PROFILES.
AN ALGEBRAIC APPROACH TO COMBINATORIAL CONNECTIVITY

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ABSTRACT. We describe an algebraic approach to combinatorial connectivity. As an application, we obtain canonical tree-decompositions distinguishing all the maximal tangles of a finite graph or matroid. For graphs we also find such decompositions which, for any given k, distinguish all their distinguishable k-blocks and tangles of order $k+1$. If we only consider robust blocks and tangles, we obtain one overall tree-decomposition which refines all these tree-decompositions, for all k simultaneously.

1. INTRODUCTION

It is an ancient and well-known problem in graph theory how to decompose a k-connected graph into its '$(k+1)$-connected components'. For $k = 1$ this is solved by the block-cutvertex tree. For $k = 2$, a similar solution was given by Tutte. For arbitrary k, two solutions have so far been proposed: a graph can either be decomposed into parts that each contain at most one maximal k-inseparable set—or *k-block*—as demonstrated in [1], or into parts that each contain at most one maximal tangle, which was proved by Robertson and Seymour [9]. There are examples of graphs for which these decompositions differ substantially.

In this paper we unify the notions of a tangle of order $k+1$ and the notion of a k-block by the notion of a *k-profile*. We shall prove decomposition theorems for these profiles that will combine, imply, and strengthen all the known decomposition results for k-blocks and tangles in graphs. As in [1], all our decompositions will be *canonical* in the sense that their construction only depends on the structure of the graph; in particular, they are invariant under the graph's automorphisms. In addition, our results will yield canonical tree-decompositions also of matroids, decompositions which likewise distinguish all those matroids' maximal tangles. Specifically, we shall prove:

Theorem 1. *Every finite graph admits, for every integer k, a canonical tree-decomposition of adhesion at most k that distinguishes all its distinguishable k-blocks and tangles of order $k+1$, and in which every k-block and every tangle of order $k+1$ inhabits a unique part.*

For tangles alone, we obtain the following strengthening of a result of Robertson and Seymour [9, 10.3]. (They construct a similar tree-decomposition which, however, depends on a fixed vertex enumeration of the graph to 'break ties' between competing separations, and hence is not canonical in our sense.)

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Theorem 2. *Every finite graph admits a canonical tree-decomposition that efficiently distinguishes all its maximal tangles, and in which every maximal tangle inhabits a unique part.*

Using profiles, we can refine this tree-decomposition further: parts that are inhabited by a unique maximal tangle but still contain more than one (robust) block can now be split into smaller parts, so that these blocks are distinguished too:

Theorem 3. *Every finite graph admits a canonical tree-decomposition which, for all k simultaneously, efficiently distinguishes all its distinguishable robust k-blocks and tangles of order $k+1$.*

Extending the Robertson-Seymour theorem cited above, Geelen, Gerards and Whittle [6, 1.1] proved recently that the maximal tangles of a matroid can be distinguished by a single tree-decomposition. Using our theory, we can do the same canonically:

Theorem 4. *Every finite matroid M has a canonical tree-decomposition that efficiently distinguishes all its maximal tangles, and in which every maximal tangle inhabits a unique part.*

The basic idea of a profile originates from the notion of a tangle. A tangle τ of order $k+1$ can be understood as a 'collection of signposts' which point, for every separation (A, B) of order at most k, either to A or to B. Wwe then consider this side, A or B, as the 'large' side of (A, B) with respect to τ.) In order to form a tangle, these signposts have to be consistent in the following sense: no three 'small' sides are allowed to cover the whole graph.

In this way, each tangle of order $k+1$ can be thought of as describing one '$(k+1)$-connected component' of the graph by telling us how to get there. The notion of a k-profile follows the same idea but weakens the consistency condition as follows. The small sides of three separations are now only forbidden to cover the whole graph if one of them is the intersection of the large sides of the other two separations: if the intersection of two large sides is the side of another separation, it must be its large side.

A k-block, too, identifies one side of every separation of order at most k as 'large': the side that contains it. While k-blocks do not necessarily induce a tangle of order $k+1$ in this way, they do induce a k-profile. This enables us to apply the techniques developed in [1] for separating blocks more generally to separate profiles, and thereby to separate both blocks and tangles at the same time.

In order to be able to apply our results to both graphs and matroids we need to work in an abstract framework general enough to allow us to express separations in either setting. For every graph or matroid there is a natural partial ordering on its set of separations, based on set-inclusion. These separation posets form lattices that are isomorphic to their own duals. The notion of such *self-dual* posets and lattices, on which our theory of separations will be based, will be introduced in Section 2. In the subsequent two sections we describe 'tree-likeness' based on self-dual posets, and show how this gives rise to formal tree-decompositions.

In Section 5 we introduce profiles and show under which conditions a given set of profiles can be distinguished in a tree-like way. This will be applied in Section 6 to *k-profiles*, which involves the notion of a *connectivity function*, to describe the order of our separations in this more general context. In Section 7 we combine the results of all the previous sections to establish our main results. In Section 8 and 9 we apply our results to graphs and matroids.

2. Basics

2.1. **Self-dual posets.** A binary relation '$\leq$' on a set P is called a *partial order* on P if it is reflexive, antisymmetric and transitive. Then the tuple $(P, \leq)$ is called a *poset*. We write '$<$' for the strict part of '$\leq$', i.e. for all $a, b \in P$ we write $a < b$ if $a \leq b$ but $a \neq b$. Given a poset $(P, \leq)$ we define the *inverse* $\geq$ by $a \geq b :\Leftrightarrow b \leq a$ for all $a, b \in P$. We call $(P, \geq)$ the *dual* of $(P, \leq)$. If there is no risk for confusion we write just P for $(P, \leq)$ and P^d for its dual. We use '$>$' in the obvious way for the strict part of '$\geq$'. Given two elements a, b of a poset P we say that b *is a successor of* a, if $a < b$ and there is no element $c \in P$ with $a < c < b$; in this case a *is a predecessor of* b. The *interval* $[a, b]$ of two elements $a, b \in P$ is defined as the set of all elements that 'lie between' a and b, i.e. $[a, b] := \{c \in P \mid a \leq c \leq b\}$. A poset is called *discrete* if each pair of its elements gives rise to a finite interval. Note that if $a \not\leq b$ then $[a, b] = \emptyset$.

An element $a \in P$ is called *minimal* if there is no element $b \in P$ with $b < a$. A minimal element l that is comparable with all elements in P is called a *least* element. If a least element exists it is clearly unique and also denoted by *the bottom* $\perp$ of P. Correspondingly an element is called *maximal* in P if it is minimal in P^d and the *largest* element of P is the least element of P^d, and also denoted by *the top* $\top$ of P.

Given two posets P and Q we call a map $\alpha : P \to Q$ a *poset homomorphism*, if $a \leq_P b \Rightarrow a^\alpha \leq_Q b^\alpha$. A bijective homomorphism whose inverse is also a homomorphism is called an *isomorphism*. An isomorphism $\alpha : P \to P$ is called an *automorphism* of P. If P and P' are posets on the same ground set and $\alpha : P \to P'$ is a bijection from P onto P', we call α *idempotent* if $\alpha = \alpha^{-1}$. If φ is an idempotent isomorphism between P and its dual P^d we call $(P, \leq, \varphi)$ a *self-dual poset* and for every $a \in P$ we call a^φ the *dual* of a.

Lemma 2.1. *Given a self-dual poset* $(P, \leq, \varphi)$ *we have:*

$$a \leq b \Leftrightarrow b^\varphi \leq a^\varphi.$$

Proof. This is due to the fact that φ is an isomorphism between P and P^d. $\square$

If $(P, \leq)$ is a poset then for every subset $S \subseteq P$ the restriction of $\leq$ onto S defines a poset $(S, \leq_{|S})$. Given a self-dual poset $(P, \leq, \varphi)$, a subset $S \subseteq P$ is called *symmetric* if it is closed under φ; then $(S, \leq_{|S}, \varphi_{|S})$ defines a self-dual poset. To simplify notation, and since there is no risk for confusion, we will always write $\leq$ instead of $\leq_{|S}$ and φ instead of $\varphi_{|S}$.

2.2. **Self-dual lattices.** Let $(L, \leq)$ be a poset and let $a, b \in L$. If there exists the largest element that is smaller than both a and b, then this element is called the *meet* of a and b, denoted by $a \wedge b$. If there is the least element that is larger than both a and b, this is called the *join* of a and b, denoted by $a \vee b$. If for all pairs of elements in L both meet and join exist, then $(L, \leq)$ is called a *lattice*. Since lattices are special posets, all terms introduced for posets also apply to lattices. In that spirit a self-dual poset $(L, \leq, \varphi)$ that is a lattice is called a *self-dual lattice*.

Example 1. Let S be an arbitrary set and let $\mathcal{P}(S)$ denote the set of subsets of S. We define $\varphi : \mathcal{P}(S) \longrightarrow \mathcal{P}(S)$ via $A^\varphi = S \setminus A$, for all $A \subseteq S$. Then $(\mathcal{P}(S), \subseteq, \varphi)$ is a self-dual lattice—the *subset lattice of* S—with $A \wedge B = A \cap B$ and $A \vee B = A \cup B$.

Lemma 2.2. *Given a self-dual lattice* $(L, \leq, \varphi)$ *and* $a, b \in L$ *we have:*

(i) $(a \wedge b)^\varphi = a^\varphi \vee b^\varphi$

(ii) $(a \vee b)^\varphi = a^\varphi \wedge b^\varphi$

Proof. Let $x := a \wedge b$. Then we have $x \leq a$ and $x \leq b$ by definition of the meet and therefore $a^\varphi \leq x^\varphi$ and $b^\varphi \leq x^\varphi$ by Lemma 2.1. So by the definition of the join we have $y := a^\varphi \vee b^\varphi \leq x^\varphi$. Now if $y < x^\varphi$ then again by Lemma 2.1 we have $x < y^\varphi \leq a$ and $x < y^\varphi \leq b$, which contradicts the definition of x. Thus we have shown (i). The dual argument yields (ii). □

While a subset of a poset is always a poset itself, this is not so for lattices. In the remainder of this paper we will often start with a (self-dual) lattice but their considered (symmetric) subsets are only treated as (self-dual) posets. So whenever we refer to the meet or join of two elements a and b of a given subset $S \subseteq L$, we actually refer to their meet or join as elements of L. This has to be mentioned since the meet (or join) in S could also exist and differ from the meet (or join) in L. But note that if $a \wedge_L b \in S$ or $a \vee_L b \in S$, respectively, then this is also the meet or join, respectively, of a and b as elements of the poset S.

2.3. **Quasi-partition lattices.** An important and natural class of self-dual lattices is given by quasi-partition lattices which we shall define in a moment. Consider an arbitrary set S and two distinct subsets $A, B \subseteq S$. Then the ordered pair (A, B) is called a *quasi-partition* of S if $A \cup B = S$. That is, contrary to the notion of a bipartition we do not require A and B to be disjoint. Let $\mathcal{Q} = \mathcal{Q}(S)$ denote the set of all quasi-partitions of S. We define a partial order on $\mathcal{Q}$ via:

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \text{ and } B \supseteq D. \tag{1}$$

Then it is clear from (1) that $\varphi : \mathcal{Q} \longrightarrow \mathcal{Q}^d$ with $(A, B)^\varphi = (B, A)$ is an idempotent isomorphism between $\mathcal{Q}$ and its dual. Furthermore it is easy to verify the following:

$$(A, B) \wedge (C, D) = (A \cap C, B \cup D); \tag{2}$$

$$(A, B) \vee (C, D) = (A \cup C, B \cap D). \tag{3}$$

Which turns $(\mathcal{Q}, \leq, \varphi)$ into a self-dual lattice. Now every subset of $\mathcal{Q}$ that is closed under $\wedge, \vee$ and φ is called a *quasi-partition lattice over* S.

Example 2. The bipartitions of S form a quasi-partition lattice over S, which we call the *bipartition lattice of* S, denoted by $\mathcal{B}(S)$.

If we recall Example 1 we note that the bipartition lattice of S is just another view of the subset lattice of S; via $\alpha : A \mapsto (A, S \setminus A)$ we clearly have an isomorphism.

Example 3. Let $G = (V, E)$ be a finite graph and consider $\big(\mathcal{Q}(V), \leq, \varphi\big)$. A *separation* of G is a quasi-partition (A, B) of V such that there is no edge in G with one endvertex in $A \setminus B$ and the other endvertex in $B \setminus A$. We shall show that the subset $\mathcal{S}_G \subseteq \mathcal{Q}(V)$ of all separations of G gives rise to a quasi-partition lattice $(\mathcal{S}_G, \leq, \varphi)$ over V, which we call the *separation lattice of* G.

Clearly $\mathcal{S}_G$ is closed under φ, so in order to show that it is also closed under $\wedge$ and $\vee$, consider $(A, B), (C, D) \in \mathcal{S}_G$. Then $(A, B) \wedge (C, D)$ is given by $(A \cap C, B \cup D)$.

Let $e = \{v, w\}$ be an edge of G with $v \in (B \cup D) \setminus (A \cap C)$. If $v \in B \setminus A$ then $w \notin A \setminus B \supseteq (A \cap C) \setminus (B \cup D)$, since (A, B) is a separation of G, and if $v \notin B \setminus A$ then $v \in D \setminus C$, such that $w \notin C \setminus D \supseteq (A \cap C) \setminus (B \cup D)$, since (C, D) also is a separation of G. Thus we have $(A, B) \wedge (C, D) \in \mathcal{S}_G$ and a similar argument yields $(A, B) \vee (C, D) \in \mathcal{S}_G$, which shows that $(\mathcal{S}_G, \le, \varphi)$ indeed is a lattice.

2.4. **Nestedness.** Let $(P, \le, \varphi)$ be a self-dual poset and let $a, b \in P$. Then a is *comparable with* b if either $a \le b$ or $b \le a$ and we say that a is *nested* with b, written as $a \parallel b$, if a is comparable with either b or b^φ. If a and b are not nested then they are *crossing*, written as $a \nparallel b$. A subset $N \subseteq P$ is called *nested* if the elements of N are pairwise nested.

Now let $(L, \le, \varphi)$ be a self-dual lattice and let $a, b \in L$. The *corners of* $\{a, b\}$ are given by $a \wedge b$, $a^\varphi \wedge b$, $a \wedge b^\varphi$ and $a^\varphi \wedge b^\varphi$. It is clear from the definition that the following lemma holds:

Lemma 2.3. *Given a self-dual lattice $(L, \le, \varphi)$ and $a, b \in L$ then both a and b are nested with every corner of $\{a, b\}$.* □

It is also true that the corners of a pair of elements are pairwise nested. More precisely we have the following:

Lemma 2.4. *Given a self-dual lattice $(L, \le, \varphi)$ and $a, b \in L$ then each corner of $\{a, b\}$ is smaller than the dual of each of the other corners.*

Proof. Applying Lemma 2.2 we get:

$$a \wedge b \le \begin{cases} a \vee b & = & (a^\varphi \wedge b^\varphi)^\varphi \\ a \vee b^\varphi & = & (a^\varphi \wedge b)^\varphi \\ a^\varphi \vee b & = & (a \wedge b^\varphi)^\varphi \end{cases} .$$

For the other corners this follows by applying φ either to a, b or both at the same time. □

The notion of nestedness clearly defines a symmetric and reflexive binary relation on L, but in general it is not transitive. So given two crossing elements $a \nparallel b$ of L there might exist an element $c \in L$ which is nested with both a and b. But as the following lemma shows, this is only possible if c relates in a specific way to the corners of $\{a, b\}$:

Lemma 2.5. *Let $(L, \le, \varphi)$ be a self-dual lattice and $a, b, c \in L$. If $a \nparallel b$ and c is nested with both a and b then c is also nested with every corner of $\{a, b\}$ and either c or c^φ is smaller than one of the corners.*

Proof. Consider $a, b, c \in L$ as in the statement. Since $c \parallel a$ we can choose $x \in \{a, a^\varphi\}$ and $z \in \{c, c^\varphi\}$ such that $z \le x$. Since c is also nested with b we can choose $y \in \{b, b^\varphi\}$ such that z is comparable with y. If $y \le z$ then by transitivity of '$\le$' we also have $y \le x$ and therefore $a \parallel b$, contradicting our assumption. Therefore we have $z \le y$ and hence $z \le x \wedge y$, by the definition of the meet. Applying Lemma 2.4 finishes the proof. □

3. Treelike posets

We assume that the reader is familiar with the concept of a graph and refer to [3] for all related terms. A *tree* is a connected (undirected) graph that does not embed any cycles. Given a tree $\mathcal{T}$ we denote its set of edges by $E(\mathcal{T})$ and its set of vertices by $V(\mathcal{T})$. Then every edge $e \in E(\mathcal{T})$ can be written as an unordered pair $e = \{v, v'\}$ with $v, v' \in V(\mathcal{T})$. An *oriented path* of length n in $\mathcal{T}$, where n is a positive integer, is a sequence $v_0 v_1 \cdots v_n$ of pairwise distinct vertices $v_i \in V(\mathcal{T})$ such that $\{v_i, v_{i+1}\} \in E(\mathcal{T})$ for all $i < n$. By $\vec{E}(\mathcal{T})$ we denote the set of *oriented edges* (v, v') where $\{v, v'\} \in E(\mathcal{T})$. It is straightforward to verify that $\mathcal{T}$ induces a partial order on $\vec{E}(\mathcal{T})$ where $(v, v') \leq (w, w')$ if and only if there is an oriented path in $\mathcal{T}$ starting with vv' and ending with ww'. From the definition it is obvious that

$$(v, v') \leq (w, w') \Leftrightarrow (w', w) \leq (v', v),$$

we just have to reverse the oriented path witnessing the one inequality, to obtain the other. So if we define $\varphi : \vec{E}(\mathcal{T}) \longrightarrow \vec{E}(\mathcal{T})$ according to $(v, w)^\varphi := (w, v)$, then $(\vec{E}(\mathcal{T}), \leq, \varphi)$ is a self-dual poset, which we call the *edge-poset of* $\mathcal{T}$.

We shall emphasize three essential properties of the edge-poset of a tree. First, since an oriented path in a tree by definition consists of pairwise distinct vertices, none of the elements of an edge-poset is comparable with its dual. Second, every two elements in an edge-poset are nested with each other: Consider a tree $\mathcal{T}$ and $(v, v'), (w, w') \in \vec{E}(\mathcal{T})$. Since $\mathcal{T}$ is a tree, there is a unique shortest path in $\mathcal{T}$ containing all the vertices v, v', w and w', with one endvertex in $\{v, v'\}$ and the other in $\{w, w'\}$. Assume v and w to be these endvertices, then $(v, v') \leq (w', w)$ and $(w, w') \leq (v', v)$. Hence, $(v, v') \parallel (w, w')$. And finally, since all vertices in a tree are connected by a path of finite length, the edge-poset of a tree is discrete.

It turns out that these properties identify precisely those self-dual posets that 'come from a tree'. To make this precise, let $(P, \leq, \varphi)$ be a self-dual poset. A symmetric subset $S \subseteq P$ is called *balanced* if none of its elements is comparable with its dual. If a subset $T \subseteq P$ is balanced and nested, it is called *treelike*. If P itself is treelike, then $(P, \leq, \varphi)$ is called a *treelike poset*. The goal of this section is to show that the discrete treelike posets are in one-to-one correspondence with the trees. The first lemma simply summarizes the previous paragraph:

Lemma 3.1. *If $\mathcal{T}$ is a tree, then $\vec{E}(\mathcal{T})$ is a discrete treelike poset.* □

The remainder of this section is devoted to the 'converse' of Lemma 3.1. The general approach will be as follows: We imagine a tree $\mathcal{T}$ and its edge-poset $\vec{E}(\mathcal{T})$ and try to re-obtain $\mathcal{T}$ from $\vec{E}(\mathcal{T})$, while on the formal level we shall only use the fact that $\vec{E}(\mathcal{T})$ is treelike and discrete. To obtain the edges of $\mathcal{T}$ (as a set) we simply identify $\{v, v'\}$ with $\{(v, v'), (v', v)\}$, so in principle we group together each element of $\vec{E}(\mathcal{T})$ with its dual. But defining (or re-obtaining) the vertices of $\mathcal{T}$ is a little bit more involved. If we consider the elements of $\vec{E}(\mathcal{T})$ as formal pairs (v, v') then we can identify the vertex v with the set of all pairs (w, w') such that $w = v$, so we are grouping together all oriented edges that share the same tail-vertex. But this grouping of oriented edges presupposes the concept of a vertex, which we are just about to deduce. Nevertheless we can use this idea of representing a vertex by a set of oriented edges, by expressing this 'tail incidence' by means of the partial order '$\leq$' on $\vec{E}(\mathcal{T})$. Note that if $v_0 v_1 v_2$ is an oriented path in $\mathcal{T}$ then (v_0, v_1) is a predecessor of (v_1, v_2) with respect to '$\leq$', and at the same time (v_1, v_0) and (v_1, v_2) shall be grouped together, since they both have v_1 as their common tail-vertex. Hence, each element of $\vec{E}(\mathcal{T})$ is grouped together with the dual of each of its predecessors.

Now we leave the intuitive level and consider a treelike poset $(T, \leq, \varphi)$. We define a binary relation '$\sim$' on T by:

$$a \sim b :\Leftrightarrow \begin{cases} a = b, \text{ or} \\ a^\varphi \text{ is a predecessor of } b. \end{cases} \tag{4}$$

We shall show in a moment that '$\sim$' indeed defines an equivalence relation, but to simplify the proof, we first prove another lemma:

Lemma 3.2. *Let $(B, \leq, \varphi)$ be a balanced poset and let $a, b \in B$. Then a cannot be comparable with both b and b^φ.*

Proof. In order to obtain a contradiction, assume a to be comparable with both b and b^φ. Up to duality there are two cases. If $a \leq b$ and $b^\varphi \leq a$ then $b^\varphi \leq b$, in contradiction to the balancedness of B. While if $a \leq b$ and $a \leq b^\varphi$ then due to Lemma 2.1 we have $b^\varphi \leq a^\varphi$ and therefore $a \leq a^\varphi$, again a contradiction. □

Lemma 3.3. *If $(T, \leq, \varphi)$ is a treelike poset then the relation '$\sim$' is an equivalence relation on T.*

Proof. By definition '$\sim$' is reflexive and symmetry is due to Lemma 2.1. To show transitivity consider three distinct elements $a, b, c \in T$ and assume $a \sim b$ and $b \sim c$. We have to show that a^φ is a predecessor of c. Let us first summarize what we know about a, b and c so far, due to (4) and Lemma 2.1:

(i) a^φ is a predecessor of b;
(ii) b^φ is a predecessor of a;
(iii) b^φ is a predecessor of c;
(iv) c^φ is a predecessor of b.

By (ii) and (iii) a is incomparable with c. Then a^φ is comparable with c, since T is nested. Now $c \leq a^\varphi$ implies

$$b^\varphi \underset{(iii)}{\leq} c \leq a^\varphi \underset{(i)}{\leq} b,$$

contradicting the balancedness of T, so we have $a^\varphi < c$, as desired.

It lasts to show that T contains no element 'between' a^φ and c. So suppose there is $x \in T$ with

$$a^\varphi < x < c. \tag{5}$$

Since T is nested we know that x is comparable with either b or b^φ. Then we have $x \not< b$, by (i), and $b^\varphi \not< x$, by (iii). If $b \leq x$ then $b < c$, due to (5), and $b^\varphi < c$, due to (iii), contradicting Lemma 3.2. On the other hand, if $x \leq b^\varphi$ then

$$a^\varphi \underset{(5)}{<} x < b^\varphi \underset{(ii)}{\leq} a,$$

contradicting the balancedness of T. Hence no such x exists and a^φ is a predecessor of c, which means $a \sim c$. □

For $a \in T$ let $[a] := \{b \sim a \mid b \in T\}$ denote the equivalence class of a with respect to '$\sim$', and let $T/\sim$ be the set of all equivalence classes of T. To show that φ induces a graph structure on $T/\sim$, we need the following lemma:

Lemma 3.4. *For every distinct $a, b \in T$ we have: $b \in [a] \Rightarrow b^\varphi \notin [a^\varphi]$.*

Proof. If $b \in [a]$ then b^φ is a predecessor of a and Lemma 2.1 yields that a^φ is a predecessor of b—not vice versa—so $b^\varphi \notin [a^\varphi]$. □

Hence, if we consider $T/\sim$ as a set of vertices and add an edge between $[a]$ and $[a^\varphi]$, for all $a \in T$, we will not construct any multiple edges. So we get a graph $\mathcal{T} = \mathcal{T}(T)$ with:

$$V(\mathcal{T}) = T/\sim$$

and

$$E(\mathcal{T}) = \big\{\{[a],[a^\varphi]\} \mid a \in T\big\}.$$

Let $e = \{v, w\}$ be an edge of $\mathcal{T}$. Then due to Lemma 3.4 there is a unique element $a \in v$ such that $a^\varphi \in w$ and we can identify the oriented edge (v, w) with a and (w, v) with a^φ. Formally, for every $v, w \in V(\mathcal{T})$ we define:

$$T(v, w) := v \cap w^\varphi,$$

where $w^\varphi = \{a^\varphi \mid a \in w\}$. So $T(v, w)$ consists of a single element of T if (v, w) is an oriented edge of $\mathcal{T}$, and it is empty otherwise.

Note that now '$\leq$' induces a partial order on the oriented edges $\vec{E}(\mathcal{T})$, since every oriented edge gets identified with a unique element of T. And it is easy to verify that $\vec{E}(\mathcal{T})$ is isomorphic to T as a poset. So we have shown that we can represent every treelike poset T as the edge-poset of a graph $\mathcal{T}$. Now we want to show that $\mathcal{T}$ itself is also 'treelike' in a graph theoretic sense.

Lemma 3.5. *If $v_0v_1v_2$ is an oriented path in $\mathcal{T}$, then there are distinct $a, b \in T$ such that $T(v_0, v_1) = \{a\}$, $T(v_1, v_2) = \{b\}$ and a is a predecessor of b.*

Proof. Since (v_0, v_1) is an oriented edge of $\mathcal{T}$, and due to Lemma 3.4, there is a unique $a \in T$ with $v_0 = [a]$ and $v_1 = [a^\varphi]$, thus $T(v_0, v_1) = \{a\}$. The same argument applied to (v_1, v_2) yields $T(v_1, v_2) = \{b\}$. Thus $a^\varphi \in [b]$ and since $v_0 \neq v_2$, which implies $a \neq b^\varphi$ and therefore $a^\varphi \neq b$, we know that a is a predecessor of b. □

Lemma 3.6. *The graph $\mathcal{T}$ is acyclic.*

Proof. Assume there is a cycle C in $\mathcal{T}$. Then we can enumerate its vertices as $v_0, v_1, \ldots, v_n$ such that (v_i, v_{i+1}) is an oriented edge of $\mathcal{T}$, for $0 \leq i \leq n$, where indices are considered modulo $(n+1)$. Applying Lemma 3.5 to $v_{i-1}v_iv_{i+1}$ for all i, we obtain elements $a_i \in T$ with $T(v_i, v_{i+1}) = \{a_i\}$ and a_i is a predecessor of a_{i+1}. But then the transitivity of '$\leq$' yields $a_0 < a_n < a_0$, a contradiction. □

If we add discreteness as an additional assumption on T we arrive at the main result of this section:

Theorem 3.7. *If $(T, \leq, \varphi)$ is a discrete treelike poset then $\mathcal{T}(T)$ is a tree.*

Proof. By Lemma 3.6 $\mathcal{T} = \mathcal{T}(T)$ is an acyclic graph. So we only have to show that $\mathcal{T}$ is connected. Let $a, b \in T$; we shall show that there is a path containing $[a]$ and $[b]$ in $\mathcal{T}$. Since T is nested, a is comparable with either b or b^φ. There is an edge between $[b]$ and $[b^\varphi]$, so if there is a path joining $[a]$ and $[b^\varphi]$ there is also one joining $[a]$ and $[b]$. Hence, we may assume that a is comparable with b. By interchanging a and b, if necessary, we may indeed assume $a \leq b$.

As T is discrete we know that the interval $I := [a, b]$ is finite. So there is a maximal chain $a = a_0 < \cdots < a_n = b$ in I where each a_i is a predecessor of a_{i+1} with respect to '$\leq$', for $0 \leq i < n$. Hence $a_i^\varphi \in [a_{i+1}]$, and therefore $[a_i^\varphi] = [a_{i+1}]$, such that $([a_i], [a_{i+1}])$ is an oriented edge in $\mathcal{T}$. Thus there is an oriented path from $[a]$ to $[b]$ in $\mathcal{T}$. □

It is important to mention that the definition of $\mathcal{T} = \mathcal{T}(T)$ is 'canonical' in the following sense:

Corollary 3.8. *If Γ is a group of automorphisms of T then Γ also acts on $\mathcal{T}$ as a group of automorphisms.*

Proof. If α is an automorphism of T, then a is a predecessor of b if and only if a^α is a predecessor of b^α. Hence, $a \sim b \Leftrightarrow a^\alpha \sim b^\alpha$, and α maps equivalence classes onto equivalence classes. For the same reason α also preserves adjacency and non-adjacency. □

Together with Lemma 3.1 we can summarize the relation between discrete treelike posets and trees as follows:

Corollary 3.9. *There is a one-to-one correspondence*[1] *between the discrete treelike posets and the trees. More precisely we have:*

(i) If T is a discrete treelike poset, then: $\vec{E}(\mathcal{T}(T)) \simeq T$;
(ii) If $\mathcal{T}$ is a tree, then: $\mathcal{T}(\vec{E}(\mathcal{T})) \simeq \mathcal{T}$. □

This correspondence goes even further as we shall show by Theorem 3.11. Before we can state it properly we need the notion of a *minor*—a graph G is called a minor of a graph G', written as $G \preccurlyeq G'$, if G can be obtained from a subgraph of G' by contracting edges [3, p. 19].

Lemma 3.10. *Let T be a discrete treelike poset, let $\mathcal{T} = \mathcal{T}(T)$, and let $a \in T$. Then $\mathcal{T}' = \mathcal{T}(T - \{a, a^\varphi\})$ arises from $\mathcal{T}$ by contracting $e = \{[a], [a^\varphi]\}$.*

Proof. Let $T' := T \setminus \{a, a^\varphi\}$ and let $\sim_T$ and $\sim_{T'}$ denote the equivalence relations obtained from T and T', respectively, according to Lemma 3.3. Let b, c be distinct elements of T'. If b^φ is a predecessor of c in T, then b^φ is also a predecessor of c in T', thus $b \sim_T c \Rightarrow b \sim_{T'} c$. On the other hand, if b^φ is a predecessor of c in T' but not in T, then either a or a^φ, but not both, due to Lemma 3.2, lies between b^φ and c. Hence, either $b \sim_T a$ and $c \sim_T a^\varphi$ or $b \sim_T a^\varphi$ and $c \sim_T a$. If we write $[b]_T$ and $[b]_{T'}$ for the equivalence class of b with respect to $\sim_T$ and $\sim_{T'}$, respectively, then we have for all $b \in T'$:

$$[b]_{T'} = \begin{cases} [b]_T & \text{if } [b]_T \cap \{a, a^\varphi\} = \emptyset \\ [a]_T \cup [a^\varphi]_T \setminus \{a, a^\varphi\} & \text{otherwise} \end{cases} . \tag{6}$$

That is, $\mathcal{T}'$ arises from $\mathcal{T}$ by deleting the edge $e = \{a, a^\varphi\}$ and merging the vertices $[a]_T$ and $[a^\varphi]_T$—in other words, by contracting e. □

Theorem 3.11. *Let T be a discrete treelike poset, let $T' \subseteq T$ be a symmetric subset of T and let $\mathcal{T}$ and $\mathcal{T}'$ be the corresponding trees, then:*

$$\mathcal{T}' \preccurlyeq \mathcal{T}.$$

Proof. Let $C = T \setminus T'$ and choose a well-order on C. The statement follows from Lemma 3.10 by transfinite induction. □

[1]Formally we have a one-to-one correspondence between the isomorphism classes of the corresponding structures.

4. Tree-decompositions of finite sets

Let S be a finite set and let $\mathcal{L} \subseteq \mathcal{Q}(S)$ be a quasi-partition lattice over S. The aim of this section is to show that every treelike subset $T \subseteq \mathcal{L}$ gives rise to a *tree-decomposition of* S, which is a pair $(\mathcal{T}, \mathcal{S})$ of a tree $\mathcal{T}$ and a family $\mathcal{S} = (S_t)_{t\in\mathcal{T}}$ of subsets $S_t \subseteq S$, indexed by the vertices of $\mathcal{T}$, such that:

(T1) $S = \bigcup_{t\in\mathcal{T}} S_t$;
(T2) $S_{t_1} \cap S_{t_3} \subseteq S_{t_2}$ whenever t_2 lies on the t_1-t_3-path in $\mathcal{T}$.

The converse is also true; every tree-decomposition of S gives rise to a treelike set of quasi-partitions. By Lemma 3.1 $\vec{E}(\mathcal{T})$ is a treelike poset and every oriented edge $e = (t_1, t_2) \in \vec{E}(\mathcal{T})$ induces a quasi-partition: Let T_i denote the component of $T - e$ containing t_i, then

$$\rho(t_1, t_2) := \Big(\bigcup_{t\in T_1} S_t, \bigcup_{t'\in T_2} S_{t'}\Big) \tag{7}$$

is a quasi-partition of S. Furthermore $(t_1, t_2) \leq (t_3, t_4)$ implies $T_1 \subseteq T_3$ and $T_4 \subseteq T_2$ and therefore $\rho(t_1, t_2) \leq \rho(t_3, t_4)$. Hence the set of quasi-partitions induced by $\vec{E}(\mathcal{T})$ inherits its treelikeness.

Note that with S also $T \subseteq \mathcal{L} \subseteq \mathcal{Q}(S)$ is finite, in particular T is discrete. So we can apply Theorem 3.7 to obtain a tree $\mathcal{T} = \mathcal{T}(T)$ and it only lasts to construct a family $\mathcal{S}$ of subsets of S such that $(\mathcal{T}, \mathcal{S})$ satisfies (T1) and (T2).

Recall that the elements of T, which is a subset of $\mathcal{Q}(S)$, are quasi-partitions of S. Furthermore the vertices of $\mathcal{T}$ are sets of elements of T, so we can define for every vertex $t \in \mathcal{T}$ a subset $S_t \subseteq S$ via

$$S_t := \bigcap\{A \mid (A, B) \in t\}, \tag{8}$$

and we define $\mathcal{S} := (S_t)_{t\in\mathcal{T}}$.

Lemma 4.1. *Let $e = \{[a], [a^\varphi]\}$ be an edge of $\mathcal{T}$ and let $\mathcal{T}_a$ denote the component of $\mathcal{T} - e$ containing $[a]$. Then*

$$a = \Big(\bigcup_{t\in\mathcal{T}_a} S_t, \bigcup_{t'\notin\mathcal{T}_a} S'_t\Big).$$

Proof. Let $a = (A, B)$. By duality it suffices to show

$$\bigcup_{t\in\mathcal{T}_a} S_t = A. \tag{9}$$

We do that by induction on $|\mathcal{T}_a|$. If $\mathcal{T}_a$ consists of a single vertex then this vertex is $[a] = \{a\}$ and therefore (9) holds by definition of S_t.

So let $\mathcal{T}_a$ consist of more than one vertex. Let $\{x_1, \dots, x_n\}$ be the set of predecessors of a, such that $[a] = \{a, x_1^\varphi, \dots, x_n^\varphi\}$. Then $[a]$ is incident in $\mathcal{T}_a$ precisely with the edges $\{[x_i], [x_i^\varphi]\}$, for $1 \leq i \leq n$, and hence the components of $\mathcal{T}_a - [a]$ are given by the $\mathcal{T}_{x_i}$. Let $x_i = (X_i, Y_i)$. Then for all i we have $|\mathcal{T}_{x_i}| < |\mathcal{T}_a|$ and hence

$$\bigcup_{t\in\mathcal{T}_{x_i}} S_t = X_i \tag{10}$$

by the induction hypothesis. This yields

$$\begin{aligned}\bigcup_{t\in\mathcal{T}_a} S_t &= S_{[a]} \cup \bigcup_{1\le i\le n}\bigcup_{t\in\mathcal{T}_{x_i}} S_t\\ &\underset{(10)}{=} S_{[a]} \cup \bigcup_{1\le i\le n} X_i\\ &\underset{(8)}{=} \Big(A \cap \bigcap_{1\le i\le n} Y_i\Big) \cup \bigcup_{1\le i\le n} X_i\end{aligned}$$

and since $x_i < a$, which means $X_i \subseteq A$, and $X_i \cup Y_i = S$, this implies (9). □

Theorem 4.2. *The pair $(\mathcal{T},\mathcal{S})$ is a tree-decomposition of S.*

Proof. Property (T1) follows immediately from Lemma 4.1. To show (T2) assume $t_1, t_2, t_3 \in \mathcal{T}$ such that t_2 lies on the t_1-t_3-path in $\mathcal{T}$. Let t_1' bet the neighbor of t_2 on the t_1-t_2-path in $\mathcal{T}$, let $a \in T$ such that $T(t_1', t_2) = \{a\}$ and let $a = (A,B)$. By Lemma 4.1 we have $S_{t_1} \subseteq A$ and $S_{t_3} \subseteq B$ and therefore $S_{t_1} \cap S_{t_3} \subseteq A \cap B$. Now by the choice of a we have $t_2 = [a^\varphi]$, such that every element $a^\varphi \neq x = (X,Y)$ of t_2 is a successor of a and therefore satisfies $A \subseteq X$. And since $a^\varphi = (B,A)$ we have $A\cap B \subseteq S_{t_2}$. Together this yields $S_{t_1} \cap S_{t_3} \subseteq A\cap B \subseteq S_{t_2}$. □

If $e = \{x,y\}$ is an edge of $\mathcal{T}$ then *contracting* e in $(\mathcal{T},\mathcal{S})$ means to contract e in $\mathcal{T}$—which merges x and y to a new vertex z—and to unify the corresponding parts, i.e. $S_z = S_x \cup S_y$. In order to obtain an extension of Theorem 3.11 to tree-decompositions we show:

Lemma 4.3. *For every $a \in T$ we have*

$$S_{[a]} \cup S_{[a^\varphi]} = \bigcap\Big\{X \mid (X,X') \in [a] \cup [a^\varphi] \smallsetminus \{a, a^\varphi\}\Big\}.$$

Proof. Let $[a] = \{a, x_1, \dots, x_m\}$, $[a^\varphi] = \{a^\varphi, y_1, \dots, y_n\}$, $a = (A,B)$, $x_i = (X_i, X_i')$ and $y_i = (Y_i, Y_i')$. Then $B \subseteq X_i$ and $A \subseteq Y_j$ for all i and j, such that

(11) $$\bigcap_{1\le i\le m} X_i = S_{[a]} \cup B \text{ and } \bigcap_{1\le j\le n} Y_j = S_{[a^\varphi]} \cup A.$$

On the other hand we have $S_{[a]} \subseteq A$ and $S_{[a^\varphi]} \subseteq B$, due to Lemma 4.1, and both $S_{[a]}$ and $S_{[a^\varphi]}$ contain $A\cap B$, due to (11) and (8). Thus

$$\bigcap\Big\{X \mid (X,X') \in [a] \cup [a^\varphi] \smallsetminus \{a, a^\varphi\}\Big\} = \bigcap_{1\le i\le m} X_i \cap \bigcap_{1\le j\le n} Y_j = S_{[a]} \cup S_{[a^\varphi]}.$$

□

Now if $(\mathcal{T}_1,\mathcal{S}_1)$ and $(\mathcal{T}_2,\mathcal{S}_2)$ are tree-decompositions, then the first is said to be a *minor of* the second, $(\mathcal{T}_1,\mathcal{S}_1) \preccurlyeq (\mathcal{T}_2,\mathcal{S}_2)$, if we obtain the first from the second by contracting edges.

Theorem 4.4. *Let $T_1 \subseteq T_2$ be treelike subsets of $\mathcal{L}(S)$ and $(\mathcal{T}_1,\mathcal{S}_1)$, $(\mathcal{T}_2,\mathcal{S}_2)$ the corresponding tree-decompositions, then*

$$(\mathcal{T}_1,\mathcal{S}_1) \preccurlyeq (\mathcal{T}_2,\mathcal{S}_2).$$

Proof. We can extend Lemma 3.10 by applying Lemma 4.3 to (6), and multiple application of this extension yields the stated relation. □

5. Profiles

Let $(L, \leq, \varphi)$ be a finite self-dual lattice. A subset $P \subseteq L$ is called a *profile* if the following two conditions hold:

(P1) For all $a \in P$ and all $b \in L$ we have: $b \leq a \Rightarrow b^\varphi \notin P$;
(P2) For all $a, b \in P$ we have: $(a \vee b)^\varphi \notin P$.

By (P1) a profile cannot contain both a and a^φ, so we can interprete[2] a profile as a function that chooses for some, but not necessary all, pairs $\{a, a^\varphi\} \subseteq L$ either a or a^φ. And these choices should be made according to some consistency conditions, given by (P1) and (P2). Let us call the subset of L for which P makes such a choice the *domain* of P, denoted by $\mathcal{D}(P)$, i.e. we have $\mathcal{D}(P) = P \cup P^\varphi$. It is obvious from the formulation of (P1) and (P2) that we cannot produce inconsistencies by considering the subset of a profile.

Lemma 5.1. *If P is a profile then $P' \subseteq P$ is also a profile.* □

We say that $a \in L$ *distinguishes* two profiles P_0, P_1 if there is $i \in \{0, 1\}$ such that $a \in P_i$ and $a^\varphi \in P_{1-i}$. For a subset S of L, a set of profiles $\mathcal{P}$ is called *S-distinguishable*, if for every pair of distinct profiles $P_0, P_1 \in \mathcal{P}$ there is an $a \in S$ that distinguishes them. Given a subset $S \subseteq L$ we call a profile *S-complete* if its domain contains S; it is called an *S-profile* if its domain is equal to S.

Our aim in this section will be the following: Given a symmetric subset $S \subseteq L$ and an S-distinguishable set of S-complete profiles $\mathcal{P}$, we want to pairwise distinguish all elements of $\mathcal{P}$ by means of a treelike subset $T \subseteq S$. There are clearly instances of that problem which do not have a solution: If L itself is not nested, consider two non-nested elements $a, b \in L$, let $S = \{a, b, a^\varphi, b^\varphi\}$ and let $\mathcal{P} = \{\{a, b\}, \{a, b^\varphi\}, \{a^\varphi, b\}, \{a^\varphi, b^\varphi\}\}$. Now $\mathcal{P}$ is an S-distinguishable set of S-complete profiles, but no treelike subset $T \subseteq S$ is able to still distinguish all elements of $\mathcal{P}$. So we have to come up with another condition on S and $\mathcal{P}$ which guarantees the existence of such a desired treelike subset.

We say that $S \subseteq L$ *separates* a set $\mathcal{P}$ of S-complete profiles *well* if $\mathcal{P}$ is S-distinguishable and for all pairs of crossing elements $a, b \in S$ and all $P_1, P_2 \in \mathcal{P}$ we have:

$$\{a, b\} \subseteq P_1 \text{ and } \{a^\varphi, b^\varphi\} \subseteq P_2 \Rightarrow \{a \vee b, a^\varphi \vee b^\varphi\} \subseteq S. \tag{12}$$

In order to proof our main result of this section, namely that this condition is indeed sufficient, we need some more preparation. First we shall show that we can restrict our attention to those elements of S that are actually useful to distinguish elements of $\mathcal{P}$. An element $a \in L$ is called *$\mathcal{P}$-relevant*, if a distinguishes two elements of $\mathcal{P}$. A subset $R \subseteq L$ is called *$\mathcal{P}$-relevant* if all of its elements are $\mathcal{P}$-relevant.

Lemma 5.2. *Let $R \subseteq S$ be the set of all $\mathcal{P}$-relevant elements in S. If S separates $\mathcal{P}$ well, then R also separates $\mathcal{P}$ well.*

Proof. Since $R \subseteq S$, all elements of $\mathcal{P}$ are also R-complete. Obviously $\mathcal{P}$ is R-distinguishable, if and only if it is S-distinguishable.

Let $a, b \in R$ and $P_1, P_2 \in \mathcal{P}$ such that the left-hand side of (12) holds. If S separates $\mathcal{P}$ well we have $\{a \vee b, a^\varphi \vee b^\varphi\} \subseteq S$. So we only have to check that they

[2] See the introduction for a more detailed motivation of this notion.

are both $\mathcal{P}$-relevant. Now $\{a,b\} \subseteq P_1$, such that $(a \vee b)^\varphi \notin P_1$ by (P2) and since P_1 is S-complete we have: $a \vee b \in P_1$. At the same time we have

$$(a \vee b)^\varphi \underset{2.2\ \text{(ii)}}{=} a^\varphi \wedge b^\varphi \leq a^\varphi,$$

and therefore $a \vee b \notin P_2$. Since P_2 is also S-complete we have $(a \vee b)^\varphi \in P_2$, thus $a \vee b \in R$. The dual argument shows $a^\varphi \vee b^\varphi \in R$. □

A maximal element of S is called *extremal in S* if it is nested with all elements of S. That is, $a \in S$ is extremal if for all b in S it holds $b \leq a$ or $b^\varphi \leq a$. The next lemma will be the cornerstone in the proof of our main theorem in this section.

Lemma 5.3. *If S separates $\mathcal{P}$ well and S is $\mathcal{P}$-relevant, then every maximal element of S is extremal in S.*

Proof. Let a be a maximal element in S and consider an arbitrary $b \in S$. If $a \parallel b$ then $b \leq a$ or $b^\varphi \leq a$ by the maximality of a. So assume $a \nparallel b$ and consider $P_1, P_2 \in \mathcal{P}$ with $a \in P_1$ and $a^\varphi \in P_2$, which exist since S is non-empty and $\mathcal{P}$-relevant. If P_1 and P_2 both contain the same element of $\{b, b^\varphi\}$ let $P_3 \in \mathcal{P}$ contain the other. If $a \in P_3$ we may assume $\{a,b\} \subseteq P_3$ and $\{a^\varphi, b^\varphi\} \subseteq P_2$ (by interchanging b and b^φ if necessary). Otherwise we may assume $\{a,b\} \subseteq P_1$ and $\{a^\varphi, b^\varphi\} \subseteq P_3$. Finally, if P_1 and P_2 contain distinct elements of $\{b, b^\varphi\}$ we may assume $\{a,b\} \subseteq P_1$ and $\{a^\varphi, b^\varphi\} \subseteq P_2$. In all cases (12) yields $a \vee b \in S$ and since a was chosen maximal we have $a = a \vee b$ and therefore $b \leq a$, a contradiction. □

Lemma 5.4. *If S is $\mathcal{P}$-relevant and $a \in S$ extremal in S, then there is a unique profile $P_a \in \mathcal{P}$ with $a \in P_a$.*

Proof. Since S is $\mathcal{P}$-relevant there is a profile $P \in \mathcal{P}$ with $a \in P$. Assume that there is another profile $P' \neq P$ in $\mathcal{P}$ with $a \in P'$. Then there is an element $b \in S$ with $b \in P$ and $b^\varphi \in P'$ because $\mathcal{P}$ is S-distinguishable. By the extremality of a we have either $b \leq a$ or $b^\varphi \leq a$, we may assume the former. But now (P1) yields $b^\varphi \notin P'$, a contradiction. Hence, $P_a := P$ is the unique profile in $\mathcal{P}$ containing a. □

If α is an automorphism of L and X a subset of L, we define $X^\alpha := \bigcup_{x \in X} x^\alpha$ and if $\mathcal{X}$ is a set of subsets of L we define $\mathcal{X}^\alpha := \{X^\alpha\}_{X \in \mathcal{X}}$. Now we can proof the main result of this section.

Theorem 5.5. *If S separates $\mathcal{P}$ well then there is a treelike subset $T(S,\mathcal{P}) \subseteq S$ such that:*

(i) $T(S,\mathcal{P})$ is $\mathcal{P}$-relevant;
(ii) $\mathcal{P}$ is $T(S,\mathcal{P})$-distinguishable;
(iii) if α is an automorphism of L, then $T(S^\alpha, \mathcal{P}^\alpha) = T(S,\mathcal{P})^\alpha$.

Proof. We proof this theorem by induction on $|\mathcal{P}|$. For $|\mathcal{P}| \leq 1$ we do not have to distinguish anything so we set $T(S,\mathcal{P}) := \emptyset$. In this case (i) and (ii) hold trivially and for (iii) note that $|\mathcal{P}^\alpha| = |\mathcal{P}|$ so we also have $T(S^\alpha, \mathcal{P}^\alpha) = \emptyset$.

Assume $|\mathcal{P}| \geq 2$. Let $R \subseteq S$ be the set of all $\mathcal{P}$-relevant elements of S and let E be the set of all extremal elements in R. Then $E \neq \emptyset$ due to Lemma 5.2 and Lemma 5.3. Let $T_E := E \cup E^\varphi$, which is nested by definition of extremality, symmetric by construction and balanced by (P1). Now by Lemma 5.4 for every $e \in E$ there is a unique profile $P_e \in \mathcal{P}$ with $e \in P_e$, such that e distinguishes P_e from all other profiles $P \neq P_e$ in $\mathcal{P}$. Let $\mathcal{P}_E := \bigcup_{e \in E} P_e$ and $\mathcal{P}' := \mathcal{P} \setminus \mathcal{P}_E$. By induction there exists a treelike subset $T(S,\mathcal{P}') \subseteq S$. Then due to (i) we have $T(S,\mathcal{P}') \subseteq R$ and therefore $T_E \parallel T(S,\mathcal{P}')$. Thus

$$T(S,\mathcal{P}) := T_E \cup T(S,\mathcal{P}')$$

is a treelike subset of S. Due to (i) and (ii) for $\mathcal{P}'$ and $T(S,\mathcal{P}')$ and the definition of T_E we also have (i) and (ii) for $\mathcal{P}$ and $T(S,\mathcal{P})$.

To verify (iii) consider an arbitrary automorphism α of L. Since R consists of all $\mathcal{P}$-relevant elements of S we know that R^α consists of all $\mathcal{P}^\alpha$-relevant elements of S^α. Furthermore e is extremal in R if and only if e^α is extremal in R^α. Together with (iii) for $\mathcal{P}'$ and $T(S,\mathcal{P}')$ this yields

$$T(S^\alpha,\mathcal{P}^\alpha) = (T_E)^\alpha \cup T(S,\mathcal{P}')^\alpha = T(S,\mathcal{P})^\alpha,$$

which finishes the proof. □

6. Profiles and connectivity functions

As in the previous section, let $(L,\leq,\varphi)$ be a finite self-dual lattice. A function $\lambda : L \to \mathbb{N}$ is called a *connectivity function* on L if it has the following two properties:

(C1) *Symmetry*: $\lambda(a) = \lambda(a^\varphi)$ for all $a \in L$;
(C2) *Sub-modularity*: $\lambda(a) + \lambda(b) \geq \lambda(a \wedge b) + \lambda(a \vee b)$ for all $a, b \in L$.

Note that for any finite set S the notion of a connectivity function on the subset lattice of S coincides in principle[3] with the common definition of a connectivity function on S (cp. [9, p. 159]) except that, for technical reasons, we require a connectivity function to be nonnegative.

So let λ be a connectivity function on L and consider a nonnegative integer k. By L_k we denote the set of all elements of L on which λ takes a value less or equal to k. By definition we have $L_k \subseteq L_{k'}$ for all $k \leq k'$ and by (C1) the sets L_k are symmetric for all k. To ease up notation we call an L_k-complete profile simply *k-complete* and an L_k-profile is called a *k-profile*. A profile that is a k-profile for some $k \in \mathbb{N}$ is called *regular*. The *order* of a regular profile P is the unique integer k for which P is a k-profile. If P is k-complete and $k' \leq k$, then $P' := P \cap L_{k'}$ is the k'-profile *induced* by P, and P' is said to *extend* to P. Two profiles P_1, P_2 are called *k-distinguishable* if both are k-complete and induce distinct k-profiles; the smallest integer k for which P_1 and P_2 are k-distinguishable is denoted by $\kappa(P_1,P_2)$. An element $a \in L$ that distinguishes P_1 from P_2 and satisfies $\lambda(a) = \kappa(P_1,P_2)$ is said to *efficiently distinguish* P_1 from P_2.

The aim of this section is to construct for every integer k and any given set of profiles $\mathcal{P}$ a treelike subset $T(\mathcal{P},k)$ of L that efficiently distinguishes each pair of k-distinguishable profiles in $\mathcal{P}$. The general approach will be as follows: first we apply Theorem 5.5 to obtain a treelike set T_0 that distinguishes all the 0-distinguishable profiles in $\mathcal{P}$. Then we consider each pair P, P' of 1-distinguishable profiles in $\mathcal{P}$ that are not distinguished by T_0. Since T_0 already distinguishes the 0-distinguishable profiles, P and P' will induce the same 0-profile Q. For each such 0-profile Q we will consider the set $\mathcal{P}_Q$ of all the extensions of Q in $\mathcal{P}$. Again with the help of Theorem 5.5 we will construct a treelike set T_Q that distinguishes all the 1-distinguishable profiles in $\mathcal{P}_Q$ and is nested with T_0. All these T_Q will also be mutually nested, such that we can define T_1 as the union of T_0 and all the T_Q.

[3] It is easy to see that every integer-valued connectivity function which also takes negative values can be shifted to a nonnegative connectivity function by adding a constant function.

Then we consider the 2-distinguishable profiles in $\mathcal{P}$ that are not distinguished by T_1 and so on. If there were no exceptions, this would define an ascending sequence $T_0 \subseteq T_1 \subseteq \dots$ such that each T_ℓ distinguishes *all* the ℓ-distinguishable profiles in $\mathcal{P}$ and we could set $T(\mathcal{P}, k) := T_k$. Unfortunately, there are exceptions.

The problem is that, in order to distinguish two ℓ-distinguishable profiles P, P' in $\mathcal{P}$, we could be forced to choose for T_ℓ an element of L that crosses all elements distinguishing some pair of k-distinguishable profiles. (An example for graphs can be found in [1].) But it turns out that, in this case, either P or P' cannot be k-complete. So we could adjust our approach, by considering at each step ℓ only the k-complete ℓ-distinguishable profiles in $\mathcal{P}$. In fact, we can be a little bit less restrictive; we can still consider all pairs of profiles in $\mathcal{P}$ that are 'robustly' ℓ-distinguishable, in a certain sense depending on k.

A profile P is called *k-robust* if for every $a \in P$ and every $b \in L_k \smallsetminus \mathcal{D}(P)$ either $a^\varphi \wedge b$ or $a^\varphi \wedge b^\varphi$ is not contained in P. A profile is called *robust* if it is k-robust for every $k \in \mathbb{N}$. Note that for every integer $h \geq k$ every h-complete profile is k-robust. Furthermore, every profile that extends to a h-complete profile is k-robust itself. More generally we have:

Lemma 6.1. *Let P be a k-robust profile and consider $P' \subseteq P$ and $k' \leq k$. Then*
- *(i) P' is a k-robust profile;*
- *(ii) P (and hence also P') is k'-robust.*

Proof. By Lemma 5.1 P' is a profile. Assume that P' is not k-robust. Then there are $a \in P'$ and $b \in L_k \smallsetminus \mathcal{D}(P')$ with $\{a^\varphi \wedge b, a^\varphi \wedge b^\varphi\} \subseteq P' \subseteq P$. Since P is k-robust we have $b \notin L_k \smallsetminus \mathcal{D}(P)$ such that $b \in \mathcal{D}(P)$. We may assume $b \in P$ which yields

$$a^\varphi \wedge b^\varphi \underset{2.2\ \text{(ii)}}{=} (a \vee b)^\varphi \underset{(P2)}{\notin} P,$$

a contradiction. Hence P' is k-robust, which shows (i). Statement (ii) follows directly from the definition of k-robustness, since $L_{k'} \subseteq L_k$. □

Given integers k and ℓ, two profiles P_1 and P_2 with $\kappa(P_1, P_2) \leq \ell$ are called *k-robustly l-distinguishable* if they induce k-robust $\kappa(P_1, P_2)$-profiles. Now we can prove the central result of this paper:

Theorem 6.2. *For every set $\mathcal{P}$ of profiles and every $k \in \mathbb{N}$ there is a treelike set $T(\mathcal{P}, k) \subseteq L$ such that:*
- *(i) each element $a \in T(\mathcal{P}, k)$ efficiently distinguishes a pair of k-robustly k-distinguishable profiles in $\mathcal{P}$;*
- *(ii) for each pair of k-robustly k-distinguishable profiles $P, P' \in \mathcal{P}$ there is an element in $T(\mathcal{P}, k)$ that efficiently distinguishes P from P';*
- *(iii) for every λ-preserving automorphism α of L we have $T(\mathcal{P}^\alpha, k) = T(\mathcal{P}, k)^\alpha$.*

Proof. For every $\ell \leq k$ let $\mathcal{P}_\ell$ denote the set of all k-robust ℓ-profiles that are induced by profiles in $\mathcal{P}$. We shall construct an ascending sequence $\big(T_\ell = T_\ell(\mathcal{P})\big)_{-1 \leq \ell \leq k}$ of treelike subsets of L such that, for every $0 \leq \ell \leq k$, we have:
- (I) $\lambda(a) = \ell$ for every $a \in T_\ell \smallsetminus T_{\ell-1}$;
- (II) T_ℓ distinguishes each pair of distinct elements of $\mathcal{P}_\ell$;
- (III) each $a \in T_\ell \smallsetminus T_{\ell-1}$ efficiently distinguishes two elements of $\mathcal{P}_\ell$;
- (IV) for every λ-preserving automorphism α of L we have $T_\ell(\mathcal{P}^\alpha) = T_\ell(\mathcal{P})^\alpha$.

Then $T(\mathcal{P},k) := T_k$ clearly satisfies (*i*) and (*iii*), due to (III) and (IV), respectively. To verify (*ii*) consider a pair of k-robustly k-distinguishable profiles $P, P' \in \mathcal{P}$. Then for $\ell = \kappa(P,P')$ they will induce distinct ℓ-profiles P_ℓ and P'_ℓ, respectively, which both are k-robust by Lemma 6.1. Hence we have $P_\ell, P'_\ell \in \mathcal{P}_\ell$, such that (II) yields an element $a \in T_\ell$ that distinguishes P_ℓ from P'_ℓ and therefore also P from P'; it does so efficiently due to (I) and the choice of ℓ.

We start with $T_{-1} = \emptyset$. Consider $\ell \geq 0$ and assume that T_m was already constructed for all $0 \leq m < \ell$, satisfying conditions (I) to (IV). Now consider a pair of profiles $P, P' \in \mathcal{P}_\ell$ that are not yet distinguished by $T_{\ell-1}$. Then, by (II) for $\ell - 1$ and Lemma 6.1, they induce the same element Q in $\mathcal{P}_{\ell-1}$. Thus, in order to distinguish all pairs of profiles in $\mathcal{P}_\ell$, we have to take a look at all profiles in $\mathcal{P}_{\ell-1}$ that extend to more than one element in $\mathcal{P}_\ell$.

For a profile $Q \in \mathcal{P}_{\ell-1}$ let $\mathcal{P}_Q \subseteq \mathcal{P}_\ell$ denote the set of all extensions of Q in $\mathcal{P}_\ell$. Furthermore, let $S_Q \subseteq L$ denote the set of all elements in L that are $\mathcal{P}_Q$-relevant and nested with $T_{\ell-1}$. Our hope is to be able to apply Theorem 5.5 in order to get a treelike subset $T_Q \subseteq S_Q$ which we could add to $T_{\ell-1}$ and which distinguishes each pair of profiles in $\mathcal{P}_Q$. So we have to check that:

($*$) *S_Q separates $\mathcal{P}_Q$ well.*

First we have to prove that $\mathcal{P}_Q$ is S_Q-distinguishable. If $|\mathcal{P}_Q| \leq 1$ there is nothing to show. So assume there are two distinct elements $P, P' \in \mathcal{P}_Q$. Then there is $a \in L_\ell$ that distinguishes P from P'. Since both P and P' induce the same $(\ell-1)$-profile Q we have $\lambda(a) = \ell$. Choose a such that it is nested with as many elements in $T_{\ell-1}$ as possible. We shall show that a is nested with $T_{\ell-1}$.

Assume not. Then there is $b \in T_{\ell-1}$ with $a \nparallel b$; let m be such that $b \in T_m \setminus T_{m-1}$. Then by (I) we have $\lambda(b) = m$ and, due to (III), there are profiles $R, R' \in \mathcal{P}_m$ such that b efficiently distinguishes R from R'. We may assume $a \in P$ and $b \in R$, such that $a^\varphi \in P'$ and $b^\varphi \in R'$, and since P and P' are not m-distinguishable, we may also assume $b \in P$ and $b \in P'$. By (P2) we know that P cannot contain $(a \vee b)^\varphi = a^\varphi \wedge b^\varphi$, while P' cannot contain the dual of this corner, since $a^\varphi \wedge b^\varphi \leq a^\varphi$. Now if $\lambda(a^\varphi \wedge b^\varphi)$ is at most ℓ, then $(a \vee b) \in P$ and $(a \vee b)^\varphi \in P'$, such that $a^\varphi \wedge b^\varphi$ distinguishes P from P'. But every element $c \in T_{\ell-1}$ that is nested with a is also nested with $a^\varphi \wedge b^\varphi$, by Lemma 2.5. And since $a^\varphi \wedge b^\varphi$ is also nested with b, by Lemma 2.3, this contradicts the choice of a. Thus we have $\lambda(a^\varphi \wedge b^\varphi) > \ell$. The same argument, with the roles of P and P' interchanged, yields $\lambda(a \wedge b^\varphi) > \ell$.

In order to obtain a contradiction, we show that one of the other two corners of $\{a, b\}$, that is $a \wedge b$ or $a^\varphi \wedge b$, distinguishes R from R' and has smaller λ-value than b. We first calculate a bound on their λ-value:

$$\lambda(a \wedge b) \underset{2.2,(C1)}{=} \lambda(a^\varphi \vee b^\varphi) \underset{(C2)}{<} m \underset{(C2)}{>} \lambda(a \vee b^\varphi) \underset{2.2,(C1)}{=} \lambda(a^\varphi \wedge b).$$

Since R is an m-profile containing b this yields $\{a \wedge b, a^\varphi \wedge b\} \subseteq R$. And since R' is a k-robust m-profile containing b^φ, it cannot contain both $a \wedge b$ and $a^\varphi \wedge b$. So R' contains the dual of one of these corners, say $(a \wedge b)^\varphi \in R'$. But then $a \wedge b$ distinguishes R from R' and satisfies $\lambda(a \wedge b) < \lambda(b)$, contradicting the fact that b efficiently distinguishes R from R'.

Hence, a is nested with $T_{\ell-1}$ and therefore contained in S_Q. So $\mathcal{P}_Q$ is S_Q-distinguishable, and it lasts to verify condition (12). Consider profiles $P, P' \in \mathcal{P}_Q$ and consider $a, b \in S_Q$ such that $\{a, b\} \subseteq P$ and $\{a^\varphi, b^\varphi\} \subseteq P'$. Then both a and b

distinguish P from P' and are nested with T_ℓ. As we have seen above, this implies $\lambda(a) = \lambda(b) = \ell$. Clearly the same holds for their duals a^φ and b^φ.

If $\lambda(a \vee b) \leq \ell$, then by (P2) we have $a \vee b \in P$ and by (P1) we have $(a \vee b)^\varphi = a^\varphi \wedge b^\varphi \in P'$. Thus, in this case, $a \vee b$ distinguishes P from P'. Since P and P' are not m-distinguishable for any $m < \ell$, this yields $\lambda(a \vee b) \geq \ell$. The dual argument yields $\lambda(a^\varphi \vee b^\varphi) \geq \ell$, and due to (C2) we have:

$$\lambda(a \vee b) = \lambda(a^\varphi \vee b^\varphi) = \ell.$$

So both $a \vee b$ and $a^\varphi \vee b^\varphi$ distinguish P from P'. By Lemma 2.5 both $a^\varphi \wedge b^\varphi$ and $a \wedge b$ are nested with $T_{\ell-1}$ and by definition of nestedness this is also true for their duals. Thus we have $\{a \vee b, a^\varphi \vee b^\varphi\} \subseteq S_Q$, which finishes the proof of $(*)$.

So we are able to apply Theorem 5.5 to obtain for every $Q \in \mathcal{P}_{\ell-1}$ a treelike subset $T_Q \subseteq S_Q$ which distinguishes each pair of profiles in $\mathcal{P}_Q$. Furthermore, each element in $a \in T_Q$ is $\mathcal{P}_Q$-relevant, so by definition of $\mathcal{P}_Q$, a efficiently distinguishes two elements of $\mathcal{P}_Q$. We shall prove that we can extend $T_{\ell-1}$ by all these T_Q, to form T_ℓ.

By construction each T_Q is nested with $T_{\ell-1}$. Now consider distinct elements $Q, Q' \in \mathcal{P}_{\ell-1}$ and the corresponding sets T_Q and $T_{Q'}$, respectively. By (II) for $\ell - 1$ there is an element $c \in T_{\ell-1}$ that distinguishes Q from Q'. We may assume $c \in Q$ and $c^\varphi \in Q'$. Now consider arbitrary elements $a \in T_Q$ and $b \in T_{Q'}$. As a is $\mathcal{P}_Q$-relevant, we have $\{a, a^\varphi\} \subseteq Q$. Since Q satisfies (P1) we have $a \not\leq c$ and $a^\varphi \not\leq c$, and since a is nested with c this yields either $a \leq c^\varphi$ or $a^\varphi \leq c^\varphi$, let us assume the former. The same argument applied to b and Q' yields $b \leq c$ or $b^\varphi \leq c$, say the latter. Together this gives $a \leq c^\varphi \leq b$, such that a is nested with b. Since both a and b were chosen arbitrarily, this yields $T_Q \parallel T_{Q'}$. Hence,

$$T_\ell := T_{\ell-1} \cup \bigcup_{Q \in \mathcal{P}_{\ell-1}} T_Q$$

is a nested set. Since every element of T_ℓ is contained in a treelike set, T_ℓ is also symmetric and balanced, and therefore treelike. We have to check that T_ℓ satisfies (I) to (IV).

To verify (I) and (III) consider an element $a \in T_\ell \setminus T_{\ell-1}$. By construction we have $a \in T_Q$, for some $Q \in T_{\ell-1}$, such that $\lambda(a) = \ell$ and a efficiently distinguishes two elements of $\mathcal{P}_Q \subseteq \mathcal{P}_\ell$. For (II) consider $P, P' \in \mathcal{P}_\ell$. If they induce distinct profiles $Q, Q' \in \mathcal{P}_{\ell-1}$, than these are efficiently distinguishes by an element $a \in T_{\ell-1}$, due to (II) for $\ell - 1$, which then also efficiently distinguishes P from P'. Otherwise, if P and P' induce the same profile $Q \in \mathcal{P}_{\ell-1}$, then there is $a \in T_Q$ that efficiently distinguishes P from P'. In both cases we have $a \in T_\ell$, which shows (II).

Finally, let α be a λ-preserving automorphism of L. Then clearly $\mathcal{P}_m^\alpha = (\mathcal{P}_m)^\alpha$, for every $0 \leq m \leq \ell$, and for $Q \in \mathcal{P}_{\ell-1}$ we have $\mathcal{P}_{Q^\alpha}^\alpha = (\mathcal{P}_Q)^\alpha$ and therefore $S_{Q^\alpha}^\alpha = (S_Q)^\alpha$. Together with Theorem 5.5 (iii) this yields

$$T_{Q^\alpha} = T(S_{Q^\alpha}^\alpha, \mathcal{P}_{Q^\alpha}^\alpha) = T((S_Q)^\alpha, (\mathcal{P}_Q)^\alpha) = T(S_Q, \mathcal{P}_Q)^\alpha = (T_Q)^\alpha,$$

and with (IV) for $\ell - 1$ we get

$$\begin{aligned} T_\ell(\mathcal{P}^\alpha) &= T_{\ell-1}(\mathcal{P}^\alpha) \cup \bigcup_{Q^\alpha \in \mathcal{P}^\alpha_{\ell-1}} T_{Q^\alpha} \\ &= T_{\ell-1}(\mathcal{P})^\alpha \cup \bigcup_{Q \in \mathcal{P}_{\ell-1}} (T_Q)^\alpha \\ &= T_\ell(\mathcal{P})^\alpha, \end{aligned}$$

which shows (IV) for ℓ, and therefore finishes the proof. □

7. Tree-decompositions of connectivity-systems

Let S be a finite set, $\mathcal{L} \subseteq \mathcal{Q}(S)$ a quasi-partition lattice over S and λ a connectivity function on $\mathcal{L}$. Then we call $(\mathcal{L}, \lambda)$ a *connectivity-system*[4] *over* S. So for the remainder of this section let $(\mathcal{L}, \lambda)$ be a connectivity-system.

A tree-decomposition $(\mathcal{T}, \mathcal{S})$ of S is called a *tree-decomposition of* $(\mathcal{L}, \lambda)$ if for every $(t, t') \in \vec{E}(\mathcal{T})$ the induced quasi-partition $\rho(t, t')$ is contained in $\mathcal{L}$. For $k \in \mathbb{N}$ we say that $(\mathcal{T}, \mathcal{S})$ has *adhesion* k if the maximal value that λ takes on the quasi-partitions induced by $(\mathcal{T}, \mathcal{S})$ is precisely k; if $\mathcal{T}$ is trivial then $(\mathcal{T}, \mathcal{S})$ has adhesion 0.

Let $(\mathcal{T}, \mathcal{S})$ be a tree-decomposition of $(\mathcal{L}, \lambda)$ and let $t \in \mathcal{T}$. We say that a profile P *inhabits* the part S_t of $(\mathcal{T}, \mathcal{S})$, if for all $t' \in \mathcal{T}$ with $(t', t) \in \vec{E}(\mathcal{T})$ we have $\rho(t', t) \in P$.

Lemma 7.1. *Let $k \in \mathbb{N}$ and let $(\mathcal{T}, \mathcal{S})$ be a tree-decomposition of adhesion at most k. Then for every k-complete profile P there is a unique vertex $t \in \mathcal{T}$ such that P inhabits the part S_t of $(\mathcal{T}, \mathcal{S})$.*

Proof. Let $(\mathcal{T}, \mathcal{S})$ and P be given as in the statement. We direct the edges of $\mathcal{T}$ by making each edge $\{t_1, t_2\}$ point from t_1 to t_2, if and only if $\rho(t_1, t_2) \in P$. Hence, P inhabits a part S_t, if and only if t has out-degree 0. Now for every vertex $t \in \mathcal{T}$ we have at most one outgoing edge. For assume there are $\{t, t'\}, \{t, t''\} \in E(\mathcal{T})$ such that $\rho(t, t'), \rho(t, t'') \in P$. By definition we have $\rho(t', t)^\varphi = \rho(t, t')$, and as we have seen on page 10, just after (7), we have $\rho(t', t) \leq \rho(t, t'')$, sucht that (P1) yields $\rho(t, t') \notin P$, a contradiction. Hence every vertex in $\mathcal{T}$ has out-degree at most 1, which implies that there is a unique vertex t with out-degree 0. □

We say that a tree-decomposition $(\mathcal{T}, \mathcal{S})$ *(efficiently) distinguishes* two profiles P and P' if there is an ordered edge (t, t') in $\mathcal{T}$ such that $\rho(t, t')$ (efficiently) distinguishes P from P'. Now we are ready to state and prove our first main result, which basically is an application of Theorem 6.2:

Theorem 7.2. *Let Γ be a group of λ-preserving automorphisms of $\mathcal{L}$ and let $\mathcal{P}$ be a Γ-invariant set of profiles of $\mathcal{L}$. Then for every integer k there is a sequence $(\mathcal{T}_\ell, \mathcal{S}_\ell)_{\ell \leq k}$ of tree-decompositions of $(\mathcal{L}, \lambda)$ such that, for all ℓ:*

- *(i) every ℓ-complete profile in $\mathcal{P}$ inhabits a unique part of $(\mathcal{T}_\ell, \mathcal{S}_\ell)$;*
- *(ii) $(\mathcal{T}_\ell, \mathcal{S}_\ell)$ efficiently distinguishes every pair of k-robustly ℓ-distinguishable profiles in $\mathcal{P}$;*
- *(iii) $(\mathcal{T}_\ell, \mathcal{S}_\ell)$ has adhesion at most ℓ;*

[4]This definition generalizes the one given by Geelen, Gerards and Whittle [6]: If S is a finite set, and λ a connectivity function on S, then λ induces a connectivity function λ' on $\mathcal{B}(S)$. Hence the connectivity-system (S, λ) induces the connectivity-system $(\mathcal{B}(S), \lambda')$.

(iv) $(\mathcal{T}_\ell, \mathcal{S}_\ell) \preccurlyeq (\mathcal{T}_{\ell+1}, \mathcal{S}_{\ell+1})$ *for* $\ell < k$*;*
(v) Γ *acts on* $\mathcal{T}_\ell$ *as a group of automorphisms.*

Proof. We apply Theorem 6.2 to obtain a treelike subset $T = T(\mathcal{P}, k) \subseteq \mathcal{L}$. Now we define for every $\ell \leq k$ the subset $T_\ell \subseteq T$ via $T_\ell := \{a \in T \mid \lambda(a) \leq \ell\}$. Since every subset of a treelike set is treelike itself, T_ℓ is treelike for all ℓ.

By Theorem 4.2, these T_ℓ give rise to tree-decompositions $(\mathcal{T}_\ell, \mathcal{S}_\ell)$ of $(\mathcal{L}, \lambda)$, which due to Lemma 4.1 satisfy *(iii)*. Then by Lemma 7.1 they also satisfy *(i)*. By construction we have $T_\ell \subseteq T_{\ell+1}$, such that Theorem 4.4 yields *(iv)*. Since Γ consists of λ-preserving automorphisms and due to 6.2 (iii), T_ℓ is Γ-invariant. Hence, Γ induces a group of automorphisms of T_ℓ and Corollary 3.8 yields *(v)*.

By 6.2 (ii), for every pair P, P' of k-robustly ℓ-distinguishable profiles in $\mathcal{P}$ there is $a \in T$ that efficiently distinguishes P from P', in particular we have $\lambda(a) \leq \ell$. So by definition of T_ℓ this implies $a \in T_\ell$, which yields *(ii)*. □

Even though Lemma 7.1 only assures that ℓ-profiles inhabit a unique part of every tree-decomposition of adhesion at most ℓ, it is still possible for an ℓ-profile to inhabit a part of a tree-decomposition of higher adhesion. This is in particular true in the context of Theorem 7.2, if we restrict our attention to sets of pairwise k-robustly k-distinguishable profiles. To simplify the proof of our second main result we prove another lemma:

Lemma 7.3. *A profile never inhabits two distinct parts of a tree-decomposition.*

Proof. Let $(\mathcal{T}, \mathcal{S})$ be tree-decomposition and assume that P inhabits two distinct parts S_{t_1} and S_{t_2}. Let t_1' and t_2' denote the neighbours of t_1 and t_2, respectively, on the unique t_1–t_2-path in $\mathcal{T}$. Then both $\rho(t_1', t_1)$ and $\rho(t_2', t_2)$ are contained in P. On the other hand we have $(t_1, t_1') \leq (t_2', t_2)$ and therefore $\rho(t_1, t_1') \leq \rho(t_2', t_2)$. Hence, $\rho(t_1', t_1) = \rho(t_1, t_1')^\varphi \notin P$ by (P2), a contradiction. □

Theorem 7.4. *Let* Γ *be a group of* λ*-preserving automorphisms of* $\mathcal{L}$*, let* $k \in \mathbb{N}$ *and let* $\mathcal{P}$ *be a* Γ*-invariant set of pairwise* k*-robustly* k*-distinguishable profiles. Then there is a tree-decomposition* $(\mathcal{T}, \mathcal{S})$ *of* $(\mathcal{L}, \lambda)$ *such that*
(i) every profile in $\mathcal{P}$ *inhabits a unique part of* $(\mathcal{T}, \mathcal{S})$*;*
(ii) $(\mathcal{T}, \mathcal{S})$ *efficiently distinguishes every pair of profiles in* $\mathcal{P}$*;*
(iii) $(\mathcal{T}, \mathcal{S})$ *has adhesion at most* k*;*
(iv) Γ *acts on* $\mathcal{T}$ *as a group of automorphisms.*

Proof. As in the proof of Theorem 7.2 we start with $T = T(\mathcal{P}, k)$ and use exactly the same notation. Then $(\mathcal{T}, \mathcal{S}) := (\mathcal{T}_k, \mathcal{S}_k)$ satisfies *(iii)* and *(iv)* as in the proof of 7.2 (iii) and 7.2 (v), respectively. Taking into account the assumptions on $\mathcal{P}$, *(ii)* is equivalent to 7.2 (ii).

To show *(i)* let P be an arbitrary profile in $\mathcal{P}$. We define

$$\kappa := \max\{\kappa(P, P') \mid P \neq P' \in \mathcal{P}\}. \tag{13}$$

By assumption we have $\kappa(P, P') \leq k$ for every P' such that $\kappa \leq k$. Then P inhabits a unique part S_t of $(\mathcal{T}_\kappa, \mathcal{S}_\kappa)$, with $t \in \mathcal{T}_\kappa$, due to 7.2 (i).

Recall from Section 3 that the vertices of $\mathcal{T}$ are subsets of T. Now assume that t (as a subset of $T_\kappa \subseteq T$) is not a vertex of $\mathcal{T}$. Then by Lemma 3.10 there must be a $c \in T \smallsetminus T_\kappa$ such that t is not a vertex of $\mathcal{T}' = \mathcal{T}(T_\kappa \cup \{c, c^\varphi\}$. By 6.2 (i) there must be k-robustly k-distinguishable profiles P_1, P_2 in $\mathcal{P}$ that are efficiently distinguished by c. And since $\lambda(c) =: l > \kappa$, this means that P_1 and P_2 are both l-complete but not κ-distinguishable. Hence P_1 and P_2 inhabit distinct parts of $(\mathcal{T}', \mathcal{S}')$ but the same part of $(\mathcal{T}_\kappa, \mathcal{S}_\kappa)$. By Lemma 3.10 again, this is only possible if both P_1 and P_2 inhabit the same part of $(\mathcal{T}_\kappa, \mathcal{S}_\kappa)$ as P, namely S_t. But then $(\mathcal{T}_\kappa, \mathcal{S}_\kappa)$ does not distinguish P from P_1 although, by the choice of κ, P and P_1 are κ-distinguishable, a contradiction to 6.2 (ii). So t is also a vertex of $\mathcal{T}$ and, by definition of the edgeset of $\mathcal{T}$ (see p. 8), still incident with the same edges, such that P inhabits the part S_t of $(\mathcal{T}, \mathcal{S})$. By Lemma 7.3, this part is unique, which shows (i). □

Adapting the corresponding definition of Geelen, Gerards, Robertson and Whittle [5], we define a *tangle of* $(\mathcal{L}, \lambda)$ *of order* $(k+1)$ as a subset $\tau \subseteq \mathcal{L}$ such that:

(θ1) for each $(A, B) \in \tau$ we have $\lambda(A, B) \leq k$;
(θ2) for each $(A, B) \in \mathcal{L}$ with $\lambda(A, B) \leq k$ either (A, B) or (B, A) is in τ;
(θ3) if $(A, B), (C, D), (E, F) \in \tau$, then $A \cup C \cup D \neq S$;
(θ4) for each $a \in S$ we have $(S - \{a\}, \{a\}) \notin \tau$.

Lemma 7.5. *Every tangle of order $k+1$ is a robust k-profile.*

Proof. Let $(A, B) \in \tau$. If $(C, D) \leq (A, B)$ then $A \cup D = S$, such that $(D, C) \notin \tau$ due to (θ3). This shows (P1). Now for every $(A, B), (C, D) \in \mathcal{L}$ we have $(B \cap D) \cup A \cup C = S$ such that (θ3) yields

$$(A, B), (C, D) \in \tau \Rightarrow (B \cap D, A \cup C) \notin \tau,$$

which shows (P2). Therefore τ is a profile and due to (θ1) and (θ2) even a k-profile. If τ is not robust then there is $(A, B) \in \tau$ and $(C, D) \in \mathcal{B}(S)$ with both $(B, A) \wedge (C, D) \in \tau$ and $(B, A) \wedge (D, C) \in \tau$. But then $A \cup (B \cap C) \cup (B \cap D) = S$ yields a contradiction to (θ3). □

A tangle τ of order k is called *maximal* if for every tangle τ' of order $k' > k$ we have $\tau \not\subset \tau'$. The following theorem extends a recent result by Geelen, Gerards and Whittle [6, Theorem 9.1]:

Theorem 7.6. *For every connectivity-system $(\mathcal{L}, \lambda)$ over a finite set S there is a tree-decomposition $(\mathcal{T}, \mathcal{S})$ of $(\mathcal{L}, \lambda)$ such that*

(i) every maximal tangle inhabits a unique part of $(\mathcal{T}, \mathcal{S})$;
(ii) $(\mathcal{T}, \mathcal{S})$ efficiently distinguishes every pair of maximal tangles;
(iii) the group Γ of all λ-preserving automorphisms of $\mathcal{L}$ acts on $\mathcal{T}$ as a group of automorphisms.

Proof. By Lemma 7.5 the set $\mathcal{P}$ of all the maximal tangles is a set of robust and regular profiles. Let k be the maximal order occurring in $\mathcal{P}$. Then the profiles in $\mathcal{P}$ are pairwise k-robustly k-distinguishable, because none is contained in an other. And clearly every λ-preserving automorphism of $\mathcal{L}$ maps a maximal tangle to a maximal tangle. So Theorem 7.4 yields the statement. □

8. Applications to graphs

Let $G = (V, E)$ be a finite graph and let $\mathcal{S}_G$ denote the separation lattice of G, as introduced in Example 3. Furthermore let $\lambda : \mathcal{S}_G \longrightarrow \mathbb{N}$ be defined as $\lambda\big((A, B)\big) := |A \cap B|$. Now λ is a connectivity function on $\mathcal{S}_G$: it is clear from the definition that λ satisfies (C1), and to verify (C2) boils down to verify that for every pair $(A, B), (C, D) \in \mathcal{S}_G$ we have

$$|(A \cap C) \cap (B \cup D)| + |(A \cup C) \cap (B \cap D)| = |A \cap B| + |C \cap D|,$$

which can be done by simple calculations. Hence, $(\mathcal{S}_G, \lambda)$ is a connectivity-system over V. It is easy to see that the notion of a tree-decomposition of $(\mathcal{S}_G, \lambda)$ coincides with the common notion of a tree-decomposition of G (as defined for example in Diestel [3, 12.3]). We refer to the profiles of $\mathcal{S}_G$ as the *profiles of* G.

Note that by definition of λ, every automorphism of G is a λ-preserving automorphism of $\mathcal{S}_G$. So Theorem 7.2 and Theorem 7.4 read out particularly well when specialised to graphs. In the remainder of this section we aim to demonstrate their usefulness by first characterising the profiles of G in more tangible and maybe more graph-theoretic terms. Afterwards we consider two special classes of profiles which have been studied before under different name.

8.1. **Profiles of graphs.** To ease up notation we write $G-S$ for the graph obtained from G by deleting a set S of vertices and all edges incident with S. With a slight looseness we also write $G-S$ instead of $V(G-S)$, which should not lead to any confusion.

Lemma 8.1. *For every set S of vertices and every profile P, we have $(V, S) \notin P$.*

Proof. Assume $(V, S) \in P$. We have $(S, V) \leq (V, S)$, such that (P1) yields $(S, V)^{\varphi} = (V, S) \notin P$, a contradiction. □

Lemma 8.2. *Let $k \in \mathbb{N}$ with $k < |V(G)|$ and let P be a k-profile of $\mathcal{S}_G$. Then for every subset $S \subset V(G)$ with $|S| \leq k$ there is a unique component C_S of $G-S$ such that $(G - C_S, S \cup C_S) \in P$.*

Proof. Consider a k-profile P and a set S with $|S| \leq k$. If $G-S$ consists of a single component C_S, we have $(G - C_S, S \cup C_S) = (S, V) \in P$, due to Lemma 8.1. So assume that there are at least two components of $G - S$ and enumerate them as $C_1, C_2, \ldots, C_n$.

For every $1 \leq i \leq n$ we have $|(G - C_i) \cap (S \cup C_i)| = |S| \leq k$, such that $(G - C_i, S \cup C_i) \in \mathcal{D}(P)$. If none of these separations is in P, then P contains the dual of each of them. But then P contains also $(S \cup C_1 \cup C_2, G - (C_1 \cup C_2))$, due to (P2), and by induction we get $(S \cup \bigcup_{1 \leq i \leq n} C_i, G - \bigcup_{1 \leq i \leq n} C_i) = (V, S) \in P$, contradicting Lemma 8.1. So without loss of generality we may assume $(G - C_1, S \cup C_1) \in P$. If C_1 is not unique with this property we may assume $(G - C_2, S \cup C_2) \in P$. But then (P2) yields $\big((G - C_1) \cup (G - C_2), (S \cup C_1) \cap (S \cup C_2)\big) = (V, S) \in P$. Hence $C_S := C_1$ is the desired unique component. □

We say that two sets of vertices of G are *touching* if they either share a vertex or there is an edge in G with one endvertex in each of the sets.

Lemma 8.3. *For every pair S_0, S_1 of sets of at most k vertices, the components C_{S_0} and C_{S_1}, as given by Lemma 8.2, are touching.*

Proof. Assume not. Then there are sets S_0, S_1 of at most k vertices such that C_{S_0} and C_{S_1} do not touch. Let S_0' denote the boundary of C_{S_0}. Then $S_0' \subseteq S_0$, such that $C_{S_0'}$ and C_{S_0} are either equal or disjoint. If they are disjoint then (P2) yields

$$\big((G - C_{S_0}) \cup (G - C_{S_0'}), (S_0 \cup C_{S_0}) \cap (S_0' \cup C_{S_0'})\big) = (V, S_0'') \in P,$$

where $S_0'' \subseteq S_0$, contradicting Lemma 8.1. Hence $C_{S_0'} = C_{S_0}$ and we may assume that S_i already is the boundary of C_{S_i}, for $i \in \{0, 1\}$. But then $S_i \cap C_{S_{1-i}} = \emptyset$, since every vertex in S_i sends an edge to a vertex in C_{S_i}. And again due to (P2) we have:

$$\big((G - C_{S_0}) \cup (G - C_{S_1}), (S_0 \cup C_{S_0}) \cap (S_1 \cup C_{S_1})\big) = (V, S_0 \cap S_1) \in P,$$

which yields a contradiction to Lemma 8.1. □

A *preference of order* $(k+1)$ *of* G is a function that maps each set S of at most k vertices to a component of $G - S$ such that all images are pairwise touching (cp. [8, p. 98]). For a k-profile P let f_P denote the function that maps each set S of at most k vertices to $f_P(S) = C_S$.

Lemma 8.4. *Let P be a k-profile and let (A, B) be a separation of order at most k. Then $(A, B) \in P$, if and only if $f_P(A \cap B) \subseteq B \smallsetminus A$.*

Proof. If $(B, A) \in P$ then $(A, B) \notin P$ and if $f_P(A \cap B) \not\subseteq B \smallsetminus A$ then $f_P(A \cap B) \subseteq A \smallsetminus B$. Thus by symmetry it suffices to show that $f_P(A \cap B) \subseteq B \smallsetminus A$ implies $(A, B) \in P$. So assume $f_P(A \cap B) \subseteq B \smallsetminus A$. Then

$$(A, B) \leq \big(G - f_P(A \cap B), (A \cap B) \cup f_P(A \cap B)\big),$$

such that $(A, B) \in P$ due to (P1). □

We call a preference f of order $(k+1)$ *tangible*, if for every pair of separations (A, B) and (C, D) of order at most k such that $f(A \cap B) \subseteq A \smallsetminus B$, $f(C \cap D) \subseteq C \smallsetminus D$ and $|S| \leq k$, for $S = A \cap C \cap (B \cup D)$, we have $f(S) \subseteq (A \cap C) \smallsetminus (B \cup D)$.

We get the following characterisation of the profiles of a graph:

Theorem 8.5. *For every $k \in \mathbb{N}$ we have:*

(i) If P is a k-profile then f_P is a tangible preference of order $k+1$;
(ii) If f is a tangible preference of order $k+1$ then there is a k-profile P_f such that $f_{P_f} = f$.

Proof. Statement (i) is due to Lemma 8.3, Lemma 8.4 and (P2). To verify (ii) consider a tangible preference f of order $(k+1)$. Let us define a set P of separations of order at most k via

$$P := \{(A, B) \mid f(A \cap B) \subseteq B \smallsetminus A\}.$$

Assume that P does not satisfy (P1). Then there are $(A, B), (D, C) \in P$ with $(C, D) \leq (A, B)$. But then $f(C \cap D) \subseteq C \smallsetminus D \subseteq A \smallsetminus B$ and $f(A \cap B) \subseteq B \smallsetminus A$, a contradiction since $A \smallsetminus B$ and $B \smallsetminus A$ do not touch.

To verify (P2) consider $(B, A), (D, C) \in P$. Then $f(A \cap B) \subseteq A \smallsetminus B$ and $f(C \cap D) \subseteq C \smallsetminus D$. Let $S := A \cap C \cap (B \cup D)$. If $|S| > k$ then $\big((B, A) \vee (D, C)\big)^{\varphi} = (A \cap C, B \cup D)$ has order bigger than k and therefore not contained in P. Otherwise, if $|S| \leq k$, we have $f(S) \subseteq (A \cap C) \smallsetminus (B \cup D)$ such that $(A \cap C, B \cup D) \notin P$.

Hence P is a k-profile and we have $f = f_P$, due to Lemma 8.4. □

8.2. **Tangles of graphs.** We now turn our attention to the tangles of a graph. Note that using the common definition by Robertson and Seymour [9], a tangle of G is not the same as a tangle of $\mathcal{S}_G$[5], but still it is a robust profile of $\mathcal{S}_G$. For a subset $A \subseteq V$ let $G[A]$ denote the subgraph of G induced by A.

[5]We could actually obtain the tangles of G as the tangles of a connectivity-system over the edgeset of G.

We say that a set $\tau \subset \mathcal{S}_G$ of separations of order at most k is a *tangle of G of order $k+1$* if

(θ1) for every $(A,B) \in \mathcal{S}_G$ with $|A \cap B| \leq k$, either (A,B) or (B,A) is in τ;
(θ2) for $(A_1,B_1),(A_2,B_2),(A_3,B_3) \in \tau$ we have $G[A_1] \cup G[A_2] \cup G[A_3] \neq G$.

It is straightforward to verify that this notion of a tangle is consistent with the one given in [9]. A tangle τ of order k is called *maximal* if for every tangle τ' of order $k' > k$ we have $\tau \not\subset \tau'$.

Lemma 8.6. *Every tangle of order $k+1$ is a robust k-profile.*

Proof. Let τ be a tangle of order $k+1$. Assume that τ violates (P1). Then there is $(A,B),(C,D) \in \tau$ with $(B,A) \leq (C,D)$. But then $G[A] \cup G[C] \supseteq G[A] \cup G[B] = G$, contradicting ($\theta$2).

Now let $(A,B),(C,D) \in \tau$. Then we have $G[A] \cup G[C] \cup G[B \cap D] = G$, and ($\theta$2) yields $\big((A,B) \vee (C,D)\big)^{\varphi} = (B \cap D, A \cup C) \notin \tau$, such that τ satisfies (P2). Together with (θ1) this shows that τ is a k-profile.

Finally assume that τ is not robust. Then there is $(A,B) \in \tau$ and $(C,D) \in \mathcal{S}_G$ such that both $(B,A) \wedge (C,D)$ and $(B,A) \wedge (D,C)$ are in τ. But since $G[A] \cup G[B \cap C] \cup G[B \cap D] = G$, this contradicts ($\theta$2). □

8.3. **Inseparable sets.** Another important class of k-profiles of G is given by the k-inseparable sets of G, where a set I of at least $(k+1)$ vertices of G is called *k-inseparable*, if for every separation (A,B) of G of order at most k we have $I \subseteq A$ or $I \subseteq B$. A maximal k-inseparable set is called a *k-block*.

Lemma 8.7. *Every k-inseparable set I induces a k-profile P_I, where*

$$P_I := \{(A,B) \in \mathcal{S}_G \mid |A \cap B| \leq k \ \wedge \ I \subseteq B\}.$$

Proof. Let I be a k-inseparable set and let P_I be defined as in the statement. Let $(A,B) \in P_I$ and $(C,D) \leq (A,B)$. Then $(D,C) \in P_I$ implies $I \subseteq B \cap C \subseteq B \cap A$ such that $|A \cap B| \geq |I| \geq (k+1)$, a contradiction to the definition of P_I. Thus $(D,C) \notin P_I$, which shows (P1). If $(A,B),(C,D) \in P_I$ then $I \subseteq B \cap D$. Now if $(B \cap D, A \cup C)$ has order at most k, then $I \not\subseteq A \cup C$, by a similar argument as before, such that $(B \cap D, A \cup C) \notin P_I$. Otherwise, if $(B \cap D, A \cup C)$ has order at least $(k+1)$, then by definition it cannot be contained in P_I. This shows (P2). Since, by definition of k-inseparability, for every separation (A,B) of order at most k, either A or B contains I, P_I is k-complete. And again by definition of P_I all its elements have order at most k, such that P_I is a k-profile. □

We call a k-inseparable set *robust* if the associated k-profile is robust. It can be shown that this notion of robustness for k-inseparable sets coincides with the one defined in [1]. In that way, Theorem 7.2 implies the main result of [1].

8.4. **Distinguishing tangles and blocks.** Now we are able to apply the results of Section 7 to tangles and blocks, in order to prove Theorem 1 to Theorem 3 from the introduction. If P_1 and P_2 each are a tangle of order $k+1$ or a k-block, then P_1 and P_2 are called *distinguishable* if they are distinguishable as k-profiles. A tree-decomposition *(efficiently) distinguishes* P_1 from P_2, if it does so for P_1 and P_2 treated as k-profiles.

Hence, a tree-decomposition distinguishes all the distinguishable k-blocks and tangles of order $k+1$ if it distinguishes each pair of distinct k-blocks, each pair of distinct tangles of order $k+1$, and every tangle τ of order $k+1$ from every k-block that is not contained in the intersection of all the large sides of τ.

Proof of Theorem 1. By Lemma 8.6 and Lemma 8.7 both a tangle of order $k+1$ and a k-block corresponds to a k-profile. Distinguishable k-blocks or tangles of order $k+1$ correspond to distinct k-profiles and each pair of distinct k-profiles is k-robustly k-distinguishable. Every automorphism of a graph is λ-preserving and maps tangles to tangles and blocks to blocks, preserving their order. Hence Theorem 1 follows from Theorem 7.4 by considering as $\mathcal{P}$ the set of all k-profiles corresponding to k-blocks or tangles of order $k+1$. □

Proof of Theorem 2. Due to Lemma 8.6, the proof of Theorem 7.6 carries over. □

Proof of Theorem 3. Let $\mathcal{P}$ be the set of all profiles arising from tangles or robust k-blocks, for some k. Then $\mathcal{P}$ only consists of robust profiles, due to Lemma 8.6 and the definition of robustness for k-inseparable sets. Let k be the maximal order occurring in $\mathcal{P}$. Then every pair of distinguishable profiles in $\mathcal{P}$ is k-robustly k-distinguishable. Furthermore $\mathcal{P}$ is, by construction, invariant under the automorphisms of G. Let $(\mathcal{T}_\ell, \mathcal{V}_\ell)$ be the series of tree-decompositions given by Theorem 7.2 and let $(\mathcal{T}, \mathcal{V}) := (\mathcal{T}_k, \mathcal{V}_k)$. Then by 7.2 (*ii*), $(\mathcal{T}, \mathcal{V})$ efficiently distinguishes all the tangles of order $\ell+1$ and robust ℓ-blocks, for every ℓ. □

9. Applications to matroids

For an introduction to matroid theory we refer to Oxley [7]. Let M be a finite matroid with ground set E, and let r denote its rank function. A *separation* of M is a bipartition of its ground set, that is, an element of the bipartion lattice of E. It is not hard to prove[6] that $\lambda : \mathcal{B}(E) \to \mathbb{N}$, with $\lambda\big((A, B)\big) := r(A)+r(B)-r(E)+1$, defines a connectivity function on $\mathcal{B}(E)$, such that $(\mathcal{B}(E), \lambda)$ is a connectivity-system. An element $(A, B) \in \mathcal{B}(E)$ with $\lambda(A, B) = k$ is called a *separation of order k of M*. By definition of λ, the automorphisms of M are in one-to-one correspondence with the λ-preserving automorphisms of $\mathcal{B}(E)$.

We call a tree-decomposition of $(\mathcal{B}(E), \lambda)$ a *tree-decomposition of M*, and a *tangle of order $k+1$ of M* is defined to be a tangle of order $k+1$ of $(\mathcal{B}(E), \lambda)$. It is easy to verify that this notion of a tangle of a matroid coincides with the one given by Geelen, Gerards, Robertson and Whittle [5]. Hence, Theorem 4 is a direct consequence of Theorem 7.6.

Just as there are k-blocks in graphs, there are *k-blocks* in a matroid: these are the maximal subsets $B \subseteq E$ such that no two elements of B are separated by a separation of M of order at most k. Every k-block defines a k-profile, so we can use Theorem 7.4 to obtain, for each k, a tree-decomposition of M that distinguishes all its k-blocks. (This generalizes the essence of the well-known decomposition theorem of Cunningham and Edmonds [2], who proved this for $k = 2$.) However, such a tree-decomposition is already provided by Theorem 4: unlike in graphs, every k-block B of M defines a tangle of M of order $k+1$. (Notice that, for every separation (X, Y) of order at most k, B lies entirely in X or entirely in Y, so no union of 'small' sides will ever cover B.)

[6]See Corollary 1.3.4 and Chapter 8 in Oxley [7].

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