \setminus S \mid \text{there exists } w \in S \text{ adjacent to } v\}$. We do not know if this is an essential feature:

Problem 1. *Is there a non-amenable counterexample to Conjecture 1.1?*

Similarly, one can ask

Problem 2. *Is there a non-amenable, hyperbolic graph with bounded-degrees,* C*-dense infinite geodesics, and the Liouville property, the hyperbolic boundary of which consists of a single point?*

Here we did not ask for transience as it is implied by non-amenability [3].

We conclude with further questions asked by I. Benjamini (private communication)

Problem 3. *Is there a uniformly transient counterexample to Conjecture 1.1? Is there an 1-ended counterexample?*

Here *uniformly transient* means that there is an upper bound on the effective resistance between any vertex of the graph and infinity.

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