Connected-homogeneous digraphs

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Chapter 1

Introduction

There are various notions of symmetry on graphs. A rather weak form is obtained if the automorphism group of a graph acts transitively on the vertices of the graph, i. e. if the graph is *transitive*. This is a very rich class, as it contains all Cayley graphs. On the other side of symmetry, that of *homogeneous* graphs is very strong, that is every isomorphism between two finite induced subgraphs extends to an automorphism of the graph. Whereas the countable homogeneous graph have been classified – these are the C_4, the C_5, the line graph of $K_{3,3}$, countably many copies of a complete graph, a generic K_n-free graph or a generic graph, see [4, 8, 10, 30, 35] –, the condition of transitivity is too weak to obtain a classification of such graphs.

As the homogeneous and the transitive graphs form the two opposite sides of symmetry notions and have quite different answers to the problem of their classification, a natural question is to ask what happens between these two notions, that is, are there natural symmetry conditions that still lead to full classification of graphs with such a kind of symmetry? So we look at symmetries that are stronger than transitivity but not as strong as homogeneity.

Considering the notion of homogeneity, one can ask, what happens if we require the finite subgraphs to be connected. Then we obtain the notion of connected-homogeneous graphs: we call a graph *connected-homogeneous*, or *C-homogeneous* for short, if every isomorphism between finite induced connected subgraphs extends to an automorphism of the whole graph. Indeed, this is a notion of symmetry that lies somewhere between transitivity and homogeneity. The countable C-homogeneous graphs have been classified, see [8, 11, 13, 20].

When we consider digraphs instead of graphs the same notions apply. But although the definitions for digraphs do not differ (much) from the undirected case, the class of digraphs for each of these conditions is in most situations harder to determine. The countable homogeneous digraphs are classified in [2, 3, 4, 28, 29]. This class contains the homogeneous tournaments, see [3, 29].

Gray and Möller [14] started tha project of classifying the countable C-homogeneous digraphs by classifying the infinite connected two-ended digraphs and offering a list of examples for the connected locally finite C-homogeneous digraphs with infinitely many ends. Here, we complete the classification of the countable C-homogeneous digraphs.

On our way, we classify the countable *C-homogeneous* bipartite graphs where we only require that isomorphisms between finite induced connected subgraphs that respects the bipartition extend to automorphisms of the bipartite graph and we classify the countable *homogeneous* 2-partite digraphs, that is, we only require that isomorphisms between finite induced subdigraphs that respects the bipartition extend to automorphisms of the 2-partite digraph.

We state our main theorem in the next section and, thereafter, we give an overview of its proof and the remaining chapters in Section 1.2.

Here, we present the content of the papers [15, 16, 17, 18].

1.1 The main result

In this section, we state our main theorem, the classification of the countable C-homogeneous digraphs (Theorem 1.1). Afterwards, we describe all the digraphs that occur in the list and that need some explanations.

Theorem 1.1. *A countable digraph is C-homogeneous if and only if it is a disjoint union of countably many copies of one of the following digraphs:*

(i) *a countable homogeneous digraph;*

(ii) $H[I_n]$ *for some* $n \in \mathbb{N}^\infty$ *and with either* $H = S(3)$ *or* $H = T^\wedge$ *for some countable homogeneous tournament* $T \neq S(2)$*;*

(iii) $X_\lambda(T)$ *for some countable homogeneous tournament* T *and* $\lambda \in \mathbb{N}^\infty$*;*

(iv) *a regular tree;*

(v) $DL(\Delta)$*, where* Δ *is a bipartite digraph such that* $G(\Delta)$ *is one of*

  (a) C_{2m} *for some integer* $m \geq 2$*,*

  (b) CP_k *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$*,*

  (c) $K_{k,l}$ *for* $k, l \in \mathbb{N}^\infty$*,* $k, l \geq 2$*, or*

  (d) *the countable generic bipartite graph;*

(vi) *$M(k,m)$ for some $k \in \mathbb{N}^\infty$ with $k \geq 3$ and some integer $m \geq 2$;*

(vii) *$M'(2m)$ for some integer $m \geq 2$;*

(viii) *Y_k for some $k \in \mathbb{N}^\infty$ with $k \geq 3$;*

(ix) *$C_m[I_k]$ for some $k, m \in \mathbb{N}^\infty$ with $m \geq 3$;*

(x) *$\mathcal{R}_m$ for some $m \in \mathbb{N}^\infty$ with $m \geq 3$;*

(xi) *$X_2(C_3)_\sim$, where $\sim$ is a non-universal $Aut(X_2(C_3))$-invariant equivalence relation on $VX_2(C_3)$; or*

(xii) *the generic orientation of the countable generic bipartite graph.*

Those countable homogeneous digraphs that are not explicitely mentioned within Theorem 1.1 will be described in Section 3.3.

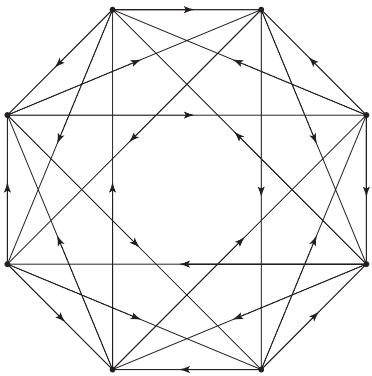

Figure 1.1: The digraph $H = C_3^\wedge$

A *tournament* is a complete digraph. For a tournament T, let T^+ be T together with a new vertex x such that $xv \in ET^+$ for all $v \in VT$. Then $T^\wedge$ is the disjoint union of two copies $T^+\varphi_1$, $T^+\varphi_2$ with isomorphisms φ_1, φ_2 and with $v\varphi_1 u\varphi_2 \in ED$ if and only if $uv \in ET^+$ and $v\varphi_2 u\varphi_1 \in ED$ if and only if $uv \in ED$. In Chapter 6, we denote by H the digraph $C_3^\wedge$, which is depicted in Figure 1.1.

Let $VS(2)$ be a dense subset of the unit circle such that the angle between any two points is rational. A vertex x is the successor of a vertex y if the angle between them is smaller than π modulo 2π (counterclockwise). The resulting tournament is $S(2)$. Similarly, let $VS(3)$ be a dense subset of the unit circle such that the angle between any two points is rational, too. Two vertices in $S(3)$ are adjacent if the angle between them is smaller than $3\pi/2$ modulo 2π (counterclockwise).

For two digraphs D, D' let the *lexicographic product* $D[D']$ be the digraph with vertex set $VD \times VD'$ and edge set

$$\{(x,x')(y,y') \mid xy \in ED \text{ or } (x = y \text{ and } x'y' \in ED')\}.$$

For a homogeneous tournament $T \neq I_1$ and a cardinal λ, let $X_\lambda(T)$ be the digraph such that every vertex is a cut vertex and lies in λ distinct blocks each of which is isomorphic to T. The digraph $X_2(C_3)$ is shown in Figure 1.2.

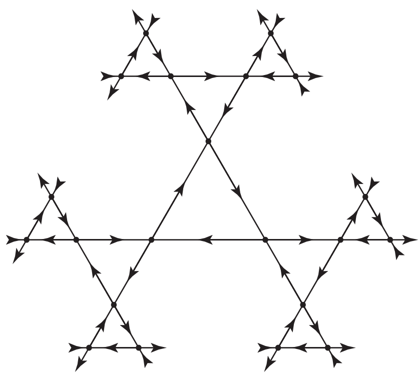

Figure 1.2: The digraph $X_2(C_3)$

For a bipartite edge-transitive digraph Δ, let $DL(\Delta)$ be the digraph such that every vertex is a cut vertex and lies in precisely two blocks each of which is isomorphic to Δ and such that the vertex has its successors in one of the two blocks and its predecessors in the other.

The complete bipartite graph with one side of size k and the other of size ℓ is $K_{k,\ell}$. The *(bipartite) complement of a perfect matching* CP_k is a complete bipartite graph $K_{k,k}$ where the edges of a perfect matching are removed. A *generic* bipartite graph is a bipartite graph with partition $\{X, Y\}$ such that for each two disjoint subsets A, B of the same side we find a vertex in the other partition set with A inside and B outside its neighbourhood.

A digraph is a *tree* if its underlying undirected graph is a tree. It is *regular* if all vertices have the same in-degree and all vertices have the same out-degree (but these two values need not coincide).

An undirected tree is *semiregular* if for the canonical bipartition $\{X, Y\}$ of the vertices of the tree the vertices in X have the same degree and the vertices in Y have the same degree. If the degree of the vertices in X is $k \in \mathbb{N}^\infty$ and those in Y is $\ell \in \mathbb{N}^\infty$, then we denote the semiregular tree by $T_{k,\ell}$.

Given integers $m \geq 2$ and $k \geq 3$ consider the tree $T_{k,m}$ and let $\{U, W\}$ be its natural bipartition such that the vertices in U have degree m. Now subdivide each edge once and endow the neighbourhood of each $u \in U$ with a cyclic order. Then for each new vertex y let u_y be its unique neighbour in U and denote by $\sigma(y)$ the successor of y in the cyclic order of $N(u_y)$. For each $w \in W$ and each $x \in N(w)$ we add an edge directed from x to all $\sigma(y)$ with $y \in N(w) \setminus \{x\}$. Finally, we delete the vertices of the $T_{k,m}$ together with all edges incident with such a vertex to obtain the digraph $M(k,m)$. The locally finite subclass of this class of digraphs coincides with those digraphs $M(k,n)$ for $k, n \in \mathbb{N}$ that are described in [14, Section 5]. In Figure 1.3 the digraph $M(3,3)$ is shown: once with its construction tree and once with the set of $\mathcal{C}$-separators for its unique basic cut system $\mathcal{C}$.[1]

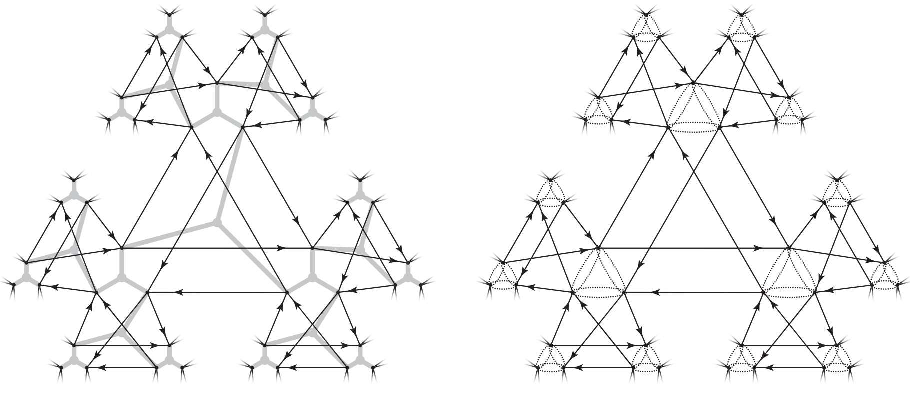

Figure 1.3: The digraph $M(3,3)$

For an integer $m \geq 2$ consider the tree $T_{2,2m}$ and let $\{U, W\}$ be its natural bipartition such that the vertices in U have degree $2m$. Now subdivide every edge once and enumerate the neighbourhood of each $u \in U$ from 1 to $2m$ in a such way that the two neighbours of each $w \in W$ have distinct parity. For each new vertex x let u_x be its unique neighbour in U and define $\sigma(x)$ to be the successor of x in the cyclic order of $N(u_x)$. For any $w \in W$ we have a neighbour a_w with even index, and a neighbour b_w with odd index. Then we add edges from both a_w and $\sigma(a_w)$ to both b_w and $\sigma(b_w)$. Finally we delete the vertices of the $T_{2,2m}$ together with all edges incident with such a vertex. By $M'(2m)$ we denote the resulting digraph. Figure 1.4 shows the digraph $M'(6)$: on the left side with its construction tree and on the right side with the separators of the two possible basic cut systems.

[1] See Section 2.4 for the definition of a cut system and related notation.

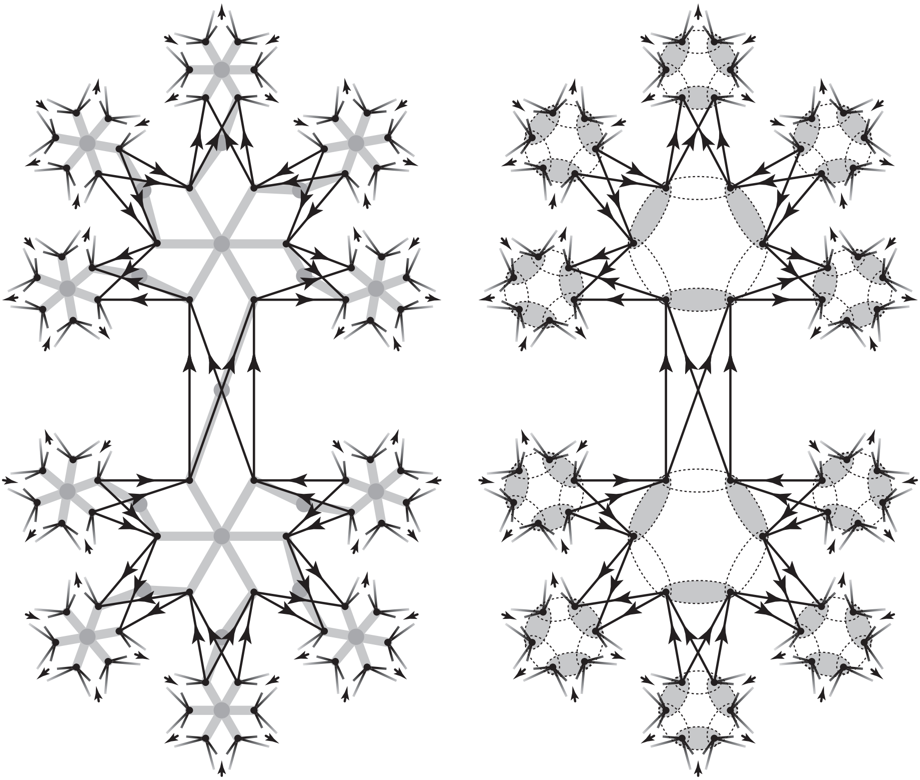

Figure 1.4: The digraph $M'(6)$

A tripartite digraph D is a digraph whose vertex set can be partitioned into three sets V_1, V_2, V_3 such that

$$VE \subseteq (V_1 \times V_2) \cup (V_2 \times V_3) \cup (V_3 \times V_1).$$

The *directed tripartite complement* of D is the digraph

$$(VD, (\bigcup_{i=1,2,3} (V_i \times V_{i+1})) \setminus ED),$$

where $V_4 = V_1$.

For $k \in \mathbb{N}^\infty$, let Y_k be the digraph with vertex set $V_1 \cup V_2 \cup V_3$ where the V_i denote pairwise disjoint independent sets of the same cardinality k such that the subdigraphs of Y_k induced by $V_i \cup V_{i+1}$ (for $i = 1, 2, 3$ with $V_4 = V_1$) are complements of perfect matchings such that all edges are directed from V_i to V_{i+1} and such that the directed tripartite complement of Y_k is the disjoint union of k copies of the directed triangle C_3.

The digraph $\mathcal{R}_m$ for $m \in \mathbb{N}^\infty$ with $m \geq 3$ is constructed as follows: take m pairwise disjoint countably infinite sets V_i for $i = 1 \dots m$ if m is finite and $i \in \mathbb{Z}$ otherwise. Then $\mathcal{R}_m$ has vertex set $\bigcup V_i$ and edges only between V_i and V_{i+1} (with $V_{m+1} = V_1$) such that the digraph induced by V_i and V_{i+1} is a countable generic bipartite digraph such that the edges are directed from V_i to V_{i+1}.

We call a 2-partite digraph D with partition $\{X, Y\}$ a *generic orientation of the countable generic bipartite graph* if for all finite $A, B, C \subseteq X$ (and all finite $A, B, C \subseteq Y$) there is a vertex $v \in X$ (a vertex $v \in Y$, respectively) with $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$ and such that v is not adjacent to any vertex of C. A back-and-forth argument shows that, up to isomorphism, there is a unique generic orientation of the countable generic bipartite graph. It is easy to verify that the underlying undirected graph of D is the countable generic bipartite graph (see Section 3.1 for the definition of a generic bipartite graph).

1.2 Overview of the proof of Theorem 1.1

First, we introduce in Chapter 2 all necessary notations for the remainder of the paper. In Chapter 3, we state and prove several classifications results that we shall need throughout our proof of Theorem 1.1. These are the homogeneous and C-homogeneous bipartite graphs and digraphs, the homogeneous 2-partite digraphs[2], the homogeneous digraphs with the subcases of the finite ones as well as the homogeneous tournaments, and last the result on C-homogeneous digraphs that have been done by Gray and Möller [14].

In Chapter 4, we prove that all digraphs listed in Theorem 1.1 are C-homogeneous. So we can concentrate in the remaining three chapters that the list in Theorem 1.1 is complete. Chapter 5 deal with those connected C-homogeneous digraphs that have more than one end. Our main tool in that section are the 'vertex cuts' defined by Dunwoody and Krön [7]. After some local analysis, we distinguish the cases whether the underlying undirected graph is C-homogeneous (Section 5.2) or not (Section 5.3).

In Chapter 6, we look at those C-homogeneous digraphs that have finite degree. We prove that the out-neighbourhood as well as the in-neighbourhood of any vertex induce (finite) homogeneous digraphs and thus, we are able to consider the list of finite homogeneous digraphs one by one and look at each case individually. If there are edges in these homogeneous digraphs, then the structure of those digraphs gives us a lot of information that we can use to complete these cases. So the only remaining case is if the out-neighbourhood

[2]Note that we shall make a difference between 2-partite and bipartite digraphs: both have a bipartition of their vertex set, but whereas the first may have edges directed in both ways between these sets, the edges of the latter all have the same direction.

as well as the in-neighbourhood of our locally finite C-homogeneous digraph is an independent vertex set. In that case, we use the reachability relation (see Section 2.2 and we shall prove that it is never universal. So we can use the reachability digraph, which turns out to be a C-homogeneous bipartite digraph. Again having just some cases, we consider them one by one and thereby complete the situation of finite degree.

In Chapter 7, we look at the case that the countable C-homogeneous digraph has at most one end (but infinite degree). The proof of this situation is similar to the one of Chapter 6, but keeping the structure of the papers that are combined to this thesis untouched we have not combined these two chapters. Once more, the out-neighbourhood of each vertex as well as the in-neighbourhood induce a (countable) homogeneous digraph. With this in mind, we consider Cherlin's classification of the countable homogeneous digraphs and investigate each of its cases one after another. If the out-neighbourhood of some vertex is not an independent set (Section 7.1), then – as in the previous case – we can prove the outcome of each case relatively easy. Interestingly, some of the ideas of the proofs of the corresponding cases for undirected C-homogeneous graphs [13] carries over but have to deal with the new situation of directed edges. These cases are for example the generic $\mathcal{H}$-free digraphs (versus generic K_n-free graphs), the generic I_n-free digraphs (versus generic I_n-free graphs), and the (semi-)generic n-partite digraphs (versus the complete n-partite graphs).

In Section 7.2, we consider the case that the out-neighbours of each vertex form an independent set and the same for the in-neighbours. Again, we can use the reachability relation and finish the situation of a bipartite reachability digraph, which is a C-homogeneous digraph, relatively easy. But in contrast to locally finite digraphs, we now can have a universal reachability relation. We treat this case in Section 7.2.2 and our main tool for that part is the classification of the homogeneous 2-partite digraphs.

Chapter 2

Definitions and Preliminaries

2.1 Basics

A *digraph* D is a pair of a non-empty set VD of *vertices* and an irreflexive and antisymmetric binary relation ED on VD, its *edges*. For a subset of vertices $X \subseteq VD$, let $D[X] := (X, ED \cap X \times X)$ be the digraph *induced by* X.[1] Two vertices $x, y \in VD$ are *adjacent* if either $xy \in ED$ or $yx \in ED$. The *out-neighbours* or *successors* of $x \in VD$ are the elements of the *out-neighbourhood* $N^+(x) := \{y \in VD \mid xy \in ED\}$ and its *in-neighbours* or *predecessors* are the elements of the *in-neighbourhood* $N^-(x) := \{y \in VD \mid yx \in ED\}$. Furthermore, let $D^+(x) := D[N^+(x)]$ and $D^-(x) := D[N^-(x)]$. If D is *vertex-transitive*, that is, if the automorphisms of D act transitively on VD, then the digraphs $D^+(x)$ and $D^+(y)$ (the digraphs $D^-(x)$ and $D^-(y)$) are isomorphic for any two vertices x, y and we denote by D^+ (by D^-, respectively) one element of their isomorphism class. For induced subdigraphs A and B of D and $x \in VD$, let $A + B$ be the digraph $D[VA \cup VB]$, let $A + x = D[VA \cup \{x\}]$, and let $A - x = D[VA \setminus \{x\}]$. If $B \subseteq A$, let $A - B = D[VA \setminus VB]$. An *independent* vertex set is a set whose elements are pairwise non-adjacent. By I_k we denote an independent vertex set of cardinality k and also a digraph whose vertex set is an independet set of cardinality k. It will always be obvious from the context, whether I_n describes a vertex set or a digraph. A *tournament* is a digraph such that each two of its vertices are adjacent.

[1] Note that if $X \subseteq VD$, then $D[X]$ is a subdigraph of D (the restiction of D onto X) and, if D' is a digraph, $D[D']$ is a new digraph (the lexicographic product).

For $k \in \mathbb{N}$, a *k-arc* is a sequence $x_0 \dots x_k$ or $k+1$ vertices with $x_i x_{i+1} \in ED$ for all $i \leq k-1$. A *path* (of *length* $\ell \in \mathbb{N}$) is a sequence $x_0 \dots x_\ell$ of $\ell+1$ distinct vertices such that for all $i \leq \ell-1$ the vertices x_i and x_{i+1} are adjacent. If we have $x_i x_{i+1} \in ED$ for all $i \leq \ell - 1$ then we call the path *directed*. Hence, a directed path of length ℓ is an ℓ-arc all whose vertices are distinct. A digraph is *connected* if each two vertices are joined by a path. A vertex, vertex set, or subdigraph *separates* a digraph if its deletion leaves more than one component. It *separates* two vertices, vertex sets, or subgraphs if these lie in distinct components after the deletion.

An *ancestor* (*descendant*) of a vertex x is any vertex y for which there exists an arc from y to x (from x to y). The *descendant-digraph* (*ancestor-digraphs*) of x is the subdigraph $\operatorname{desc}(x) \subseteq D$ (the subdigraph $\operatorname{anc}(x) \subseteq D$) that is induced by the set of all its descendants (its ancestors, respectively).

A *cycle* (of *length* $\ell \geq 3$) is a path of length $\ell - 1$ whose end vertices are joined by an edge. A *directed* cycle, denoted by C_ℓ, is a cycle $x_1 \dots x_{\ell-1}$ either with $x_i x_{i+1} \in ED$ and $x_{\ell-1} x_1 \in ED$ or with $x_{i+1} x_i \in ED$ and $x_1 x_{\ell-1} \in ED$. Triangles are cycles of length 3. Up to isomorphism, there are two distinct kinds of triangles. We call those triangles that are not directed *transitive*. We also denote graphs that are cycles of length ℓ by C_ℓ. It will always be clear from the context whether C_ℓ is a graph or a digraph.

For an equivalence relation $\sim$ on VD let $D_\sim$ be the digraph whose vertices are the equivalence classes of $\sim$ and where $XY \in ED_\sim$ if and only if there are $x \in X$ and $y \in Y$ with $xy \in ED$. We call $D_\sim$ a *quotient digraph* of D (induced by $\sim$). In general, this is not a digraph since it may have loops as well as edges XY and YX. However, we only consider equivalence relations $\sim$ such that $ED_\sim$ is an irreflexive and antisymmetric relation. But in each situation in which we consider quotient digraphs $D_\sim$ we will prove that $ED_\sim$ is irreflexive and antisymmetric.

The *underlying undirected graph* of a digraph $D = (V, E)$ is the graph $G = (V, \{\{x, y\} \mid xy \in E\})$. A *tournament* is a digraph whose underlying undirected graph is a complete graph.

Let $\mathbb{N}^\infty = \mathbb{N} \cup \{\omega\}$. The *diameter* of D is defined by

$$\operatorname{diam}(D) = \inf\{n \in \mathbb{N}^\infty \mid d(x, y) \leq n \text{ for all } x, y \in VD\}.$$

A *ray* in a graph is a one-way infinite path and a *double ray* is a two-way infinite path. Two rays are *equivalent* if for every finite vertex set S both rays lie eventually in the same component of $G - S$. This is an equivalence relation whose classes are the *ends* of the graph. *Rays*, *double rays*, and *ends* of a digraph are those of its underlying undirected graph. For abbreviation, we denote by C_∞ the directed double ray.

If the underlying undirected graph of a digraph D is bipartite then D is 2-*partite*. If in addition all edges are directed from the same partition set to the other then we call D *bipartite*.

2.2 Reachability relation

A digraph D is called k-*arc-transitive* if its automorphisms act transitively on the k-arcs of D and it is called *highly-arc-transitive* if it is k-arc-transitive for every $k \in \mathbb{N}$.

Let D be a digraph. A *walk* is a sequence $x_0 \dots x_k$ of vertices such that x_i and x_{i+1} are adjacent for all $0 \leq i < k$. If $x_{i-1} \in N^+(x_i) \Leftrightarrow x_{i+1} \in N^+(x_i)$ for all $0 < i < k$ then the walk is called *alternating*. Two edges on a common alternating walk are *reachable* from each other. This defines an equivalence relation, the *reachability relation* $\mathcal{A}$. For an edge $e \in ED$, let $\mathcal{A}(e)$ be the equivalence class of e and let $\langle \mathcal{A}(e) \rangle$ be the *reachability* digraph of D that contains e, that is, the vertex set incident with some edge in $\mathcal{A}(e)$ and edge set $\mathcal{A}(e)$. If D is 1-arc transitive, then the digraphs $\langle \mathcal{A}(e) \rangle$ are isomorphic for all $e \in ED$ and we denote by $\Delta(D)$ one digraph of their isomorphism class.

The following proposition is due to Cameron et al.

Proposition 2.1. [1, Proposition 1.1] *Let D be a connected 1-arc transitive digraph. Then $\Delta(D)$ is 1-arc transitive and connected. Furthermore, either*

(a) *$\mathcal{A}$ is the universal relation on ED and $\Delta(D) \cong D$, or*

(b) *$\Delta(D)$ is a bipartite reachability digraph.* □

We say that a cycle C *witnesses that $\mathcal{A}$ is universal* if C contains an induced 2-arc and if there is an edge e on C such that C without the edge e is an alternating walk.

Lemma 2.2. *Let D be a non-empty vertex-transitive and 1-arc transitive digraph whose reachability relation $\mathcal{A}$ is universal. Then D contains a cycle that witnesses that $\mathcal{A}$ is universal.*

Proof. As D is non-empty, it contains some edge xy and, since D is vertex-transitive, it also has some edge yz. Hence, D contains a (not necessarily induced) 2-arc xyz. By universality of $\mathcal{A}$, there must be a minimal alternating walk P in D whose first edge is xy and whose last edge is yz. Either this walk is a cycle or there is a vertex incident with at least three edges of that walk. If the walk is a cycle, then it obviously witnesses that $\mathcal{A}$ is universal. If the walk contains a vertex v incident with three edges of the walk, then one edge incident with v is directed towards v and one is directed away from v, as otherwise we have a contradiction to the minimality of the alternating walk. So v is the middle vertex of two 2-arcs uvw and either $u'vw$ or uvw' in the digraph (VP, EP), say $u'vw$. Then we find a shorter alternating walk – a proper subwalk of P – either between uv and vw or between $u'v$ and vw and we are done by induction. (Note that this is not necessarily a contradiction since, e.g., xyz might be an induced 2-arc but uvw induces a triangle.) □

Lemma 2.2 just tells us that we find some cycle witnessing that $\mathcal{A}$ is universal. Next, we show that we can even find an induced cycle with the same property.

Lemma 2.3. *Let D be a non-empty vertex-transitive and 1-arc transitive digraph whose reachability relation $\mathcal{A}$ is universal. If D contains some cycle witnessing that $\mathcal{A}$ is universal, then it contains an induced such cycle of at most the same length.*

Proof. Let us suppose that none of the minimal cycles witnessing the universality of $\mathcal{A}$ is induced. Let C be such a cycle of minimal length. This exists by Lemma 2.2. Let $xy \in EC$ such that C without the edge xy is an alternating walk P. Since C is not induced, it has a chord uv. If u and v lie in the same set of the canonical bipartition of VP, then the subwalk uPv together with the edge uv is a smaller cycle witnessing that $\mathcal{A}$ is universal. By minimality of C, this cannot be. So u and v lie in distinct sets of the canonical bipartition of P. But then we also find a smaller cycle in C together with the edge uv: if the out-degree of v in P is 0, then we take uv together with the subwalk of C that contains xy, and otherwise we take uv together with uPv. This contradiction to the minimality of C shows the lemma. □

2.3 Group actions

Let Γ be a group acting on a digraph D and let $U \subseteq VD$. We denote by Γ_U the *(pointwise) stabilizer* of U, that is the subgroup of Γ that fixes each element of U. The same notion holds for an edge $e \in ED$ or a single vertex $x \in VD$. If Γ fixes the set U setwise, then we denote by Γ^U the group of all automorphisms of U that are obtained by restricting elements of Γ to U.

We will use the following theorem on subgroups of the symmetric group S_n.

Theorem 2.4. [22, Satz II.5.2] *Every proper subgroup of S_n with $n \neq 4$ is equal to A_n or has index at least n. If $n = 4$, then, except for A_n, the Sylow 2-subgroups are the only proper subgroups of index less than n.* □

2.4 Structure trees

In this section we introduce the terms of cuts and structure trees that were developed by Dunwoody and Krön in [7]. Compared to [7] we use a different notation for the cut systems in order to indicate the relation of cut systems with the well-known graph theoretic concept of separations, see [5].

Let G be a connected graph and let $A, B \subseteq V(G)$ be two vertex sets. The pair (A, B) is a *separation* of G if $A \cup B = V(G)$ and $E(G[A]) \cup E(G[B]) = E(G)$. The *order* of a separation (A, B) is the cardinality of its *separator* $A \cap B$ and the subgraphs $G[A \setminus B]$ and $G[B \setminus A]$ are the *wings* of (A, B). With $(A, \sim)$ we refer to the separation $(A, (V(G) \setminus A) \cup N(V(G) \setminus A))$. A *cut* is a separation (A, B) of finite order with non-empty wings such that the wing $G[A \setminus B]$ is connected and such that no proper subset of $A \cap B$ separates the wings of (A, B). A *cut system* of G is a non-empty set $\mathcal{S}$ of separations (A, B) of G satisfying the following three properties.

1. If $(A, B) \in \mathcal{S}$ then there is an $(X, Y) \in \mathcal{S}$ with $X \subseteq B$.

2. Let $(A, B) \in \mathcal{S}$ and C be a component of $G[B \setminus A]$. If there is a separation $(X, Y) \in \mathcal{S}$ with $X \setminus Y \subseteq C$, then the separation $(C \cup N(C), \sim)$ is also in $\mathcal{S}$.

3. If $(A, B) \in \mathcal{S}$ with wings X, Y and $(A', B') \in \mathcal{S}$ with wings X', Y' then there are components C in $X \cap X'$ and D in $Y \cap Y'$ or components C in $Y \cap X'$ and D in $X \cap Y'$ such that both C and D are wings of separations in $\mathcal{S}$.

Two separations $(A_0, A_1), (B_0, B_1) \in \mathcal{S}$ are *nested* if there are $i, j \in \{0, 1\}$ such that one wing of $(A_i \cap B_j, \sim)$ does not contain any connected component C with $(C \cup N(C), \sim) \in \mathcal{S}$ and $A_{1-i} \cap B_{1-j}$ contains $(A_0 \cap A_1) \cup (B_0 \cap B_1)$. A cut system is *nested* if each two of its cuts are nested.

Remark 2.5. *The following two assertions hold.*

1. *If, for two $\mathcal{C}$-cuts $(A_0, A_1), (B_0, B_1)$, the separator $A_0 \cap A_1$ contains vertices of both wings of (B_0, B_1), then the two cuts are not nested.*

2. *In any transitive graph G with an* $\mathrm{Aut}(G)$*-invariant cut system $\mathcal{C}$, any two nested cuts (A_0, A_1) and (B_0, B_1) with $(A_0 \cap A_1) \cup (B_0 \cap B_1) \subseteq A_{1-i} \cap B_{1-j}$ have the property that $A_i \cap B_j$ is empty by* [7, Lemma 3.5].

A cut in a cut system $\mathcal{S}$ is *minimal* if its order in $\mathcal{S}$ is minimal. A *minimal cut system* is a cut system all whose cuts are minimal and thus have the same order.

Let us describe two minimal cut systems one of which was introduced by Dunwoody and Krön [7, Example 2.2]. Both will be used in our proofs.

Example 2.6. Let G be a connected infinite graph with at least two ends. Let n be the smallest cardinality of a finite vertex set X such that there are at least two components in $G - X$ that contain a ray each. Let $\mathcal{S}$ be the set of all cuts (A, B) with order n such that both $G[A]$ and $G[B]$ contain a ray. Then $\mathcal{S}$ is a minimal cut system.

An $\mathcal{S}$-*separator* is a vertex set S that is a separator of some separation in $\mathcal{S}$. Let $\mathcal{W}$ be the set of $\mathcal{S}$-separators. An $\mathcal{S}$-*block* is a maximal induced subgraph X of G such that

(i) for every $(A, B) \in \mathcal{S}$ there is $V(X) \subseteq A$ or $V(X) \subseteq B$ but not both;

(ii) there is some $(A, B) \in \mathcal{S}$ with $V(X) \subseteq A$ and $A \cap B \subseteq V(X)$.

Let $\mathcal{B}$ be the set of $\mathcal{S}$-blocks. For a nested minimal $\mathrm{Aut}(G)$-invariant cut system $\mathcal{S}$ let $\mathcal{T}$ be the graph with vertex set $\mathcal{W} \cup \mathcal{B}$. Two vertices X, Y of $\mathcal{T}$ are adjacent if and only if either $X \in \mathcal{W}$, $Y \in \mathcal{B}$, and $X \subseteq Y$ or $X \in \mathcal{B}$, $Y \in \mathcal{W}$, and $Y \subseteq X$. Then $\mathcal{T} = \mathcal{T}(\mathcal{S})$ is called the *structure tree* of G and $\mathcal{S}$.

Lemma 2.7. [7, Lemma 6.2] *Let G be a connected graph, and let $\mathcal{S}$ be a nested minimal cut system. Then the structure tree of G and $\mathcal{S}$ is a tree.* □

A cut system $\mathcal{S}$ of a connected graph G is *basic* if $\mathcal{S}$ is minimal, nested, $\mathrm{Aut}(G)$-invariant, if $\mathcal{S}$ is a subsystem of the minimal cut system given in Example 2.6 and if all separators $A \cap B$ with $(A, B) \in \mathcal{S}$ belong to the same $\mathrm{Aut}(G)$-orbit.

We state here that part of Theorem 7.2 of [7] that we shall use here.

Theorem 2.8. *For every graph G with at least two ends there is a basic cut system $\mathcal{S}$ of G.* □

If we take, for a connected graph G, all those cuts whose separators consist of one vertex, each, then we obtain as the structure tree the well-known block-cutvertex tree. So in this case the two obtained trees coincide, but have different notations. That is, in the proof of Theorem 5.9 we could argue using the block-cutvertex tree instead of the structure tree. But for consistency reasons we have not done so.

Lemma 2.9. *Let G be a graph and let $\mathcal{C}$ be a nested cut system of G such that no $\mathcal{C}$-separator contains any edge. For any path P that has both its end vertices in the same $\mathcal{C}$-separator S, there is a $\mathcal{C}$-block with maximal distance to S in $\mathcal{T}(\mathcal{C})$ that contains edges of P. This $\mathcal{C}$-block contains at least two edges of P.*

Proof. Any two vertices that are not in a common $\mathcal{C}$-block, are separated by some $\mathcal{C}$-separator. So we conclude that for each edge of G there is a unique

$\mathcal{C}$-block that contains this edge, as it is not contained in any $\mathcal{C}$-separator. The path P has only finitely many edges, so there are just finitely many $\mathcal{C}$-blocks that contain edges of P and we may pick one, X say, with maximal distance to S in $\mathcal{T}(\mathcal{C})$. Let xy be an edge on P that lies in X. Then either x or y does not lie in that $\mathcal{C}$-separator S' that separates X from S and lies in X. We assume that this is y. Let z be the other neighbour of x on P. The edge yz cannot lie further away from S than X in the structure tree, but since $y \notin S'$, we have $yz \in EX$. So X contains two edges of P. □

In the context of a digraph D all concepts introduced in this section are related to the underlying undirected graph G of D except for one definition: We call a cut system $\mathcal{C}$ for a digraph D *basic* if it has the following properties.

(i) $\mathcal{C}$ is non-empty, minimal, nested and $\mathrm{Aut}(D)$-invariant.

(ii) $\mathrm{Aut}(D)$ acts transitively on $\mathcal{S}$.

(iii) For each $\mathcal{C}$-cut (A, B) both A and B contain an end of D and there is no separation of smaller order that has this property.

Then Theorem 2.8 does not only hold for any graph but also for any digraph by the results in [7]. We have to define the property of being basic differently, because we know in general only that we may consider $\mathrm{Aut}(D)$ as a subgroup of $\mathrm{Aut}(G)$, but we do not know whether it is a proper subgroup or not. Thus, our cut system could have more than one $\mathrm{Aut}(D)$-orbit of separators which would be more difficult to deal with.

Chapter 3

Some homogeneous structures

3.1 Homogeneous and connected-homogeneous bipartite graphs and digraphs

In this section, we cite the classifications of the countable homogeneous bipartite graphs and classify the C-homogeneous bipartite graphs. Then, for countable (C-)homogeneous bipartite digraphs, the analogous theorems hold.

A bipartite graph G (with bipartition $\{X, Y\}$) is *homogeneous bipartite* if every isomorphism between two isomorphic finite induced subgraphs A and B of G that preserves the bipartition (that means that $VA \cap X$ is mapped onto $VB \cap X$ and $VA \cap Y$ is mapped onto $VB \cap Y$) extends to an automorphism of G that preserves the bipartition. We call G *connected-homogeneous bipartite*, or simply *C-homogeneous bipartite*, if every isomorphism between two isomorphic finite induced connected subgraphs A and B of G that preserves the bipartition extends to an automorphism of G that preserves the bipartition. The same notions apply to bipartite and 2-partite digraphs.

We begin with the classification of the homogeneous bipartite graphs.

Theorem 3.1. [12, Remark 1.3] *A countable bipartite graph is homogeneous if and only if it is isomorphic to one of the following graphs:*

(i) *a complete bipartite graph;*

(ii) *an empty bipartite graph;*

(iii) *a perfect matching;*

(iv) *the bipartite complement of a perfect matching; or*

(v) *the countable generic bipartite graph.* □

The *generic* bipartite graph is the bipartite graph G with bipartition $\{X, Y\}$ such that for every two finite subsets $U_X, W_X \subseteq X$ and every two finite subsets $U_Y, V_Y \subseteq Y$ there exists $x \in X$ and $y \in Y$ with $U_X \subseteq N(y)$ and $V_X \cap N(y) = \emptyset$ and with $U_Y \subseteq N(x)$ and $V_Y \cap N(x) = \emptyset$. Next, we complete the classification of connected C-homogeneous bipartite graphs, which was already done for locally finite graphs, by Gray and Möller [14]. They already mentioned that their work should be extendable with not too much effort – and indeed this section has essentially the same structure.

The proof of the locally finite analog [14, Lemma 4.4] of Lemma 3.2 is self contained and does not use the local finiteness of the graph. Thus we can omit the proof here.

Lemma 3.2. *Let G be a connected C-homogeneous bipartite graph with bipartition $X \cup Y$. If G is not a tree and has at least one vertex with degree greater than 2 then G embeds C_4 as an induced subgraph.* □

Let G be a bipartite graph with bipartition $X \cup Y$. Then for each edge $xy \in EG$ we define the neighbourhood graph to be:

$$\Omega(x, y) := G[N(x) + N(y) - \{x, y\}]$$

A C-homogeneous graph G is, in particular, edge-transitive. Hence there is a unique neighbourhood graph $\Omega(G)$.

Lemma 3.3. *Let G be a connected C-homogeneous bipartite graph. Then $\Omega(G)$ is a homogeneous bipartite graph, and therefore is one of: an edgeless bipartite graph, a complete bipartite graph, a complement of a perfect matching, a perfect matching, or a homogeneous generic bipartite graph.*

Proof. If we do not ask $\Omega(G)$ to be finite, the proof of the locally finite analogue [14, Lemma 4.5] carries over. Compared to the locally finite case, we only have to deal with one other 'type' of graph, due to [12, Remark 1.3] □

Lemma 3.4. *Let G be a C-homogeneous generic bipartite graph. Then G is homogeneous bipartite.*

Proof. Let $VG = A \cup B$ be the natural bipartition of G, let X and Y be two isomorphic induced finite subgraphs of G, and let $\varphi : X \to Y$ be an isomorphism. Let $a \in A \setminus X$ be a vertex adjacent to all the vertices of $X \cap B$ and let $b \in B \setminus X$ be a vertex adjacent to all the vertices of $X \cap A$ and to a. Let a', b' be the corresponding vertices for Y. Since G is bipartite, both $G[X + a + b]$ and $G[Y + a' + b']$ are connected induced subgraphs of G that are isomorphic to each other. Furthermore there is an isomorphism $\psi : G[X + a + b] \to G[Y + a' + b']$ such that the restriction of ψ to X is φ. As there is an automorphism of G that extends ψ, this automorphism also extends φ and G is homogeneous. $\square$

Now we are able to prove the classification result of the C-homogeneous bipartite graphs.

Theorem 3.5. *A connected graph is a C-homogeneous bipartite graph if and only if it belongs to one of the following classes:*

(i) $T_{\kappa,\lambda}$ *for cardinals* κ, λ*;*

(ii) C_{2m} *for* $m \in \mathbb{N}$*;*

(iii) $K_{\kappa,\lambda}$ *for cardinals* κ, λ*;*

(iv) CP_κ *for a cardinal* κ*;*

(v) *homogeneous generic bipartite graphs.*

Proof. The nontrivial part is to show that this list is complete. So consider an arbitrary connected C-homogeneous bipartite graph G with bipartition $X \cup Y$. If G is a tree then it is obviously semi-regular and hence a $T_{\kappa,\lambda}$. So suppose G contains a cycle. Then, since G is C-homogeneous, each vertex lies on a cycle. Now G is either a cycle, which is even since G is bipartite, or at least one vertex in G has a degree greater than 2 and G embeds a C_4, due to Lemma 3.2. Thus $\Omega(G)$ contains at least one edge and by Lemma 3.3 we have to consider the following cases:

Case 1: $\Omega(G)$ *is complete bipartite.* Suppose that there is an induced path $P = uxyv$ in G. Then $\Omega(x, y)$ gives rise to an edge between u and v, a contradiction. Hence G is complete bipartite.

Case 2: $\Omega(G)$ *is the complement of a perfect matching.* Consider $x \in X$ and $y \in Y$ such that $\{x, y\}$ is an edge of G. Since $\Omega(x, y)$ is the complement of a perfect matching and G is not a cycle, there is an index set $I \supseteq \{1, 2\}$ such that $N(x) = \{y\} \cup \{y_i | i \in I\}$, $N(y) = \{x\} \cup \{x_i | i \in I\}$ and for $i \in I$ the vertex x_i is nonadjacent to y_i but adjacent to all y_j with $j \in I \setminus \{i\}$. Since $\Omega(x, y_1)$ is also the complement of a perfect matching there is a unique vertex $a \in N(y_1) \setminus N(y)$. Since x_i with $i \neq 1$ is adjacent to y_1 it is contained in $\Omega(x, y_1)$ and therefore y_i is adjacent to a. Thus for all $i \in I$ we have $N(y_i) = N(y) - x_i + a$. Now by symmetry there is a unique vertex b adjacent to all x_i with $i \in I$ but non-adjacent to x and for all $i \in I$ there is $N(x_i) = N(x) - y_i + b$. If we look at

$\Omega(x_1, y_2)$ we have $x, a \in N(y_2)$ and $y, b \in N(x_1)$ which implies $\{a, b\} \in EG$ and hence $N(a) = N(x) - y + b$ and $N(b) = N(y) - x + a$. Because G is connected we have $X = N(y) + a$ and $Y = N(x) + b$ which means that G is itself the complement of a perfect matching.

Case 3: *$\Omega(G)$ is a perfect matching.* For the same reason as for locally finite graphs this case cannot occur (cp. [14, Theorem 4.6]).

Case 4: *$\Omega(G)$ is homogeneous generic bipartite.* Let U and W be two disjoint finite subsets of X (of Y). Since G is connected there is a finite connected induced subgraph $H \subset G$ that contains both U and W. By genericity, we find an isomorphic copy H_Ω of H in $\Omega(G)$. Because G is C-homogeneous there is an automorphism φ of G with $H_\Omega^\varphi = H$. Now there is a vertex v in Y (in X) that is adjacent to all vertices in $U^{\varphi^{-1}}$ and non-adjacent to all vertices in $W^{\varphi^{-1}}$. Hence v^φ is adjacent to all vertices in U and none in W which implies that G is generic bipartite. Furthermore G is homogeneous bipartite by Lemma 3.4, as it is C-homogeneous. □

Note that Theorems 3.1 and 3.5 also apply to homogeneous and C-homogeneous bipartite digraphs, but not to 2-partite digraphs. The 2-partite digraphs are in the case of homogeneity subject of the next section.

3.2 Homogeneous 2-partite digraphs

As mentioned before, a 2-partite digraph D with partition $\{X, Y\}$ is *homogeneous* if every isomorphism φ between finite induced subdigraphs A and B with $(VA \cap X)\varphi \subseteq X$ as well as $(VA \cap Y)\varphi \subseteq Y$ extends to an automorphism α of D with $X\alpha = X$ and $Y\alpha = Y$. Let us state the classification of the countable 2-partite digraphs:

Theorem 3.6. *Let D be a countable 2-partite digraph with partition $\{X, Y\}$. Then D is homogeneous if and only if one of the following cases holds:*

(i) *D is a homogeneous bipartite digraph;*

(ii) *$D \cong CP'_k$ for some $k \in \mathbb{N}^\infty$ with $k \geq 2$;*

(iii) *D is the countable generic 2-partite digraph; or*

(iv) *D is the generic orientation of the countable generic bipartite graph.*

For $k \in \mathbb{N}^\infty$ with $k \geq 2$, let CP'_k be the 2-partite digraph with partition $\{X, Y\}$ such that $ECP'_k \cap (X \times Y)$ induces a CP_k on VCP'_k and $ECP'_k \cap (Y \times X)$ induces a perfect matching on VCP'_k. Note that its underlying undirected graph is a complete bipartite graph.

We call a 2-partite digraph D with partition $\{X, Y\}$ *generic* if for every finite $A, B \subseteq X$ (for every finite $A, B \subseteq Y$) there is a vertex $v \in Y$ (a vertex $v \in X$, respectively) with $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$. A back-and-forth argument shows that there is a unique countable generic 2-partite digraph (up to isomorphism). Similarly, we call a 2-partite digraph D with partition $\{X, Y\}$ a *generic orientation of a generic bipartite graph* if for all pairwise disjoint finite subsets $A_X, B_X, C_X \subseteq X$ and $A_Y, B_Y, C_Y \subseteq Y$ there are vertices $y \in Y$ and $x \in X$ with $A_X \subseteq N^+(y)$, $B_X \subseteq N^-(y)$ and $C_X \subseteq y^\perp$ as well as with $A_Y \subseteq N^+(x)$, $B_Y \subseteq N^-(x)$ and $C_Y \subseteq x^\perp$. It is easy to verify that its underlying undirected graph is a generic bipartite graph.

To prove Theorem 3.6, we change our notation a bit: a *bipartite graph* is a triple $G = (X, Y, E)$ of pairwise disjoint sets such that every $e \in E$ is a set consisting of one element of X and the one element of Y. We call $VG = X \cup Y$ the *vertices* of G and E the *edges* of G. So $(X \cup Y, E)$ is a bipartite graph in the usual sense with bipartition $\{X, Y\}$ A *2-partite digraph* is a triple $D = (X, Y, E)$ of pairwise disjoint sets with $E \subseteq (X \times Y) \cup (Y \times X)$ and such that $(u, v) \in E$ implies $(v, u) \notin E$. Again, $VD = X \cup Y$ are the *vertices* of D and E are the *edges* of D. So $(X \cup Y, E)$ is a digraph in the usual sense, whose underlying undirected graph is bipartite. We write uv instead of (u, v) for edges of D. A 2-partite digraph (X, Y, E) is *bipartite* if either $E \subseteq X \times Y$ or $E \subseteq Y \times X$. The *underlying undirected bipartite graph* of a 2-partite digraph (X, Y, E) is defined by

$$(X, Y, \{\{u, v\} \mid uv \in E\}).$$

Two vertices u, v of a 2-partite digraph $D = (X, Y, E)$ are *adjacent* if either $uv \in E$ or $vu \in E$. The *successors* of $u \in VD$ are the elements of the *out-neighbourhood* $N^+(u) := \{w \in VD \mid uw \in E\}$ and its *predecessors* are the elements of the *in-neighbourhood* $N^-(u) := \{w \in VD \mid wu \in E\}$. For $x \in X$, we define

$$x^\perp = \{y \in Y \mid y \text{ not adjacent to } x\}$$

and, for $y \in Y$, we define

$$y^\perp = \{x \in X \mid x \text{ not adjacent to } y\}.$$

A bipartite graph $G = (X, Y, E)$ is *homogeneous* if every isomorphism φ between finite induced subgraphs A and B with

$$(VA \cap X)\varphi \subseteq X \qquad \text{and} \qquad (VA \cap Y)\varphi \subseteq Y$$

extends to an automorphism α of G with $X\alpha = X$ and $Y\alpha = Y$. Similarly, a 2-partite digraph $D = (X, Y, E)$ is *homogeneous* if every isomorphism φ between finite induced subdigraphs A and B with

$$(VA \cap X)\varphi \subseteq X \qquad \text{and} \qquad (VA \cap Y)\varphi \subseteq Y$$

extends to an automorphism α of D with $X\alpha = X$ and $Y\alpha = Y$.

A first step towards the classification of the homogeneous 2-partite digraphs was already done when Goldstern et al. [12] classified the homogeneous bipartite graphs, cp. Theorem 3.1: for bipartite digraphs (X, Y, E), Theorem 3.1 applies analogously in the following sense: as we have either $E \subseteq X \times Y$ or $E \subseteq Y \times X$, the underlying undirected bipartite graph is homogeneous, so belongs to some class of the list in Theorem 3.1. Conversely, every orientation of a homogeneous bipartite graph that results in a bipartite digraph gives a homogeneous bipartite digraph. Note that homogeneous bipartite digraphs are in particular homogeneous 2-partite digraphs. Hence, the above classification gives us a partial classification in the case of the homogeneous 2-partite digraphs in that it gives a full classification of the homogeneous bipartite digraphs. In the remainder of this note we extend this partial classification by classifying those homogeneous 2-partite digraphs that are not bipartite.

Theorem 3.7. *A 2-partite digraph is homogeneous if and only if it is isomorphic to one of the following 2-partite digraphs:*

(i) *a homogeneous bipartite digraph;*

(ii) *an M_κ for some cardinal $\kappa \geq 2$;*

(iii) *a generic 2-partite digraph;*

(iv) *a generic orientation of a generic bipartite graph.*

For a cardinal $\kappa \geq 2$, let M_κ be a bipartite digraph (X, Y, E) with $|X| = \kappa = |Y|$ such that either $(X, Y, E \cap (X \times Y))$ or $(X, Y, E \cap (Y \times X))$ is a perfect matching and the other is the bipartite complement of a perfect matching. In particular, the underlying undirected bipartite graph is a complete bipartite graph.

We call a 2-partite digraph (X, Y, E) *generic* if its underlying undirected bipartite graph is a complete bipartite graph and if for all pairwise disjoint finite subsets $A_X, B_X \subseteq X$ and $A_Y, B_Y \subseteq Y$ there are vertices $y \in Y$ and $x \in X$ with $A_X \subseteq N^+(y)$ and $B_X \subseteq N^-(y)$ as well as $A_Y \subseteq N^+(x)$ and $B_Y \subseteq N^-(x)$. Similarly, we call a 2-partite digraph (X, Y, E) a *generic orientation of a generic bipartite graph* if for all pairwise disjoint finite subsets $A_X, B_X, C_X \subseteq X$ and $A_Y, B_Y, C_Y \subseteq Y$ there are vertices $y \in Y$ and $x \in X$ with $A_X \subseteq N^+(y)$, $B_X \subseteq N^-(y)$ and $C_X \subseteq y^\perp$ as well as with $A_Y \subseteq N^+(x)$, $B_Y \subseteq N^-(x)$ and

$C_Y \subseteq x^\perp$. It is easy to verify that its underlying undirected graph is a generic bipartite graph.

Note that standard back-and-forth arguments show that, up to isomorphism, there are a unique countable generic 2-partite digraph and a unique countable generic orientation of the (unique) countable generic bipartite graph.

It is worthwhile noting that by Theorem 3.6 the underlying undirected bipartite graph of a homogeneous 2-partite digraph is always homogeneous, which is false for arbitrary homogeneous digraphs and their underlying undirected graphs.

The fact that the listed 2-partite digraphs in Theorem 3.6 are homogeneous is already discussed in the previous section for case (i), while in case (ii) it is a consequence of the fact that the bipartite complement of a perfect matching is homogeneous. The cases (iii) and (iv) can be easily verified by the above mentioned back-and-forth argument. (This can also be applied if they are not countable to show that they are homogeneous.) Before we start with the remaining direction of the proof of Theorem 3.6, we show some lemmas.

Lemma 3.8. *Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph. If $N^+(v)$ and $N^-(v)$ are infinite and $v^\perp$ is finite for some $v \in VD$, then $v^\perp = \emptyset$.*

Proof. Let $x \in X$. First, let us suppose that $m := |x^\perp| = 1$. We note that any automorphism of D that fixes x must also fix the unique element $x_Y \in x^\perp$. Indeed, since D is homogeneous and each of the two sets $\{y_1, y_2\}$ and $\{y, x_Y\}$ induces a digraph without any edge, we can extend every isomorphism between them to an automorphism α of D and, if x' is the common predecessor of y_1 and y_2, then $x'\alpha$ is the common predecessor of y and x_Y. Let y be a successor of x. As $N^+(x)$ is infinite, we find two vertices y_1, y_2 in Y that have a common predecessor. Homogeneity then implies that the two vertices y and x_Y in Y have a common predecessor z. Let z' be a successor of x_Y. By homogeneity, we find an automorphism β of D that fixes x and maps z to z'. As mentioned above, β must fix x_Y as it fixes x. But we have $zx_Y \in E$ and $(x_Y z)\alpha = x_Y z' \in E$, which is impossible.

Now let us suppose that $|x^\perp| \geq 2$. By homogeneity and as m is finite, we find for any subset A of Y of cardinality m a vertex $a \in X$ with $a^\perp = A$. As Y is infinite, there are two subsets A_1, A_2 of Y of cardinality m with $|A_1 \cap A_2| = m-1$ and two such subsets B_1, B_2 with $|B_1 \cap B_2| = m-2$. Let $a_i, b_i \in X$ with $a_i^\perp = A_i$ and $b_i^\perp = B_i$, respectively. Then there is no automorphism of D that maps a_1 to b_1 and a_2 to b_2 even though D is homogeneous as the number of vertices that are not adjacent to a_1 and a_2 is larger than the corresponding number for b_1 and b_2. Analogous contradictions for any vertex in Y instead of $x \in X$ show the assertion. $\square$

Lemma 3.9. *Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph. If $N^+(v)$ and $N^-(v)$ are infinite and $v^\perp = \emptyset$ for all $v \in VD$, then D is a generic 2-partite digraph.*

Proof. It suffices to show that for any two disjoint finite subsets A and B of X we find a vertex $v \in Y$ with $A \subseteq N^+(v)$ and $B \subseteq N^-(v)$. Indeed, the corresponding property for subsets of Y then follows analogously. Note that we find for every $y \in Y$ two sets $A_y \subseteq N^+(y)$ and $B_y \subseteq N^-(y)$ with $|A| = |A_y|$ and $|B| = |B_y|$. As D is homogeneous and as $A \cup B$ and $A_y \cup B_y$ induce (empty) isomorphic finite subdigraphs of D, there exists an automorphism α of D that maps A_y to A and B_y to B. So $y\alpha$ is a vertex we are searching for. □

Lemma 3.10. *Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph. If the sets $N^+(v)$, $N^-(v)$, and $v^\perp$ are infinite for all $v \in VD$, then D is a generic orientation of a generic bipartite graph.*

Proof. Similarly to the proof of Lemma 3.9, it suffices to show that for any three pairwise disjoint finite subsets A, B, C of X we find a vertex $v \in Y$ with $A \subseteq N^-(v)$ and $B \subseteq N^+(v)$ and $C \subseteq v^\perp$. For every $y \in Y$, we find subsets $A_y \subseteq N^+(y)$ and $B_y \subseteq N^-(y)$ and $C_y \subseteq y^\perp$ with $|A| = |A_y|$ and $|B| = |B_y|$ and $|C| = |C_y|$. Note that each of the two sets $A \cup B \cup C$ and $A_y \cup B_y \cup C_y$ has no edge. Applying homogeneity, we find an automorphism α of D that maps A_y to A and B_y to B and C_y to C. So $y\alpha$ is a vertex that has the desired properties. □

Now we are able to prove Theorem 3.6.

Proof of Theorem 3.6. Let $D = (X, Y, E)$ be a homogeneous 2-partite digraph that is not bipartite. Then we find in X some vertex with a predecessor in Y and some vertex with a successor in Y. By homogeneity, we can map the first onto the second and conclude the existence of a vertex in X that has a predecessor and a successor in Y. Analogously, we obtain the same for some vertex of Y. By homogeneity, every vertex of D has predecessors and successors. In particular, we have $|X| \geq 2$ and $|Y| \geq 2$.

Let us suppose that two vertices $u, v \in X$ have the same successors, that is, $N^+(u) = N^+(v)$. By homogeneity, we can fix u and map v onto any vertex w of $X \setminus \{u\}$ by some automorphism of D and thus obtain $N^+(w) = N^+(u)$ for every $w \in X$. So no vertex in $N^+(u)$ has successors in X, which is impossible as we saw earlier. Hence, we have $N^+(u) \neq N^+(v)$ for each two distinct vertices $u, v \in X$. Analogously, the same holds for each two distinct vertices in Y and also for the set of predecessors of every two vertices either in X or in Y. Thus, we have shown

$$N^+(u) \neq N^+(v) \quad \text{and} \quad N^-(u) \neq N^-(v) \quad \text{for all } u \neq v \in X \tag{3.1}$$

and

$$N^+(u) \neq N^+(v) \quad \text{and} \quad N^-(u) \neq N^-(v) \quad \text{for all } u \neq v \in Y. \tag{3.2}$$

Let us assume that $n := |N^+(u)|$ is finite for some $u \in X$. Note that, for any subset A of Y of cardinality n, we find a vertex $a \in X$ with $N^+(a) = A$ by homogeneity. If $|Y| > n+1$ and $n \geq 2$, then we find two subsets of Y of cardinality n whose intersection has $n-1$ elements and two such sets whose intersection has $n-2$ elements. So we find two vertices in X with $n-1$ common successors and we also find two vertices in X with $n-2$ common successors. This is a contradiction to homogeneity, because we cannot map the first pair of vertices onto the second pair. Thus, we have either $n=1$ or $|Y| = n+1$. If $|Y| = n+1$, then we directly obtain $D \cong M_{n+1}$ since every vertex in X also has some predecessor in Y. So let us assume $n = 1$. If we have $1 < k \in \mathbb{N}$ for $k := |N^-(u)|$, then we obtain $D \cong M_{k+1}$, analogously. So let us assume that either $|N^-(u)| = 1$ or $N^-(u)$ is infinite. First, we consider the case that $N^-(u)$ is infinite. An empty set $u^\perp$ directly implies $D \cong M_{|Y|}$. So let us suppose $u^\perp \neq \emptyset$. Let u^+ be the unique vertex in $N^+(u)$. Since $u^\perp \neq \emptyset$, we find for some and hence by homogeneity for every vertex in Y some vertex in X it is not adjacent to. Let $w \in (u^+)^\perp$ and let $v \in N^+(u^+)$. By homogeneity, we find an automorphism α of D that fixes u and maps v to w. Since α fixes u, it must also fix u^+. But since $u^+v \in E$ and $(u^+v)\alpha = u^+w \notin E$, this is not possible. Hence, if $N^+(u)$ is finite, it remains to consider the case $n = 1 = k$. Due to (3.1), no two vertices of X have a common predecessor or a common successor. Thus, also every vertex in Y has precisely one predecessor and one successor. Let $v \in Y$ and $w \in X$ with $uv, vw \in E$. Then we can map the pair (u, w) onto any pair of distinct vertices of X, as D is homogeneous. Thus, for all $x \neq z \in X$, there exists $y \in Y$ with $xy, yz \in E$. This shows $|X| = 2$ as every vertex of D has precisely one successor. Hence, D is a directed cycle of length 4, which is isomorphic to M_2.

Analogous argumentations in the cases of finite $N^-(u)$, $N^+(v)$ or $N^-(v)$ with $u \in X$ and $v \in Y$ show that the only remaining case is that every vertex in D has infinite in- and infinite out-neighbourhood. Due to Lemma 3.8, we know that $|u^\perp|$ is either 0 or infinite and that $|v^\perp|$ is either 0 or infinite. Since $x^\perp \neq \emptyset$ if and only if $y^\perp \neq \emptyset$ for all $x \in X$ and $y \in Y$, the assertion follows from Lemmas 3.9 and 3.10. □

3.3 Homogeneous digraphs

In this section, we state Cherlin's classification of the countable homogeneous digraphs.

Theorem 3.11. [4, 5.1] *A countable digraph is homogeneous if and only if it is isomorphic to one of the following digraphs:*

(i) I_n *for some* $n \in \mathbb{N}^\infty$*;*

(ii) $T[I_n]$ *for some homogeneous tournament* $T \neq I_1$ *and some* $n \in \mathbb{N}^\infty$*;*

(iii) $I_n[T]$ *for some homogeneous tournament* $T \neq I_1$ *and some* $n \in \mathbb{N}^\infty$*;*

(iv) *the countable generic* $\mathcal{H}$*-free digraphs for some set* $\mathcal{H}$ *of finite tournaments;*

(v) *the countable generic* I_n*-free digraphs for some integer* $n \geq 3$*;*

(vi) $T^\wedge$ *for some tournament* $T \in \{I_1, C_3, \mathbb{Q}, T^\infty\}$*;*

(vii) *the countable generic* n*-partite digraph for some* $n \in \mathbb{N}^\infty$ *with* $n \geq 2$*;*

(viii) *the countable semi-generic* ω*-partite digraph;*

(ix) $S(3)$*;*

(x) *the countable generic partial order* $\mathcal{P}$*; or*

(xi) $\mathcal{P}(3)$. □

For Chapter 6, it is convenient to have a list of the finite homogeneous digraphs. This partial result of Theorem 3.11 is due to Lachlan.

Theorem 3.12. [28, Theorem 1] *A finite digraph is homogeneous if and only if it is isomorphic to one of the following digraphs:*

(i) C_4*;*

(ii) I_n *for some* $n \geq 1$*;*

(iii) $I_n[C_3]$ *for some* $n \geq 1$*;*

(iv) $C_3[I_n]$ *for some* $n \geq 1$*;*

(v) *the digraph* H. □

Furthermore, we state Lachlan's classification of the countable homogeneous tournaments.

Theorem 3.13 ([3, Theorem 3.6]). *A countable tournament is homogeneous if and only if it is the trivial tournament on one vertex, the directed triangle, the generic tournament on countably many vertices, the tournament $\mathbb{Q}$, or the tournament $\mathcal{P}$.* □

The generic tournament T^∞ that is the Fraïssé limit (see [32] for more on these limits) of all finite tournaments, so the unique homogeneous tournament that embeds all finite tournaments. The tournament $\mathbb{Q}$ has vertex set $\mathbb{Q}$ and edges xy if and only if $x < y$.

For a set $\mathcal{H}$ of finite tournaments, the countable *generic* $\mathcal{H}$-free digraph is the Fraïssé limit of the class of all finite $\mathcal{H}$-free digraph. Similarly, for $n \in \mathbb{N}$, the countable *generic* I_n-free digraph is the Fraïssé limit of the class of all finite I_n-free digraphs and the countable *generic* n-partite digraph is the Fraïssé limit of all orientations of finite complete n-partite graphs (where some partition classes may have no element).

The countable *semi-generic* ω-partite digraph is the Fraïssé limit of those finite complete ω-partite digraphs that have the additional property that

for each two pairs $(x_1, x_2), (y_1, y_2)$ from distinct classes, the number of edges from $\{x_1, x_2\}$ to $\{y_1, y_2\}$ is even. (3.3)

By $\mathcal{P}$ we denote the countable *generic* partial order, the Fraïssé limit of all finite partial orders. Every partial order P is in a canonical way a digraph: for two elements x, y of P we have $xy \in EP$ if and only if $x < y$. We call digraphs that are obtained from partial orders in this way also partial orders. Note that no partial order contains an induced 2-arc.

It remains to define the variant $\mathcal{P}(3)$ of $\mathcal{P}$. This digraph was first described in [4]. A subset X of $V\mathcal{P}$ is *dense* if for all $a, b \in V\mathcal{P}$ with $ab \in E\mathcal{P}$ there is a vertex $c \in X$ with $ac, cb \in E\mathcal{P}$. Let $\{P_0, P_1, P_2\}$ be a partition of $V\mathcal{P}$ into three dense sets. For this definition, let $x \perp y$ if x and y are not adjacent. Let $\mathbb{H} = (P_0, P_1, P_2)$ be the digraph on $V\mathcal{P}$ such that for all $x, y \in P_i$ we have

$$xy \in E\mathbb{H} \text{ if and only if } xy \in E\mathcal{P}$$

and such that for all $x \in P_i$ and $y \in P_{i+1}$ we have

$$\begin{aligned} xy \in E\mathbb{H} &\text{ if and only if } yx \in E\mathcal{P}, \\ yx \in E\mathbb{H} &\text{ if and only if } x \perp y \in E\mathcal{P}, \text{ and} \\ x \perp y \in E\mathbb{H} &\text{ if and only if } xy \in E\mathcal{P}. \end{aligned}$$

Let p be an element not in $V\mathcal{P}$. Then $\mathcal{P}(3)$ is the digraph on the vertex set

$V\mathcal{P} \cup \{p\}$ such that $(p^\perp, p^\rightarrow, p^\leftarrow) = \mathbb{H}$, where

$$\begin{aligned} p^\perp &:= V\mathcal{P} \setminus N(p), \\ p^\rightarrow &:= N^+(p), \text{ and} \\ p^\leftarrow &:= N^-(p). \end{aligned}$$

3.4 Previous results on connected-homogeneous digraphs

We cite the classification of locally finite connected C-homogeneous digraphs with precisely two ends that is due to Gray and Möller [14], whose paper was the starting point of the classification process of the C-homogeneous digraphs.

Theorem 3.14. [14, Theorem 6.2] *A connected locally finite digraph with precisely two ends is C-homogeneous if and only if it is isomorphic to $C_\infty[I_n]$ for some $n \in \mathbb{N}$.*

Note that it should be possible without too much effort to obtain this previous result with our methods from Chapter 5.

Chapter 4

Verification of C-homogeneity

Within this chapter, we verify that all digraphs that are listed in Theorem 1.1 are C-homogeneous.

It is straight forward to see that the digraphs $DL(\Delta)$ with Δ as specified in Theorem 1.1 are C-homogeneous. Let $D \cong M(\kappa, m)$ for an $m \in \mathbb{N}$ with $m \geq 2$ and a cardinal κ. Let $\mathcal{C}$ be a basic cut system of D. Let A and B be two connected induced finite and isomorphic subdigraphs of D and let φ be an isomorphism from A to B. Let us first consider the case that A contains no 2-arc. Then both A and B lie in a reachability digraph, each. Without loss of generality we may assume that they lie in the same reachability digraph Δ of D. But, as the reachability-digraphs are obviously C-homogeneous, it is straight forward to see that the isomorphism φ from A to B first extends to an automorphism of Δ and then also to an automorphism of D. So let us assume that A contains a 2-arc. Let us consider the case that A is a k-arc for some $k \geq 2$. Let A_1, A_2 be two induced subdigraphs of A that have one common vertex, are both connected, and whose union is A. Then both are shorter arcs and, by induction, we can extend both restrictions, $\varphi|_{A_1}$ and $\varphi|_{A_2}$, to automorphisms ψ_1, ψ_2 of D, respectively. Let S be a $\mathcal{C}$-separator that contains the common vertex of A_1 and A_2. There are two possibilities for S if $m \geq 3$, and one possibility if $m = 2$. If $m = 2$, then it is an immediate consequence that $S^{\psi_1} = S^{\psi_2}$ and that we can combine the two automorphisms to one that extends φ by setting $\varphi|_{K_i} = \psi_i|_{K_i}$, where K_i is the component of $D - S$ that contains vertices of A_i, and $\varphi|_S = \psi_1|_S$. So we assume that $m \geq 3$. We choose in this case S so that it lies in a common $\mathcal{C}$-block with an edge of A_1.

Let $S' = S^{\psi_1}$. As we had just two possibilities for the choice of S, the image of S under ψ_2 has to be S', too. In the same way as above, we can combine appropriate restrictions of ψ_1 and ψ_2 to an automorphism of D that extends φ.

Now let us assume that A is no k-arc. Then there is a $\mathcal{C}$-block X that contains two edges of A that have a common vertex. Let us first assume that X contains three edges of A. Then, since $\Delta \cong CP_\kappa$, we know that $X \cap A$ is connected. Thus, $(X \cap A)\varphi$ lies in a $\mathcal{C}$-block Y of B and we have $(X \cap A)\varphi = B \cap Y$. We have already shown that we can extend $\varphi|_{A\cap X}$ to an automorphism ψ_X of D. If each component of $D - X$ contains at most one component of A, then we have the extensions of the restriction of φ to these components and we can construct, as in the case of k-arcs, an automorphism of D. So we assume that there is at least one component C of $D - X$ such that, for the $\mathcal{C}$-separator $S \subseteq X$ that separates X and C, the digraph $A' = A \cap (C \cup S)$ contains at least two components. As the $\mathcal{C}$-separators have cardinality 2, A' consists of precisely two components. Let $Z \neq X$ be the second $\mathcal{C}$-block that contains S. If Z contains edges, that means $m = 2$, then $A \cap Z$ consists of precisely two edges that have their other incident vertices again in a common separator. Since the same must be true for $Z^{\psi_X} \cap B$, we may assume inductively that we have extended $\varphi|_{A\cap X}$ so that ψ_X coincides with $\varphi|_{Z\cap A}$ on $Z \cap A$. Thus, we can consider the case that Z does not contain any edge. There is an enumeration $z_1, \ldots, z_m$ of the vertices of Z such that $\{z_m, z_1\}$ and for all $i \leq m$ also $\{z_i, z_{i+1}\}$ are all the $\mathcal{C}$-separators in X. We may assume that $S = \{z_1, z_m\}$. Let C_i be the subdigraph induced by z_i, z_{i+1} and that component of $D - \{z_i, z_{i+1}\}$ that contains no other z_j. If $C_i \cap A$ consists of one component and contains z_i and z_{i+1}, then we can extend the restriction of φ to that component to an automorphism ψ_i of D and we may suppose that we have chosen ψ_X so that they are equal on C_i. If there is one C_i that has at least two components of $A \cap C_i$, then it is unique and we can suppose that $\psi_X|_{C_i} = \psi_i|_{C_i}$ on all C_i such that $A \cap C_i$ is connected. By induction, we can assume that the same holds also for a component C_i such that $A \cap C_i$ is not connected. So the only remaining case is if $C_i \cap A$ is connected but contains only one of the vertices z_i, z_{i+1}. But in this case we know that this situation occurs in at most one other C_j with $i \neq j$. Then $\varphi|_{A\cap C_k}$ with $k \in \{i, j\}$ extends to an automorphism ψ_k of D by induction. Because these two automorphisms exist, we know that $S_i^{\psi_k}$ contains only one vertex of B, and hence we can assume that ψ_X and ψ_k coincide on C_k. Thus, if we extend this to all the components of $D - X$, we know that ψ_X extends φ.

The final case that remains is when the block X contains only two edges. Then it might be the case that $X \cap A$ is not connected. If it is not connected, then there has to be a $\mathcal{C}$-block that contains at least three edges, so we assume that $X \cap A$ is connected. If, for the $\mathcal{C}$-block Y that contains $(X \cap A)\varphi$, we have that $Y \cap B$ is connected, then we can construct an automorphism that extends φ as in the case where X contained three edges of A. On the other hand, if $Y \cap B$ is not connected, there has to be a $\mathcal{C}$-block that contains three edges of B, and the same must be true for a $\mathcal{C}$-block and A. Since we know that in this case there is an automorphism of D that extends φ, we have proved that $M(\kappa, m)$ is C-homogeneous.

In the case that $D \cong M'(2m)$ for an $m \in \mathbb{N}$ the arguments used are analog ones as in the case $D \cong M(\kappa, m)$ and therefore we omit that proof here.

We will postpone the proof for the quotients of $X_2(C_3)$ to the proof of Theorem 6.16.

That $H[I_n]$ is C-homogeneous, follows from the fact that H is homogeneous. Obviously, C_m is C-homogeneous for all $m \geq 3$ (even for $m = \infty$), so the same is true for $C_m[I_n]$ as its reachability digraph is a complete bipartite digraph.

Next, we consider the digraphs Y_k with $k \geq 3$. Let A and B be two isomorphic connected induced subdigraphs of $D := Y_k$. Let V_1, V_2, V_3 be the three vertex sets as in the proof of Lemma 6.14 and let $\Delta_1, \Delta_2, \Delta_3$ be the corresponding reachability digraphs such that $\Delta_i = D[V_i \cup V_{i+1}]$ with $V_4 = V_1$. Let α be an isomorphism from A to B. It is straightforward to see that $(VA \cap V_i)\alpha$ is precisely the intersection of VB with some V_j: consider an undirected path between two vertices of VA and subtract from the number of forward directed edges on that path the number of backward directed edges. The resulting number is divisible by 3 if and only if the end vertices of the path lie in the same V_i. Hence, we may assume that $(VA \cap V_i)\alpha = VB \cap V_i$ for all $i \leq 3$. Let us first assume that $\Delta_i \cap A$ is connected for some $i \leq 3$, say for $i = 1$. Let Δ_1' be a minimal subdigraph of Δ_1 isomorphic to some CP_ℓ with $\ell \leq k$ such that $A \cap \Delta_1 = A \cap \Delta_1'$. By replacing B by $B\gamma$, for an automorphism γ of D, we may assume that also $B \cap \Delta_1 = B \cap \Delta_1'$ holds. Since $G(CP_\ell)$ is a C-homogeneous bipartite graph, we can extend every isomorphism from $\Delta_1' \cap A$ to $\Delta_1' \cap B$, in particular the restriction of α, to an automorphism of Δ_1'. Let α' be the automorphism of Δ_1' that extends the above restriction of α. Let $V_3' \subseteq V_3$ be the set of those vertices that are non-adjacent to at least one vertex of Δ_1'. As each vertex in V_3' is uniquely determined by two non-adjacent vertices one of which lies in $V_1 \cap V\Delta_1'$ and the other in $V_2 \cap V\Delta_1'$, the isomorphism α' has precisely one extension β on $D' := D[V\Delta_1' \cup V_3']$. By the construction of β it is easy to see that the restriction of α to $A \cap D'$ is again an isomorphism from $A \cap D'$ to $B \cap D'$ and is equal to the restriction of β to $A \cap D'$. Since all vertices of $A \cap (V_3 \setminus V_3')$ are adjacent to all vertices of $A \cap (V_1 \cup V_2)$ and since the same holds for B instead of A, the isomorphism β can be extended to an automorphism of D whose restriction to A is α.

If no $\Delta_i \cap A$ is connected, then we have $|V_i \cap VA| \leq 2$ for all $i \leq 3$. In particular, we have $|VA| \leq 6$. As $|VA| \leq 4$ also leads to some connected $\Delta_i \cap A$, we have $5 \leq |VA| \leq 6$. Hence, we may assume that $|VA \cap V_1| = 2 = |VA \cap V_2|$ and $|VA \cap V_3| \in \{1, 2\}$. As $\Delta_1 \cap A$ is not connected, it is a perfect matching. Either the same holds for $\Delta_2 \cap A$ and $\Delta_3 \cap A$ and we conclude that $A \cong C_6$, or $|V_3 \cap VA| = 1$ and A is a directed path of length 4. In both cases it is easy to verify that α extends to an automorphism of D.

The only remaining digraphs of Theorem 1.1 we have to consider are the digraphs $\mathcal{R}_m$ and the generic orientation of the countable generic bipartite graph. Whereas the latter is a direct consequence of the fact, that it is a homogeneous 2-partite digraph that has an automorphism that switches its partition sets, we only have to consider the digraphs $\mathcal{R}_m$. But this is an easy consequence of the fact that the subdigraph of $\mathcal{R}_m$ induced by $V_i \cup V_{i+1}$ is the countable homogeneous bipartite digraph and that two vertices in finite induced subdigraphs lie in the same set V_i if and only if any path between them has the same number of forward and backward directed edges modulo m.

Chapter 5

The case: more than one end

This chapter deals with infinite connected C-homogeneous digraphs with more than one end, that need not be countable. The chapter is structured as follows. First, we determine the infinite connected C-homogeneous digraphs with infinitely many ends whose underlying undirected graph is a C-homogeneous graph those of *Type I* (Section 5.2), followed by those with infinitely many ends whose underlying undirected graph is not a C-homogeneous graph that are the digraphs of *Type II* (Section 5.3).

It is well known that a transitive connected locally finite graph either contains one, two, or infinitely many ends. For arbitrary transitive connected infinite graphs, this was proved by Diestel, Jung and Möller [6]. Since the underlying undirected graph of a transitive digraph is also transitive, the same holds for infinite transitive digraphs. As two-ended connected transitive graphs are locally finite [6, Theorem 7] we refer to Gray and Möller [14, Theorem 6.2] for the classification result in the case of two-ended C-homogeneous digraphs. Consequently, we complete in this chapter the classification of the locally finite C-homogeneous digraphs and of the connected C-homogeneous digraphs with more than one end.

Due to [19], the only connected C-homogeneous graphs with more than one end are the graphs $X_{\kappa,\lambda}$ where κ and λ are cardinals larger than 1. These are the graphs in which every vertex is a cut vertex and lie in λ distinct blocks all of which are complete graphs on κ vertices.

5.1 Local structure of C-homogeneous digraphs of Type I and Type II

In this section we summarize some preliminary results of the relation between a C-homogeneous digraph and a basic cut system $\mathcal{C}$ of this digraph. In particular we investigate the local structure around $\mathcal{C}$-separators.

Lemma 5.1. *Let D be a connected C-homogeneous digraph with more than one end. Let $\mathcal{C}$ be a basic cut system and let S be a $\mathcal{C}$-separator. Then there is no edge xy in D with both vertices in S. In particular, no two $\mathcal{C}$-blocks can share an edge.*

Proof. Let $(A, B) \in \mathcal{C}$ with $A \cap B = S$ and let us suppose that there is $xy \in ED$ with $x, y \in S$. By the minimality of $\mathcal{C}$ each vertex in a $\mathcal{C}$-separator has neighbours in both wings of the corresponding separation. Let $a \in A \setminus B$ and $b \in B \setminus A$ be such neighbours of y. Then there are different possibilities for the direction of their connecting edges. Let us first consider the case that $ay, by \in ED$. Then there is an automorphism α that maps xy onto by. Then $S\alpha$ lies in B, since $\mathcal{C}$ is nested and $b \in S\alpha$, and we have either $A \subseteq A\alpha$ or $A \subseteq B\alpha$ by the nestedness of $\mathcal{C}$. So we have a vertex b_0, which is either $a\alpha$ or $b\alpha$, that lies either in $B \cap B\alpha$ or in $B \cap A\alpha$ such that $b_0 y \in ED$ and $S_0 := S\alpha$ separates a and b_0. Let $\{A_0, B_0\} = \{A\alpha, B\alpha\}$ such that $a \in A_0$ and $b_0 \in B_0$.

Now let α_0 be an automorphism of D such that $a^{\alpha_0} = a$ and $(by)^{\alpha_0} = b_0 y$. Hence the vertex $b_1 := b_0^{\alpha_0}$ lies in $B_1 \setminus A_1$, where $A_1 := A_0^{\alpha_0}$ and $B_1 := B_0^{\alpha_0}$, and we have $b_1 y \in ED$. Since A_0 meets A_1, $S_1 := S_0^{\alpha_0} \neq S_0$ lies in B_0, and $\mathcal{C}$ is nested, we know that A_0 is a proper subset of A_1, B_1 is a proper subset of B_0 and S_0 lies in A_1, which implies $x \in A_1$. Furthermore b_1 and a are separated from each other by both S_0 and S_1. By repeating this process recursively, we have an α_i that fixes a and y and maps b_{i-1} onto b_i and we get a further vertex $b_{i+1} = b_i^{\alpha_i} \in B_{i+1} \setminus A_{i+1}$ that is separated by $S_{i+1} := S_i^{\alpha_i}$ from $a \in A_{i+1} \setminus B_{i+1}$. And with the same argument as before we have that A_i is a proper subset of $A_{i+1} := A_i^{\alpha_i}$, that $B_{i+1} := B_i^{\alpha_i}$ is a proper subset of B_i and that $b_i \in A_{i+1}$. Hence, $b_j \in A_{i+1}$ for all $j \leq i$, which implies $b_i \neq b_j$ for all $i \neq j$.

Thus, after the step $m := |S|+1$ there has to be some $k < |S|$ such that b_k is not contained in S_m and therefore lies in $A_m \setminus B_m$. That is, b_k is also separated from b_m by S_m and $I := \{a, b_k, b_m\}$ forms an independent set. Note that by construction all elements in I have y as a common out-neighbour. Hence, due to the C-homogeneity of D, there is an automorphism β of D which interchanges a and b_k and fixes y and b_m. Note that $a \in A \setminus B \subset A_{k+1} \setminus B_{k+1}$ and $b_m \in B_m \setminus A_m \subset B_{k+1} \setminus A_{k+1}$. Thus, S_{k+1} is a separator containing b_k that separates a and b_m, which implies that $S_{k+1}\beta$ is a separator containing a that separates

b_k and b_m. But due to the minimality of $\mathcal{C}$, there is a b_k-b_m-path in B_{k+1} that meets S_{k+1} only in b_k and therefore $S_{k+1}\beta$ meets both $A_{k+1} \setminus B_{k+1}$ and $B_{k+1} \setminus A_{k+1}$, contradicting the nestedness of $\mathcal{C}$ (confer Remark 2.5).

So let us suppose $by, ya \in ED$. Let α be an automorphism of D with $(xy)\alpha = ya$ and choose $\{X, Y\} = \{A\alpha, B\alpha\}$ such that $b \in X \setminus Y$. Then there is a neighbour c of y in $Y \setminus X$, which is separated from b by $S\alpha$. Note that by nestedness, and since both X and Y meet A, we have that either $X \cap B$ or $Y \cap B$ is empty. So $b \in X \cap B$ yields $c \in A$. If $cy \in ED$ then we may take the vertices c, b instead of a, b and get a contradiction by the first case above. Thus we may assume that $yc \in ED$. But then we can map the digraph $D[b, y, a]$ onto $D[b, y, c]$ such that the number of separators that separate b from a without containing one of them equals the number of separators that separate b and c without containing one of them. Due to [7, Lemma 4.1] this number is finite. Because of the nestedness of $\mathcal{C}$, every separator that separates the vertices b and a lies entirely in X, and since a and c are joined by a path that lies except for a in $Y \setminus X$, it also separates b and c. But $S\alpha$ contains a and separates b from c, a contradiction. The case $ay, yb \in ED$ is analogous.

Let us finally suppose that $ya, yb \in ED$. By considering the digraph D^{-1} instead of D we also may assume that there are $a' \in A \setminus B$ and $b' \in B \setminus A$ with $a'x, b'x \in ED$. Let α be an automorphism of D with $(xy)\alpha = yb$. Then there is a vertex $b'' \in B \setminus A$ that is separated by $S\alpha$ from a and such that $b''y \in ED$. But then we have the situation of the previous case and thus we know that no such edge xy exists.

Let X and Y be distinct $\mathcal{C}$-blocks, then there is $x \in X \setminus Y$. Since Y, being a $\mathcal{C}$-block, is maximally $\mathcal{C}$-inseparable, there is a $\mathcal{C}$-separator S that separates x from Y. As X is also $\mathcal{C}$-inseparable, we have $X \cap Y \subseteq S$. Therefore X and Y cannot share an edge. □

Lemma 5.2. *Let D be a connected C-homogeneous digraph with more than one end and let $\mathcal{C}$ be a basic cut system. Then for each 2-arc P in D we have $|P \cap S| \leq 1$ for all $\mathcal{C}$-separators S.*

Proof. Let $P = xay$ be a 2-arc in D and S a $\mathcal{C}$-separator. By Lemma 5.1 we only have to show that S cannot contain both x and y. So assume $\{x, y\} \subseteq S$. Let $(A, B) \in \mathcal{C}$ with $A \cap B = S$ and $a \in A$. Since D is transitive there is an arc zx in D. If z lies in A consider a neighbour z' of x in B. Now either zxa, zxz' or $z'xa$ is an induced 2-arc in D, which we denote by Q, with one vertex in $A \setminus B$ and one vertex in $B \setminus A$. Because D is connected-homogeneous there is an automorphism α with $P\alpha = Q$. Then $S\alpha$ contains vertices of both wings of (A, B). By Remark 2.5, this contradicts the nestedness of $\mathcal{C}$. □

Lemma 5.3. *Let D be a connected C-homogeneous digraph with more than one end, let $\mathcal{C}$ be a basic cut system of D, and let S be a $\mathcal{C}$-separator. Then there is no directed path in D with both endvertices in S.*

Proof. Suppose that there is such a path P. We may choose the path such that it has minimal length. Then all of the vertices of P lie in the same $\mathcal{C}$-block X. By Lemma 5.2 the endvertices of any directed path of length 2 are separated by a $\mathcal{C}$-separator. Hence no directed path of length at least 2 can lie in any $\mathcal{C}$-block. □

Lemma 5.4. *Let D be a connected C-homogeneous triangle-free digraph with more than one end, and let $\mathcal{C}$ be a basic cut system. Then for any cut $(A, B) \in \mathcal{C}$ there is no path xyz in $D[A]$ with $y \in A \cap B$.*

Proof. By Lemma 5.1 we only have to show that given a cut $(A, B) \in \mathcal{C}$ there is no 2-arc xyz in D such that $y \in S := A \cap B$ and $x, z \in A \setminus B$. So let us suppose there is such a path. Then y has a neighbour $b \in B \setminus A$. We may assume that their connecting edge is pointing towards y, since otherwise changing the direction of each edge gives a digraph D' which is C-homogeneous and has this property.

Suppose that there is a second neighbour $c \in B \setminus A$ of y. If $yc \in ED$, then there is an $\alpha \in \mathrm{Aut}(D)$ that fixes b, y, z and with $x\alpha = c$, $c\alpha = x$, as D is triangle-free. But then the separations (A, B) and $(A\alpha, B\alpha)$ are not nested. Thus we may assume that $cy \in ED$. In this situation let β be an automorphism of D that fixes x, y, b and maps z onto c and vice versa – a contradiction as before.

So b is the unique neighbour of y in B. We may assume that there is another vertex a, say, that lies in S, since otherwise we could map the 2-arc byz onto xyz, as D is C-homogeneous and triangle-free, and, thus, y would seperate x from z, contradicting the fact that x and z lie in the same component of $D - S$. Now consider a path P in D connecting a and y and let $\mathcal{T}$ denote the structure tree of D and $\mathcal{C}$. Let $\mathcal{M}$ be the set of $\mathcal{C}$-blocks containing edges of P. Since $\mathcal{C}$-separators do not contain any edge, distinct blocks cannot contain a common edge. Thus we choose a block $M \in \mathcal{M}$ whose distance to S in $\mathcal{T}$ is maximal with respect to $\mathcal{M}$.

Now each nontrivial component of $P \cap M$ has to contain exactly two edges: An isolated edge would either be contained in a separator, in contradiction to Lemma 5.1, or it would connect M to two distinct neighbours in $\mathcal{T} \cap \mathcal{M}$, contradicting the choice of M. If there is a segment of P in M with a length of at least three, then it contains either a directed subsegment, isomorphic to byz, or a subsegment isomorphic to $by \cup xy$. In each case there exists an isomorphism

φ such that $S\varphi$ separates the endvertices of this subsegment, which is impossible since M is a $\mathcal{C}$-block.

Considering an arbitrary nontrivial component of $P \cap M$, its two edges have a common vertex which we denote by m. With an analogous argument as above, both edges are directed away from m. Let us denote their heads by u and v, respectively. By construction, u and v lie both in the separator $S_M \subset M$ that lies on the unique shortest path between M and S in $\mathcal{T}$. Consider an arbitrary cut with seperator S_M. Then u has a neighbour u' in the wing not containing m. Let ψ be an automorphism with $(mu)\psi = by$ and either $(uu')\psi = yz$, if $uu' \in ED$ or $(u'u)\psi = xy$, if $u'u \in ED$. Since $\mathcal{C}$ is nested we have $S_M\psi \subset B$ which means that x and z are separated from b by $S_M\psi$. By relabeling $S := S_M\psi$ and $a := v\psi$, if neccessary, we may assume that ba is an edge.

Then there is a neighbour z' of b in $B \setminus A$, and we can find an automorphism γ with $(by)\gamma = ba$ and either $x\gamma = z'$ or $z\gamma = z'$, depending on the orientation of the edge between b and z'. Again by the nestedness of $\mathcal{C}$ we have $S\gamma \subset B$ and also $B\gamma \subseteq B$. And since x is separated from b by $S\gamma$ we have $y \in S\gamma$. But that implies that y and a both have b as their unique neighbour in $B\gamma$. Hence, $S\gamma \setminus \{y, a\} \cup \{b\}$ is a seperator in D that seperates ends and has smaller cardinality, contradicting the fact that $\mathcal{C}$ is basic. ☐

Lemma 5.5. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Let S be a $\mathcal{C}$-separator and let $s \in S$. Then there is precisely one $\mathcal{C}$-block that contains s and all edges directed away from s, and there is precisely one $\mathcal{C}$-block that contains s and all edges directed towards s. Furthermore there is $d^+(s) > 1$ and $d^-(s) > 1$.*

Proof. By Lemma 5.4 there is at most one kind of neighbours in each $\mathcal{C}$-block. Suppose first that there is a $\mathcal{C}$-block Z with only one neighbour a of s. We may assume that $as \in ED$. By C-homogeneity, we can map each edge xs onto as. As there is by Lemma 5.1 precisely one $\mathcal{C}$-block Y that contains xs, Y contains no other neighbour of s, because the same holds for s and Z. Thus each component of each $\mathcal{C}$-block is either a single vertex or a star the edges of which are directed towards the leaves of the star. If each $\mathcal{C}$-block is a tree and every $\mathcal{C}$-separator consists of one vertex, then the digraph D has to be a tree. Since we excluded this case, there is a second vertex $t \in S$. For every component C of $D - S$, there is an (undirected) s-t-path P with all its vertices but s and t in C. Let X be a $\mathcal{C}$-block with maximal distance to S in the structure tree of G and $\mathcal{C}$ such that there are at least two edges from P in X. This $\mathcal{C}$-block exists by Lemma 2.9. As each component of X that contains edges is a star, the longest subpath of P that lies completely in X has length 2. Let xyz be such a subpath. Then due to

Lemma 5.2 we have $xy, zy \in ED$ and y is the only X-neighbour of both x and z. Let S' be the $\mathcal{C}$-separator in X that separates X from S. Then, S' contains x and z. But, as in the previous lemma, $S' \setminus \{x, z\} \cup \{y\}$ would be a separator of smaller cardinality separating two ends, a contradiction.

Thus a $\mathcal{C}$-block cannot contain s together with a single neighbour of s and by C-homogeneity there has to be one $\mathcal{C}$-block that contains all in-neighbours of s and one that contains all out-neighbours of s. □

Lemma 5.6. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Then each $\mathcal{C}$-separator has degree two in the structure tree $\mathcal{T}$ for D and $\mathcal{C}$.*

Proof. Let S be a C-separator. Then for each component X of $\mathcal{T} - S$ the vertex set $(\bigcup X) \setminus S$ is the union of components of $D - S$. Since each $s \in S$ has a neighbour in each component of $D - S$, it also has at least one neighbour in each component of $\mathcal{T} - S$. With Lemma 5.5 we have $d_{\mathcal{T}}(S) = 2$. □

If we combine Lemma 5.5 and Lemma 5.6 we get the following

Corollary 5.7. *Let D be a connected C-homogeneous triangle-free digraph that is not a tree and that has more than one end, and let $\mathcal{C}$ be a basic cut system of D. Let B be a $\mathcal{C}$-block, $S \subset B$ a $\mathcal{C}$-separator and $s \in S$. If s has no neighbour in B, then there is exactly one $\mathcal{C}$-separator $S' \subset B$ such that $s \in S' \cap S$. If s has a neighbour in B, then S is the only $\mathcal{C}$-separator in B that contains s.* □

Lemma 5.8. *Let D be a connected C-homogeneous digraph with more than one end that embeds a triangle, and let $\mathcal{C}$ be a basic cut system of D. Then every $\mathcal{C}$-block that contains edges is a tournament and D has connectivity* 1.

Proof. Let S be a $\mathcal{C}$-separator and let $x \in S$. Then x has adjacent vertices in both wings of each cut $(A, B) \in \mathcal{C}$ with $A \cap B = S$. As D contains triangles, each edge lies on a triangle. We know that each wing of (A, B) contains both an in- and an out-neighbour of x, as any triangle contains a 2-arc and D is edge-transitive. Thus every induced path of length 2 in D can be mapped on a path crossing S, i.e. a path both end vertices of which lie in distinct wings of (A, B). Hence no two vertices in the same $\mathcal{C}$-block can have distance 2 from each other and, in particular, every component of every $\mathcal{C}$-block has diameter 1.

To prove that each $\mathcal{C}$-block has diameter 1 we just have to show that each $\mathcal{C}$-block is connected. So let us suppose that this is not the case. Let X be a $\mathcal{C}$-block and let P be a minimal (undirected) path in D from one component of X to another. Let Y be a $\mathcal{C}$-block with maximal distance in the structure tree of D and $\mathcal{C}$ to X that contains edges of P. By Lemma 5.1 the block Y has to contain at least two edges and there are two non-adjacent vertices in the same component of Y. This contradicts the fact that these components are complete graphs. Hence each $\mathcal{C}$-block that contains edges has precisely one component which has diameter 1.

For any $\mathcal{C}$-block X, there is a $\mathcal{C}$-separator S with $S \subseteq X$. By Lemma 5.1, S contains no edge and thus precisely one vertex. □

5.2 C-homogeneous digraphs of Type I

In this section we shall completely classify the countable connected C-homogeneous digraphs of Type I with more than one end and give – apart from the classification of infinite uncountable homogeneous tournaments – a classification of uncountable such digraphs. As a part of the countable classification we apply the classification of Lachlan [29], see also [3], of the countable homogeneous tournaments (see Theorem 3.13).

The underlying undirected graph of a digraph $X_\lambda(T)$ for a homogeneous tournament T is some $X_{\kappa,\lambda}$, which is a distance-transitive graph[1] as described in [19, 31, 34]. Thus, if a digraph $X_\lambda(T)$ is C-homogeneous, then so is its underlying undirected graph.

Theorem 5.9. *Let D be a connected digraph with more than one end. Then D is C-homogeneous of Type I if and only if one of the following statements holds:*

(1) *D is a tree with constant in- and out-degree;*

(2) *D is isomorphic to a $X_\lambda(T^\kappa)$, where κ and λ are cardinals with $\lambda \geq 2$ and κ either 3 or infinite and T^κ is a homogeneous tournament on κ vertices.*

Proof. Let us first assume that D is a C-homogeneous digraph of Type I. Then the underlying undirected graph is isomorphic to a $X_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$. If $\kappa = 2$, then D is a tree with constant in- and out-degree, so we may assume $\kappa \geq 3$. As each block is a complete digraph, it is homogeneous and, thus, we conclude from Theorem 3.13 that the cardinal κ has to be either 3 or infinite. This proves the necessity-part of the statement.

Since the digraphs of part (1) are obviously C-homogeneous of Type I, we just have to assume for the remaining part that D is isomorphic to $X_\lambda(T^\kappa)$ for a cardinal $\lambda \geq 2$ and a homogeneous tournament T^κ on κ vertices for a cardinal κ that is either 3 or infinite. Let $\mathcal{C}$ be a basic cut system of D. Let X and Y be two connected induced finite and isomorphic subdigraphs of D. Let φ be the isomorphism from X to Y. If X has no cut vertex, then X lies in a subgraph

[1] A graph G is called *distance-transitive* if for each two pair (x_1, x_2) and (y_1, y_2) of vertices with $d(x_1, x_2) = d(y_1, y_2)$ there is an automorphism α of G with $x_i\alpha = y_i$.

of D that is a homogeneous tournament and the same is true for Y, so φ extends to an automorphism of D. So let $x \in VX$ be a cut vertex of X. Hence $x\varphi$ is a cut vertex of Y. It is straight forward to see that for any $\mathcal{C}$-block B the image of $X \cap B$ in Y is precisely the intersection of Y with a $\mathcal{C}$-block A. Since the $\mathcal{C}$-blocks are all isomorphic homogeneous tournaments, the isomorphism from $X \cap B$ to $Y \cap A$ extends to an isomorphism from X to Y. Thus the isomorphism from X to Y easily extends to an automorphism of D. Since the underlying undirected graph is C-homogeneous (see [13]), D is C-homogeneous of Type I. $\square$

Lachlan's theorem together with Theorem 5.9 enables us to give a complete classification of countable connected C-homogeneous digraphs of Type I and with more than one end:

Corollary 5.10. *Let D be a countable connected digraph with more than one end. Then D is C-homogeneous of Type I if and only if one of the following assertions holds:*

(1) *D is a tree with constant countable in- and out-degree;*

(2) *D is isomorphic to a $X_\lambda(Y)$, where κ is a countable cardinal greater or equal to 2 and Y is one of the four non-trivial homogeneous tournaments of Theorem 3.13.* $\square$

5.3 C-homogeneous digraphs of Type II

5.3.1 Reachability and descendant digraphs

In this subsection we prove that, if a connected C-homogeneous digraph D with more than one end contains no triangles, then D is highly-arc-transitive, each reachability digraph of D is bipartite, and, if furthermore D has infinitely many ends, then the descendants of each vertex in D induce a tree. All these properties were proved to be true in the case that D is locally finite, see [14, Theorem 4.1].

Theorem 5.11. *Let D be a connected C-homogeneous triangle-free digraph with more than one end. Then D is highly-arc-transitive.*

Proof. Let $\mathcal{C}$ be a basic cut system. It suffices to show that each directed path is induced. Suppose this is not the case. Then there is a smallest k such that there is a k-arc $A = x_0 \ldots x_k$ that is not induced. Hence there is an edge between x_0 and x_k. Consider a $\mathcal{C}$-separator S that contains x_1. By Lemma 5.3 we have $x_k \notin S$ and by Lemma 5.1 we have $x_0 \notin S$; hence x_0 and x_k lie on the same side of S. But then the same holds for x_{k-1} and so on. So finally x_0 and x_2 have to lie on the same side of S, in contradiction to Lemma 5.4. □

Theorem 5.12. *Let D be a connected C-homogeneous triangle-free digraph with more than one end. Then $\Delta(D)$ is bipartite and if D is not a tree, then each Δ_e with $e \in ED$ is a component of a $\mathcal{C}$-block. Furthermore, if D has infinitely many ends, then every descendant digraph* $\mathrm{desc}(x)$ *with $x \in VD$ is a tree.*

Proof. Let $\mathcal{C}$ be a basic cut system. We first show that either D is a tree or any Δ_e with $e \in ED$ is a component of a $\mathcal{C}$-block. Let us assume that D is not a tree. Lemma 5.5 immediately implies that Δ_e for any $e \in ED$, cannot be separated by any $\mathcal{C}$-separator and, thus, each Δ_e lies in a $\mathcal{C}$-block. As there are induced paths of length 2 crossing some $\mathcal{C}$-separator and as D contains no triangle, a component of a $\mathcal{C}$-block X cannot contain more vertices than Δ_e with $e \in E(D[X])$ contains. Thus Δ_e is a component of a $\mathcal{C}$-block.

Suppose that $\Delta(D)$ is not bipartite. Then there is a cycle of odd length in $\Delta(D)$. Thus there has to be a directed path of length at least 2 on that cycle. By Lemma 5.2 this path lies in distinct $\mathcal{C}$-blocks. This is not possible as shown above and thus $\Delta(D)$ has to be bipartite.

Now suppose that there is $x \in VD$ such that $\mathrm{desc}(x)$ contains a cycle. So by transitivity there is a descendant y of x such that there are two x-y-arcs that are apart from x and y totally disjoint. Thus, since we are C-homogeneous, any two out-neighbours of x have a common descendant. Assume that there are two distinct $\mathcal{C}$-separators S, S' such that both $Y := S \setminus S'$ and $Y' := S' \setminus S$ contain an out-neighbour of x. Then it exists a vertex z in D with Y-z- and Y'-z-arcs. But by the Lemmas 5.3 and 5.4 the vertices x and z cannot lie on the same side of S and S', respectively, hence S and S' meet on both sides, a contradiction to the nestedness of $\mathcal{C}$. Thus there is a $\mathcal{C}$-separator S_{+1} that contains the whole out-neighbourhood of x. This implies that all descendants of distance k are contained in a common $\mathcal{C}$-separator S_{+k}, since either all distinct k-arcs originated at x are disjoint, and we can apply the same argument as above, or each two of those k-arcs intersect in a vertex x' in D that has the same distance to x on both arcs by Lemma 5.3, and we are home by induction.

With a symmetric argument we get that each k-arc that ends in x has to start in a common $\mathcal{C}$-separator S_{-k}. For a path P in D that starts in x, let $\sigma(P)$ denote the difference of the number of edges in P that are directed away from x (with respect to P) minus the number of edges of the other type. Then one easily checks that the endvertex of P lies in $S_{\sigma(P)}$. Since all $\mathcal{C}$-separators have the same finite order s, say, there can be at most $2s$ rays that are eventually pairwise disjoint. Hence D has finitely many ends, which proves the last statement of the theorem. □

Lemma 5.13. *Let D be a connected C-homogeneous triangle-free digraph with more than one end and let $\mathcal{C}$ be a basic cut system of D. Then for each $\mathcal{C}$-separator S of order at least 2 there is a reachability digraph Δ_e and a $\mathcal{C}$-block K such that $|S \cap \Delta_e| \geq 2$, $\Delta_e \subseteq K$, and $S \subseteq K$.*

Proof. Let S be a $\mathcal{C}$-separator with $|S| \geq 2$. Suppose that there is no reachability digraph Δ_e with $|S \cap \Delta_e| \geq 2$. Let $x, y \in S$ and let P be an x-y-path in a component of $D - S$. Let B be a $\mathcal{C}$-block that contains edges of P and such that $d_{\mathcal{T}}(S, B)$ is maximal with this property. Then the $\mathcal{C}$-separator $S_B \subseteq B$ that separates S and B in $\mathcal{T}$ has the desired property and thus each $\mathcal{C}$-separator has it, in contradiction to the assumption. □

We have roughly described the global structure of C-homogeneous digraphs. To investigate the local structure of these graphs, we show that the underlying undirected graph of each reachability digraph is a connected C-homogeneous bipartite graph. Such graphs were already described in Section 3.1.

Lemma 5.14. *Let D be a connected C-homogeneous digraph such that $\Delta(D)$ is bipartite. Then the underlying undirected graph of $\Delta(D)$ is a connected C-homogeneous bipartite graph.* □

5.3.2 The classification

As a first result we prove that no connected C-homogeneous digraph of Type II with more than one end contains any triangle.

Lemma 5.15. *Let D be a connected C-homogeneous digraph of Type II with more than one end. Then D contains no triangle.*

Proof. Let $\mathcal{C}$ be a basic cut system and suppose that D contains a triangle. By Lemma 5.8, every $\mathcal{C}$-block of D that contains an edge is a tournament and D has connectivity 1. Hence, each $\mathcal{C}$-block contains edges and the $\mathcal{C}$-blocks have to be homogeneous tournaments. Thus, D is of Type I in contradiction to the assumption. □

Now we are able to classify the connected C-homogeneous digraphs of Type I with at least two ends and connectivity 1.

Lemma 5.16. *Let D be a connected C-homogeneous digraph of Type II with more than one end. If D has connectivity 1, then D is isomorphic to $DL(\Delta(D))$.*

Proof. This is direct consequence of Lemma 5.15 and Lemma 5.5. □

In the next two theorems we prove that in the cases that the reachability digraph is either isomorphic to CP_κ or to $K_{2,2}$ the digraph has connectivity at most 2. Thus, in this case it remains to determine those with connectivity exactly 2.

Theorem 5.17. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and with $\Delta(D) \cong CP_\kappa$ for a cardinal $\kappa \geq 3$. If D has connectivity more than one, then D is isomorphic to $M(\kappa, m)$ for an $m \in \mathbb{N}$ with $m \geq 2$.*

Proof. By Lemma 5.15 the digraph D contains no triangle. Let $\mathcal{C}$ be a basic cut system and let $\mathcal{T}$ be the structure tree of D and $\mathcal{C}$. Let S^0 be a $\mathcal{C}$-separator, let $X^0 = \Delta_e$ for an $e \in ED$ such that $|S^0 \cap X^0| \geq 2$, and let K^0 be a $\mathcal{C}$-block with $S^0 \subseteq K^0$ and $\Delta_e \subseteq K^0$, which all exists by Lemma 5.13. Let $A \cup B$ be the natural bipartition of X^0 such that its edges are directed from A to B. For each $a \in A$ let us denote by b_a the unique vertex in B such that ab_a is not an edge in X^0. By symmetry we may assume that $A \cap S^0 \neq \emptyset$, so let $a \in A \cap S^0$.

First we will show that $X^0 \cap S^0 = \{a, b_a\}$. Since S^0 contains no edges by Lemma 5.1 it suffices to show that $A \cap S^0 = \{a\}$. So let us suppose that there is another vertex $a' \neq a$ in $A \cap S^0$. Since any two vertices in A have a common successor in B, we have $A \subseteq S^0$ by C-homogeneity. Let $a' \in A$ be distinct from a and P an induced a-a'-path whose interior is contained in $D - K^0$. Denote the unique neighbour of a on P by c. Taking into account that X^0 is a CP_κ, there is a common successor for each pair of A-vertices; let b be such a common successor of a and a'. Since S^0 separates both, b and b_a, from the interior of P, the paths cPb and cPb_a are isomorphic and,by C-homogeneity, we can map cPb onto cPb_a by an automorphism φ of D. Then $a\varphi$ is a successor of c that sends an edge to b_a. Hence $a\varphi$ lies in A and is distinct from a, contradicting the fact that $\operatorname{desc}(c)$ is a tree. Thus we know that $X^0 \cap S^0 = \{a, b_a\}$ for a vertex $a \in A$.

For the remainder let $X^0 \cap S^0 = \{x_0, x_1\}$. Because each vertex clearly lies in exactly two distinct reachability digraphs, there is a unique reachability digraph $X^1 \neq X^0$ that contains x_1. If $x_0 \in X^1$ then it is straight forward to see that $D \cong M(\kappa, 2)$. So assume $x_0 \notin X^1$ and let ψ be an automorphism of D mapping X^0 onto X^1 and x_0 to x_1. Let S^1, K^1 denote the image under ψ of S^0, K^0, respectively, and let $x_2 = x_1\psi$. Since $\mathcal{C}$ is basic there is an induced x_0-x_1-path P the interior of which lies in $D - K^0$. We shall show that P contains x_2.

Suppose that P does not contain x_2 and has minimal length with this property. Let u be the neighbour of x_1 on P, which clearly lies in X^1, and let v be a neighbour of u in X^1 distinct from x_1. If v does not lie on P, then Puv is a path of the same length as P which is induced by the minimality of P and Theorem 5.12, contradicting the fact that x_0 and v cannot lie in a common reachability digraph. On the other hand, if v lies on P then consider a neighbour w of x_2 in X^1 distinct from v. Remark that since X^1 is a CP_κ there is an edge between v and x_2. Thus by the choice of P the path Pvx_2w is induced and of the same length as P, which is impossible since x_0 and w do not belong to a common reachability digraph. Hence P contains x_2.

We have just proved that $\{x_1, x_2\}$ separates x_0 from any neighbour of x_1 in X^1. Hence all $\mathcal{C}$-separators have order 2 and thus the blocks which contain edges consist each of a single reachability digraph. Now we repeat the previous construction to continue the sequences $(X^i)_{i\in\mathbb{N}}$, $(S^i)_{i\in\mathbb{N}}$, $(K^i)_{i\in\mathbb{N}}$ and $(x_i)_{i\in\mathbb{N}}$, respectively. Since Px_2 is an induced x_0-x_2-path the interior of which lies in $D - K^1$, we can apply the same argument as above to assure that P contains x_3. Hence by induction we have $x_i \in P$ for all $i \in \mathbb{N}$, and since P is finite there is an $m \in \mathbb{N}$ such that $x_m = x_0$. Furthermore we have $X^m = X^0$, $S^m = S^0$ and $K^m = K^0$. One can verify that $\{x_0, x_1, \ldots, x_{m-1}\}$ forms a maximal $\mathcal{C}$-inseparable set – a $\mathcal{C}$-block – which means that D is isomorphic to $M(\kappa, m)$. □

Theorem 5.18. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and with $\Delta(D) \cong K_{2,2}$. If D has connectivity more than one, then D is isomorphic to $M'(2m)$ for $2 \leq m \in \mathbb{N}$.*

Proof. Lemma 5.15 implies that D contains no triangle. Let $\mathcal{C}$ be a basic cut system of D. Let S^0 be a $\mathcal{C}$-separator and let $X^0 = \Delta_e$ for an $e \in ED$ such that $|S^0 \cap X^0| \geq 2$. Such an X^0 exists by Lemma 5.13. As $\Delta(D) \cong K_{2,2}$ and as no $\mathcal{C}$-separator contains any edge by Lemma 5.1, there is $|S^0 \cap X^0| = 2$. So let x_0, x_1 be the two vertices in $X^0 \cap S^0$. Let X^1 be the other reachability digraph that contains x_1 and let x_2 be the unique vertex in X^1 that is not adjacent to x_1. Let ψ be an automorphism of D that maps X^0 onto X^1 and let S^1 be the image of S^0 under ψ.

With the same technique as in the previous proof, we can verify that $\{x_1, x_2\}$ separates D and so $S^0 = \{x_0, x_1\}$. We can continue the sequences $(x_i)_{i\in\mathbb{N}}$ and $(S^i)_{i\in\mathbb{N}}$ so that $S^1 = \{x_1, x_2\}$ and $S^i = \{x_i, x_{i+1}\}$, and there is an $n \in \mathbb{N}$ such that $x_n = x_0$. Since D has infinitely many ends we have $n \geq 3$, and as $x_i \in S^i$ only holds for all even integers i we have $n = 2m$ with $m \geq 2$. Now analog as in the proof of Theorem 5.17 $\bigcup_i S^i$ forms a $\mathcal{C}$-block that contains no edges. Hence there are precisely two $\mathrm{Aut}(D)$-orbits on the $\mathcal{C}$-blocks and D is isomorphic to $M'(2m)$. □

If we assume $\Delta(D)$ to be one of the other possibilities as described in Theorem 3.5, then the C-homogeneous digraphs have – in contrast to the other two cases – connectivity 1.

Lemma 5.19. *Let D be a connected C-homogeneous digraph of Type II with infinitely many ends and such that $\Delta(D)$ is isomorphic to a $T_{\kappa,\lambda}$ for cardinals κ, λ, a C_{2m} with $4 \leq m \in \mathbb{N}$, a $K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$, or an infinite homogeneous generic bipartite digraph. Then D has connectivity* 1.

Proof. Since D is of Type II, it contains no triangle by Lemma 5.15. Let us suppose that D has connectivity at least 2 and let $\mathcal{C}$ be a basic cut system of D. Let S be a $\mathcal{C}$-separator and let X be a reachability digraph with $|S \cap X| \geq 2$ as in Lemma 5.13. We investigate the given reachability digraphs one by one and get in each case a contradiction and, thereby, we get a contradiction in general to the assumption that D has connectivity at least 2. So let us assume that $X \cong T_{\kappa,\lambda}$ for cardinals κ, λ. By Lemma 5.5 we know that $\kappa, \lambda \geq 2$, as D is not a tree. Let $x, y \in S \cap X$ such that $d_X(x, y)$ is maximal. Such vertices exist as S is finite. Let e_1 be the first edge on the path from x to y in X and let e_2 be another edge incident with x. There is an $\alpha \in \mathrm{Aut}(D)$ with $e_1\alpha = e_2$. But then $y\alpha$ lies in a common separator with x, as $x\alpha = x$. By Corollary 5.7 the separator $S\alpha$ has to be the same as S. But this contradicts the maximality of $d_X(x, y)$, as $d_X(y\alpha, y) > d_X(x, y)$.

Let us now assume that $X \cong C_{2m}$ for a $4 \leq m \in \mathbb{N}$ and let x, y be distinct vertices in $S \cap X$. Then there is an induced path P from x to y that lies apart from x and y in a component of $D - S$ that intersects trivially with X. We first show that we may assume that $d_X(x, y) \geq 4$. Let e_1, e_2 be the two edges in $D[X]$ that are incident with x. If $d_X(x, y) = k \leq 3$, then let $\alpha \in \mathrm{Aut}(D)$ with $e_1\alpha = e_2$. Then there is $d_X(y, y\alpha) = 2k$, as $m \geq 4$. Thus we have shown that there are $x, y \in S \cap X$ with $d_X(x, y) \geq 4$. Let s_1 and s_2 be the vertices in X that are adjacent to y and let t be a vertex in X that is adjacent to x. Since $d_X(x, y) \geq 4$, the graphs $txPys_i$ for $i = 1, 2$ are induced paths. Hence there is an automorphism α of D that maps $txPys_1$ onto $txPys_2$ and thus $d_X(s_1, x) = d_X(s_2, x)$ and $d_X(s_1, t) = d_X(s_2, t)$, a contradiction as X is a cycle.

For the next case let us assume that $X \cong K_{\kappa,\lambda}$ for cardinals $\kappa, \lambda \geq 2$. Let $A \cup B$ be the natural bipartition of X. Since $|S \cap X| \geq 2$, the vertices in $S \cap X$ lie in the same partition set, A say. By the C-homogeneity it is an immediate consequence that $A \subseteq S$. As the $\mathcal{C}$-separators have minimal cardinality with respect to separating ends, there is $|A| \leq |B|$. If there is a $\mathcal{C}$-separator S' with $|S' \cap B| \geq 2$, then $B \subseteq S'$. If in addition the intersection of B with another reachability digraph distinct from X is B, then it is a direct consequence that $\kappa = \lambda$ is finite and that D has two ends. Thus there are two distinct reachability digraphs X_1, X_2 that intersect with B non-trivially and that are distinct from X. Let A_1, B_1, A_2, B_2 be the natural bipartitions of X_1, X_2, respectively. Let P be an induced path from $A_1 \cap B$ to $A_2 \cap B$ in a component of $D - S'$ that intersects non-trivially with X. Let a be the vertex on P that is adjacent to the vertex in $P \cap A_1$ and let b be a vertex in $B \cap A_1$ not on P. Then there is an automorphism α of D that maps P onto baP. But this contradicts the fact that the endvertices of P lie both in B but the endvertices of baP do not lie in in any common reachability digraph as $|A_1 \cap B| = 1$. Thus we conclude that $|B \cap S'| = 1$. So let $x, y, z \in B$ be three distinct vertices. There is a shortest induced path P from x to y in that component of $D - S$ that contains B. Let $a \in A$ and let b be the vertex on P with distance 2 to y. Then there is an automorphism α of D that maps $zaxPb$ onto $yaxPb$. Thus we conclude that $d(b, z) = 2$. But then z has to have incident edges that are directed both towards or both from distinct $\mathcal{C}$-blocks. This contradicts Lemma 5.5.

Let us finally assume that X is isomorphic to an infinite homogeneous generic bipartite digraph. Let again $A \cup B$ be the natural bipartition of X. Since X is homogeneous, all vertices in the same set A or B have distance 2 to each other. We conclude that $|S \cap A| \geq 2$ immediately implies $A \subseteq S$ which contradicts the finiteness of S. Conversely we also know $|B \cap S| \leq 1$. Since D has connectivity at least 2, there is $|A \cap S| = 1 = |B \cap S|$. Let a, b be the vertices in $A \cap S, B \cap S$, respectively, and let $ab'a'b$ be a path of length 3 from a to b. This path exists because each two vertices in the same set A or B have distance 2 to each other as before. Since there are infinitely many vertices in A that are adjacent to b' but not to b, all these vertices have to lie in S, a contradiction. Thus we conclude that D has connectivity 1. $\square$

Let us summarize the conclusions of this section in the following theorem. In its proof we will finally prove that all the candidates for C-homogeneous digraphs are really C-homogeneous.

Theorem 5.20. *Let D be a connected digraph of Type II with infinitely many ends. Then D is C-homogeneous if and only if one of the following holds:*

(1) $\Delta(D) \cong CP_\kappa$ *for a cardinal* $\kappa \geq 3$ *and* $D \cong DL(\Delta(D))$.

(2) $\Delta(D) \cong C_{2m}$ *for* $2 \leq m \in \mathbb{N}$ *and* $D \cong DL(\Delta(D))$.

(3) $\Delta(D) \cong K_{\kappa,\lambda}$ *for cardinals* $\kappa, \lambda \geq 2$ *and* $D \cong DL(\Delta(D))$.

(4) $\Delta(D)$ *is isomorphic to an infinite homogeneous generic bipartite digraph and* $D \cong DL(\Delta(D))$.

(5) $\Delta(D) = CP_\kappa$ *and* $D \cong M(\kappa, m)$ *for a cardinal* $\kappa \geq 3$ *and* $2 \leq m \in \mathbb{N}$.

(6) $\Delta(D) = K_{2,2}$ *and* $D \cong M'(2m)$ *for* $2 \leq m \in \mathbb{N}$.

Proof. The theorem follows from Lemmas 5.15, 5.16, and 5.19 and by Theorems 5.17 and 5.18 as we have already shown in Chapter 4 that the digraphs mentioned are C-homogeneous. □

5.3.3 Line digraphs of C-homogeneous digraphs

It is well known (see [1]) that line digraphs of highly-arc-transitive digraphs are again highly-arc-transitive. In some cases also C-homogeneity is preserved under taking the line digraph: Gray and Möller [14] stated that the line digraph of a $DL(C_{2m})$ is C-homogeneous. In terms of our classification:

Remark 5.21. *For each $m \in \mathbb{N}$ we have $L(DL(C_{2m})) \cong M'(2m)$.*

Proof. Consider the digraph $D = DL(C_{2m})$ for a $m \in \mathbb{N}$. By construction the deletion of each single vertex v of D splits the digraph into two components such that v has two out-neighbours in the one and two in-neighbours in the other component. Thus the four edges that are incident with v form a $K_{2,2}$ in $L(D)$ whose independent vertex sets separate $L(D)$. Furthermore the edges of each C_{2m} in D form an independent set in $L(D)$ so that any two adjacent edges lie in a common $K_{2,2}$ in $L(D)$. One can easily verify that this digraph is indeed isomorphic to $M'(2m)$. □

Interestingly, our classification of the C-homogeneous digraphs with infinitely many ends implies that C-homogeneity is not generally preserved under taking line digraphs. Indeed, for all $m \in \mathbb{N}$ the line digraph of $M'(2m)$ is triangle-free, has infinitely many ends, and has connectivity 4, hence it is not of Type II. Thus, by Theorem 5.20, we know that $L(M'(2m)) \cong L(L(DL(C_{2m})))$ is not C-homogeneous. This had remained an open question in [14].

Chapter 6

The case: finite degree

In the chapter we look at the C-homogeneous digraphs that have finite degree and at most one end. The first results in this part are obtained in the case that the out-neighbourhood (or in-neighbourhood) of any vertex is not independent (Section 6.1) and then we look at the more difficult case, where the out-neighbourhood and also the in-neighbourhood is independent (Section 6.2 and Section 6.3). This latter case also divides into two parts: either the directed triangle embeds into the digraph or not. Note that the next chapter has basically the same structure. It just deals with countable C-homogeneous digraphs of arbitrary degree. This chapter is based on [15] and Chapter 7 is based on [16].

6.1 The non-independent case for C-homogeneous digraphs with at most one end

It is a straightforward argument that the out-neighbourhood as well as the in-neighbourhood of any vertex of a C-homogeneous digraph have to be homogeneous digraphs: extend any two finite isomorphic induced subdigraphs in $D^+(x)$ (in $D^-(x)$) for $x \in VD$ with the aid of x to connected such digraphs. As any of their isomorphisms extend to automorphisms of the whole digraph, so do the isomorphisms between the two original subdigraphs. Let us fix this as a lemma.

Lemma 6.1. *Let D be a C-homogeneous digraph. Then D^+ and D^- are homogeneous digraphs.* □

We investigate which of the homogeneous digraphs of Theorem 3.12 may occur as a subdigraph D^+ or D^-. In this section we take a look at those cases that contain an edge and show that there is precisely one such case that may occur. This case is a generalization of the digraph H that occurs in the case (v) of Theorem 3.12. Our first aim is to show that neither D^+ nor D^- is isomorphic to H.

Lemma 6.2. *Let D be a connected locally finite C-homogeneous digraph. Then $D^+ \ncong H$ and $D^- \ncong H$.*

Proof. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may suppose that $D^+(x) \cong H$ for every $x \in VD$. Let $z \in N^+(x)$. As $D^+(x) \cong H$, the digraph $D^+(x) \cap D^+(z)$ consists of a directed triangle. Let v_1, v_2, v_3 be three vertices in $N^+(z) \setminus N^+(x)$ such that v_1 has precisely two neighbours in $N^+(x) \cap N^+(z)$, such that $N^+(x) \cap N^+(z) \subseteq N^+(v_2)$, and such that $N^+(x) \cap N^+(z) \subseteq N^-(v_3)$. These vertices exist because $D^+(z) \cong H$. Then there are two vertices v_i, v_j $(i \neq j)$ such that they are both either in the in-neighbourhood of x or not adjacent to x. This implies that $D[z, x, v_i] \cong D[z, x, v_j]$. As D is C-homogeneous, there is an automorphism of D mapping the first onto the second subdigraph that fixes x and z. But this is a contradiction to the choice of v_i and v_j as they behave differently to $N^+(x) \cap N^+(z)$. □

The next case that we exclude is that the out- or the in-neighbourhood induces a subdigraph isomorphic to C_4.

Lemma 6.3. *Let D be a connected locally finite C-homogeneous digraph. Then $D^+ \ncong C_4$ and $D^- \ncong C_4$.*

Proof. Analogously to the proof of Lemma 6.2, we may suppose that $D^+(x) \cong C_4$. Let us denote by $v_1, \ldots, v_4$ the four vertices in $N^+(x)$ such that $v_i v_{i+1} \in ED$ for $1 \leq i \leq 3$ and $v_4 v_1 \in ED$. According to Lemma 6.2, we know that $D^-(v_1) \ncong H$.

Let us suppose that there is a vertex $y \in N^-(v_1) \cap N^-(v_2)$ distinct from x. An immediate consequence of C-homogeneity is $N^+(x) = N^+(y)$. Indeed, we can extend the isomorphism from $D[x, y, v_1]$ to $D[x, y, v_2]$ that fixes x and y to an automorphism of D, which implies that $v_3 \in N^+(y)$. Analogously, we have $v_4 \in N^+(y)$, too, so $N^+(x) = N^+(y)$. Hence, neither xy nor yx can be an edge of D. The subdigraph $D[x, y, v_4]$ is a subdigraph of $D^-(v_1)$ and thus, by Theorem 3.12, we have $D^-(v_1) \cong C_3[I_n]$ for some $n > 1$. As$x \in N^-(v_1)$, there is a vertex in $N^+(x) \cap N^-(v_1)$ which is distinct from v_4. As this is impossible, we have proved

$$N^-(v_1) \cap N^-(v_2) = \{x\}. \tag{6.1}$$

Due to C-homogeneity, we know that (6.1) holds for each two adjacent vertices v_i and v_j in $N^+(x)$.

The next step in the proof is to show

$$N^-(v_1) \cap N^+(v_2) = \emptyset. \tag{6.2}$$

Let us suppose that there is a vertex $y \in N^-(v_1) \cap N^+(v_2)$. If y is neither adjacent to x nor to v_4, then by Theorem 3.12 $D^-(v_1)$ has to be isomorphic to $I_n[C_3]$ for some $n > 1$. So there is a vertex $z \in N^-(v_1)$ that lies in $N^+(v_4) \cap N^-(x)$. As $xv_2 \in ED$ and as v_2 and v_4 are not adjacent, C-homogeneity implies that we must have $v_2z \in ED$. Indeed, otherwise we could map z either to x or to v_4 and fix v_1 and v_2 by an automorphism of D. But both cases imply that then the whole directed triangle $D[x, v_4, z]$ in $D^-(v_1)$ must have the same adjacency to v_2 which is impossible. Both digraphs $D[z, v_1, v_2]$ and $D[y, v_1, v_2]$ are directed triangles. Hence, there is an automorphism α of D that maps z to y and fixes v_1 and v_2. But as x and y are not adjacent, we know that $x \neq x\alpha$. Since also $x\alpha$ lies in $N^-(v_1) \cap N^-(v_2)$, this contradicts (6.1). So y is adjacent to at least one of x and v_4.

If y is adjacent to x but not to v_4, then yx lies in ED as $y \notin \{v_1, \ldots, v_4\} = N^+(x)$. Since an induced 2-arc embeds into $N^-(v_1)$, we know that $D^-(v_1) \cong C_4$, as the only other possible case $D^-(v_1) \cong H$ is not possible due to Lemma 6.2. Hence, there is a vertex $z \in N^-(v_1)$ that lies in $N^+(v_4) \cap N^-(y)$ and that is not adjacent to x. As a consequence of (6.1) we know that zv_2 is not an edge in D. If z and v_2 are not adjacent, we also obtain a contradiction. Indeed, then there is an automorphism β of D that maps v_4 to z and fixes v_1 and v_2. So $x\beta \neq x$ but both lie in $N^-(v_1) \cap N^-(v_2)$, which is impossible. Hence, we know that $v_2z \in ED$. So there is an automorphism β of D that maps y to z and fixes v_1 and v_2. As x and y are adjacent but x and z are not, we have again two distinct vertices, x and $x\beta$ in $N^-(v_1) \cap N^-(v_2)$ which is impossible by (6.1).

If y is adjacent to v_4 but not to x, then we know by (6.1) applied to v_4 and v_1 that $yv_4 \notin ED$. So v_4y is an edge of D. This implies as above that $D^-(v_1) \cong C_4$. Hence, there is a vertex $z \in N^-(v_1) \setminus \{v_4, x, y\}$. If z is not adjacent to v_2, then there is an automorphism of D that maps z to v_4 and fixes v_1 and v_2. Since this automorphism cannot fix x, the image of x is a second vertex in $N^-(v_1) \cap N^-(v_2)$ contrary to (6.1). Hence, z and v_2 are adjacent. Due to (6.1), zv_2 is no edge of D, so we have $v_2z \in ED$. Then there is an automorphism of D that maps y to z and fixes v_1 and v_2. Again, x and its image under that automorphism are distinct. But both lie in $N^-(v_1) \cap N^-(v_2)$ in contradiction to (6.1).

Thus, we conclude that both x and v_4 are adjacent to y. Due to (6.1), we have $v_4y \in ED$ and not $yv_4 \in ED$, and because of $y \notin N^+(x)$ we have $yx \in ED$.

By C-homogeneity, there is an automorphism γ of D that maps v_2 to v_4 and fixes y and x. Hence, we have $v_1\gamma = v_3$ and $yv_3 \in ED$. But then $D[v_1, x, v_3]$ is a subdigraph of $N^+(y)$ that cannot be embedded into a C_4. This contradiction shows that (6.2) is true.

Let us suppose that there exists a vertex $y \in N^-(v_1) \cap N^+(v_4)$. Due to (6.2), we have $yv_3 \notin ED$. The existence of an edge v_3y in D implies that there is an automorphism α of D that maps v_3 to v_1 and fixes x and y. But then, we have $v_4\alpha = v_2$ and hence $v_2y \in ED$ contrary to (6.2). So we have $v_3y \notin ED$. Thus, there is an automorphism β of D that maps v_1 to y and fixes v_3 and v_4. Since $y \notin N^+(x)$, we have $x \neq x\beta \in N^-(v_3) \cap N^-(v_4)$ and thus a contradiction to (6.1). This shows

$$N^-(v_1) \cap N^+(v_4) = \emptyset. \tag{6.3}$$

Since there is a vertex in $N^-(v_1) \cap N^+(x)$, the same is true for $N^-(v_1) \cap N^+(v_4)$ due to C-homogeneity. This contradiction to (6.3) shows that $D^+(x)$ cannot be isomorphic to C_4. □

Lemma 6.4. *Let D be a connected locally finite C-homogeneous digraph such that $D^+ \cong I_n[C_3]$ and $D^- \cong I_m[C_3]$ with $m, n \geq 1$. Then $m = n = 1$.*

Proof. Let $xy \in ED$. Then there exists $z \in N^-(y) \cap N^-(x)$. By considering $D^-(y)$, we obtain a vertex $a \in N^-(y) \cap N^+(x)$ with $az \in ED$. Let b be the third vertex of $N^+(x)$ in that isomorphic image of C_3 that contains y and a. If either zb or bz lies in ED, then we have either $by \in E(D^+(x) \cap D^+(z))$ or $ab \in E(D^+(x) \cap D^-(z))$. This is a contradiction as each of $N^+(x) \cap N^+(z)$ and $N^+(x) \cap N^-(z)$ consists of precisely one vertex by the assumption $D^+(x) \cong I_n[C_3]$. Hence, z and b are not adjacent. So in the isomorphic copy $D[y, a, b]$ of C_3 in $D^+(x)$, there is an in- and an out-neighbour of z and one vertex not adjacent to z.

Let us suppose that $n > 1$. Then there exists a vertex $y' \in N^+(x)$ that is distinct from a, b, and y. So there is a vertex $v \in \{a, b, y\}$ and an automorphism of D that maps v to y' and fixes x and z. Hence, the isomorphic image of C_3 in $D^+(x)$ that contains y' contains a vertex of $N^+(z)$. We may suppose that this is y'. But then $D[y, x, y']$ is a digraph that cannot be embedded into $D^+(z)$. This contradiction shows $n = 1$. By a symmetric argument we also have $m = 1$. □

Lemma 6.5. *Let D be a connected locally finite C-homogeneous digraph. If either $D^+ \cong C_3[I_n]$ or $D^- \cong C_3[I_n]$ for some $n \geq 1$, then $D \cong H[I_n]$.*

Proof. Analogously to the proof of Lemma 6.2, we may suppose that $D^+(x) \cong C_3[I_n]$ for some $n \geq 1$. Let $xy \in ED$. Then x and n independent vertices of

$N^+(x)$ lie in $N^-(y)$ and hence either $n = 1$ and $D^-(y) \cong I_m[C_3]$ for some $m \geq 1$ or $D^-(y) \cong C_3[I_m]$ for some $m \geq n$. In the first case, we have $m = 1$ according to Lemma 6.4. So in both cases, we have $D^-(y) \cong C_3[I_m]$ for some $m \geq n$. With a symmetric argument we conclude $m = n$. Hence, there is a vertex $z \in N^-(x) \cap N^-(y)$. As $D^+(z) \cong C_3[I_n]$ and $x \in N^+(z)$ and as $D^-(x) \cong C_3[I_n]$ and $z \in N^-(x)$, we have that

$N^+(x) \cap N^+(z)$ and $N^-(x) \cap N^-(z)$ are independent sets of cardinality n. (6.4)

An immediate consequence of the C-homogeneity of D is $N^+(x) \cap N^-(z) \neq \emptyset$ as D contains some directed triangle. Our next aim is to show that

$N^+(x) \cap N^-(z)$ is an independent set of cardinality n. (6.5)

Let us suppose that there is an edge ab with its two incident vertices in $N^+(x) \cap N^-(z)$. Then the digraphs $D[x, z, a]$ and $D[x, z, b]$ are isomorphic and there is an automorphism α of D mapping a to b and fixing x and z. As a consequence of (6.4), both a and b have to be adjacent to all the vertices in $N^+(x) \cap N^+(z)$. Since $D^+(x) \cong C_3[I_n]$ and $a, b \in N^+(x)$, we have $y'a \in ED$ and $by' \in ED$ for all $y' \in N^+(x) \cap N^+(z)$. Indeed, an edge ay' would imply that y' and b are not adjacent and the same would be true for an edge $y'b$. Thus, the automorphism α cannot exist and we conclude that no such edge ab exists. So $N^+(x) \cap N^-(z)$ is an independent set. Since every edge lies on at least n distinct directed triangles, there are at least n vertices in $N^+(x) \cap N^-(z)$ and, as a largest independent set in $N^+(x)$ consists of n vertices, we have proved (6.5).

As a further step in this proof, we prove the following:

Each two non-adjacent vertices in $N^+(x)$ have the same in-neighbors. (6.6)

Let $a, b \in N^+(x)$ be non-adjacent and $x' \in N^-(a)$ with $x' \neq x$. Let us first assume that x and x' are adjacent. In each of the two sets $N^+(x)$ and $N^+(x')$ there is precisely one maximal independent set that contains a as $D^+(x) \cong C_3[I_n]$. Due to (6.4) applied to x and x' instead of x and z, these two maximal sets must be $N^+(x) \cap N^+(x')$. Hence, also b must lie in $N^+(x')$. So let us assume that x and x' are not adjacent. Then there is a third vertex x'' in $N^-(a)$ that is adjacent to both x and x'. Applying the previous case, we know that $x'' \in N^-(b)$ and hence also $x' \in N^-(b)$. This shows (6.6).

The remaining step in the proof is to show the following:

There is an equivalence relation $\sim$ on VD, each of whose equivalence classes has precisely n independent vertices, such that $D_\sim$ is isomorphic to H and $D_\sim[I_n]$ is isomorphic to D. (6.7)

Let us define a relation $\sim$ on VD via

$$a \sim b \quad :\Longleftrightarrow \quad N^-(a) = N^-(b).$$

Obviously, $\sim$ is an equivalence relation. First, we note that every equivalence class must be an independent vertex set due to the definition of the relation $\sim$. Hence, there are more than one equivalence classes. Let A and B be two distinct equivalence classes, $a_1, a_2 \in A$, and $b_1, b_2 \in B$ such that $a_1b_1 \in ED$. According to the definition of $\sim$, we have $a_1b_2 \in ED$ and thus, $B \subseteq N^+(a_1)$. As B is an independent set and $D^+(a_1) \cong C_3[I_n]$, there are at most n vertices in B. On the other side, (6.6) with x replaced by a_1 implies that there are n vertices in B, so B is the maximal independent set in $N^+(a_1)$ that contains b_1. The vertex b_1 has a successor c that is a predecessor of a_1. By definition of $\sim$, we have $ca_2 \in ED$. Since $|A| = n$, we conclude by (6.5) with x, z replaced by c, b_1 that $a_2b_1 \in ED$. So we also have $a_2b_2 \in ED$. Thus, $D_\sim$ is a digraph with $D \cong D_\sim[I_n]$. The digraph $D_\sim$ is C-homogeneous, since D is C-homogeneous and since we can lift any connected induced subdigraph F of $D_\sim$ to a connected induced subdigraph of D that has as its vertices the union of the vertices of F – note that the vertices of F are equivalence classes of vertices of D. It remains to show that $D_\sim \cong H$. As $D_\sim$ is a C-homogeneous digraph with $D^+(v) \cong C_3$ for all $v \in VD_\sim$, it suffices to assume $n = 1$ and to show that $D \cong H$.

Let $x \in VD$. We know that $D^+(x) \cong C_3 \cong D^-(x)$. Let $N^+(x) = \{v_1, v_2, v_3\}$ and $N^-(x) = \{u_1, u_2, u_3\}$ with $v_iv_{i+1} \in ED$ and $v_iu_{i+1} \in ED$ (where $v_4 = v_1$ and $u_4 = u_1$). As xv_1 is an edge in $D[x, v_1, v_2]$, also u_1x must lie in the same position in some triangle. Thus, there is an edge from u_1 to one of the vertices v_i, say to v_1. Then $N^-(v_1) = \{u_1, x, v_3\}$ and hence, we have $v_3u_1 \in ED$. As $N^+(u_1) = \{u_2, x, v_1\}$, we have $v_1u_2 \in ED$. Now we can apply similar arguments and obtain that v_2u_3, u_2v_2, and u_3v_3 lie in ED. Let y be the third out-vertex of v_3 distinct from v_1 and u_1. Notice that y cannot be u_2. Because of $D^+(v_3) \cong C_3$, we have $yu_1 \in ED$ and $v_1y \in ED$. By $D^+(v_1) \cong C_3$ we conclude $v_2y \in ED$ and $yu_2 \in ED$ and $D^-(u_1) \cong C_3$ implies $yu_3 \in ED$. The constructed digraph has the correct out- and in-degree at every vertex and is isomorphic to H. This finishes the proof of Lemma 6.5. □

Let us combine the results of this section with Theorem 3.12:

Theorem 6.6. *Let D be a connected locally finite C-homogeneous digraph and $x \in VD$. Either $N^+(x)$ and $N^-(x)$ are independent vertex sets or there is an $n \geq 1$ such that $D^+ \cong C_3[I_n] \cong D^-$ and $D \cong H[I_n]$.* □

6.2 The independent case for C-homogeneous digraphs with at most one end

In this section, we consider the situation that every out-neighbourhood – and hence due to Theorem 6.6 also every in-neighbourhood – is independent. Let us briefly outline the content of this section. First, we show that if either the out-degree or the in-degree is 1, then the connected locally finite C-homomgeneous digraph is a tree (Lemma 6.7). Thereafter, we show in Lemmas 6.8 and 6.11 that the reachability relation is not universal in our situation. So due to Proposition 2.1, the reachability digraphs are bipartite. That is why we turn our attention towards connected (locally finite) C-homogeneous bipartite graphs. We use their classification (Theorem 3.5) to obtain a complete classification in the case of connected locally finite C-homogeneous digraphs with at most one end if the digraphs contain no directed triangle (Lemma 6.14) and then a partial classification of such digraphs if they contain a directed triangle (Lemma 6.15). We continue the investigation of this situation in Section 6.3.

Lemma 6.7. *Let D be a connected vertex-transitive digraph and let $x \in VD$. If $N^+(x)$ or $N^-(x)$ consists of precisely one vertex, then D is either an infinite tree or a directed cycle.*

Proof. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $N^+(x)$ consists of precisely one vertex. Let us assume that D is not a tree. Then there is a cycle C in D. If C is not a directed cycle, then there is a vertex with out-degree at least 2 on that cycle. Hence, we may assume that C is a directed cycle. For every vertex on C, its descendants must lie on C, so they induce a subdigraph that is a cycle. If $D \neq C$, then there must be a vertex u outside C that is adjacent to some vertex v on C. The edge between u and v cannot be vu as we already mentioned, so it must be uv. So the descendants of u do not induce a directed cycle, as they contain u and all vertices of C. But as D is vertex-transitive, the descendants of u and those of v induce isomorphic digraphs. This contradiction shows that $D = C$ is a directed cycle. □

Notice that C-homogeneous digraphs are edge-transitive and hence Lemma 6.7 holds for them. Let us now look at the reachability relation of C-homogeneous digraphs. The proof that this relation is not universal splits into two cases: whether a directed triangle embeds into D or not. We start with the latter case:

Lemma 6.8. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$ and such that D contains no directed triangle. Then the reachability relation of D is not universal.*

Proof. Let $x \in VD$. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $d^+(x) \geq d^-(x)$ and due to Lemma 6.7, we may also assume that $d^-(x) \geq 2$. Let $y \in N^+(x)$ and $\Omega = N^+(y)$. Since D is C-homogeneous and contains no directed triangle and since Ω and $N^+(x)$ are independent sets of vertices, the group $\Gamma := \mathrm{Aut}(D)_{xy}$ acts on Ω like S_Ω, the symmetric group on Ω, i.e. $\Gamma^\Omega \cong S_\Omega$. By induction, we will show $(\Gamma_Q)^\Omega = \Gamma^\Omega$ for all alternating walks Q with initial edge xy. Let P be such an alternating walk. Let us assume that $(\Gamma_P)^\Omega = \Gamma^\Omega$ and let $e \in ED$ such that Pe is an alternating walk. Let z be the vertex incident with e but distinct from the end vertex of P. We will show that $(\Gamma_{Pe})^\Omega = \Gamma^\Omega$, and hence, $(\Gamma_z)^\Omega = \Gamma^\Omega$. There are at most $|\Omega| - 1$ vertices in $\{z\alpha \mid \alpha \in \Gamma_P\}$, as this set is contained either in the out- or in the in-neighbourhood of z', the other vertex that is incident with e, but it does not contain the neighbour of z' on P. So we have $|\Gamma_P : \Gamma_{Pe}| < |\Omega|$. Since $\Gamma^\Omega = (\Gamma_P)^\Omega$, we have either $|\Omega| = 2$ or

$$|\Gamma^\Omega : (\Gamma_z)^\Omega| \leq |\Gamma : \Gamma_z| = |\{z\alpha \mid \alpha \in \Gamma\}| < |\Omega|.$$

Let us first assume that $|\Omega| \neq 2$. Then, due to Theorem 2.4, $(\Gamma_z)^\Omega$ is either Γ^Ω or isomorphic to A_Ω, the alternating group on Ω, or $|\Omega| = 4$ and $(\Gamma_z)^\Omega$ is a Sylow 2-subgroup of Γ^Ω. In each of these cases, the group $(\Gamma_z)^\Omega = (\Gamma_{Pe})^\Omega$ acts transitively on Ω. But then, as Ω is an independent set, for any $A, B \subseteq \Omega$ with $|A| = |B|$, the digraph D_1 induced by Pe and A must be isomorphic to the subdigraph D_2 induced by Pe and B and any bijection from A to B extends to an isomorphism from D_1 to D_2 fixing Pe. Hence, $(\Gamma_z)^\Omega$ must be the full symmetric group S_Ω.

Let us now consider the case that $|\Omega| = 2$. Then we have $d^+(x) = d^-(x) = 2$. Hence, the orbit of z under Γ_P contains only z and we conclude $\Gamma = \Gamma_z$. As for $a \in \Omega$ the orbit of a under Γ contains both successors of y, the vertex z cannot lie in Ω.

In both cases, no vertex of Ω can lie on an alternating walk that contains the edge xy and thus, the reachability relation of D cannot be universal. □

Before we turn our attention to investigate the reachability relation if D contains directed triangles, we prove some lemmas.

Lemma 6.9. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If C_3 embeds into D, then $d^+(x) = d^-(x)$.*

Proof. Let n be the number of directed triangles that contain a fixed edge xy of D. As D is C-homogeneous, we conclude for the number of directed triangles that contain x:

$$|N^+(x)|n = |N^-(x)|n.$$

Hence, we have $d^+(x) = d^-(x)$. □

Lemma 6.10. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If D contains a directed triangle, then the number of directed triangles that contain a given edge $xy \in ED$ is either 1 or at least $(d^+ - 1)$.*

Proof. Let Ω_1 be the set of all vertices in $N^+(y)$ that lie on a common directed triangle with xy, let $\Omega_2 = N^+(y) \setminus \Omega_1$, and let $\Omega_3 := N^+(x) \setminus \{y\}$. Note that $\Omega_3 \cap N^+(y) = \emptyset$ as $N^+(x)$ is an independent set. Let $d_1 = |\Omega_1|$ and $d_2 = |\Omega_2|$. Then we have $d = d_1 + d_2$ where $d := d^+$ which is the same as d^- by Lemma 6.9.

Let us suppose that d_1 and d_2 are both at least 2, so we have $|\Omega_3| \geq 3$. We consider the action of $\Gamma := \mathrm{Aut}(D)_{xy}$ on Ω_3. Since $N^+(x)$ is an independent set and since D is C-homogeneous, Γ acts on Ω_3 like S_{Ω_3}, the symmetric group on Ω_3. For every $z \in \Omega_i$, $i = 1, 2$, we have $|\Gamma : \Gamma_z| = d_i < d^+ - 1 = |\Omega_3|$. Thus and due to Theorem 2.4, we have either $(\Gamma_z)^{\Omega_3} \cong S_{\Omega_3}$, or $(\Gamma_z)^{\Omega_3} \cong A_{\Omega_3}$, or $|\Omega_3| = 4$ and $|\Gamma : \Gamma_z| = 3$. In each case, Γ_z acts transitively on Ω_3. As Ω_3 is an independent set, the subdigraph D_1 induced by x, y, z, and A is isomorphic to the subdigraph D_2 induced by x, y, z, and B for any two subsets A and B of Ω_3 with $|A| = |B|$ and, furthermore, any bijection from A to B extends to an isomorphism from D_1 to D_2 fixing x, y, and z. As D is C-homogeneous, each of these isomorphisms extends to an automorphism of D, so $(\Gamma_z)^{\Omega_3}$ cannot be a proper subgroup of S_{Ω_3} and we have that Γ_z acts on Ω_3 like S_{Ω_3}. But then for all $y_1, y_2 \in N^+(x) \setminus \{y\}$ we have $N^+(y) \subseteq N^+(y_1)$ if and only if $N^+(y) \subseteq N^+(y_2)$ and we have either $N^+(y) \subseteq N^+(y_1)$ or $N^+(y) \cap N^+(y_1) = \emptyset$. So we conclude that each edge lies either on precisely one or on d distinct directed triangles. This contradicts the assumptions that $d_1 \geq 2$ and $d_2 \geq 2$ and hence shows the assertion. □

Now we are able to prove also for connected locally finite C-homogeneous digraphs that contain directed triangles that their reachability relation is not universal.

Lemma 6.11. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If D contains a directed triangle, then the reachability relation of D is not universal.*

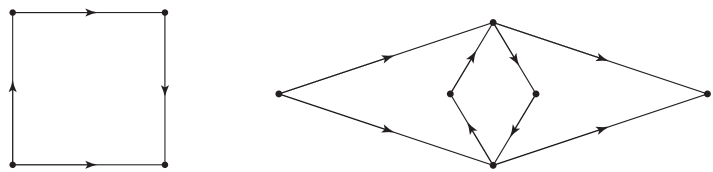

Figure 6.1: On the left side the digraph D_1 and on the right side the digraph D_2.

Proof. For this proof, we use two certain digraphs D_1 and D_2 depicted in Figure 6.1.

Let $d = d^+$. By Lemma 6.9 we have $d = d^-$. Let us suppose that the reachability relation $\mathcal{A}$ of D is universal. The digraph D_1 is an example of a cycle witnessing that $\mathcal{A}$ is universal (removing the uppermost edge leaves an alternating walk of length 3) and up to isomorphism D_1 is the only such cycle of length 4. Remember that Lemma 2.2 tells us that

there is a cycle in D witnessing that $\mathcal{A}$ is universal (6.8)

and that Lemma 2.3 tells us that

if D contains a cycle witnessing that $\mathcal{A}$ is universal, then it contains an induced such cycle of shorter or equal length. (6.9)

The next step is to show:

If D contains an induced cycle C of even length witnessing that $\mathcal{A}$ is universal, then each edge lies on precisely one directed triangle. (6.10)

Let xyz be a directed path of length 2 on C. Then $C - y$ has an automorphism that interchanges x and z. This automorphism of $C - y$ extends to an automorphism α of D. As C is induced, the same holds for $C\alpha$. Thus and since y and $y\alpha$ cannot be adjacent because $N^+(y)$ and $N^-(y)$ are independent, we obtain that y and $y\alpha$ are not adjacent. Hence, y is the first vertex of at least two directed paths of length 2 that share the edge yz: one is $yzy\alpha$ and the other is yzu where u is the second neighbour of z on C. Thus, the edge yz lies on at most $d^+(z) - 2$ directed triangles which directly implies (6.10) due to Lemma 6.10 and as $\mathrm{Aut}(D)$ acts transitively on the edges of D.

Let us show:

If D contains an induced cycle of length 4 witnessing that $\mathcal{A}$ is universal, then it contains an isomorphic copy of D_2. (6.11)

Let u, v, x, y be the vertices of D_1 such that $uv, vx, xy \in ED$. Then there is an automorphism α of D that fixes u and interchanges v and y. As the out-and the in-neighbourhood of x is independent, the vertices x and $x\alpha$ are not adjacent and D_1 together with $x\alpha$ forms all but the rightmost vertex of D_2. An automorphism β that maps u to v, v to x and x to y gives us the remaining vertex z of D_2: take $z = y\beta$. An edge between z and u either contradicts (6.10) or leads to an out- or an in-neighbourhood that is not independent – depending on its direction. Similarly, z and x are not adjacent. This shows (6.11).

Now we exclude the existence of induced cycles witnessing that $\mathcal{A}$ is universal step by step: first we exclude such cycles if they have precisely four vertices, then we exclude odd such cycles of length at least 5 and last we exclude even such cycles of length at least 6. When we have shown that none of these cases occur, we have a contradiction to the assumption that $\mathcal{A}$ is universal.

No induced cycle of length 4 *in* D *witnesses that* $\mathcal{A}$ *is universal.* (6.12)

To show (6.12), let us suppose for a contradiction that there is an induced cycle of length 4 witnessing that $\mathcal{A}$ is universal. Due to (6.11), D contains an isomorphic copy D' of D_2. Let x be the leftmost and y the rightmost vertex and let a, b, u, v the vertices of the inner cycle such that x and y are adjacent to a and u and such that $uv \in ED$. Since D contains a directed triangle, there is a vertex $a' \in N^+(a) \cap N^-(x)$. Then a' is adjacent neither to b, nor to v, nor to y, since the only directed triangle that contains aa' is $D[x, a, a']$ and since the in- and the out-neighbourhoods of every vertex are independent sets. Hence, there is an automorphism α of D that fixes a', x, and u, and maps v onto y. Then α also has to fix a, since it fixes together with x and a' the unique vertex in the directed triangle that contains the edge $a'x$. As $va \in ED$ but $ay \in ED$, this is a contradiction that shows (6.12).

No induced odd cycle of length at least 5 *in* D *witnesses that* $\mathcal{A}$ *is universal.* (6.13)

Let us suppose that D contains an induced odd cycle C of length at least 5 that witnesses that $\mathcal{A}$ is universal. Let xy be an edge on C such that either $d^+_C(x) = 2$ and $d^+_C(y) = 1$ or $d^+_C(x) = 1$ and $d^+_C(y) = 0$. Let z be the second neighbour of y on C. Then $C - x$ and $C - y$ are isomorphic and hence, there is an automorphism α of D that maps $C - x$ onto $C - y$. The digraph $D[x, y, z, x\alpha]$ is isomorphic to D_1 because $N^-(z)$ and $N^+(z)$ are independent sets. This contradicts (6.12). So we proved (6.13).

The next claim will finish the proof of Lemma 6.11.

No induced even cycle in D *witnesses that* $\mathcal{A}$ *is universal.* (6.14)

Let us suppose that D contains an induced even cycle C of minimal length witnessing that $\mathcal{A}$ is universal. Due to (6.12), the length of C is at least 6. As its length is even, there is a directed path $xyzu$ on C. Due to C-homogeneity, D has an automorphism α that maps $C - y$ onto itself with $x\alpha = z$. Hence, the path xyz lies on a directed cycle of length 4, the cycle induced by x, y, z, and $y\alpha$. Note that y and $y\alpha$ cannot be adjacent as y has independent out- and independent in-neighbourhood. Let a be the neighbour of u on C that is not z. As every edge lies on precisely one directed triangle due to (6.10), there are uniquely determined vertices a' and z' such that a, a', and u induce a directed triangle and the same holds for z, z', and u. Furthermore, the vertex a' is not adjacent to z or z' and z' is also not adjacent to a because of the independent out- and in-neighbourhoods and due to (6.10). The induced 2-arc zua' lies on a directed cycle of length 4 as the same holds for xyz. Let y' be the fourth vertex on that cycle. Then y' cannot be adjacent to a as otherwise the in-neighbourhood of a' is not independent. We shall show that $a'y \in ED$. This is true if $y' = y$, so let us assume that $y' \neq y$. Then the digraphs $D[a, u, z, y]$ and $D[a, u, z, y']$ are isomorphic. Hence, there is an automorphism β of D that fixes a, u, and z and maps y' to y. As a and u lie on precisely one common directed triangle, β must also fix a', so $y = y'\beta$ must be adjacent to $a'\beta = a'$. Then the digraph induced by a' and all the vertices of C but u and z contains a cycle C' witnessing that $\mathcal{A}$ is universal and this cycle C' has smaller length than C. Due to (6.9), there is also an induced such cycle C'' of at most the same length as C'. If the length of C'' is either 4 or odd, then we obtain the claim by (6.12) or (6.13), and if the length of C'' is even and at least 6, then we obtain a contradiction to the minimality of the length of C. This shows (6.14) and finishes the proof of the lemma. □

As the reachability relation is not universal for any locally finite C-homogeneous digraph D if $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$, we conclude with Proposition 2.1 that D has a bipartite reachability digraph. That is, why we are interested in the classification of the locally finite C-homogeneous bipartite graphs.

Now, we use the classification result of the C-homogeneous bipartite graph (Theorem 3.5) to continue our classification of the connected locally finite C-homogeneous digraphs. Remember that Lemma 5.14 tells us that $G(\Delta(D))$ is a connected C-homogeneous bipartite graph, if $\Delta(D)$ is bipartite. At this place the assumption that the digraphs have at most one end will be used for the first time in this chapter and the remaining lemmas of this section will also build on it.

Lemma 6.12. *Let D be a locally finite connected C-homogeneous digraph with at most one end such that $N^+(x)$ and $N^-(x)$ are independent sets for all vertices $x \in VD$. Then either $\Delta(D)$ is a finite digraph or D contains a directed triangle and $G(\Delta(D)) \cong T_{2,2}$.*

Proof. Due to Lemmas 6.8 and 6.11, we know that the reachability relation of D is not universal and hence that the reachability digraphs are bipartite by Proposition 2.1 and that we can apply Theorem 3.5. Let us suppose that $\Delta(D)$ is not finite. Since D is locally finite, we conclude from Theorem 3.5 that $G(\Delta(D)) \cong T_{k,\ell}$ for integers $k, \ell \geq 2$. Let us first assume that $k \geq 3$. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $k = d^+(x)$. Let $u \in VD$ and $x, y, z \in N^+(u)$. As there is a ray in $G(\Delta(D))$ and as D has at most one end, it has precisely one end. Hence, removing the (finite) set S of all vertices with distance at most 3 to u separates D into components such that precisely one of them is infinite, because D is locally finite. Let C be this infinite component. Let R_x, R_y be rays that start at u and contain x, y, respectively, and that lie in the same reachability digraph Δ that contains u and x. Since D is locally finite, there are vertices a, b on R_x, R_y, respectively, that lie in C. So we have $d(a, x) \geq 3$ and $d_\Delta(a, x) = d_\Delta(a, u) - 1$ as well as $d(b, y) \geq 3$ and $d_\Delta(b, y) = d_\Delta(b, u) - 1$, where d_Δ denotes the distance in Δ. Let P be a path (not necessarily directed) in C from a to b, and let Q be the path in Δ between a and x. Note that neither P nor Q has y or z as a neighbour. Indeed, for P this follows from the fact $P \subseteq D - S$ and for Q it is a consequence of the fact that Δ is an induced subdigraph and Q contains no neighbour of y in Δ by the choice of a. Then the digraph induced by P, Q, u, and y is isomorphic to the digraph induced by P, Q, u, and z, but there is no automorphism of D that maps one onto the other by fixing P, Q, and u and mapping y to z since $d_\Delta(b, y) = d_\Delta(b, z) - 2$, which follows from $d_\Delta(b, y) = d_\Delta(b, u) - 1$ as Δ is a tree. This shows $k = 2$. The case $\ell \geq 3$ is analogous, so we conclude $k = \ell = 2$ and $d^+ = d^- = 2$.

It remains to show that D contains a directed triangle. So let us suppose that there is no directed triangle in D. Let $z \in VD$, let x and y be the two predecessors of z and let z_1 be a successor of z. Due to the assumptions, $D[x, z, z_1]$ and $D[y, z, z_1]$ are induced 2-arcs and we conclude with C-homogeneity that $\Gamma := \mathrm{Aut}(D)_{zz_1}$ acts transitively on $\{x, y\}$. Let $z_1 z_2, \ldots$ be the ray with $z_2 \neq z$ in that reachability digraph that contains z and z_1. The group Γ must fix z_2 as $d^- = 2$ and, inductively, it fixes every z_i as also $d^+ = 2$. Let z_i be a vertex on that ray that has distance at least 3 to z. As above, there is a path P from z_i to a vertex a that lies in the same reachability digraph as the edge xz and has distance at least 3 to z such that every vertex of P has distance at least 3 from z. So neither x nor y has a neighbour on P. Furthermore, $\Gamma = \Gamma_{zz_1 \ldots z_i}$ acts transitively on $\{x, y\}$, so any successor or predecessor of x is also a successor or predecessor of y, respectively. Hence, the digraphs $D_1 := D[\{x, z, z_1, \ldots, z_i\} \cup VP]$ and $D_2 := D[\{y, z, z_1, \ldots, z_i\} \cup VP]$ are isomorphic. So the isomorphism that maps x to y and fixes all other vertices of D_1 extends to an automorphism α

of D that fixes $\langle \mathcal{A}(xz) \rangle = \langle \mathcal{A}(yz) \rangle$. Hence, $a = a\alpha$ has the same distance to x and to $x\alpha = y$. But because of $d^+ = d^- = 2$ the unique path in $\langle \mathcal{A}(xz) \rangle$ from z to a contains either x or y but not both. Thus, a has distinct distance to x and to y. This contradiction shows that D contains a directed triangle if $G(\Delta(D)) \cong T_{2,2}$. □

Lemma 6.13. *Let D be a locally finite connected C-homogeneous digraph with at most one end such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If $\Delta(D)$ is finite and if the intersection of any two reachability digraphs does not separate each of them, then no reachability digraph separates D.*

Proof. As in the proof of Lemma 6.12, we know that $\Delta(D)$ is bipartite. Let us suppose that there is a reachability digraph Δ_1 that separates D but that there are no two reachability digraphs whose intersection separates each of them. Let Δ_2 be a reachability digraph with $V\Delta_1 \cap V\Delta_2 \neq \emptyset$, let $x \in V\Delta_1 \cap V\Delta_2$, y be a neighbour of x in Δ_2, and z be a neighbour of y outside Δ_2. Note that z exists as otherwise every neighbour of y lies in Δ_2, which implies by C-homogeneity that every neighbour of x lies in a unique reachability digraph in contradiction to the choice of Δ_1 and Δ_2. Let D_i, $i = 1, 2$, be the component of $D - \Delta_i$ that contains y or contains z, respectively. If D_2 does not contain any vertex of Δ_1, then we have $D_2 \subseteq D_1$ with $D_2 \neq D_1$. So both D_1 and D_2 must be infinite since they are isomorphic – there is an automorphism α of D with $x\alpha = y$ and $y\alpha = z$, and this automorphism maps Δ_1 to Δ_2 and D_1 to D_2. As D is locally finite, it has one end in $D_1 \cap D_2$ and symmetrically also another one in $(D - D_1) \cap (D - D_2)$ contrary to the assumptions. Hence, D_2 contains a vertex of Δ_1 and $D_2 \not\subseteq D_1$. But then, as $\Delta_1 - \Delta_2$ is connected and $D - \Delta_2$ is not connected, there is another component D_2' of $D - \Delta_2$ that is completely contained in D_1 and contains no vertex of Δ_1. The component D_2' need not be isomorphic to D_1, but since there is a reachability digraph $\Delta_3 \neq \Delta_2$ in $D[VD_2' \cup N(VD_2')]$, we obtain a component D_3 of $D - \Delta_3$ with $D_3 \subseteq D_2'$ and so on. Because the degree of any vertex is finite, there are $m, n \in \mathbb{N}$ with $m \leq n$ such that D_m and D_n – or D_2' if m is 2 – are isomorphic and we obtain an analogous contradiction as before: we conclude as above that there is one end in D_1. On the other side the component E_n of $D - \Delta_n$ that contains $\Delta_m - \Delta_n$ contains an isomorphic component E_m of $D - \Delta_m$ and we obtain a second end of D in E_m, which is impossible due to our assumptions. □

The following lemma is the main lemma for the case that there is no isomorphic copy of C_3 in the C-homogeneous digraph.

Lemma 6.14. *Let D be a locally finite connected C-homogeneous digraph with at most one end that contains no directed triangle. If $N^+(x)$ and $N^-(x)$ are* *independent sets for all $x \in VD$, then D is isomorphic to $C_m[I_n]$ for some $m \geq 4$, $n \geq 1$.*

Proof. As in the previous proofs, we know that $\Delta(D)$ is bipartite. By Lemma 6.13, $\Delta(D)$ is finite. So due to Lemma 6.7, we may assume that $d^+ \geq 2$ and $d^- \geq 2$. Let $x \sim y$ for $x, y \in VD$ if x and y *lie on the same side* of a reachability digraph, that is, both have the same out-degree and the same in-degree in that reachability digraph and one of these two values is 0. If x and y lie in a common reachability digraph but not on the same side they lie *on distinct sides* of a reachability digraph. Remark that, a priori, $\sim$ is not an equivalence relation. But we shall show later that it is an equivalence relation in our situation.

Let us show for any two distinct reachability digraphs Δ_1 and Δ_2 with non-empty intersection the following property:

Either $\Delta_1 \cap \Delta_2$ lies on the same side of Δ_1 or $\Delta(D) \cong CP_k$ for some $k \geq 3$ and the intersection consists of precisely one unmatched pair in CP_k. (6.15)

Let us suppose that (6.15) does not hold. As reachability digraphs are induced subdigraphs, D consists of at least three reachability digraphs and $\Delta(D)$ cannot be a complete bipartite digraph because Δ_2 contains vertices on distinct sides of Δ_1. Therefore, D is either the complement of a perfect matching or a directed cycle according to Theorem 3.5 and Lemma 6.12. Let $x, y \in \Delta_1 \cap \Delta_2$ be on distinct sides of Δ_1 with minimal distance in Δ_1. So either all predecessors of x and all successors of y lie in Δ_1 or the other way round. Thus, x and y lie also on distinct sides of Δ_2. The distance between x and y in Δ_1 is at least 3 as they are not adjacent and as they do not lie on the same side of Δ_1. If $\Delta(D) \cong C_{2m}$ for some $m \geq 4$, then we choose a minimal path P in Δ_2 from x to y. Let x' be a neighbour of x in Δ_1, let y_1, y_2 be the two neighbours of y in Δ_1, and let y' be the neighbour of y on P. The subdigraphs induced by y', y, y_1 and by y', y, y_2 are isomorphic, as D contains no triangles at all – neither directed nor the unique second kind of triangles, as $N^+(x)$ is an independent set. Thus, there is an automorphism α of D that fixes y' and y and maps y_1 to y_2. This automorphism must fix the reachability digraph that contains the edge between y and y', which is Δ_2, and hence it fixes Δ_1, the only other reachability digraph that contains y, too. As $y\alpha = y$ and $y'\alpha = y'$, the automorphism α fixes one edge of Δ_2 and hence the whole digraph Δ_2 because of $\Delta_2 \cong C_{2m}$. In particular, we have $x\alpha = x$. Let P_i be the unique path in Δ_1 from y to x containing y_i, respectively. As α fixes x and y and maps y_1 to y_2, we conclude $P_1\alpha = P_2$. Thus, they have the same length, which must be m. Hence, x and y are the only vertices in $\Delta_1 \cap \Delta_2$. We conclude that the subdigraphs induced by $x'xPyy_1$

and $x'xPyy_2$ are isomorphic: if $x'xPyy_1$ is not a path, then y_1 is adjacent to a vertex z on P but also $y_1\alpha = y_2$ must be adjacent to $z\alpha = z$. As $x'xPyy_1$ and $x'xPyy_2$ are isomorphic via an isomorphism that fixes $x'xPy$, we conclude as before for y and x using the two paths Q_i in Δ_1 from y to x' such that y_i lies on Q_i that the distance between y and x' in Δ_1 is m, a contradiction. So $\Delta(D)$ is isomorphic to CP_k for some $k \geq 3$ and $\Delta_1 \cap \Delta_2$ consists of precisely two vertices that are not matched. This shows (6.15).

For $x, y \in VD$, let $x \approx y$ if x and y lie on the same side of two reachability digraphs. As every vertex lies in precisely two reachability digraphs, $\approx$ is an equivalence relation. The next aim is to show that $\sim$ and $\approx$ are (despite their different definition) the same relation, that is:

For all $x, y \in VD$, we have $x \sim y$ if and only if $x \approx y$. (6.16)

Let $x, y \in VD$. In order to prove (6.16), it suffices to show that $x \sim y$ implies $x \approx y$. So let us suppose that $x \sim y$ but $x \not\approx y$ and let Δ be the reachability digraph that contains x and y on the same side. If for every vertex each two of its successors are $\approx$-equivalent, then one whole side of Δ lies in a second reachability digraph Δ' on the same side. If $G(\Delta(D)) \not\cong K_{k,\ell}$, then its sides have the same size due to Theorem 3.5 as $\Delta(D)$ is finite by Lemma 6.12 and $d^+ \geq 2$ and $d^- \geq 2$, so $\sim$ and $\approx$ are the same relation in contradiction the choice of x and y. Hence for some vertex, two of its successors are not $\approx$-equivalent. Thus, we assume $G(\Delta(D)) \cong K_{k,\ell}$ but then some vertex in Δ' has two predecessors in $\Delta \cap \Delta'$ and by C-homogeneity each two of its predecessors, and hence one whole side of Δ', lie in $\Delta \cap \Delta'$. This shows also in the remaining situation that for some vertex two of its successors are not $\approx$-equivalent. Similarly, there is a vertex with two predecessors that are not $\approx$-equivalent. By C-homogeneity, for every vertex and each two of its successors z_1 and z_2, we have $z_1 \sim z_2$ but $z_1 \not\approx z_2$ and the same for any two of its predecessors. Thus, we may assume that x and y have distance 2 in $G(\Delta)$.

The next step is to show that no reachability digraph separates D. Let us suppose that the converse holds. Due to Lemma 6.13, there are two reachability digraphs whose intersection separates at least one of them. As there is a 2-arc in these two reachability digraphs, we can map them onto each two reachability digraphs with non-trivial intersection. Since there is no separating vertex in any of the possible reachability digraphs given by Theorem 3.5 as due to Lemma 6.12 and because of $d^+ \geq 2$ and $d^- \geq 2$ the digraph $\Delta(D)$ is not a tree, we conclude that each two reachability digraphs with at least one common vertex have at least two common vertices. Thus and due to (6.15), either the intersection of each two reachability digraphs is contained on the same side of each of them or

$\Delta(D) \cong CP_3$ since no two vertices in CP_k, for $k \geq 4$, separate that digraph. Let us first assume that $\Delta(D) \ncong CP_3$. Note that no two vertices with a common successor can lie in the intersection of two reachability digraphs, as otherwise C-homogeneity implies $x \approx y$. Thus, Theorem 3.5 implies that $G(\Delta(D))$ is a cycle of length $2m$ for some $m \geq 4$, as $\Delta(D)$ is finite by Lemma 6.12. Let a and b be two vertices in the same two reachability digraphs Δ_1 and Δ_2 of minimal distance to each other and let P be a minimal path between a and b in Δ_1. Let w_1, w_2 be the neighbours of b in Δ_2, let u_1 be the vertex on P that is adjacent to a, and let u_2 be a vertex in Δ_2 that is adjacent to a. With an analogous argument as in the proof of (6.15), we know that $aPbw_1$ and $aPbw_2$ induce isomorphic digraphs. Hence, there is an automorphism of D that maps the first onto the second one and fixes P. Thus, the distance in Δ_2 from a to w_1 is the same as the one from a to w_2. Because of $m \geq 4$, also the digraphs induced by u_2aPbw_1 and by u_2aPbw_2 are isomorphic. Thus, u_2 and w_1 have the same distance in Δ_2 like u_2 and w_2. But this cannot be true. Let us now assume that $\Delta(D) \cong CP_3$. Then the intersection of two reachability digraphs Δ_1, Δ_2 is, if it is not empty, precisely one unmatched pair a, b in each of the two reachability digraphs since $x \sim y$ but $x \not\approx y$. Let $uavw$ be a 3-arc in D. Let us assume that $ua \in E\Delta_1$ and $av \in E\Delta_2$. We cannot have $w \in V\Delta_2$, because Δ_2 contains no 2-arc. Since D contains no directed triangle, w cannot lie on the same side of Δ_1 as b since otherwise $wa \in E\Delta_1$. Since $v \notin V\Delta_1$, we have $vw \notin E\Delta_1$, so w cannot lie on the same side of Δ_1 as a. This shows $w \notin V\Delta_1$. Analogous arguments show that the same holds for any vertex w' in $\Delta_3 - \Delta_2$, where Δ_3 is the reachability digraph that contains the edge vw, because we find for w' either a 3-arc that has its first edge in Δ_1 and w' as its last vertex or a 3-arc whose first vertex is w' and whose last edge lies in Δ_1 where we may assume that this 3-arc contains b and the neighbour of v in the directed bipartite complement of Δ_2. Since $D[(V\Delta_2 \setminus V\Delta_1) \cup V\Delta_3]$ is connected, we know that $\Delta_2 - \Delta_1$ lies in one component of $D - \Delta_1$ and we can apply the proof of Lemma 6.13 to show that Δ_1 does not separate D. Hence, we proved in each case that no reachability digraph separates D.

Let v_1 be a vertex in the same reachability digraph as x and y that is adjacent to both x and y. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $xv_1, yv_1 \in ED$. Due to what we just showed, we find a second induced (aside from the edge yv_1) path R from v_1 to y whose only vertices in Δ are v_1 and y and that does not use the edge yv_1. We may choose R so that the only vertices on R that are adjacent to x are v_1 or the neighbour of y on R by applying C-homogeneity to an automorphism that fixes v_1 and maps x and y onto each other. Let v_3, v_2, y be the last three vertices on R. So we have $v_2y \in ED$. Since $v_2 \notin V\Delta$ and $x \not\approx y$, the vertices x

and v_2 are not adjacent. So v_1 is the only neighbour of x on R. If $v_3 \sim y$, then we have $v_2v_3 \in ED$. If $v_3 \sim x$, then as $v_2 \notin V\Delta$ their common reachability digraph must be the one that contains x and its predecessors. By definition of $\sim$, it must be $\langle \mathcal{A}(v_2v_3)\rangle = \langle \mathcal{A}(v_2y)\rangle$. So we have $x \approx y$ in contradiction to their choice. Thus, we have $v_3 \not\sim x$. By C-homogeneity, yv_1Rv_3 can be mapped onto xv_1Rv_3 by an automorphism of D that fixes v_1Rv_3 and thus, we obtain $v_3 \sim x$, a contradiction.

So we have $v_3 \not\sim y$ and hence $v_3v_2 \in ED$. As D is C-homogeneous, there is as above an automorphism α of D that maps yv_1Rv_3 onto xv_1Rv_3 and fixes v_1Rv_3. We conclude that there is a vertex $v_4 = v_2\alpha$ in D with $v_3v_4 \in ED$ and $v_4x \in ED$. Let v_0 be the neighbour of v_1 on R. Since $v_0 \notin \Delta$, we have $v_1v_0 \in ED$. As D contains no directed triangle and $N^+(x)$ is an independent set, D contains no triangle at all. Hence, there is an automorphism β of D that maps $v_3v_2yv_1$ onto $v_2yv_1v_0$. Let $y' = v_4\beta$ and $v_1' = x\beta$. The vertices $v_1, v_0, v_1', y', v_2, y$ induce a cycle. So if neither y' nor v_1' lies in Δ, then we could have chosen $R' = v_1v_0v_1'y'v_2y$ instead of R and obtain a contradiction as above since $y \sim y'$. Thus, either y' or v_1' lies in Δ. If y' lies in Δ, then we have that y and y' must lies on the same side of Δ since v_2 lies not in Δ. So we have $y \approx y'$ and hence also $v_1 \approx v_1'$. As v_1 and v_1' have a common successor, C-homogeneity implies that any two predecessors of any vertex are $\approx$-equivalent. In particular, we have $x \approx y$. Thus, y' does not lie in Δ, but v_1' does. If v_1' lies on the same side of Δ as v_1, then we obtain again $v_1 \approx v_1'$ and $x \approx y$. So v_1' lies on the same side as y and x. But then v_0 lies on the same side of Δ as v_1 and there is an edge between vertices of that side in contradiction to the assumption that $\Delta(D)$ is bipartite. This shows (6.16).

Since $\approx$ is an equivalence relation on VD, we conclude from (6.16) that the same is true for $\sim$. Let us define a digraph Γ on the equivalence classes of $\sim$ as vertices such that there is a directed edge from one class X_1 to a second class X_2 if and only if there are vertices $x_1 \in X_1$ and $x_2 \in X_2$ with $x_1x_2 \in ED$. By (6.16) each vertex of Γ has precisely one successor and one predecessor. Every equivalence class of $\sim$ is finite, since $\Delta(D)$ is finite by Lemma 6.12. If $G(\Gamma)$ is a double ray, then this implies that D has at least two ends. Since this is false, Γ must be a directed cycle C_n for some $n \geq 3$.

An edge e of Γ corresponds to a reachability digraph Δ of D in that the two equivalence classes of $\sim$ in Δ are the two vertices that are incident with e. Thus, it remains to show that $\Delta(D)$ is a complete bipartite digraph because then we have $D \cong C_n[I_n]$. Let $V_1, \ldots, V_n$ denote the equivalence classes of $\sim$ such that $V_iV_{i+1} \in E\Gamma$ for $i < n$ and $V_nV_1 \in E\Gamma$. Due to Theorem 3.5 and Lemma 6.12 and as $d^+ \geq 2$ and $d^- \geq 2$, we just have to show that $G(\Delta(D))$ is neither an undirected cycle C_{2m} nor the complement of a perfect matching CP_k.

First, let us suppose that $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 4$. Let $x \in V_1$ and let a, b be its successors. Let a_1 and a_2 be the successors of a. As D contains no directed triangle and as Γ is a directed cycle, x is adjacent neither to a_1 nor to a_2. Thus, there is an automorphism α of D that maps a_1 to a_2 and fixes a and x. Hence, also b must be fixed by α and the two a-b paths in $G(D[V_2 \cup V_3])$ must have the same length, which must be m. But then the same holds for the second predecessor $y \neq x$ of a with its two successors instead of x and its two successors. Thus, y also has to be adjacent to b and we have $m = 2$, a contradiction.

Let us now suppose that $\Delta(D) \cong CP_k$ for some $k \geq 3$. Let $x \in V_1$. If $n = 3$, then there is a directed triangle in D, as $k \geq 3$, which is impossible. So we conclude $n \geq 4$. There exists a unique vertex in V_2 that is not adjacent to x and this vertex itself has a unique vertex $y \in V_3$ to which it is not adjacent. Let P be a path that consists of x and of one vertex from every V_i for $i \geq 4$ such that the vertex in V_4 is the only vertex incident with all of V_3 but y. This path exists since $k \geq 3$. Let $X = D[(V_3 \setminus \{y\}) \cup VP]$, let x' be another vertex of V_1 that is adjacent to the predecessor of x on P and let $Y = D[(VX \setminus \{x\}) \cup \{x'\}]$. Then the finite subdigraphs X and Y are isomorphic but there is no automorphism of D that maps the first onto the second one, since there is a unique vertex in V_2 that is not adjacent to x and y, but for x' and y there is no such vertex. Hence, we have $\Delta(D) \not\cong CP_k$. So $\Delta(D)$ is a complete bipartite digraph. As D is C-homogeneous, it is transitive and thus, all equivalence classes have the same size, that is $\Delta(D) \cong K_{k,k}$ for some $k \geq 1$. As D contains no directed triangle, we also conclude that $n \geq 4$, which proves the assertion. □

Having completed the case that the locally finite connected C-homogeneous digraph with at most one end contains no directed triangle, we look at those that contain directed triangles. The following lemma is the main lemma for this situation. The case (iv) of the conclusions of Lemma 6.15 will be investigated in more detail in Section 6.3.

Lemma 6.15. *Let D be a locally finite connected C-homogeneous digraph that contains a directed triangle. If $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$, then one of the following cases holds.*

(i) *The digraph D has at least two ends.*

(ii) *The reachability digraph $\Delta(D)$ is isomorphic to a complete bipartite digraph $K_{k,k}$ for some $k \geq 3$ and D is isomorphic to $C_3[I_k]$.*

(iii) *The reachability digraph $\Delta(D)$ is isomorphic to CP_k for some $k \geq 4$ and D is isomorphic to Y_k.*

(iv) *The underlying undirected graph of the reachability digraph $\Delta(D)$ is isomorphic either to C_{2m} for some $m \geq 2$ or to $T_{2,2}$.*

Proof. Due to Lemma 6.11 and Proposition 2.1, the reachability digraph $\Delta(D)$ is bipartite. Let us assume that D has at most one end and that $G(\Delta(D))$ is neither isomorphic to C_{2m} for some $m \geq 2$ nor isomorphic to $T_{2,2}$. Due to Lemma 6.7, we may assume $d^+ \geq 2$ and $d^- \geq 2$. According to Lemma 6.12 and Theorem 3.5, we know that $\Delta(D)$ is finite and either a complete bipartite digraph or the directed complement of a perfect matching.

Let us first assume that $\Delta(D)$ is a complete bipartite digraph $K_{k,l}$ for $k, l \in \mathbb{N}$ but not $K_{2,2}$ as that is a cycle. By Lemma 6.9, we know that $k = l$. If we have $|\Delta \cap \Delta'| \geq 2$ for two distinct reachability digraphs Δ and Δ', then $\Delta \cap \Delta'$ lies on one side of Δ and it is a direct consequence of C-homogeneity that $\Delta \cap \Delta'$ is a complete side of Δ and hence of Δ' since some two vertex in $\Delta \cap \Delta'$ have a common predecessor x in either Δ or Δ' and by C-homogeneity each two successors of x lie in $\Delta \cap \Delta'$. But then, as shown in the proof of Lemma 6.14, we know that (ii) holds in this case. So let us suppose that there are two reachability digraphs Δ and Δ' with $|\Delta \cap \Delta'| = 1$. If an edge lies in more than one directed triangle, then it lies in at least $k-1$ distinct such triangles due to Lemma 6.10. So the intersection $\Delta \cap \Delta'$ has to contain at least $k-1$ elements which is a contradiction. Hence, every edge lies in a uniquely determined directed triangle.

To show that this situation cannot occur, let x and y be two vertices on the same side of Δ such that their out-degree in Δ is 0. Let u be a common predecessor of x and y. As every edge lies on a unique directed triangle, we find successors a, b of x, y, respectively, such that they are predecessors of u. As $k \geq 3$ and as every edge lies on precisely one directed triangle, there is a successor c of a and b such that neither $D[x, a, c]$ nor $D[y, b, c]$ are triangles, in particular, we may choose any successor of a except for u. As $k \geq 3$, there is a second predecessor z of b such that z and c as well as z and u are not adjacent. The vertices a and z cannot be adjacent because otherwise either y and x have to lie in two common reachability digraphs (if $za \in ED$) which we supposed to be false or z and c lie in a common reachability digraph (if $az \in ED$) and then it is not a bipartite reachability digraph because zbc is a 2-arc in that reachability digraph. Furthermore, zx cannot be an edge of D, because then the edge yb would have its two incident vertices on the same side of a reachability digraph. Let us suppose that xz is an edge of D. Then there is an automorphism α of D that maps $D[x, a, c, b]$ onto $D[z, b, c, a]$. We conclude that there is a vertex $z' = z\alpha \in N^-(a)$ with $zz' \in ED$. But the edge zz' has the wrong direction: in a bipartite reachability digraph all edges are directed from one side to the other, but zz' is directed the other way round compared with the edges xa, xz, and $z'a$. This contradiction shows that x and z cannot be adjacent. Hence, we have shown that the subdigraphs $D[x, a, c, b, y]$ and $D[x, a, c, b, z]$ are isomorphic. But there is no automorphism of D that maps one onto the other by fixing all of x, a, c, b, since x and y lie on the same side of a reachability digraph but x and z do not because of $uz \notin ED$. Thus, we showed that there are no two reachability digraphs whose intersection consists of precisely one vertex. This completes the case $\Delta(D) \cong K_{k,l}$.

The next and only remaining situation which we consider is that $\Delta(D) \cong CP_k$ for some $k \geq 3$. If $k = 3$, then $G(\Delta(D))$ is a cycle, so we may assume $k \geq 4$. Let Δ_1 and Δ_2 be two distinct reachability digraphs of D with non-trivial intersection. Let us suppose that $|\Delta_1 \cap \Delta_2| = 1$. Then this holds for any two distinct reachability digraphs with non-trivial intersection. Let $a, b, c, v, w \in V\Delta_1$ such that $b, v, w \in N^+(a)$ and $b, w \in N^+(c)$ but $cv \notin ED$. Such vertices exist as $k \geq 4$. Since any edge lies in a directed triangle, there are $x, y \in N^-(a)$ with $x \in N^+(v)$ and $y \in N^+(w)$. Because of $|\Delta_1 \cap \Delta_2| = 1$, no other edges than the described ones lie in $D[a, b, c, v, w, x, y]$. Then the digraphs $D_1 := D[a, b, c, x]$ and $D_2 := D[a, b, c, y]$ are isomorphic but there is no automorphism of D that maps D_1 onto D_2 because such an automorphism has to map v, the unique predecessor of x in Δ_1, onto w, the unique predecessor of y in Δ_1, but w is adjacent to c and v is not. Thus, we have proved

$$|\Delta_1 \cap \Delta_2| \geq 2. \tag{6.17}$$

Let us suppose that $\Delta_1 \cap \Delta_2$ is not contained in any of the sides of Δ_1. Then $\Delta_1 \cap \Delta_2$ consists of precisely two vertices that are adjacent in the directed bipartite complement of Δ_1 and, furthermore, any edge lies in at most two directed triangles (because of $|\Delta_1 \cap \Delta_2| = 2$) and by Lemma 6.10 any edge lies in precisely one directed triangle (because of $k \geq 4$). Let us consider the subdigraph of Δ_1 with vertices a, b, c, d and edges ba, bc, dc such that $\{a, d\} = V(\Delta_1 \cap \Delta_2)$. Let z be the vertex on the unique directed triangle that contains ba and let x and y be two predecessors of d in Δ_2 such that x is the neighbour of z in the directed bipartite complement of Δ_2 and such that neither x nor y is adjacent to c. We can choose them in this way as $k \geq 4$ and as dc lies in precisely one directed triangle. Furthermore, neither x nor y can be adjacent to b, as b and d do not lie in two common reachability digraphs. Hence, the subdigraphs $D[b, c, d, x]$ and $D[b, c, d, y]$ are isomorphic to each other, so there is an automorphism α of D that fixes each of b, c, and d and maps x to y. Then also a must be fixed by α, as it is the unique neighbour of d in the directed bipartite complement of Δ_1, and hence, we also have $z\alpha = z$ by the choice of z.

But this is impossible because y and z are adjacent in contrast to x and z. Thus, we proved that $\Delta_1 \cap \Delta_2$ is contained in one side of Δ_1. C-homogeneity directly implies that $\Delta_1 \cap \Delta_2$ is a whole side of Δ_1, as we can map any two vertices of $\Delta_1 \cap \Delta_2$ with a common neighbour in Δ_1 onto any other two vertices on the same side as $\Delta_1 \cap \Delta_2$ of Δ_1 with a common neighbour in Δ_1. Thus, we have

$$|\Delta_1 \cap \Delta_2| = k. \tag{6.18}$$

Now, we are able to prove $D \cong Y_k$. Due to (6.18) and as every edge lies in a directed triangle, D consists of precisely three reachability digraphs Δ_1, Δ_2, and Δ_3. Let $V_i := V\Delta_i \cap V\Delta_{i+1}$ with $\Delta_4 = \Delta_1$ and let $\overline{D}$ denote the directed tripartite complement of D. Since $\Delta(D) \cong CP_k$, the digraph $\overline{D}$ is a union of directed cycles. We shall show that every component of $\overline{D}$ is a directed cycle of length 3. So let us suppose that this is not the case. Then there are $x, y \in V_1$ that lie in a common directed cycle of length at least 6 in $\overline{D}$ and have distance 3 on that cycle. Since $k \geq 4$, there is a vertex $a \in V_2$ that is adjacent in D to both x and y. We conclude by C-homogeneity that for every vertex $z \in V_1$, distinct from x, we have that x and z lie on a common directed cycle in $\overline{D}$ and have distance 3 on that cycle. It is a direct consequence that $k \leq 3$ in contrast to the assumption $k \geq 4$. Hence, we have shown $D \cong Y_k$. □

6.3 An imprimitive case

In this section, we investigate the situation from Lemma 6.15 (iv): we look at locally finite connected C-homogeneous digraphs that contain directed triangles, all whose vertices have independent out- and in-neighbourhood and for whose reachability digraph the underlying undirected graph is either $T_{2,2}$ or C_{2m} for some $m \geq 2$. In [14], Gray and Möller showed the existence of such a digraph, in that they showed that $X_2(C_3)$ has all these properties. It has infinitely many ends. But although we are interested only in digraphs with at most one end, this particular digraph turns out to be very important in our situation: we shall show that every digraph with the above described properties and with at most one end is a homomorphic image of $X_2(C_3)$. More precisely, we prove:

Theorem 6.16. *The following assertions are equivalent for any locally finite connected digraph D all whose vertices have independent out- and in-neighbourhood.*

(i) *The digraph D is C-homogeneous and contains a directed triangle. If $D \not\cong C_3$, then the underlying undirected graph of its reachability digraph is either $T_{2,2}$ or C_{2m} for some $m \geq 2$.*

(ii) *There is a non-universal* $\mathrm{Aut}(X_2(C_3))$*-invariant equivalence relation* $\sim$ *on* $VX_2(C_3)$ *such that* $X_2(C_3)_\sim$ *is a digraph that is isomorphic to* D.

Furthermore, D *has at most one end if and only if one, and hence every, equivalence class of* $\sim$ *consists of more than one element.*

Proof. To see that (i) implies (ii), we may assume that D is not isomorphic to C_3: otherwise take any labeling of the vertices of $X_2(C_3)$ with labels in $\{0,1,2\}$ such that no two adjacent vertices have the same label and such that out-neighbours of vertices labeled by i are labeled by $i+1 \pmod 3$. This labeling induces an $\mathrm{Aut}(X_2(C_3))$-invariant equivalence relation $\sim$ on $VX_2(C_3)$ such that $X_2(C_3)_\sim$ is a directed triangle.

Therefore, every vertex of D has out-degree 2. So every edge lies in at most two directed triangles. Let us first assume that every edge of D lies in precisely two directed triangles. For an edge xy, the two successors of y are the two predecessors of x. So the other successor of x must have the same successors as y. The analogous statements hold for the second predecessor of y. It is a direct consequence that $G(\Delta(D)) \cong C_4 \cong K_{2,2}$ and that $D \cong C_3[I_2]$. Let x_i, y_i, z_i for $i = 1,2$ be the vertices of D such that x_iy_j, y_iz_j and z_ix_j, for all $i,j \in \{1,2\}$, are the edges of D. We label the vertices of $X_2(C_3)$ with labels from $V(D)$ so that for every vertex labeled by x_i its successors obtain different labels from $\{y_1, y_2\}$ and its predecessors obtain different labels from $\{z_1, z_2\}$ and so that the analogue statements hold for vertices labeled by y_i and by z_i. Starting with a triangle labeled by $x_1y_1z_1$, there is a unique way to extend its labelling to the whole digraph $X_2(C_3)$ such that the just described property holds. Two vertices are $\sim$-equivalent if they have the same label. Then by definition, $X_2(C_3)_\sim$ is a digraph and isomorphic to D. Furthermore, the $\mathrm{Aut}(X_2(C_3))$-invariance is a consequence of the unique extension property of the labeling by starting it at a directed triangle.

Let us now assume that every edge of D lies in precisely one directed triangle. As $d^+ = 2$, every vertex lies in precisely two. Let $xy \in ED$ and $ab \in EX_2(C_3)$. For every vertex u in $X_2(C_3)$ there exists a unique shortest path $P = a_1 \dots a_n$ from a to u. In D there are precisely two walks $x_1 \dots x_n$ and $y_1 \dots y_n$ starting at x (i.e. with $x_1 = x = y_1$) such that $D[x_i, x_{i+1}, x_{i+2}]$ and $D[y_i, y_{i+1}, y_{i+2}]$ are isomorphic to $D[a_i, a_{i+1}, a_{i+2}]$ for all $i \leq n-2$ in the canonical way (i.e. such that x_i and y_i are mapped to a_i and so on). That there are precisely two such walks in D follows from the fact that every vertex of D lies in precisely two directed triangles and in the middle of precisely two induced 2-arcs. In particular, no two end vertices of any subpath of length 2 of the walks in D are adjacent. If $a_2 = b$ or if a_2 is adjacent to b, then let Q be that one of the two above described walks in D whose second vertex is y or is adjacent to y, and in the other case for a_2 let Q be the other described walk in D. Let u_D denote the last vertex of Q. Thereby, we define for every vertex v of $X_2(C_3)$ a vertex v_D in D.

We are now able to define the equivalence relation $\sim$: let $u \sim v$ for two vertices $u, v \in VX_2(C_3)$ if $u_D = v_D$. Obviously, this is a non-universal equivalence relation. It remains to show that $X_2(C_3)_\sim$ is a digraph, that $D \cong X_2(C_3)_\sim$ and that $\sim$ is $\mathrm{Aut}(X_2(C_3))$-invariant. Let us first show that $\sim$ is $\mathrm{Aut}(X_2(C_3))$-invariant. Let π be the map from $X_2(C_3)$ to D that maps z to z_D, let $u, v \in VX_2(C_3)$ with $u \sim v$ and let ψ be an automorphism of $X_2(C_3)$. It suffices to show $u\psi \sim v\psi$. First, let us consider the case that the shortest path $P = u_1 \dots u_n$ from u to v does not contain any other vertex of the equivalence class that contains u. If we have shown this, then the assertion follows by an easy induction on the number of elements on P that are equivalent to u. We look at the images of P and $P\psi$ under π. These are walks due to the definition of π, because adjacent vertices in $X_2(C_3)$ are mapped to adjacent vertices of D. As $u \sim v$, the walk $P\pi$ starts and ends at the same vertex u_D. For every $i \leq n$, we can map $(u_1 \dots u_i)\pi$ onto $(u_1 \dots u_i)\psi\pi$ inductively, since D is C-homogeneous and since $(u_{i+1})_D$ is uniquely determined in D by the two walks $(u_1 \dots u_i)\pi$ and $(u_1 \dots u_i)\psi\pi$. We conclude that also the walk $P\psi\pi$ has the same end vertices. So we have $u\psi \sim v\psi$. Hence, $\sim$ is $\mathrm{Aut}(X_2(C_3))$-invariant.

Next, we show that $X_2(C_3)_\sim$ is a digraph. That there are no loops in $X_2(C_3)_\sim$ is a direct consequence of the definition of $\sim$, as we do not have $a'_D = a_D$ for any neighbour a' of a and as D is $\mathrm{Aut}(X_2(C_3))$-invariant. The only other obstacle for $X_2(C_3)_\sim$ being a digraph is that the edge set contains loops or is not antisymmetric. Another consequence of the definition of $\sim$ is that no two neighbours of a are $\sim$-equivalent, as every vertex of D and every vertex of $X_2(C_3)$ lies in precisely two directed triangles. Let us suppose that there are vertices a_1, a_2, b_1, and b_2 in $X_2(C_3)$ with a_1a_2, $b_1b_2 \in EX_2(C_3)$ and $a_1 \sim b_2$ and $a_2 \sim b_1$. Due to transitivity of $X_2(C_3)$, there is an automorphism α of $X_2(C_3)$ that maps a_2 to b_1. Since $\sim$ is $\mathrm{Aut}(X_2(C_3))$-invariant, there is also an in-neighbour of b_1 in the same equivalence class as b_2, which is impossible as we already saw. Thus, we have shown that $X_2(C_3)_\sim$ is a digraph.

That D and $X_2(C_3)_\sim$ are isomorphic is a direct consequence of the definition of $\sim$, since they have the same in- and out-degree. This shows (ii).

Let us now assume that (ii) holds, more precisely, that $D = X_2(C_3)_\sim$. We shall prove (i). As $X_2(C_3)$ is vertex-transitive so is D. Let us assume that D is not a directed triangle. So every vertex of D has two successors and, as every edge lies in a directed triangle since they do so in $X_2(C_3)$, every vertex of D lies in at least two directed triangles and no two neighbours of a vertex of $X_2(C_3)$ are $\sim$-equivalent. Thus, for every $uv \in ED$ and every $x \in X_2(C_3)$ whose equivalence class is u, there is a vertex $y \in N^+(x)$ whose equivalence class is v, as $d^+(x) = 2 = d^-(x)$. We also obtain that $\Delta(D)$ is a homomorphic image of $\Delta(X_2(C_3))$, so its underlying undirected graph is either $T_{2,2}$ or C_{2m} for some $m \geq 2$. To show that D is C-homogeneous, let A and B be isomorphic induced connected subdigraphs of D and let $\varphi : A \to B$ be an isomorphism. Let T_A be a spanning tree of A. Then we can map T_A by an injective homomorphism π_A to $X_2(C_3)$ such that a is the equivalence class of $\pi_A(a)$ for all $a \in VA$. Notice that π_A is uniquely determined by the image of one vertex of A. Analogously, we define T_B and π_B such that $T_B = T_A\varphi$. The subdigraphs of $X_2(C_3)$ induced by $A' := T_A\pi_A$ and $B' := T_B\pi_B$ are isomorphic by an isomorphism that induces on the equivalence classes of the vertices of A' and of B' the isomorphism φ. As $X_2(C_3)$ is C-homogeneous, this isomorphism extends to an automorphism ψ of $X_2(C_3)$. Since $\sim$ is $\mathrm{Aut}(X_2(C_3))$-invariant, this automorphism induces an automorphism ϕ of D that extends φ. So D is C-homogeneous.

The only remaining part to show is the additional claim on multi-ended digraphs which is a direct consequence of [14, Theorem 7.1], because $X_2(C_3)_\sim$ is not isomorphic to $X_2(C_3)$ as soon as each equivalence class contains at least two elements. □

Figure 6.2 shows two C-homogeneous digraphs that arise as quotient digraphs in Theorem 6.16 one of which is finite and the other being infinite and one-ended. In the finite digraph the edges of each reachability digraph, which is isomorphic to C_{10}, are drawn in different styles. The reachability digraphs of the infinite digraph are the cycles of length 6.

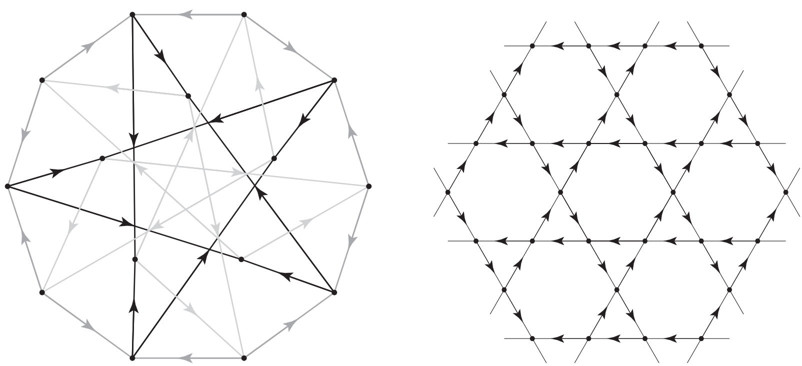

Figure 6.2: A finite and an infinite one-ended C-homogeneous digraph

As the automorphism group of $X_2(C_3)$ is a free product of the cyclic groups

C_2 and C_3, it is isomorphic to the modular group. Let us consider the Cayley digraph Λ of $\Gamma := C_2 * C_3 = \langle x\rangle * \langle y\rangle$ with respect to the two canonical generators x and y. If we contract the edges in Λ that correspond to the involution x, then we obtain the digraph $X_2(C_3)$. Let $\sim$ be an $\mathrm{Aut}(X_2(C_3))$-invariant equivalence relation on $VX_2(C_3)$ and let X be the equivalence class that contains the vertex that arose from 1 and x in Λ by contracting the edges labeled by x. It is straight-forward to show that X corresponds to vertices of Λ that coincide with a subgroup of Γ that contains x. Conversely, the cosets of any subgroup of Γ that contains x induce in a canonical way a partition of $VX_2(C_3)$ and hence an equivalence relation of $VX_2(C_3)$ that is $\mathrm{Aut}(X_2(C_3))$-invariant. Therefore, instead of giving a precise list of the digraphs that may occur as quotients in Theorem 6.16, it is equivalent to describe all those subgroups of $C_2 * C_3$ that contain x. By Kurosh's Subgroup Theorem [27], every subgroup of the modular group is a free product of cyclic groups of orders 2, 3, or ∞ and the involutions form a conjugacy class in Γ. Thus, any subgroup of Γ that contains an involution is – up to conjugation – an example of a subgroup that corresponds to a C-homogeneous digraph in Theorem 6.16. As the number of cosets of a subgroup of Γ coincides with the number of vertices in the C-homogeneous digraph to which it corresponds in the above sense, the subgroups of finite index correspond to the finite and the subgroups of infinite index correspond to the infinite C-homogeneous digraphs in Theorem 6.16. There are numerous papers written on the subgroups of the modular group. Some of them deal with those of finite index, see [23, 33], and some with those of infinite index, see [36, 37, 38].

6.4 The classification result for locally finite C-homogeneous digraphs with at most one end

Let us now state our main result. We shall prove it by applying all the results of the previous sections.

Theorem 6.17. *Let D be a locally finite connected digraph with at most one end. Then D is C-homogeneous if and only if one of the following cases holds.*

(i) $D \cong C_m[I_n]$ *for integers* $m \geq 3, n \geq 1$*;*

(ii) $D \cong H[I_n]$ *for an integer* $n \geq 1$*;*

(iii) $D \cong Y_k$ *for an integer* $k \geq 3$*;*

(iv) *there exists a non-trivial* $\mathrm{Aut}(T_2(C_3))$*-invariant equivalence relation* $\sim$ *on* $VT_2(C_3)$ *such that* $D \cong T_2(C_3)_\sim$*.*

Proof. Let us assume that D is C-homogeneous and that D has at least one edge. If the out-neighbourhood (or symmetrically the in-neighbourhood) of any vertex of D is not independent, then we conclude from Theorem 6.6 that D is finite and isomorphic to $H[I_n]$ for some $n \geq 1$. So we may assume that the out-neighbourhood of each vertex is independent. Then, it is a direct consequence of Lemma 6.14, Lemma 6.15, and Theorem 6.16 that either (ii), (iv), or (v) holds. As we already proved that all digraphs mentioned in the theorem are C-homogeneous, we completed this classification result. $\square$

Chapter 7

The case: countably infinite degree and one end

7.1 The case: $D^+ \ncong I_n \ncong D^-$

In this section we will investigate the situation that D^+ contains some edge. Before we tackle this situation, we first show some general lemmas. Remember that we showed in Lemma 6.1 that D^+ and D^- are homogeneous digraphs if D is a C-homogeneous digraph. Thus, we are able to go through the list of countable homogeneous digraphs and look at each of them one by one, which is the general strategy for the proof of our main theorem.

Lemma 7.1. *Let D be a countable connected C-homogeneous digraph with infinite out-degree. Then either the in-degree is also infinite or D^+ is isomorphic to either I_ω or $I_\omega[C_3]$.*

Proof. The claim follows directly from Theorem 3.11 and Lemma 6.1. □

As the locally finite C-homogeneous digraphs have already been classified by Theorem 6.17, the previous lemma allows to concentrate (mostly) on digraphs with infinite D^+.

Lemma 7.2. *Let D be a C-homogeneous digraphs such that it contains isomorphic copies of every orientation of C_5. Then the diameter of D is 2.*

Proof. Since D contains some orientation of C_5, it contains two non-adjacent vertices. Hence, the diameter of D is at least 2. Let us suppose that D does not have diameter 2. Let x and y be vertices of distance 3 in D and P be a shortest path between them. Then there is an injection from P into one of the orientations of C_5. Let C be a copy of this orientation in D. By C-homogeneity, we find an automorphism α of D that maps P into C. Let z be the vertex on C that is adjacent to the end vertices of $P\alpha$. Then $z\alpha^{-1}$ is adjacent to x and y, which is a contradiction to the choice of these two vertices. Thus, D has diameter 2. □

Lemma 7.3. *Let D be a C-homogeneous digraph such that it contains isomorphic copies of all orientations of C_4. Then each two non-adjacent vertices of D have a common successor and a common predecessor.*

Proof. Let a and b be two non-adjacent vertices of D. By Lemma 7.2 there is a vertex x that is adjacent to a and b. Since every orientation of C_4 embeds into D, there is one such copy that has an isomorphic image of $D[a, x, b]$ in D such that the images of a and b are the predecessors of the fourth vertex y, and there is one such image such that the images of a and b are the successors of the fourth vertex y'. By C-homogeneity, we can map $D[a, x, b]$ onto such copies by automorphisms α, β of D. Then $y\alpha^{-1}$ and $y'\beta^{-1}$ verify the assertion. □

7.1.1 Generic I_n-free digraphs as D^+

Throughout this section, let D be a countable connected C-homogeneous digraph such that D^+ is isomorphic to the countable generic I_n-free digraph for some integer $n \geq 3$. (Note that $n = 2$ implies that D^+ is a tournament. We consider this case in a later section.) Our first step is to show that D^+ and D^- are isomorphic.

Lemma 7.4. *We have $D^+ \cong D^-$.*

Proof. Let F be any finite I_n-free digraph. Then we find an isomorphic copy of F in D^+ and, in addition, we find a vertex $x \in VD^+$ with $yx \in ED$ for all $y \in VF$. Hence, D^- contains an isomorphic copy of F.

Since D^- contains every finite I_n-free digraph, it is a direct consequence of Theorem 3.11 that D^- is either a generic I_m-free digraph for some $m \geq n$ or a generic $\mathcal{H}$-free digraph with $\mathcal{H} = \emptyset$. The latter or the first with $m > n$ is impossible since they contain a vertex with n independent successors. So D^- is also the countable generic I_n-free digraph. □

Our next aim is to show that every finite induced I_n-free subdigraph of D lies in $D^+(x)$ for some $x \in VD$. We do this in two steps and begin with the case that the subdigraph is some I_m with $m < n$.

Lemma 7.5. *If $H \subseteq D$ is an isomorphic copy of I_m for some $m < n$, then there exist vertices $x, y \in VD$ with $H \subseteq D^+(x)$ and with $H \subseteq D^-(y)$.*

Proof. Note that $d^- \neq 0$. So for $m = 1$ the assertion is obvious and for $m = 2$ it follows from Lemma 7.3. Let $m \geq 3$ and let $H \cong I_m$ be a subdigraph of D with $VH = \{x_1, \ldots, x_m\}$. By induction, we find $a, b \in VD$ with $\{x_1, \ldots, x_{m-1}\} \subseteq N^+(a)$ and $\{x_2, \ldots, x_m\} \subseteq N^+(b)$. The digraph $F := H + a + b$ is connected because of $m \geq 3$. Since F is I_n-free, $D^+(a)$ contains an isomorphic copy F' of F. Applying C-homogeneity, we find an automorphism α of D that maps F' to F. So we have $H \subseteq F' \subseteq D^+(a\alpha)$.

By an analogous argument, we find $y \in VD$ with $H \subseteq D^-(y)$. □

Lemma 7.6. *If $H \subseteq D$ is a finite induced I_n-free digraph, then there exist vertices $x, y \in VD$ with $H \subseteq D^+(x)$ and $H \subseteq D^-(y)$.*

Proof. If $H \cong I_m$ for some $m < n$, then the assertion follows from Lemma 7.5. So we may assume that H has a vertex a with $N^+(a) \cap VH \neq \emptyset$. By induction, there is a vertex u in D with $H - a \subseteq D^+(u)$. Thus, $H + u$ is connected and I_n-free. Applying an analogous argument as in the proof of Lemma 7.5, we find a vertex x in D with $H \subseteq H + u \subseteq D^+(x)$.

The existence of y follows analogously. □

Our next aim is to show that, for any two disjoint finite induced I_n-free digraphs A and B, we find a vertex x in D with $A \subseteq D^+(x)$ and $B \subseteq D^-(x)$. We do not know whether this is true, even if we assume that A is maximal I_n-free in $A + B$. In particular, we need more structure on $D[N(x)]$ than we have till now. But if we make the additional assumption that we find an isomorphic situation *somewhere* in D, that is, if we find subdigraphs A' and B' such that there exists an isomorphism $\varphi\colon A + B \to A' + B'$ with $A\varphi = A'$ and $B\varphi = B'$ and if $A' + B'$ has the claimed property, then we find such a vertex x without any further knowledge on the structure of $D[N(x)]$.

Lemma 7.7. *Let A be a finite induced I_n-free subdigraph of D and let $z \in VD$ such that z has a predecessor in A. Then there exists a vertex $x \in VD$ with $A \subseteq D^-(x)$ and $z \in N^+(x)$.*

Proof. Due to Lemma 7.6, we find a vertex $v \in VD$ with $A \subseteq D^+(v)$. Let $a \in VA$ be a predecessor of z. Let $x_1, \ldots, x_{n-1}$ be $n - 1$ independent vertices in $N^+(v)$ with $a'x_i \in ED$ for all $a' \in VA$. These vertices exist as $D[A, x_1, \ldots, x_{n-1}]$ is I_n-free by construction and as D^+ is the generic I_n-free digraph. All the vertices $z, x_1, \ldots, x_{n-1}$ lie in $N^-(a)$, so they cannot be independent. By the choice of the x_i, we know that z must be adjacent to at least one of them, say x_i. As $A+z$ is not I_n-free, we do not have $zx_i \in ED$. Hence, we have $x_iz \in ED$ and x_i is a vertex we are searching for. $\square$

Lemma 7.8. *Let A, B, A', B' be finite induced I_n-free subdigraphs of D such that an isomorphism $\varphi\colon A' + B' \to A + B$ with $A'\varphi = A$ and $B'\varphi = B$ exists. If A is maximal I_n-free in $A+B$ and if D has a vertex v with $A' \subseteq D^+(v)$ and $B' \subseteq D^-(v)$, then there exists $x \in VD$ with $A \subseteq D^+(x)$ and $B \subseteq D^-(x)$.*

Proof. If $A+B$ is connected, then the assertion is a direct consequence of C-homogeneity and, if B has no vertex, then the assertion follows from Lemma 7.6. So we may assume that there is some $z \in VB$. Let $z' = z\varphi^{-1}$.

By induction, we find a vertex w with $A \subseteq D^+(w)$ and $B - z \subseteq D^-(w)$. Hence, we can map $A' + B' - z' + v$ onto $A + B - z + w$ by an automorphism α of D with $u\alpha = u\varphi$ for all $u \in V(A + B - z)$. Taking $A'\alpha$, $B'\alpha$, $z'\alpha$, and $v\alpha$ instead of A', B', z', and v shows that

$$\text{we may assume } A' = A \text{ and } B' - z' = B - z. \tag{7.1}$$

Let $u \in VA$ be in a component of $A+B$ that does not contain z. Because of $n \geq 3$ and $z'v \in ED$, the subdigraph $D[v, z, z']$ is I_n-free. So by Lemma 7.7 we find a vertex y with $v, z, z' \in N^-(y)$ and $u \in N^+(y)$. The digraphs $(A+y)+B$ and $(A+y)+B'$ are isomorphic and have less components than $A+B$. As $A'+y \subseteq D^+(v)$ and $B' \subseteq D^-(v)$, we find $x \in VD$ with $A+y \subseteq D^+(x)$ and $B \subseteq D^-(x)$ by induction on the number of components, which finishes the proof. $\square$

Now we are able to prove the main result of this section:

Proposition 7.9. *Let D be a countable connected C-homogeneous digraph such that D^+ is the countable generic I_n-free digraph for some $n \geq 3$. Then D is homogeneous.*

Proof. Let A and B be two finite isomorphic induced subdigraphs of D and let $\varphi\colon A \to B$ be an isomorphism. If A is conntected, then φ extends to an automorphism of D by C-homogeneity. So let us assume that A is not connected. Let $A_1 \subseteq A$ be maximal I_n-free with vertices from at least two distinct components of A and let $A_2 \subseteq A - A_1$ be maximal I_n-free such that for some $x \in VD$ there is an isomorphic copy of $D[A_1, A_2]$ in $D[N(x)]$ such that the image of A_1 lies in $D^+(x)$ and the image of A_2 lies in $D^-(x)$. For $B_1 := A_1\varphi \subseteq B$ and $B_2 := A_2\varphi \subseteq B$, the corresponding statements hold. According to Lemma 7.8, we find two vertices x_A and x_B with $A_1 \subseteq D^+(x_A)$ and $A_2 \subseteq D^-(x_A)$ and with $B_1 \subseteq D^+(x_B)$ and $B_2 \subseteq D^-(x_B)$. Then x_A has no neighbour in $A - (A_1 + A_2)$ by the maximalities of A_1 and A_2 and, analogously, x_B has no neighbour in

$B - (B_1 + B_2)$. Hence, φ extends to an isomorphism φ' from $A + x_A$ to $B + x_B$ and these two subdigraphs of D have less components than A and B. By induction on the number of components, φ' extends to an automorphism of D and so does φ. □

7.1.2 Generic $\mathcal{H}$-free digraphs as D^+

In the following, let D be a countable connected C-homogeneous digraph such that D^+ is the countable generic $\mathcal{H}$-free digraph for some set $\mathcal{H}$ of finite tournaments on at least three vertices. (If we exclude the tournament on two vertices, then D^+ is an edgeless digraph. These will be investigated in Section 7.2.) In this section, we investigate the largest class of homogeneous digraphs: the class of the countable generic $\mathcal{H}$-free digraphs contains uncountably many elements, as Henson [21] proved, whereas all the other classes contain only countably many elements.

Lemma 7.10. *There is a set $\mathcal{H}'$ of finite tournaments on at least three vertices such that D^- is the generic $\mathcal{H}'$-free digraph.*

Proof. With a similar argument as in the proof of Lemma 7.4, the assertion follows from Theorem 3.11. □

For the remainder of this section, let $\mathcal{H}'$ be the finite set of tournaments we obtain from Lemma 7.10.

Our next aim is to show that every finite induced $\mathcal{H}$-free subdigraph of D lies in $D^+(x)$ for some $x \in VD$.

Lemma 7.11. *For every two disjoint finite induced $\mathcal{H}$-free tournaments A and B in D, there exists a vertex x with $A + B \subseteq D^+(x)$.*

Proof. If $|VA| = 1 = |VB|$, then the assertion follows directly from Lemma 7.3, because D^+ embeds every orientation of C_4. So we may assume $|VA| \geq 2$ and $|VA| \geq |VB|$. Let $a \in VA$ such that a has a successor in $A^- := A - a$. By induction on $|VA| + |VB|$, we find a vertex v with $A^- + B \subseteq D^+(v)$. Since D^+ is generic $\mathcal{H}$-free, there is a vertex $w \in N^+(v)$ that has precisely one successor a' in A^- and one successor b in B. If a and w are not adjacent, then $A + B + w$ is connected and $\mathcal{H}$-free. Hence, the out-neighbourhood of some vertex of D contains an isomorphic copy of $A + B + w$ and, by C-homogeneity, there exists a vertex x with $A + B + w \subseteq D^+(x)$. So we assume in the following that w and a are adjacent. Note that the only triangle in $A + B + w$ is the transitive triangle $D[a, a', w]$. Hence, if $\mathcal{H}$ does not contain the transitive triangle, then $A + B + w$ is $\mathcal{H}$-free and connected and we find a vertex x with $A + B + w \subseteq D^+(x)$. So we assume for the remainder of this proof that $\mathcal{H}$ contains the transitive triangle.

First, we consider the case $|VA| = 2$ and $|VB| = 1$. If $wa \in ED$, then $A + B \subseteq D^+(w)$ and w is a vertex we are searching for. If $aw \in ED$, let $w' \in N^+(w)$ with $w'a', w'b \in ED$, which exists as $D^+(w)$ is generic $\mathcal{H}$-free. If $aw' \in ED$, then $D[w, w', a']$ is a transitive triangle that lies in $D^+(a)$, which is impossible by the choice of $\mathcal{H}$. Hence, either $w'a \in ED$ or a and w' are not adjacent, and the assertion follows as before, just with w instead of w'.

The next case that we look at is $|VA| = 2 = |VB|$. Let b' be the second vertex in B. As $D^+(v)$ is generic $\mathcal{H}$-free, we find a vertex $c \in N^+(v)$ with $cb' \in ED$ but that is adjacent to neither a' nor b. If a and c are adjacent, then $D[a, a', c, b', b]$ is connected and $\mathcal{H}$-free, so we find $x \in VD$ with $A+B+c \subseteq D^+(x)$. Thus, let us assume that a and c are not adjacent. Then let $d \in N^+(v)$ with $da', dc \in ED$ and $b, b' \notin N(d)$, which exists as $D^+(v)$ is generic $\mathcal{H}$-free. If a and d are not adjacent, then $D[a, a', d, c, b', b]$ is connected and $\mathcal{H}$-free, so we find $x \in VD$ with $A + B + c + d \subseteq D^+(x)$ as before. Hence, we may assume that a and d are adjacent. Considering the edge between b and b' and the edge between a and d, we find by induction a vertex v' with $a, b', d \in N^+(v')$. The connected subdigraphs $D[a', d, v', b', b]$ and $D[a', a, v', b', b]$ are isomorphic, so we find by C-homogeneity and automorphism α of D that fixes a', v', b', and b and maps d to a. Then $A+B = (A^- + B + d)\alpha \subseteq D^+(v\alpha)$ proves the assertion in this case.

The only remaining case is $|VA| \geq 3$. Let $\hat{a}$ be a vertex in $N^+(v)$ such that there exists an isomorphism from $A^- + B + \hat{a}$ to $A + B$ that fixes $A^- + B$. This vertex exists as $A + B$ is $\mathcal{H}$-free and hence has an isomorphic copy in the generic $\mathcal{H}$-free digraph $D^+(v)$. As $D^+(v)$ is homogeneous, we then may assume that this copy conincides with $A + B$ on $A^- + B$. Note that we may have chosen w such that w and $\hat{a}$ are not adjacent. Let $c \in VA^-$ be a vertex that is not adjacent to w. If a and $\hat{a}$ are adjacent, let $F = D[\hat{a}, a, w, b]$ and let $F = D[\hat{a}, c, a, w, b]$ otherwise. Then F is connected contains no triangle, so it is $\mathcal{H}$-free and we find a vertex x with $F \subseteq D^+(x)$. Then there is an isomorphism from $A^- + B + x + \hat{a}$ to $A + B + x$ that fixes $A^- + B + x$. This isomorphism extends to an automorphism α of D by C-homogeneity. Then $A + B = (A^- + B + \hat{a})\alpha \subseteq D^+(v\alpha)$ shows the remaining case of the lemma. □

Lemma 7.12. *For every finite induced $\mathcal{H}$-free subdigraph A of D, there exists a vertex x with $A \subseteq D^+(x)$.*

Proof. If A is connected, then we find an isomorphic copy of A in some $D^+(y)$, as D^+ is generic $\mathcal{H}$-free. So C-homogeneity implies the assertion. Next, let us assume that A has precisely two components A_1 and A_2. If both these components are tournaments, then Lemma 7.11 implies the assertion. So we may assume that A_1 has two non-adjacent vertices a_1 and a_2. Furthermore, we may assume that $A_1^- := A_1 - a_1$ is connected. By induction, there exists a vertex $v \in VD$ with $A_1^- + A_2 \subseteq D^+(v)$. As $D^+(v)$ is generic $\mathcal{H}$-free, we find a vertex $w \in N^+(v)$ with precisely one neighbour in A_2 and such that a_2 is its only neighbour in A_1^-. As a_1 and a_2 are not adjacent, the digraph $A + w$ is connected and $\mathcal{H}$-free. So we find a vertex x of D with $A \subseteq A + w \subseteq D^+(x)$.

Let us now assume that A consists of more than two components $A_1, \ldots, A_n$ with $n \geq 3$. Let $a \in VA_1$. By induction, we find a vertex $v \in VD$ with $A - a \subseteq D^+(v)$. As $D^+(v)$ is generic $\mathcal{H}$-free, there is a vertex $w \in N^+(v)$ that has no neighbour in $A_1 - a$ and precisely one neighbour in each A_i for $2 \leq i \leq n$. Then $A + w$ is $\mathcal{H}$-free and has at most two components. By the previous cases, we find a vertex x with $A \subseteq A + w \subseteq D^+(x)$ as claimed. □

Note that we also obtain with the same arguments as in the proofs of Lemma 7.11 and 7.12 that for every finite induced $\mathcal{H}'$-free subdigraph A of D we find some $x \in VD$ with $A \subseteq D^-(x)$.

Lemma 7.13. *Let A and A' be finite induced $\mathcal{H}$-free subdigraphs of D and let B and B' be finite induced $\mathcal{H}'$-free subdigraphs of D such that an isomorphism $\varphi\colon A' + B' \to A + B$ with $A'\varphi = A$ and $B'\varphi = B$ exists. If A is maximal $\mathcal{H}$-free in $A + B$ and if D has a vertex v with $A' \subseteq D^+(v)$ and $B' \subseteq D^-(v)$, then there exists a vertex $x \in VD$ with $A \subseteq D^+(x)$ and $B \subseteq D^-(x)$.*

Proof. If $A + B$ is connected, then the assertion is a direct consequence of C-homogeneity and, if $|VB| = 0$, then the assertion follows from Lemma 7.12. So let us assume that $A + B$ is not connected and that B has some vertex z. Let $z' = z\varphi$. As in the proof of Lemma 7.8,

$$\textit{we may assume } A' = A \textit{ and } B' - z' = B - z. \tag{7.2}$$

By maximality of A in $A + B$ being $\mathcal{H}$-free, we conclude $z \notin N^+(v)$ and that A contains from each component of $A + B$ at least one vertex. Let $a \in VA$ be in a component of $A + B$ that does not contain z. Note that we may assume $z \notin N^-(v)$, as otherwise v is a vertex we are searching for. Hence, z and v are not adjacent. So $D[a, v, z, z']$ is $\mathcal{H}'$-free and we find $y \in VD$ with $D[a, v, z, z'] \subseteq D^-(y)$ due to the corresponding statement of Lemma 7.12 for $\mathcal{H}'$ instead of $\mathcal{H}$. Because of $vy \in ED$, we know that $A' + y$ is $\mathcal{H}$-free. Note that there exists an isomorphism from $(A + y) + B$ to $(A' + y) + B'$ extending φ and that $A' + y \subseteq D^+(v)$ and $B' \subseteq D^-(v)$. By induction on the number of components of $A + B$, we find a vertex $x \in VD$ with $A + y \subseteq D^+(x)$ and $B \subseteq D^-(x)$. This shows the assertion. □

Now we are ready to prove the main result of this section:

Proposition 7.14. *Let D be a countable connected C-homogeneous digraph such that D^+ is the countable generic $\mathcal{H}$-free digraph for some set $\mathcal{H}$ of finite tournaments. Then D is homogeneous.*

Proof. Let A and B be two isomorphic finite induced subdigraphs of D and let $\varphi\colon A \to B$ be an isomorphism. Let A^+ be a maximal induced $\mathcal{H}$-free subdigraph of A. Note that A^+ contains at least one vertex from each component of A. Let $A^- \subseteq A - A^+$ be maximal $\mathcal{H}'$-free such that for some vertex v of D there exists an embedding $\psi\colon A^+ + A^- \to D[N(v)]$ with $A^+\psi \subseteq D^+(v)$ and $A^-\psi \subseteq D^-(v)$. According to Lemma 7.13, there is a vertex $x \in VD$ with $A^+ \subseteq D^+(x)$ and $A^- \subseteq D^-(x)$. By the maximimal choices of A^+ and A^-, we conclude that x is not adjacent to any vertex of A outside $A^+ + A^-$. Let $B^+ = A^+\varphi$ and $B^- = A^-\varphi$. By the same argument as above, there is also a vertex y with $B^+ \subseteq D^+(y)$ and $B^- \subseteq D^-(y)$ such that no other vertex of B is adjacent to y. So φ extends to an isomorphism φ' from $A + x$ to $B + y$. Since $A + x$ is connected, we can extend φ', and thus also φ, to an automorphism of D by C-homogeneity. $\square$

7.1.3 Generic n-partite or semi-generic ω-partite digraph as D^+

Within this section, let us assume that D is a countable connected C-homogeneous digraph such that D^+ is either a countable generic n-partite digraph for some $n \in \mathbb{N}^\infty$ with $n \geq 2$ or the countable semi-generic ω-partite digraph.

Lemma 7.15. *We have $D^+ \cong D^-$.*

Proof. First, let us assume that D^+ is either generic n-partite for some $n \geq 3$ or semi-generic ω-partite. Since for every $k < n$ every finite complete k-partite digraph (with the property (3.3) if D^+ is semi-generic ω-partite) lies in $D^-(y) \cap D^+(x)$ for some edge $xy \in ED$, we conclude from Theorem 3.11 that D^- is either a countable generic $\mathcal{H}$-free digraph for some set $\mathcal{H}$ of finite tournaments or a countable generic m-partite digraph for some $m \geq n-1$ or the countable semi-generic ω-partite digraph. The first digraph is excluded by Section 7.1.2.

If D^+ is generic n-partite, then we can also exclude the countable semi-generic ω-partite digraph for D^-, since D^- contains *every* finite complete k-partite digraph. For $xy \in ED$, we find some $(k+1)$-partite digraph in $D^-(y)$: the digraph $A+x$ where A is an arbitrary complete k-partite digraph in $D^+(x) \cap D^-(y)$. Hence, we have $m \geq n$ and by symmetry we also have $n \geq m$, so $D^+ \cong D^-$.

If D^+ is semi-generic ω-partite, then we find for every $k < \omega$ some complete k-partite digraph in D^-, so D^- is either generic or semi-generi ω-partite. We exclude the first possibility by our previous situation. Thus, we have also $D^+ \cong D^-$ in this case.

Now we consider the remaining situation, that is, that D^+ is the countable generic 2-partite digraph. Then, for every edge $xy \in ED$, the digraph $D^-(y)$ contains the complete 2-partite digraph with x on one side and with infinitely many successors of x on the other side. Due to Theorem 3.11, we conclude that the only possibilities for D^- are $\mathcal{P}$, $\mathcal{P}(3)$, $T[I_\omega]$ for some homogeneous tournament $T \neq I_1$, the generic $\mathcal{H}$-free digraphs, which are excluded by Section 7.1.2, or the (semi-)generic n-partite digraph, which must be the generic 2-partite digraph due to our previous situations. If D^- is either $\mathcal{P}$ or $\mathcal{P}(3)$, then D^- has a vertex with three successors in D^- that induce an edge with an isolated vertex. Since this digraph does not lie in the countable generic 2-partite digraph, D^- is neither $\mathcal{P}$ nor $\mathcal{P}(3)$.

If $D^- \cong T[I_\omega]$ for an infinite homogeneous tournament T, then D^+ contains an arbitrarily large tournament, which cannot lie in any 2-partite digraph. Let us suppose $D^- \cong C_3[I_\omega]$. Let $x \in VD$ and $D[v_1, v_2, v_3]$ be a directed triangle in $D^-(x)$. Considering $D^+(v_i)$, we know that v_i has successors in precisely one set of the 2-partition of $D^+(x)$. Hence for two v_i, these sets coincide. Applying C-homogeneity to fix x and rotate $D[v_1, v_2, v_3]$ by an automorphism of D, we conclude that these sets coincide for all v_i and, applying C-homogeneity once more, we know that the same holds for all directed triangles in $D^-(x)$. Thus, all vertices in $N^-(x)$ have their successors in $N^+(x)$ in the same partition set of $D^+(x)$, which contradicts C-homogeneity, as we can fix x and map one vertex of $N^+(x) \cap N^+(v_1)$ onto one of its successors in $D^+(x)$ by an automorphism of D since D is C-homogeneous. So we have $D^- \ncong C_3[I_\infty]$. Hence, we have shown the assertion in this case, too. □

Now we are able to prove the main result of this section:

Proposition 7.16. *Let D be a countable connected C-homogeneous digraph such that D^+ is either the countable generic n-partite digraph for some $n \in \mathbb{N}^\infty$ with $n \geq 2$ or the countable semi-generic ω-partite digraph. Then D is homogeneous.*

Proof. Let $x \in VD$ and $a, b \in N^+(x)$ with $ab \in ED$. As $D^- \cong D^+$ holds by Lemma 7.15, we have

$$N^-(b) \setminus N(x) \subseteq N(a). \tag{7.3}$$

Note that all partition sets of $D^-(b)$ except for the one containing x have elements in $N^+(x)$. A direct consequence is the following:

For every maximal tournament in $D^+(x)$ that contains b and has no edge directed away from b, this tournament has vertices of each partition set of $D^-(b)$ except for the one containing x. (7.4)

Let us show that also

$$N^+(b) \setminus N(x) \subseteq N(a) \tag{7.5}$$

holds. Let us suppose that (7.5) does not hold. Then we find $y \in N^+(b)$ that is adjacent to neither a nor x. As an induced directed cycle of length 4 embeds into D^+, C-homogeneity implies the existence of a vertex $c \in N^+(y) \cap N^-(a)$ such that b and c are not adjacent and, furthermore, we find a vertex $z \in VD$ with $D[a, b, y, c] \subseteq D^+(z)$ by C-homogeneity. The structure of $D^-(a)$ implies that x is adjacent to either c or z. First, let us assume that x and z are adjacent. Since $D[a, x, y]$ does not embed into $D^+(z)$, we have $xz \in ED$ and, as $D[a, b, z]$ is a triangle in $D^+(x)$, we have $n \geq 3$ if D^+ is generic n-partite. Let $\{v_i \mid i \in I\}$ be a maximal set in $N^+(z)$ such that $X := \{a, b, v_i \mid i \in I\}$ induces a tournament and such that $D[a, b, c, y] \subseteq D^+(v_i)$ for all $i \in I$. By its maximality and due to the structure of D^+, the set X contains vertices from each maximal independent set in $N^+(z)$. Due an analogue of (7.4) for z instead of x, we know that X meets every maximal independent set of $N^-(b)$ but the one that contains z. So x must be non-adjacent to some v_i. As $D[v_i, c, x] \subseteq D^-(a)$, we conclude that x and c are adjacent. So if we replace z by v_i if necessary, we may assume that x and c are adjacent but x and z are not.

Because $D[x, y, z]$, a digraph on three vertices with precisely one edge, cannot lie in $D^-(c)$, we have $xc \notin ED$. So $cx \in ED$ and $D[x, b, y, c]$ is an induced directed cycle. As C_4 embeds into D^+, we find $z' \in VD$ with $D[x, b, y, c] \subseteq D^+(z')$ by C-homogeneity. Considering $D^-(b)$, we conclude that z and z' are adjacent. The corresponding edge is not $z'z$, as $D[x, y, z]$ cannot lie in $D^+(z')$. Hence, we have $zz' \in ED$. Because $D[a, y, z']$ lies in $D^+(z)$, we know that a and z' are adjacent and, as $D[a, x, y]$ cannot lie in $D^+(z')$, the corresponding edge must be az'. Since $D^+(z)$ contains the triangle $D[b, y, z']$, we have $n \geq 3$ if D^+ is generic n-partite. Similarly as above, we choose a maximal set $\{w_i \mid i \in I\}$ in $N^+(z')$ such that the set $X = \{b, y, w_i \mid i \in I\}$ induces a tournament and such that $D[b, y, c, x] \subseteq D^+(w_i)$ for all $i \in I$. By its maximality, the set X contains vertices from each maximal independent set in $N^+(z')$. Then an analogue of (7.4) for z' instead of x implies that X meets every maximal independent set of $N^-(b)$ but the one that contains z'. So a must be non-adjacent to some w_i and z is adjacent to every w_j, in particular to w_i. But $zw_i \in ED$ is impossible, as $D[a, w_i, y]$ does not embed into $D^+(z)$, and $w_i z \in$

ED is impossible, as $D[x, y, z]$ does not embed into $D^+(w_i)$. This contradiction proves (7.5).

Now we have shown $N(b) \setminus N(x) \subseteq N(a)$. For an induced directed cycle $x_1x_2 \dots x_m$ (with $m \leq 5$) in $N^+(x)$ with $x_{m-1} = a$ and $x_1 = b = x_m$, we use C-homogeneity to find an automorphism that fixes x and rotates the cycle backwards so that we can conclude inductively

$$N(x_m) \setminus N(x) \subseteq N(x_{m-1}) \setminus N(x) \subseteq \dots \subseteq N(x_2) \setminus N(x) \subseteq N(x_1) \setminus N(x).$$

Because of $x_1 = x_m$, all inclusions are equalities of the involved sets. In particular, we have $N(a) \setminus N(x) = N(b) \setminus N(x)$. Note that any two vertices in $N^+(x)$ lie on an induced directed cycle of length at most 4. Hence, we can apply the above argument and obtain

$$N(u) \setminus N(x) = N(v) \setminus N(x) \text{ for all } u, v \in N^+(x). \tag{7.6}$$

By symmetry and as $D^+ \cong D^-$ due to Lemma 7.15, we have

$$N(u) \setminus N(x) = N(v) \setminus N(x) \text{ for all } u, v \in N^-(x). \tag{7.7}$$

Let us show for $A := N(a) \setminus N(x)$ the following:

A is an independent set. (7.8)

Let us suppose that there are two vertices $u, v \in A$ with $uv \in ED$. Note that b is adajcent to u and v by (7.6). We find $w \in N^+(u) \cap N^+(v)$. The analogue of (7.6) for u instead of x gives us $N(v) \setminus N(u) = N(w) \setminus N(u)$, which shows that w is not adjacent to x. If $av \in ED$, then we obtain a contradiction to an analogue of (7.7) as x lies in $N(a) \setminus N(v)$ but not in $N(u) \setminus N(v)$. Thus, we have $va \in ED$ and we conclude $vb \in ED$ analogously. Due to the structure of $D^+(v)$ we know that w has to be adjacent to either a or b. First, let us assume that a and w are adjacent. If $aw \in ED$, then we conclude $x \in N(a) \setminus N(w) = N(v) \setminus N(w)$ by an analogue of (7.7), which contradicts $v \in A$, and if $wa \in ED$, then x is not adjacent to both end vertices of vw, which is impossible in $D^-(a)$. We obtain analogous contradictions if w and b are adjacent. Hence, we have shown (7.8).

Let us show

$$VD = A \cup N(x). \tag{7.9}$$

First, let $y \in N(x)$ and let u be a neighbour of y. If u lies outside $N(x)$, then we find a vertex v with $D[x, y, u] \subseteq D^-(v)$ due to C-homogeneity and as D^- contains an isomorphic copy of $D[x, y, u]$. So we conclude $u \in A$ due to (7.6). Now let $y \in A$ and let u be a neighbour of y. If u is adjacent to a, then $u \in A \cup N(x)$. So let us assume that a and u are not adjacent. Then we find by C-homogeneity a vertex v with $D[a, y, u] \subseteq D^-(v)$. As v is adjacent to a, it lies in $N(x) \cup A$ and as it is adjacent to y, it cannot lie in A due to (7.8). So v lies in $N(x)$ and by the first case we conclude that u lies in $A \cup N(x)$. This shows (7.9).

Our last step, before we show the homogeneity of D, is to show that

$$D \text{ is complete } m\text{-partite for some } m \in \mathbb{N}^\infty. \tag{7.10}$$

Let $\mathcal{I}$ be the set of maximal independent sets in $N^+(x)$. Let $A' = A \cup \{x\}$ and, for every $I \in \mathcal{I}$, let I' be a maximal independent set in D that contains I. Due to (7.6), every vertex of A' is adjacent to all vertices of $N^+(x)$. As $D[x, a, a']$ with $a' \in A$ embeds into $D^-(x)$, we find by C-homogeneity a vertex v with $D[x, a, a'] \subseteq D^-(v)$. So every vertex of A' is adjacent to some vertex of $N^-(x)$ and hence by (7.7) to every vertex of $N^-(x)$. So by (7.9), every vertex of A' is adjacent to every vertex outside A'. As D is vertex-transitive, the same holds for every maximal independent vertex set of D. Thus, (7.10) holds.

To show that D is homogeneous, let F and H be two isomorphic induced subdigraphs of D. If they are connected, then C-homogeneity implies that every isomorphism from F to H extends to an automorphism of D. So we may assume that they are not connected. As D is complete m-partite, we conclude that VF is an independent set and the same is true for VH. Then we find u_F and u_H with $VF \subseteq N(u_F)$ and $VH \subseteq N(u_H)$. Note that due to the structure of $D^+(x)$, we find subdigraphs F' and H' of $D^+(x)$ that are isomorphic to $F + u_F$ and $H + u_H$, respectively. By C-homogeneity, we find an automorphism φ_F of D that maps $F + u_F$ to F' and an automorphism φ_H that maps $H + u_H$ to H'. Then $F + x\varphi_F^{-1}$ and $H + x\varphi_H^{-1}$ are connected and every isomorphism from F to H extends to an isomorphism from $F + x\varphi_F^{-1}$ to $H + x\varphi_H^{-1}$, so C-homogeneity implies the assertion. □

7.1.4 The digraphs $T^\wedge$ as D^+

In this section, we investigate countable connected C-homogeneous digraphs D with $D^+ \cong T^\wedge$ for some $T \in \{I_1, C_3, \mathbb{Q}, T^\infty\}$. If T is either I_1 or C_3, then we obtain from Lemma 7.1 that D is locally finite and due to Lemmas 6.2 and 6.3 we obtain that no such C-homogeneous digraph exists. Hence, it suffices to consider only the cases $T \cong \mathbb{Q}$ and $T \cong T^\infty$ in the proof of Proposition 7.17.

Proposition 7.17. *No countable connected C-homogeneous digraph D satisfies $D^+ \cong T^\wedge$ for any $T \in \{I_1, C_3, \mathbb{Q}, T^\infty\}$.*

Proof. Let us suppose that some countable connected C-homogeneous digraph D with $D^+ \cong T^\wedge$ exists for some $T \in \{\mathbb{Q}, T^\infty\}$. Note that it was already proven in [15] that no such digraph exists if $T \in \{I_1, C_3\}$, as we have already mentioned earlier. Due to Theorem 3.11 and the previous sections, the only possibilities for D^- are $I_n[T_0]$, $T_0[I_n]$, $S(3)$, $T_0^\wedge$, $\mathcal{P}$, or $\mathcal{P}(3)$, where $n \in \mathbb{N}^\infty$ and T_0 is some homogeneous tournament. Because the latter two digraphs contain the complete bipartite digraph $K_{1,3}$, but $T^\wedge$ contains no three independent vertices, we know that D^- is one of the first four digraphs. Since the first three digraphs in that list do not contain the digraph D' depicted in Figure 7.1, we have the following:

if D' embeds into D^-, then $D^- \cong T_0^\wedge$ for some infinite homogeneous tournament T_0. (7.11)

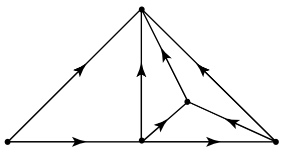

Figure 7.1: The digraph D'

Let $xy \in ED$. Note that $D^+(x) \cap D^+(y) \cong T$. The first statement that we shall show is the following:

There is a unique pair of vertices $v, \mathring{v}$ in $D^+(y)$ that are not adjacent and each of which is not adjacent to x. (7.12)

For each $z \in N^+(y)$, let $\mathring{z}$ denote the unique vertex in $D^+(y)$ that is not adjacent to z. For every $z \in N^+(x) \cap N^+(y)$, either $\mathring{z} \in N^-(x)$ or $\mathring{z}$ is not adjacent to x, since $D^+(x) \cap D^+(y)$ is a tournament. Let us suppose that $\mathring{z}$ is not adjacent to x. By C-homogeneity, the same holds for every $\mathring{u}$ with $u \in N^+(x) \cap N^+(y)$. Let $u_1, u_2, u_3 \in N^+(x) \cap N^+(y)$ with $u_iu_j \in ED$ for $i < j \leq 3$ and with $u_iz \in ED$ for all $i \leq 3$. These vertices exist as every vertex of $\mathbb{Q}$ and T^∞ contains the directed triangle in its in-neighbourhood, so the same holds for z in $D^+(x) \cap D^+(y)$. The digraph $D[x, y, u_1, \mathring{z}, \mathring{u}_3]$ is isomorphic to D' and lies in $D^-(u_2)$. Due to (7.11), we have $D^- \cong T_0^\wedge$ for some infinite homogeneous tournament T_0. Hence, $D^-(u_2)$ contains a unique vertex that is not adjacent to x which contradicts the fact that $\mathring{z}$ and $\mathring{u}_3$ are not adjacent to x even though they lie in $N^-(u_2)$. This contradiction shows $\mathring{z}x \in ED$. By C-homogeneity, we conclude that for any $w \in N^+(y)$ that is not adjacent to x also the vertex $\mathring{w}$ is not adjacent to x. Indeed, if not, then we have $\mathring{w}x \in ED$ by the previous situation. Hence, some automorphism of D fixes x and y and maps $\mathring{z}$ to $\mathring{w}$ and we obtain $xw \in ED$, contrary to the choice of w. Since D^+ contains an induced

2-arc, there is a vertex in $N^+(y)$ that is not adjacent to x, which shows the existence of a pair of vertices as described in (7.12). It remains to show that this pair is unique.

Let us suppose that $N^+(y) \setminus N(x)$ contains two vertices v, w with $vw \in ED$. Among the vertices $v, \mathring{v}, w$, and $\mathring{w}$, we find two adjacent ones, say v and w with $vw \in ED$ such that there are two vertices $u_1, u_2 \in N^+(x) \cap N^+(y)$ with $u_1, u_2 \in N^-(v) \cap N^-(w)$ and with $u_1u_2 \in ED$. The digraph $D[x, y, u_1, \mathring{v}, \mathring{w}]$ is isomorphic to D' and lies in $D^-(u_2)$. Due to (7.11), we have $D^- \cong T_0^\wedge$ for some infinite homogeneous tournament T_0. Note that $T_0^\wedge$ does not contain a subdigraph on three vertices with precisely one edge. But $D[\mathring{v}, \mathring{w}, x]$ is such a digraph, which lies in $D^-(u_2) \cong T_0^\wedge$. This contradiction shows the uniqueness of the vertex pair in (7.12), as every maximal independent vertex set in $D^+(y)$ has precisely two vertices.

Let $N = N^+(x) \cap N^+(y)$. In the following, let v and $\mathring{v}$ be the vertices of (7.12). Our next step is to show

$$N \subseteq N^+(v) \quad \text{or} \quad N \subseteq N^-(v). \tag{7.13}$$

Let us suppose that we find vertices $a \in N^+(v) \cap N$ and $b \in N^-(v) \cap N$. Note that a and b are adjacent, since both lie in the tournament $D^+(x) \cap D^+(y)$. Since T contains a transitive triangle, let $c \in N$ such that $D[a, b, c]$ is a transitive triangle. Then either $c \in N^+(v)$ or $c \in N^-(v)$. If $c \in N^+(v)$, then we find an automorphism of D that fixes x and y and maps the edge between a and b to the edge between a and c by C-homogeneity. If $c \in N^-(v)$, then we find an automorphism of D that fixes x and y and maps the edge between a and b to the edge between b and c. Any of these automorphisms can neither fix v nor map it to $\mathring{v}$ even though its image must lie in $\{v, \mathring{v}\}$ by (7.12). This contradiction shows (7.13).

By symmetry, we may assume $N \subseteq N^+(v)$ and hence $N \subseteq N^-(\mathring{v})$. Since D is C-homogeneous, we find an automorphism α of D that fixes x and y and maps v to $\mathring{v}$. Since α fixes x and y, we have $N\alpha = N$ and hence

$$N\alpha = N \subseteq N^+(v) = (N^+(\mathring{v}))\alpha.$$

Thus, we have $N \subseteq N^+(\mathring{v})$. This is a contradiction to $N \subseteq N^-(\mathring{v})$, which shows the assertion. □

7.1.5 The digraph $S(3)$ as D^+

In this section, we show that no countable connected C-homogeneous digraphs D has the property $D^+ \cong S(3)$. Our strategy in the proof is to exclude all countable homogeneous digraphs for D^-.

Proposition 7.18. *No countable connected C-homogeneous digraph D satisfies $D^+ \cong S(3)$.*

Proof. Let us suppose that some countable connected C-homogeneous digraph D with $D^+ \cong S(3)$ exists. Since $D^+ \cong S(3)$, we have $D^+(x) \cap D^+(y) \cong \mathbb{Q}$ for every edge $xy \in ED$. Let $v \in N^+(x) \cap N^+(y)$. As D^+ contains a transitive triangle, C-homogeneity implies the existence of some $z \in VD$ with $D[x, y, v] \subseteq D^+(z)$. In $D^+(z)$ we find a vertex u with $u \in N^+(y) \cap N^+(v)$ that is not adjacent to x. By C-homogeneity, we can map xyu onto any other induced 2-arc xya and obtain

$$N^-(a) \cap N^+(x) \cap N^+(y) \neq \emptyset \textit{ for every } a \in N^+(y) \setminus N(x). \tag{7.14}$$

As $D^+(x) \cap D^+(y) \cong \mathbb{Q}$ is a proper subdigraph of $D^+(y) \cong S(3)$, we find a predecessor w of v in $N^+(y)$ that lies outside $N^+(x)$ and has only successors in $N^+(x) \cap N^+(y)$. The vertices x and w are adjacent due to (7.14). As $w \notin N^+(x)$, we have $wx \in ED$. Thus, $D^-(v)$ contains the directed triangle $D[x, y, w]$.

Note that v has some predecessor w' in $N^+(x) \cap N^+(y)$. This vertex must be adjacent to w as each two predecessors of v are adjacent by the structure of $S(3)$. As $N^-(w)$ contains no vertex of $D^+(x) \cap D^+(y)$, we have $w' \in N^+(w)$. Note that we also have $D[x, y, w, w'] \subseteq D^-(v)$.

Since D^- contains a copy of $D[x, y, w, w']$ and a copy of $\mathbb{Q}$, Theorem 3.11 implies that the only possibilities for D^- are either $\mathcal{P}(3)$, $I_n[T^\infty]$, or $T^\infty[I_n]$ for some $n \in \mathbb{N}^\infty$ by the previous sections. We cannot have $D^- \cong \mathcal{P}(3)$, since $\mathcal{P}(3)$ contains a vertex with three independent successors, but D^+ contains no independent set of three vertices. So we have $D^- \cong I_n[T^\infty]$ or $D^- \cong T^\infty[I_n]$. But then D^- contains a vertex with a directed triangle in its out-neighbourhood. This is impossible, since $S(3)$ contains no directed triangle. As no possibility is left for D^-, we have shown the assertion. □

7.1.6 The digraph $\mathcal{P}(3)$ as D^+

In this section, we show that no countable connected C-homogeneous digraph D has the property $D^+ \cong \mathcal{P}(3)$.

Proposition 7.19. *No countable connected C-homogeneous digraph D with $D^+ \cong \mathcal{P}(3)$ exists.*

Proof. Let us suppose that there is a countable connected C-homogeneous digraph D with $D^+ \cong \mathcal{P}(3)$. Since the in-neighbourhood of any vertex contains every finite partial order, we have $D^- \cong \mathcal{P}$ or $D^- \cong \mathcal{P}(3)$. Furthermore, we have $D^-(y) \cap D^-(x) \cong \mathcal{P}$ for every edge $xy \in ED$. As D^+ contains a directed triangle, C-homogeneity implies the existence of a vertex $a \in N^+(y)$ such that $D[x, y, a]$ is a directed triangle. Let

$$\begin{aligned} a^\perp &:= \{b \in N^+(y) \mid a \text{ not adjacent to } b\}, \\ a^\rightarrow &:= N^+(a) \cap N^+(y), \text{ and} \\ a^\leftarrow &:= N^-(a) \cap N^+(y). \end{aligned}$$

So we have $\mathbb{H}(a) := (a^\perp, a^\rightarrow, a^\leftarrow) \cong \mathbb{H}$. Note that $D^+(x)$ has an edge with both its incident vertices in the same set $a^\perp$, $a^\rightarrow$, or $a^\leftarrow$, as $D^+(x) \cap D^+(y)$ contains a tournament on four vertices. If either $a^\perp$ or $a^\leftarrow$ contains an edge uv of $D^+(x)$, then we find an edge $u'v'$ in $D^-(u) \cap D^-(v)$ with $u', v' \in a^\rightarrow$ due to the structure of $\mathcal{P}(3)$. If either u' or v' does not lie in $N^+(x)$, then xy together with this vertex induce either a 2-arc or a directed triangle in $D^-(u) \cap D^-(v) \cong \mathcal{P}$, which is impossible. So we may assume that there are two adjacent vertices b and c of $N^+(x)$ in $a^\rightarrow$. Then $D[a, x, y]$ is a directed triangle in $D^-(b) \cap D^-(c)$, which is impossible. □

7.1.7 Generic partial order $\mathcal{P}$ as D^+

Within this section, let D be a countable connected C-homogeneous digraph with $D^+ \cong \mathcal{P}$. Before we are able to prove that D is homogeneous in this situation, we will prove several lemmas. Our first one determines D^-.

Lemma 7.20. *We have $D^- \cong \mathcal{P}$.*

Proof. Since, for every edge $xy \in ED$, the digraph $D^+(x) \cap D^-(y)$ contains every finite partial order, the assertion follows from Theorem 3.11 together with the previous sections. □

Our general strategy to prove that D is homogeneous is similar to those of the Sections 7.1.1 and 7.1.2. In particular, one step is to show that every finite partial order in D lies in $D^+(x)$ for some $x \in VD$ (Lemma 7.22). As in the other two cases, we prove it by induction. In this situation, the base case (Lemma 7.21) turns out to be the most complicated part of the proof.

Lemma 7.21. *Any two vertices in D have a common predecessor.*

Proof. If D contains no induced 2-arc, then any induced path is an alternating walk and lies in the out-neighbourhood of some vertex by C-homogeneity. Hence, any two vertices have a common predecessor.

Thus, we assume that D contains induced 2-arcs. Our first aim is to show that

the end vertices of any induced 2-arc have a common predecessor or a common successor. (7.15)

In order to prove (7.15) we investigate for $xy \in ED$ the three sets:

$$\begin{aligned} x^\perp &:= \{z \in N^+(y) \mid x \text{ not adjacent to } z\}, \\ x^\rightarrow &:= N^+(y) \cap N^+(x), \text{ and} \\ x^\leftarrow &:= N^+(y) \cap N^-(x). \end{aligned}$$

If $ba \in ED$ for some $a \in x^\rightarrow$ and some $b \in x^\perp$, then xyb is an induced 2-arc in $D^-(a)$. As $D^- \cong \mathcal{P}$ by Lemma 7.20 and $\mathcal{P}$ contains no induced 2-arc, we have shown:

$$\textit{no vertex in } x^\perp \textit{ has successors in } x^\rightarrow. \tag{7.16}$$

If $ba \in ED$ for some $a \in x^\rightarrow$ and some $b \in x^\leftarrow$, then the directed triangle $D[x, y, b]$ lies in $D^-(a) \cong$, which is not possible. Thus, we have

$$\textit{no vertex in } x^\leftarrow \textit{ has successors in } x^\rightarrow. \tag{7.17}$$

Let us suppose that no $a \in x^\rightarrow$ and $b \in x^\perp$ are adjacent. In $D^+(y)$, we find a common predecessor c and a common successor c' of a and b. Since neither of them can lie in $x^\perp$ or in $x^\rightarrow$ by assumption, both lie in $x^\leftarrow$. Any predecessor of c in $D^+(y)$ is also a predecessor of a and b and thus must lie in $x^\leftarrow$. By C-homogeneity, we find an automorphism α of D that fixes x and y and maps c to c'. This is impossible, as $c' = c\alpha$ has predecessors in $D^+(y)$ that lie outside $x^\leftarrow = (x^\leftarrow)\alpha$. Thus, we have shown that some vertex of $x^\rightarrow$ has a neighbour in $x^\perp$. By C-homogeneity and due to (7.16), we have

$$\textit{every vertex in } x^\perp \textit{ has a predecessor in } x^\rightarrow \textit{ and every vertex in } x^\rightarrow \textit{ has a successor in } x^\perp. \tag{7.18}$$

If any vertex a in $x^\perp$ has a predecessor in $x^\leftarrow$, then the end vertices of the induced 2-arc xya have a common predecessor. Thus, we have shown:

$$\textit{if (7.15) does not hold, then no vertex of } x^\perp \textit{ has a predecessor in } x^\leftarrow. \tag{7.19}$$

Let us assume that we have $ab \in ED$ for all $a \in x^\rightarrow$ and all $b \in x^\perp$. Because of $D^+ \cong \mathcal{P}$, we find a vertex $z \in N^+(y)$ that is adjacent to neither a nor b. Hence, z lies neither in $x^\perp$ nor in $x^\rightarrow$. Thus, we have $z \in x^\leftarrow$. Let u be a common successor of z and b in $D^+(y)$. We have $u \notin x^\rightarrow$ by (7.16) because of $bu \in ED$. By (7.19), the edge zu implies that either (7.15) holds or $u \notin x^\perp$. So we may assume $u \in x^\leftarrow$. Then (7.19) implies (7.15) as $b \in x^\perp$ has the predecessor $u \in x^\leftarrow$. Due to (7.16), we have shown

$$\textit{if (7.15) does not hold, then for every vertex in } x^\perp \textit{ there is some vertex in } x^\rightarrow \textit{ such that these two vertices are not adjacent.} \tag{7.20}$$

Since every two vertices in $N^+(y)$ have a common predecessor, the existence of a vertex z_1 in $x^{\leftarrow}$ and a vertex z_2 in $x^{\rightarrow}$ that are not adjacent implies that the end vertices of the induced 2-arc z_1xz_2 have a common predecessor. Together with (7.17), this implies that

if (7.15) *does not hold, then every vertex of* $x^{\rightarrow}$ *is a predecessor of every vertex of* $x^{\leftarrow}$. (7.21)

Let $ab \in ED$ with $a \in x^{\rightarrow}$ and $b \in x^{\perp}$. This edge exists due to (7.18). By (7.20), we may assume that there is some vertex $c \in x^{\rightarrow}$ with $cb \notin ED$. Then (7.16) implies that c and b are not adjacent.

If a and c are adjacent, then $ca \notin ED$ because we have $cb \notin ED$ and $D^+(y)$ contains no induced 2-arc. So let us assume $ac \in ED$. In $D^+(y)$, we find a vertex $c' \in N^-(c)$ that is adjacent to neither a nor b. We have $c' \notin x^{\perp}$ due to (7.16) because of $c \in x^{\rightarrow}$. By (7.21), either (7.15) holds or $c' \notin x^{\leftarrow}$. Thus, we may assume $c' \in x^{\rightarrow}$. Taking c' instead of c, we may assume that a and c are not adjacent. Thus, the end vertices of $D[c, x, a, b]$ lie in $D^+(y)$ and hence have a common predecessor. By a symmetric argument, we obtain that

if (7.15) *does not hold, then the end vertices of any induced path isomorphic to either* $D[c, x, a, b]$ *or the digraphs obtained from* $D[c, x, a, b]$ *by reversing the directions of all its edges have a common predecessor.* (7.22)

Let α be an automorphism of D that fixes x and y and interchanges a to c. For $b' := b\alpha$ we have $b \neq b' \in (x^{\perp})\alpha = x^{\perp}$. Since $ab' \notin ED$ and $D^+(y) \cong \mathcal{P}$, we have $bb' \notin ED$ and, symmetrically, we have $b'b \notin ED$. Hence, b and b' are not adjacent. Let $u \in N^+(y)$ with $a, b, c \in N^-(u)$ and such that u and b' are not adjacent. If $u \in x^{\perp}$, then (7.22) applied to $D[x, c, u, b]$ implies (7.15), since x and b are the end vertices of the induced 2-arc xyb. Due to (7.16), the vertex u does not lie in $x^{\rightarrow}$. Hence, we may assume $u \in x^{\leftarrow}$. Let v be a predecessor of b' in $D^+(y)$ that has no neighbour in $\{a, b, c, u\}$. Since v and u are not adjacent, (7.21) implies either (7.15) or $v \notin x^{\rightarrow}$. By (7.19) and as $vb' \in ED$, either (7.15) holds or $v \notin x^{\leftarrow}$. Thus, we may assume $v \in x^{\perp}$. Then (7.22) applied to $D[x, c, b', v]$ shows that the end vertices of the induced 2-arc xyv have a common predecessor. This shows (7.15).

Due to (7.15), every two vertices of distance 2 have a common successor or a common predecessor. If they have a common successor, then these three vertices induce a connected finite partial order and, by C-homogeneity, we find a common predecessor of all three vertices. Hence, we have shown

any two vertices of distance 2 *have a common predecessor.* (7.23)

To show the lemma, it thus suffices to show

$$\mathrm{diam}(D) = 2. \tag{7.24}$$

We consider all possible induced paths P of length 3, not necessarily directed, one by one and show that the end vertices of such a path have distance 2. If P is an alternating walk, then it is a partial order and, for every $x \in VD$, the subdigraph $D^+(x)$ contains an isomorphic copy of P. By C-homogeneity, we find a vertex z with $P \subseteq D^+(z)$ and the claim follows directly.

Let a_1, a_2, a_3, a_4 be the vertices of P. Let us assume that a_1a_2, a_2a_3, and a_4a_3 are the edges on P. Since $D[a_2, a_3, a_4]$ is a connected partial order, we find a vertex x with $a_2, a_3, a_4 \in N^+(x)$. If a_1 and x are adjacent, then we have $d(a_1, a_4) = 2$. If a_1 and x are not adjacent, then $D[a_1, a_2, x, a_4]$ is a connected partial order that lies in $D^+(z)$ for some $z \in VD$ by C-homogeneity. Thus, also in this case, a_1 and a_4 have a common neighbour. Similar orientations like in this case (e.g., with edges a_2a_1, a_2a_3, and a_3a_4) follow by symmetric arguments.

The only remaining case is that P is an induced 3-arc. Then we find a common predecessor of the first and the third vertex on P and obtain – either directly or by the previous case – that the end vertices of P have distance 2. This shows (7.24) and, as previously mentioned, the lemma. □

Lemma 7.22. *For every finite partial order A in D, there exists some $x \in VD$ with $A \subseteq D^+(x)$.*

Proof. If A is connected, then the assertion is a direct consequence of C-homogeneity, as for every $x \in VD$ the subdigraph $D^+(x)$ contains an isomorphic copy of A. So let us assume that A is not connected. If $|VA| = 2$, then the assertion follows from Lemma 7.21. So we may assume $|VA| \geq 3$. If VA is an independent set, let a be an arbitrary vertex of A. If A has an edge, let $a \in VA$ such that a has a successor in A but no predecessor. By induction on $|A|$, we find $x \in VD$ with $A - a \subseteq N^+(x)$. If $xa \in ED$, then x is the vertex we are searching for. So let us assume either that $ax \in ED$ or that a and x are not adjacent. In each case, $A + x$ is a partial order and it has less components than A. Thus, the assertion holds by induction on the number of components of A. □

Lemma 7.23. *Let A, A', B, B' be finite induced partial orders in D such that an isomorphism $\varphi\colon A' + B' \to A + B$ with $A'\varphi = A$ and $B'\varphi = B$ exists. If A is a maximal partial order in $A + B$ and if D has a vertex v with $A' \subseteq D^+(v)$ and $B' \subseteq D^-(v)$, then there exists $x \in VD$ with $A \subseteq D^+(x)$ and $B \subseteq D^-(x)$.*

Proof. If $A + B$ is connected or if B is empty, then the assertion follows either by C-homogeneity or by Lemma 7.22. So let us assume that $A + B$ has at least two components and that B is not empty. By induction and similar to the proof of Lemma 7.8, we may assume that there are $z \in VB$ and $z' \in VB'$ such that $A = A'$ and $B - z = B' - z'$. Furthermore, we may assume that z does not lie in $N^-(v)$, because the assertion follows directly in that case. Since A is a maximal partial order in $A + B$, we know that A contains vertices from each component of $A+B$. Let $a_1, \ldots, a_n \in VA$ such that $\{a_1, \ldots, a_n, z\}$ has precisely one vertex from each component of $A + B$. By Lemma 7.22, we find a vertex y with $\{a_1, \ldots, a_n, z, z'\} \subseteq N^+(y)$. The digraphs $A + B + y$ and $A' + B' + y$ are connected and isomorphic to each other. By C-homogeneity, there is an automorphism α of D that fixes y and all vertices of A and $B - z$ and maps z to z'. Hence, $v\alpha$ is a vertex we are searching for. □

Proposition 7.24. *Let D be a countable connected C-homogeneous digraph with $D^+ \cong \mathcal{P}$. Then D is homogeneous.*

Proof. Let A and B be isomorphic finite induced subdigraphs of D and $\varphi\colon A \to B$ be an isomorphism. Let A_1 be a maximal partial order of A and A_2 be a maximal partial order of $A \setminus A_1$ such that for some vertex $x \in VD$ there is an embedding τ from $A_1 + A_2$ to $D^+(x) + D^-(x)$ such that $A_1\tau \subseteq D^+(x)$ and $A_2\tau \subseteq D^-(x)$. Note that A_1 contains vertices from each component of A by its maximality. Let $B_1 = A_1\varphi$ and $B_2 = A_2\varphi$. Due to Lemma 7.23, we find a vertex y with $A_1 \subseteq D^+(y)$ and $A_2 \subseteq D^-(y)$ and a vertex z with $B_1 \subseteq D^+(z)$ and $B_2 \subseteq D^-(z)$. By maximalities of A_1 and A_2, we know that no vertex of $A \setminus (A_1 + A_2)$ is adjacent to y and, similarly, no vertex of $B \setminus (B_1 + B_2)$ is adjacent to z. The isomorphism φ extends canonically to an isomorphism $\varphi'\colon A + y \to B + z$. Since $A + y$ and $B + z$ are connected, we can extend φ', and hence also φ, to an automorphism α of D by C-homogeneity. □

7.1.8 The digraphs $T[I_n]$ as D^+

In this section, let D be a countable connected C-homogeneous digraph with $D^+ \cong T[I_n]$ for some countable homogeneous tournament $T \neq I_1$ and some $n \in \mathbb{N}^\infty$. Our first aim in this section is to determine D^-.

Lemma 7.25. *If $n \geq 2$, then $D^- \cong T'[I_m]$ for some countable homogeneous tournament $T' \neq I_1$ and some $m \in \mathbb{N}^\infty$.*

Proof. Let $xz \in ED$. Note that VD^- is not an independent set, since z has a predecessor in $D^+(x)$. As $n \geq 2$, there are two non-adjacent vertices $y_1, y_2 \in N^+(x) \cap N^-(z)$. Since the digraph $D[x, y_1, y_2] \subseteq D^-(z)$ cannot be embedded into $I_k[T']$ for any countable homogeneous tournament $T' \neq I_1$ and any $k \in \mathbb{N}^\infty$, Theorem 3.11 together with the previous sections imply the assertion. □

Lemma 7.26. *If $D^- \cong T'[I_m]$ for some countable homogeneous tournament $T' \neq I_1$ and some $m \in \mathbb{N}^\infty$, then $D^+ \cong D^-$.*

Proof. To show $m = n$, let $x \in VD$. As $T \neq I_1$, any vertex in $D^+(x)$ has n independent predecessors in $D^+(x)$. Hence, we conclude $m \geq n$. By a symmetric argument we also have $n \geq m$. To show $D^+ \cong D^-$ it thus suffices to show $T = T'$.

Note that $T = C_3$ implies $T' = C_3$ and vice versa because in any countable infinite homogeneous tournament, we have arbitrarily large finite tournaments in the out- and in the in-neighbourhood of every vertex.

Let us now show $T = T'$ in the case $T = T^\infty$. Let $x \in VD$ and let F be a finite tournament in $D^+(x)$. As T^∞ is homogeneous and embeds every finite tournament, we find a vertex $y \in N^+(x)$ with $F \subseteq D^-(y)$. Thus, T' contains every finite tournament. So we have $T' = T^\infty = T$.

Next, we assume $T = \mathbb{Q}$. Let us suppose $T \neq T'$. Then we obtain from the previous cases $T' = S(2)$. Let $xy \in ED$. As x has a predecessor in $D^-(y)$, let $a \in N^-(x) \cap N^-(y)$. Since $D^-(x)$ contains a directed triangle and is homogeneous, we find $b, c \in N^-(x)$ with $ab, bc, ca \in ED$. Since $D^+(a) \cong \mathbb{Q}[I_n]$, we have $by \in ED$. Similarly, we conclude $cy \in ED$. The digraph $D[x, a, b, c]$ cannot be embedded into $S(2)[I_m]$ even though it lies in $D^-(y)$. This contradiction shows $T = T'$ if $T = \mathbb{Q}$ and finishes the proof of the lemma. □

We remark that we will see in Section 7.1.9, that the assumption $D^- \cong T'[I_m]$ in Lemma 7.26 is not only satisfied if $n \geq 2$ (due to Lemma 7.25) but also if $n = 1$ (due to Lemma 7.36).

If either $n \geq 2$ or $D^+ \cong T \cong D^-$, then the next lemma will exclude the possibility $T = S(2)$:

Lemma 7.27. *If $D^+ \cong D^-$, then $T \neq S(2)$.*

Proof. Let us suppose $T = S(2)$. Let $x \in VD$ and let $a, b, c \in N^+(x)$ with $ab, bc, ca \in ED$. Since $D[x, a, b]$ can be embedded into D^+, we find a vertex $y \in VD$ with $D[x, a, b] \subseteq D^+(y)$ by C-homogeneity. Since $D^-(a) \cong S(2)[I_n]$ and c and y do not both lie either in $D^-(x)$ or in $D^+(x)$, these two vertices must be adjacent. Because $D[x, a, b, c]$ does not embed into D^+, this edge cannot be yc, so it is cy. In $D^-(b)$ we find a vertex z with $z \in N^+(a) \cap N^+(x) \cap N^-(y)$.

Since D is C-homogeneous, we find an automorphism α of D that fixes x and y and maps ca to zb. Since b lies in $N^+(a) \cap N^-(c)$, its image $b\alpha$ lies in $N^+(b) \cap N^-(z)$. Considering $D^+(x)$, we know that $b\alpha$ cannot lie in $N^+(a)$ as $D^+(x) \cap D^+(a)$ contains no directed triangle $D[b, b\alpha, z]$ but $b\alpha$ must be adjacent to a. So we have $b\alpha a \in ED$. But then $D[a, b, b\alpha, x]$ is a digraph which lies in

$D^+(y)$ even though it cannot be embedded into $S(2)[I_n]$. This contradiction shows the assertion. $\square$

The following lemma shows that we can restrict ourselves to the situation $n = 1$ in the remainder of this section: all the other C-homogeneous digraphs that satisfy the assumptions of this section and that have the property $n \geq 2$ arise from those with $n = 1$ in a canonical way.

Lemma 7.28. *If $D^+ \cong D^-$, then there is a countable connected C-homogeneous digraph D' with $D'^+ \cong T \cong D'^-$ and with $D'[I_n] \cong D$.*

Proof. Let $x \in VD$. Let us first show that

$$N^-(a) = N^-(b) \text{ for each two non-adjacent vertices } a, b \in N^+(x). \tag{7.25}$$

Let $y \in N^-(a)$. First, let us assume that x and y are adjacent. If $y \in N^+(x)$, then it is an immediate consequence of $D^+(x) \cong T[I_n]$ that y lies in $N^-(b)$. So let us assume $yx \in ED$. If D contains no directed triangle, then it contains a transitive triangle and, by C-homogeneity, we find a vertex $z \in VD$ with $D[x, y, a] \subseteq D^-(z)$. Then $D^+(x)$ shows $bz \in ED$ and $D^-(z)$ shows $yb \in ED$. If T contains a directed triangle, let $z \in N^-(a)$ such that $D[x, y, z]$ is a directed triangle. Let $a^\perp$ be the set of vertices in D that are not adjacent to a. Due to the structure of $D^+(y)$, we observe $N^+(y) \cap a^\perp \subseteq N^+(x) \cap a^\perp$ and conclude

$$N^+(y) \cap a^\perp \subseteq N^+(x) \cap a^\perp \subseteq N^+(z) \cap a^\perp \subseteq N^+(y) \cap a^\perp.$$

So all inclusions are equalities, which shows $yb \in ED$.

Now we assume that x and y are not adjacent. Then we find $z \in N^-(a)$ with $x, y \in N^+(z)$. So we have due to the previous situation that z lies in $N^-(b)$ and hence that y lies in $N^-(b)$. This shows (7.25).

Let us define a relation $\sim$ on VD via

$$u \sim v \;:\Longleftrightarrow\; N^-(u) = N^-(v) \text{ for all } u, v \in VD. \tag{7.26}$$

Then $\sim$ is obviously an $\mathrm{Aut}(D)$-invariant equivalence relation with no two adjacent vertices in the same equivalence class. Let A, B be two equivalence classes and let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $a_1b_1 \in ED$. By definition, we know $a_1b_2 \in ED$. Let $c \in N^-(a_1) \cap N^-(b_1)$. By definition of $\sim$, we conclude $ca_2, cb_1 \in ED$. So we have $D[a_1, a_2, b_1, b_2] \subseteq D^+(c)$. Due to the structure of $D^+(c)$ and as a_1 and a_2 are not adjacent, a_2 is a predecessor of b_1 and of b_2. Thus, we have shown that

each two equivalence classes induce either a complete or an empty bipartite digraph. (7.27)

Thus, $D_\sim$ is a digraph. Note that (7.27) implies that $D_\sim$ inherits C-homogeneity from D. By (7.25), we conclude $D \cong D_\sim[I_n]$ and $D^+ \cong T$. $\square$

Now we are able to complete the investigation for D if $D^+ \cong C_3[I_n] \cong D^-$.

Lemma 7.29. *If $D^+ \cong C_3[I_n] \cong D^-$, then $D \cong C_3^\wedge[I_n]$.*

Proof. By Lemma 7.28, it suffices to show $D \cong C_3^\wedge$ if $n = 1$. Note that D is locally finite, if $n = 1$. So we obtain the assertion from [15, Lemma 4.5]. $\square$

In the following we only have to look closer at the cases $T = T^\infty$ and $T = \mathbb{Q}$. So we assume for the remainder of this section that T is one of those two tournaments. In both cases we obtain (among others) digraphs that are similar to those that we obtain in the case of $T = C_3$: the digraphs $T^\wedge[I_n]$. The situation in which they occur (in the case $n = 1$) is that every edge lies on precisely two induced 2-arcs, once as the first edge and once as the last edge:

Lemma 7.30. *If $n = 1$, if $D^+ \cong D^-$, and if every edge of D is on precisely one induced 2-arc the first edge and on precisely one induced 2-arc the last edge, then $D \cong T^\wedge$.*

Proof. Let $x \in VD$. We first show that

there exists a unique vertex $x^\perp$ such that every induced 2-arc that starts at x ends at $x^\perp$. (7.28)

Suppose (7.28) does not hold. Then we find two distinct 2 arcs xyz and xuv in D. By assumption, we have $y \neq u$. Since y and u lie in the tournament $D^+(x)$, they are adjacent. So we may assume $yu \in ED$. Because there is a unique induced 2-arc whose second edge is uv, we know that y and v are adjacent. As x and v are not adjacent, v cannot lie in $D^-(y)$, so we have $v \in N^+(y)$. But then the edge xy lies on the two induced 2-arcs xyz and xyv. This contradiction to the assumption shows (7.28).

Next, we show

$$(x^\perp)^\perp = x. \tag{7.29}$$

Let $xyx^\perp$ be an induced 2-arc. Let $a \in N^+(y) \cap N^-(x^\perp)$. Since xya cannot be an induced 2-arc by assumption, a and x are adjacent. This edge must be xa because of $D^+(a) \cong T$. So there exists $b \in VD$ with $x, a \in N^+(b)$ Since $xax^\perp$ is an induced 2-arc, the edge $ax^\perp$ cannot lie on a second induced 2-arc $bax^\perp$. Hence, b and $x^\perp$ are adjacent. Note that $x^\perp$ does not lie in $N^+(b)$ because of $D^+(b) \cong T$ and $x \in N^+(b)$. So $x^\perp b \in ED$ and $x^\perp bx$ is an induced 2-arc that shows (7.29).

Let us show that

$$\text{diam}(D) = 2 \textit{ and } x^\perp \textit{ is the only vertex in } D \textit{ that is not adjacent to } x. \tag{7.30}$$

Since D contains induced 2-arcs, its diameter is at least 2. Let $xux^\perp$ and $x^\perp vx$ be induced 2-arcs. Any neighbour of $x^\perp$ except for u and v must be adjacent to either u or v because of $D^+ \cong T \cong D^-$, so its distance to x is at most 2. Because of $D^+ \cong T \cong D^-$, any two vertices a, b with $d(a, b) = 2$ must be the end vertices of an induced 2-arc. Hence, (7.28) and (7.29) show that every neighbour of $x^\perp$ must be adjacent to x. This shows (7.30).

Now we are able to show $D \cong T^\wedge$. Due to (7.30), we know that D is the union of $D_1 := D^+(x) + x$ and $D_2 := D^-(x) + x^\perp$. Furthermore, we have $D^-(x) = D^+(x^\perp)$ because x and $x^\perp$ have no common successor and no common predecessor. Let us define

$$\varphi\colon D_1 \to D_2, \quad y \mapsto y^\perp.$$

Since D_1 and D_2 are tournaments, $y^\perp$ does not lie in D_1 for any $y \in VD_1$, so φ is well-defined. Similarly, φ is surjective. Due to (7.28) and (7.29), we also have that φ is injective. Let $uv \in ED_1$. Then $vu^\perp \in ED$ and $u^\perp v^\perp \in ED$ as D^- is a tournament. This shows that φ is an isomorphism. Let $a \in D_1$ and $b \in D_2$. If $ab \in ED$, then $ba^\perp \in ED$ and, if $ba \in ED$, then $a^\perp b \in ED$. Thus, we have shown $D \cong T^\wedge$. □

Now we determine D in the case $T = T^\infty$ if $D^+ \cong D^-$.

Lemma 7.31. *If $D^+ \cong T^\infty[I_n] \cong D^-$, then either $D \cong (T^\infty)^\wedge[I_n]$ or $D \cong T'[I_n]$ for some countable homogeneous tournament T'.*

Proof. Let us assume that $n = 1$ and that D is not a homogeneous tournament. As any induced subdigraph of a tournament is connected, C-homogeneity implies that D is no tournament at all.

Since D^+ and D^- are tournaments, we find between each two vertices x and y of distance 2 an induced 2-arc xyz in D. Our aim is to apply Lemma 7.30. Therefore, we prove that

$$\textit{there is no } z' \neq z \textit{ in } VD \textit{ such that } xyz' \textit{ is an induced 2-arc.} \tag{7.31}$$

Let us suppose that we find a vertex $z' \neq z$ such that xyz' is an induced 2-arc. Since $D^+(y) \cong T^\infty$, the vertices z and z' are adjacent, say $zz' \in ED$. Let $a \in N^+(y) \cap N^+(x)$. Because $D^-(a)$ is a tournament, neither z nor z' lies in $N^-(a)$. Since $D^+(y)$ is a tournament, a is adjacent to z and to z'. Thus, z and z' lie in $N^+(a)$. In $D^+(y) \cong T^\infty$, we find a vertex b with $ba, bz, z'b \in ED$.

Considering $D^-(a)$, we know that b and x are adjacent, but neither bx nor xb is an edge of D since neither $D^+(b)$ can contain x and z nor $D^-(b)$ can contain x and z'. This contradiction shows (7.31).

By an analogous proof as above, there is precisely one induced 2-arc whose second edge is xy. Thus, the assertion follows from Lemma 7.30. □

It remains to determine D in the case $T = \mathbb{Q}$.

Lemma 7.32. *If $D^+ \cong \mathbb{Q}[I_n] \cong D^-$, then D is isomorphic to one of the following digraphs:*

(i) $\mathbb{Q}^\wedge[I_n]$;

(ii) $S(3)[I_n]$; *or*

(iii) $T'[I_n]$ *for some countable homogeneous tournament T'.*

Proof. As in the proof of Lemma 7.31, we assume $n = 1$ and that D is not a (homogeneous) tournament. If for every edge xy there is precisely one induced 2-arc whose first edge is xy and precisely one induced 2-arc whose second edge is xy, then Lemma 7.30 implies $D \cong \mathbb{Q}^\wedge$. By symmetry, let us assume that xy lies on two induced 2-arcs xyz and xyz'.

Considering $D^+(y)$, the vertices z and z' are adjacent. We may assume $zz' \in ED$. Let $z'' \in N^+(y) \cap N^+(x)$. Note that $D^-(z'') \cong \mathbb{Q}$ implies that neither z nor z' lies in $N^-(z'')$. But as $z, z', z'' \in N^+(y)$, the vertex z'' is adjacent to z and to z'. Hence, we have $z''z \in ED$ and $z''z' \in ED$. By C-homogeneity, we find an automorphism α of D that fixes y and z' and maps z'' to z. Since $x \in N^-(z'')$ but $x \notin N(z')$, we conclude $x' := x\alpha \neq x$. Note that x and x' must be adjacent as both vertices lie in $D^-(y)$ but $x'x \notin ED$ because not both of the two non-adjacent vertices x and z can lie in $D^+(x')$. Thus, we have $xx' \in ED$ and $xx'zz'$ is an induced 3-arc. Thus, we have shown that

the end vertices of any induced 2-arc are also end vertices of an induced 3-arc. (7.32)

Let us show the following:

D contains either an induced directed cycle or an induced directed double ray. (7.33)

First, let us assume that there is an integer m such that D contains an induced m-arc but no induced $(m + 1)$-arc. Let m be smallest possible. Due to (7.32), we have $m \geq 3$. Let $a_0 \ldots a_m$ be an induced m-arc and $a_{m+1} \in VD$ such that $a_1 \ldots a_{m+1}$ is also an induced m-arc. To see that such a vertex a_{m+1} exists, take an automorphism α of D that maps $a_0 \ldots a_{m-1}$ to $a_1 \ldots a_m$, which exists by C-homogeneity, and set $a_{m+1} := a_m\alpha$. By the choice of m, we know that $a_0 \dots a_{m+1}$ is not an induced $(m+1)$-arc. If $a_0 = a_{m+1}$, then $a_0 \dots a_m$ is an induced directed cycle. So a_0 and a_{m+1} are distinct but adjacent. As $m \geq 2$, the vertices a_0 and a_m are not adjacent. Hence, a_0 cannot lie in the tournament $D^-(a_{m+1})$. Thus, we have $a_{m+1}a_0 \in ED$ and the vertices $a_0, \dots, a_{m+1}$ form an induced directed cycle.

If no such m exists, then D contains an induced n-arc for every $n \in \mathbb{N}$, as it contains an induced 3-arc by (7.32). Hence, D contains an induced directed double ray: by C-homogeneity, we can enlarge every n-arc $a_1 \dots a_{n+1}$ to an $(n+2)$-arc $a_0 \dots a_{n+2}$ in a similar way we enlarged the m-arc in the previous case. Continuing in this way we obtain an induced directed double ray, which shows (7.33).

Next, we show that

D contains no induced 4-arc. (7.34)

Let us suppose that D contains an induced 4-arc $a_0 \dots a_4$. By (7.32) and C-homogeneity, we find a vertex b such that a_0ba_3 in an induced 2-arc. Since $D^-(b)$ does not contain two non-adjacent vertices, we have $a_4b \notin ED$. So either a_0ba_4 is an induced 2-arc or $a_0ba_3a_4$ is an induced 3-arc and we find by (7.32) a vertex c such that a_0ca_4 is an induced 2-arc. For simplicity, set $c := b$ if a_0ba_4 is an induced 2-arc. Considering $D^+(a_0)$ we know that a_1 and c are adjacent. As an edge ca_1 is a contradiction to $D^+(c) \cong \mathbb{Q}$, we have $a_1c \in ED$ and we conclude as before that a_2 and c are adjacent. But an edge a_2c implies that $D^-(c)$ contains the two non-adjacent vertices a_0 and a_2 and an edge ca_2 implies that $D^+(c)$ contains the two non-adjacent vertices a_2 and a_4. This contradiction shows (7.34).

A direct consequence of (7.34) is that

D contains neither an induced directed double ray nor an induced directed cycle of length at least 6. (7.35)

The next step is to show that

D contains no directed triangle. (7.36)

Let $xy \in ED$. For every $a \in N^-(y)$, we define

$$\begin{aligned} a^{\rightarrow} &= \{v \in N^+(y) \mid av \in ED\}, \\ a^{\leftarrow} &= \{v \in N^+(y) \mid va \in ED\}, \text{ and} \\ a^{\perp} &= \{v \in N^+(y) \mid a \text{ not adjacent to } v\}. \end{aligned}$$

Let $a_1 \in a^{\rightarrow}$, $a_2 \in a^{\leftarrow}$ and $a_3 \in a^{\perp}$. These three vertices form a transitive triangle as they lie in $D^+(y) \cong \mathbb{Q}$. Since $D^+(a_2)$ is a tournament and $a \in N^+(a_2)$, we have $a_3a_2 \in ED$ and, since $D^-(a_1)$ is a tournament and $a \in N^-(a_1)$, we have $a_1a_3 \in ED$. As $D[a_1, a_2, a_3]$ is transitive, we conclude $a_1a_2 \in ED$. So we have $a^{\rightarrow} \cup a^{\perp} \subseteq N^-(a_2)$ and $a^{\leftarrow} \cup a^{\perp} \subseteq N^+(a_1)$.

Let us suppose that D contains some directed triangle. Let $z, z' \in x^{\perp}$ with $zz' \in ED$, let $u \in x^{\rightarrow}$, and let $v \in x^{\leftarrow}$. As D contains a directed triangle, we find a vertex w such that $D[w, y, u]$ is such a triangle. As we have $w^{\rightarrow} \cup w^{\perp} \subseteq N^-(w')$ for every $w' \in w^{\leftarrow}$ and as $u \in w^{\leftarrow}$, we conclude $N^+(y) \cap N^+(u) \subseteq w^{\leftarrow}$. In particular, we have $x^{\perp} \subseteq N^+(y) \cap N^+(u) \subseteq w^{\leftarrow}$. In particular, we have $z'w \in ED$. By C-homogeneity, we find an automorphism α of D that fixes y and z and maps v to z'. Then we have $x\alpha \neq x$, as $v\alpha = z' \in x^{\perp}$ but $v \notin x^{\perp}$. Since w and $x\alpha$ lie in $D^-(y)$, they are adjacent to x. But neither of them lies in $N^+(x)$, because both lie in $N^+(z')$ and D^- is a tournament. Note that $z \in x^{\perp} \cap (x\alpha)^{\perp}$. Thus, we have $x^{\perp} \nsubseteq (x\alpha)^{\leftarrow}$ and hence we do not find any automorphism of D that fixes x and y and maps w to $x\alpha$. This contradiction to C-homogeneity shows (7.36).

We know by (7.33)–(7.36) that the only induced directed cycles in D have length either 4 or 5. Next, we show that

D contains a directed cycle of length 4. (7.37)

If D contains no induced directed cycle of length 4, then D contains only induced directed cycles of length 5. Let $a_1 \ldots a_5a_1$ be such a cycle. Due to (7.32), there is an induced 2-arc a_1ua_4 in D. Since $a_1ua_4a_5a_1$ is not an induced directed cycle of length 4, the vertices u and a_5 must be adjacent. But an edge ua_5 implies that $a_1ua_5a_1$ is a directed triangle and an edge a_5u implies that ua_4a_5u is a directed triangle. These contradictions to (7.36) show (7.37).

Let us show that

for every directed cycle C of length 4 every vertex of D outside C has a predecessor u and a successor w on C with $uw \in ED$. (7.38)

First, let v be a vertex outside C that has a neighbour on C. If v has a predecessor on C, then there are at most two predecessors of v on C, since $D^-(v)$ is a tournament. Let u be that predecessor of v on C whose successor w on C does not lie in $N^-(v)$. Since v and w lie in $N^+(u)$, they are adjacent and by the choice of u we have $vw \in ED$. If v has a successor on C, then an analogous argument shows the assertion for v. Since $D^+ \cong \mathbb{Q} \cong D^-$, any neighbour of v that does not lie on C must be adjacent to some neighbour of v on C – either to a predecessor or a successor. Thus, every vertex of $D \setminus C$ is adjacent to some vertex of C and we have shown (7.38).

A consequence of (7.38) is the following:

the vertices that are not adjacent to a given vertex induce a tournament. (7.39)

Let $C = x_1x_2x_3x_4x_1$ be a directed cycle of length 4, which exists by (7.37), and let u and v be two vertices that are not adjacent to x_1. By (7.38) we know that each of u and v has a predecessor on C, which cannot be x_4 since $D^+(x_4)$ is a tournament. Furthermore, each of u and v has a successor on C, which cannot be x_2 since $D^-(x_2)$ is a tournament. If u and v are not adjacent, then we may assume that $x_3u, ux_4 \in ED$ and $x_2v, vx_3 \in ED$ as D^+ and D^- are tournaments. Note that neither vx_4 nor x_2u lies in ED as u and v are not adjacent. Thus, $ux_4x_1x_2v$ is an induced 4-arc. This contradiction to (7.34) proves (7.39).

We are now able to show $D \cong S(3)$. To show this, it suffices to show that D is homogeneous, because the only homogeneous digraph with $D^+ \cong \mathbb{Q}$ that has two distinct induced 2-arcs xyz and xyz' is $S(3)$.

Let A and B two isomorphic finite induced subdigraphs of D and $\varphi\colon A \to B$ be an isomorphism. If A is connected, then φ extends to an automorphism of D by C-homogeneity. So let us assume that A has at least two components. Then (7.39) shows that A has precisely two components A_1 and A_2 both of which are tournaments. Furthermore, each component can be embedded into $\mathbb{Q}$ since D contains no directed triangle by (7.36). Let $a_1 \in VA_1$ such that $A_1 - a_1 \subseteq D^+(a_1)$ and let $a_2 \in VA_2$ such that $A_2 - a_2 \subseteq D^-(a_2)$. Let C be a directed cycle of length 4. This exists by (7.37). By C-homogeneity, we may assume $a_1 \in VC$. Due to (7.38), we know that D contains either an induced 2-arc from a_1 to a_2 or an induced 2-arc from a_2 to a_1. Indeed, if $auvw$ is the cycle C, then a_2 has a predecessor on C by (7.38) which cannot be w since $D^+(w)$ does not contain two non-adjacent vertices. Similarly, u is not a successor of a_2. Hence, either a_1ua_2 or a_2wa_1 is the induced 2-arc we are searching for. Since a_1 and a_2 lie on an induced 2-arc, C-homogeneity implies that we may also assume $a_2 \in VC$. So we find a vertex $a \in VC \cap N^+(a_1) \cap N^-(a_2)$. Note that $a \notin VA$. Then every vertex $a_1' \in A_1 \setminus \{a_1\}$ must be adjacent to a since a and a_1' lie in $D^+(a_1) \cong \mathbb{Q}$. If a is a predecessor of a_1', then $D^+(a)$ contains the two non-adjacent vertices a_2 and a_1', which is impossible. Hence, a is a successor of a_1'. Similarly, we obtain that a is a predecessor of every vertex $a_2' \in VA_2$. So we have $A_1 \subseteq D^-(a)$ and $A_2 \subseteq D^+(a)$. Similarly, we find a vertex $b \in VD$ with $A_1\varphi \subseteq D^-(b)$ and $A_2\varphi \subseteq D^+(b)$. Then φ extends to an isomorphism from $A + a$ to $B + b$ and hence by C-homogeneity to an automorphism of D. So we obtain that D is homogeneous and hence isomorphic to $S(3)$. $\square$

Let us summarize the results of this section:

Proposition 7.33. *Let D be a countable connected C-homogeneous digraph with $D^+ \cong T[I_n]$ for some countable homogeneous tournament T and some $n \in \mathbb{N}^\infty$. If either $n \geq 2$ or if $D^+ \cong T \cong D^-$, then D is isomorphic to one of the following digraphs:*

(i) *$T^\wedge[I_n]$ if $T \in \{C_3, \mathbb{Q}, T^\infty\}$; or*

(ii) *$S[I_n]$, where either $S = S(3)$ or S is some countable homogeneous tournament.*

Proof. Note that $D^+ \cong D^-$ also holds if $n \geq 2$ due to Lemmas 7.25 and 7.26. Then the assertion directly follows from Lemmas 7.27, 7.29, 7.30, 7.31, and 7.32. □

We will see in Section 7.1.9 (Lemma 7.36) that $D^+ \cong T$ implies $D^- \ncong I_m[T']$ for any $m \in \mathbb{N}^\infty$ with $m \geq 2$ and any countable homogeneous tournament $T' \neq I_1$. Thus, we have $D^- \cong T'[I_m]$ for some countable homogeneous tournament $T' \neq I_1$ and some $m \in \mathbb{N}^\infty$. So Lemma 7.26 implies $D^+ \cong D^-$ and hence Proposition 7.33 covers this situation.

7.1.9 $D^+ \cong I_n[T]$ with $T \neq I_1$

In this section, let D be a countable connected C-homogeneous digraph with $D^+ \cong I_n[T]$ for some countable homogeneous tournament $T \neq I_1$ and some $n \in \mathbb{N}^\infty$ with $n \geq 2$. A direct consequence of the previous sections together with the fact that T contains some edge is the following lemma:

Lemma 7.34. *We have $D^- \cong I_m[T']$ for some $m \in \mathbb{N}^\infty$ and some countable homogeneous tournament $T' \neq I_1$.* □

Our next lemma says that T and T' are infinite tournaments. Note that we do not know so far whether $m > 1$ or not. We will see this in Lemma 7.36.

Lemma 7.35. *We have $T \neq C_3 \neq T'$.*

Proof. Seeking for a contradiction, let us suppose $T = C_3$. Let $xy \in ED$ and let $a, b \in N^+(x)$ with $ya, ab, by \in ED$. Let z be a common predecessor of x and y. Considering $D^-(y)$, the vertices z and b lie in the same component, which is a tournament. Thus, they are adjacent. As an edge zb gives us a transitive triangle $D[x, y, b]$ in $D^+(z)$ and as this is not possible, we have $bz \in ED$. Hence, the directed triangle $D[a, b, y] \subseteq D^+(x)$ contains one successor and one predecessor of z. So if the third vertex is either a successor or a predecessor of z, then we can find an automorphism of D that fixes x and z and rotates the directed triangle $D[a, b, y]$. More precisely, the automorphism maps a to either b or y and hence it must leave the component $D[a, b, y]$ of $D^+(x)$ invariant. Applying the same automorphism once more, we obtain that the whole triangle $D[a, b, y]$ lies either in $D^+(z)$ or in $D^-(z)$. As neither of these two cases can occur, the third vertex of $D[a, b, y]$ is not adjacent to z.

Thus, in the directed triangle $D[a, b, y] \subseteq D^+(x)$, we find a predecessor of z, a successor of z and a vertex not adjacent to z. By C-homogeneity, we find the same in each directed triangle in $D^+(x)$. Indeed, if u is a vertex in another directed triangle in $D^+(x)$, then we have $D[u, x, z] \cong D[v, x, z]$ for some $v \in \{a, b, y\}$. Thus, x together with $n \geq 2$ independent successors lies in $D^+(z)$, which is impossible. This shows $T \neq C_3$. So T is an infinite tournament and $D^+(x) \cap D^-(y)$ contains a transitive triangle. Thus, we also have $T' \neq C_3$. □

Now we can describe the structure of the neighbourhood of any vertex:

Lemma 7.36. *For every $x \in VD$, the digraph $D^+(x) + D^-(x)$ is a disjoint union of isomorphic homogeneous tournaments. Each of its components consists of one component of $D^+(x)$ and one component of $D^-(x)$.*

In particular, we have $m = n$.

Proof. For every $u \in N^-(x)$, there is a unique component of $D^+(x)$ that contains successors of u because of $D^+(u) \cong I_n[T]$. We denote this component by A_u.

The first step is to show

$$A_u = A_v \text{ for all adjacent vertices } u, v \in N^-(x). \tag{7.40}$$

We may assume $uv \in ED$. Since T is infinite by Lemma 7.35, it contains a transitive triangle. Hence, there is a vertex $y \in N^+(x) \cap N^+(v)$ in $D^+(u)$. This vertex y already shows us $A_u = A_v$.

By C-homogeneity, there is for every component C of $D^+(x)$ some vertex $v \in D^-(x)$ with $C = A_v$. Thus, (7.40) implies $n \leq m$. Symmetrically, we obtain $m \leq n$. Hence, we have $n = m$.

Let us show

$$N(v) \cap N^+(x) \subseteq A_v \text{ for every } v \in N^-(x). \tag{7.41}$$

Since $D^+(v) \cap D^+(x)$ is a tournament, we have $N^+(v) \cap N^+(x) \subseteq A_v$. Let us suppose $N(v) \cap N^+(x) \not\subseteq A_v$. Then we find a vertex $y \in N^+(x) \cap N^-(v)$ that lies outside A_v. Let C_v be the component of $D^-(x)$ that contains v. Note that y has no predecessor in C_v as $y \notin A_v$ and due to (7.40). If v is the unique successor of y in C_v, then we can find an automorphism of D that fixes x and y

and maps some predecessor v^- of v in C_v to some successor v^+ of v in C_v. Note that neither v^- nor v^+ is adjacent to y as we already mentioned. This automorphism fixes C_v setwise, so it must fix v, the unique neighbour of y in C_v. But we have $(v^-v)\alpha = v^+v \notin ED$ even though $v^-v \in ED$. This shows that y has a second successor $u \neq v$ in C_v. As u and v are adjacent, we have $A_u = A_v$ by (7.40). Hence, we may assume $uv \in ED$. By C-homogeneity, we find an automorphism α of D that maps yu to vx. Then v has a predecessor $x\alpha$ in $N^+(x)$ that is adjacent to $v\alpha \in A_v$. As A_v contains some predecessor of v, C-homogeneity implies that it contains every predecessor of v in $N^+(x)$ in contradiction to $y \notin A_v$. Indeed, we find an automorphism that fixes x and v and maps $x\alpha$ to y and this automorphism does not fix A_v setwise even though it fixes x and v. This shows (7.41).

Next, we show

$$A_v = N(v) \cap N^+(x) \text{ for every } v \in N^-(x). \tag{7.42}$$

If A_v contains some vertex y that is not adjacent to v, then, by C-homogeneity, some automorphism of D maps y to some vertex z in $N^+(x) \setminus A_v$ and fixes x and v. Note that z exists because of $n \geq 2$. But then this automorphism does not fix A_v setwise even though it fixes x and v. This contradiction shows (7.42).

By symmetric arguments, there is for every $u \in N^+(x)$ a component B_u of $D^-(x)$ with $B_u = N(u) \cap N^-(x)$ and for each two vertices u, v in the same component of $D^+(x)$, the components B_u and B_v coincide. Thus, $D^+(x) + D^-(x)$ is a disjoint union of isomorphic tournaments and each component of $D^+(x) + D^-(x)$ consists of precisely one component of $D^+(x)$ and one component of $D^-(x)$. That every component of $D^+(x) + D^-(x)$ is homogeneous is a direct consequence of C-homogeneity. □

Note that with Lemma 7.36, we have completed the analysis of Section 7.1.8. Furthermore, we have all lemmas we need to finish the situation if D^+ is isomorphic to $I_n[T]$ for some $n \in \mathbb{N}^\infty$ with $n \geq 2$ and some countable homogeneous tournament $T \neq I_1$. (Note that the case $n = 1$ was already completed in Section 7.1.8.)

Proposition 7.37. *If D is a countable connected C-homogeneous digraph with $D^+ \cong I_n[T]$ for some countable homogeneous tournament $T \neq I_1$ and some $n \in \mathbb{N}^\infty$ with $n \geq 2$, then $D \cong X_\lambda(T')$ for some countable infinite homogeneous tournament T' and for some countable cardinal $\lambda \geq 2$.*

Proof. For $x \in VD$, let $D_x := D^+(x) + D^-(x)$. Due to Lemma 7.36, the digraph D_x is a disjoint union of isomorphic infinite tournaments. First, we show that

$$\text{for every } x \in VD, \text{ no two components of } D_x \text{ lie in the same component of } D - x. \tag{7.43}$$

Let us suppose that we find a path in $D-x$ between vertices in distinct components of D_x. Let P be such a path of minimal length and let u and v be its end vertices. If $ux \in ED$, let a and b two vertices in $N^+(u)$ such that $a \in N^-(x)$ and $b \in N^+(x)$. If $xu \in ED$, we choose a and b in $N^-(u)$ such that $a \in N^-(x)$ and $b \in N^+(x)$. These vertices exist as $D^+(u)$ and $D^-(u)$ are disjoint unions of homogeneous tournaments. If a or b has a neighbour c on P other than u, this neighbour must be the neighbour of u on P by the minimality of P. But then a, b, c, and x lie in the same component of D_u, which is a tournament. So c is already adjacent to x, which contradicts the minimality of P. Hence, the paths $vPua$ and $vPub$ are isomorphic and, by C-homogeneity, we can find an automorphism α of D that maps the first onto the second path by fixing P pointwise and mapping a to b. Since a lies in $N^-(x)$ and b lies in $N^+(x)$, we have $x \neq x\alpha$. But as $x\alpha$ is adjacent to u and to b, it lies in the same component of D_u as x. So x and $x\alpha$ are adjacent and $x\alpha$ lies in the same component of D_x as a and b. Since $x\alpha$ is a neighbour of $v = v\alpha$, also v lies in the same component of D_x as $x\alpha$ and thus the vertices u and v are adjacent. This contradiction to the choice of u and v shows (7.43).

For every $x \in VD$, each component of D_x is an infinite tournament and hence contains a ray. Rays from distinct components of D_x cannot be equivalent as they lie in distainct components of $D - x$ due to (7.43). Hence, D has at least two ends. Thus, the assertion follows from Corollary 5.10 and Theorem 5.20, the classification result of connected C-homogeneous digraphs with more than one end. □

7.1.10 Another partial result

By summarizing the propositions of the previous sections together with Cherlin's classification of the homogeneous digraphs, Theorem 3.11, we obtain the following theorem:

Theorem 7.38. *Let D be a countable connected C-homogeneous digraph. Then one of the following cases holds:*

(i) *D is homogeneous;*

(ii) *$D \cong T^\wedge[I_n]$ for some $n \in \mathbb{N}^\infty$ and some tournament $T \in \{C_3, \mathbb{Q}, T^\infty\}$;*

(iii) *$D \cong S(3)[I_n]$ for some $n \in \mathbb{N}^\infty$;*

(iv) *$D \cong X_\lambda(T)$ for some countable infinite homogeneous tournament T and for some countable cardinal $\lambda \geq 2$; or*

(v) *$D^+ \cong I_n$ and $D^- \cong I_m$ for some $m, n \in \mathbb{N}^\infty$.* □

7.2 The case: $D^+ \cong I_n$ and $D^- \cong I_m$

Throughout this section, let D be a countable connected C-homogeneous digraph with $D^+ \cong I_n$ for some $n \in \mathbb{N}^\infty$. By the previous sections, we also have $D^- \cong I_{n'}$ for some $n' \in \mathbb{N}^\infty$.

Due to Lemma 6.7 and Chapter 5, we may assume $d^+ > 1$ and $d^- > 1$ and we may assume for the remainder of this section that D contains at most one end.

Due to Lemmas 6.8 and 6.11, the reachability relation of every locally finite C-homogeneous digraph with at most one end and whose out-neighbourhood is independent is not universal. If we consider such digraphs of arbitrary degree, this does no longer hold. For example, the countable generic 2-partite digraph is a C-homogeneous digraph with independent out-neighbourhood and with precisely one end and its reachability relation is universal. In the following, we distinguish the two cases whether the reachability relation $\mathcal{A}$ of D is universal or not.

7.2.1 Non-universal reachability relation

Within this section, let D be a countable connected C-homogeneous digraph with $D^+ \cong I_n$ for some $n \in \mathbb{N}^\infty$, with $D^- \cong I_{n'}$ for some $n' \in \mathbb{N}^\infty$, with at most one end. We assume that $\mathcal{A}$ is not universal and, due to Lemma 6.7, that $n, n' \geq 2$. Hence, we obtain by Proposition 2.1 that $\Delta(D)$ is bipartite. That is the reason, why we turn our attention towards the classification of the C-homogeneous bipartite graphs. Remember that we are interested in the C-homogeneous bipartite graphs due to Lemma 5.14: It tells us that $G(\Delta(D))$ belongs to one of the five classes described in Theorem 3.5. In the following, we will treat these five possibilities one by one. Let us start with the case $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 2$, where we notice that D must be locally finite as every vertex lies in at most two reachability digraphs:

Lemma 7.39. *If $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 2$, then D is locally finite.* □

Thus, if $G(\Delta(D))$ is an even cycle, then we obtain this part of the classification due to Chapter 6. In the following, we assume $G(\Delta(D)) \ncong C_{2m}$ for any $m \in \mathbb{N}$. Since locally finite C-homogeneous digraphs have already been classified, we may assume in the following that either $d^+ = \omega$ or $d^- = \omega$. By reversing the directions of each edge if necessary, we may assume $d^+ = \omega$.

For a reachability digraph Δ of D, two vertices or a set of vertices of Δ *lie on the same side of* Δ if their out-degree, and hence also their in-degree, in Δ is the same.

Lemma 7.40. *For each two reachability digraphs Δ_1 and Δ_2 of D we have either $\Delta_1 \cap \Delta_2 = \emptyset$ or $|V(\Delta_1 \cap \Delta_2)| \geq 2$.*

Proof. Let us suppose that the intersection of two distinct reachability digraphs Δ_1 and Δ_2 consists of precisely one vertex. Since every vertex lies in precisely two reachability digraphs and since D is vertex-transitive, each two distinct reachability digraphs either have trivial intersection or share precisely one vertex.

We distinguish the cases whether C_3 embeds into D or not. First, we assume that D contains no directed triangle. Let $xy \in ED$ and $\Delta = \langle \mathcal{A}(xy) \rangle$. If $G(\Delta) \ncong CP_k$, let P be any path of minimal length from any successor u of y to x avoiding y. Such a path exists as the one-ended digraph D cannot contain any cut-vertex. If $G(\Delta) \cong CP_k$, let P be any path of minimal length from any successor u of y to x that avoids y and the unique neighbour $\bar{y}$ of y in the bipartite complement of Δ. As $k = d^+ = \omega$, both of the two reachability digraphs $\langle \mathcal{A}(yu) \rangle$ and Δ contain rays that avoid y and $\bar{y}$ and hence y and $\bar{y}$ separate neither these rays nor u from x. Thus, we also know in this situation that P exists.

By the minimality of P, the only successor of y on P is u. If y has a predecessor x' on P, then xyu and $x'yu$ are induced 2-arcs, so we find an automorphism of D that maps one onto the other and we obtain a contradiction to the minimality of P. Thus, y has no neighbour on P except for u and x. At most $|VP|$ vertices of Δ that lie on the same side as y can have successors on P, since any two such vertices with a common successor on P would lie in two common reachability digraphs. Since $N^+(x)$ contains infinitely many vertices, all of which lie on the same side of Δ as y, we find one such vertex z that has no successor on P. If $G(\Delta)$ is either complete bipartite or the bipartite complement of a perfect matching, then every predecessor of z on P is also a predecessor of y by the assumption that in the case $G(\Delta) \cong CP_k$ the path P does not contain $\bar{y}$. Hence, P contains predecessors of z only if $G(\Delta)$ is the generic bipartite graph or a tree $T_{k,\ell}$. Note that any predecessor of z on P is a predecessor in Δ of z. Thus, in these two cases we may have chosen z among the infinitely many vertices of $N^+(x)$ that have no predecessor on P. Let v be the neighbour of u on P. Then both vertices y and z have only one neighbour on vPx, the vertex x. By C-homogeneity, we find an automorphism α of D that fixes vPx and interchanges y and z. Let $w = u\alpha$.

If $vu \in ED$, then v and y lie on the same side of $\langle \mathcal{A}(yu) \rangle$ and on this side lies also $y\alpha = z$ as $(vu)\alpha = vw$. But then y and z lie in two common reachability digraphs which contradicts the assumption. Hence, we have $uv \in ED$ and $wv \in ED$. The two 2-arcs xyu and xzw induce a digraph that consists only of these two 2-arcs: as z and u are not adjacent, neither are $y = z\alpha$ and $w = u\alpha$. Note that no successor of y can have w or z as a predecessor because otherwise either w or z lies in the two reachability digraphs $\langle\mathcal{A}(yu)\rangle$ and either $\langle\mathcal{A}(wv)\rangle$ or Δ, which is impossible by assumption. By the same assumption and similar as above, only finitely many successors of u have successors on the 1-arc zw. Since $d^+ = \omega$, we find a vertex $u' \in N^+(y)$ that is adjacent to neither w nor z. Note that u' and x are not adjacent since D contains no triangle. Hence, we find by C-homogeneity an automorphism β of D that fixes $D[x, y, z, w]$ pointwise and maps u to u'. So u' and w have a common successor $v\beta$ and thus u and u' lie on the same side of $\langle\mathcal{A}(uv)\rangle$ and of $\langle\mathcal{A}(yu)\rangle$. This contradiction shows the assertion in the situation that C_3 does not embed into D.

Now we consider the case that D contains a directed triangle. For every edge xy those successors of y that are predecessors of x lie in two common reachability digraphs. As the intersection of two distinct reachability digraphs contains at most one vertex, we obtain that

$$\textit{every edge lies on precisely one directed triangle.} \tag{7.44}$$

We distinguish whether $G(\Delta(D))$ is a semi-regular tree or not. First, we consider the case $G(\Delta(D)) \cong T_{k,\ell}$ for some $k, \ell \in \mathbb{N}^\infty$ with $k, \ell \geq 2$. Let $x \in VD$ and let P be a shortest path in $G - x$ between any two successors y and z of x. Since P must contain some edge that does not lie in $\langle\mathcal{A}(xy)\rangle$ and since any two distinct reachability digraphs intersect in at most one vertex, P contains some vertex outside $\langle\mathcal{A}(xy)\rangle$. Thus and by the assumption on the intersection of any two distinct reachability digraphs, P has at least three edges. Let z_2, z_1, z be the last three vertices of P. Let a be a third successor of x. This vertex exists as $d^+ = \omega$. By minimality of P, it contains no neighbour of a as otherwise we find a shorter path between a and either y or z, since neither a and y nor a and z have a common predecessor, as they lie in only one common reachability digraph. Hence, the connected subdigraphs $zxyPz_2$ and $axyPz_2$ are isomorphic and we find an automorphism α of D that fixes $xyPz_2$ and interchanges a and z, as D is C-homogeneous. So we obtain that $D' := D[z, z_1, z_2, z_1\alpha, a]$ consists of four edges and, with $z_1' := z_1\alpha$, we have $zz_1 \in ED$ if and only if $az_1' \in ED$ and the same for z_1z_2 and $z_1'z_2$. Since the intersection of any two reachability digraphs contains at most one vertex, the path D' is not an alternating walk. Thus, D' consists of two induced 2-arcs. If these are z_2z_1z and $z_2z_1'a$, then z_1 and z_1' lie in the intersection of the two reachability digraphs $\langle\mathcal{A}(xz)\rangle$ and $\langle\mathcal{A}(z_2z_1)\rangle$. Thus, these 2-arcs must be zz_1z_2 and $az_1'z_2$. If x and z_1 are adjacent, then the edge between them must be z_1x since $N^+(x)$ is independent. But then, we have $z_1'x \in ED$, too, and $D[x, z_1, z_2, z_1']$ is a cycle in $\langle\mathcal{A}(xz)\rangle$, which is impossible. Similarly, x and z_1' are not adjacent. Hence, the digraph $D[x, z, z_1, a, z_1']$ consists of only the two induced 2-arcs xzz_1 and xaz_1' and we can proceed as in the case that C_3 does not embed into D to obtain a contradiction with the additional requirement that u' is not adjacent to x, which is possible as only one successor of z is adjacent to x by (7.44) and $d^+ = \omega$.

It remains to consider the case that $G(\Delta(D))$ is not a semi-regular tree. Due to the structure of $G(\Delta(D))$, both sides of each reachability digraph have the same cardinality. As $d^+ = \omega$, we also have $d^- = \omega$. Let $x \in VD$ and y and z be two vertices in $N^+(x)$. Let u and v be the unique successors of y and z, respectively, that lie on a common directed triangle with x, see (7.44). Since each edge lies on a unique (directed) triangle, every common successor $w \neq x$ of u and v is adjacent to neither y nor z. As $d^- = \omega$ and due to (7.44), we find $a \in N^-(v)$ that is adjacent to neither w nor x. An edge au implies that u and v lie in two common reachability digraphs and an edge ua leads to a cycle $D[a, u, w, v]$ witnessing that $\mathcal{A}$ is universal. As both situations are impossible, a and u are not adjacent. Furthermore, az cannot be an edge because then $D[a, v, y, x, z]$ is a cycle witnessing that $\mathcal{A}$ is universal. As this is not the case, we have $az \notin ED$. Let us suppose that za is an edge of D. Then by C-homogeneity, we find an automorphism α of D that fixes w and maps zu to av and v to u. Note that $b := a\alpha \neq z$ since $za \in ED$ but $ba = (az)\alpha \notin ED$. As $bu \in ED$, the digraph $D[a, b, u, z]$ is a cycle witnessing that $\mathcal{A}$ is universal. This contradiction shows that z and a are not adjacent. So we find an automorphism β of D that fixes z, u, w, v and maps y to a, as D is C-homogeneous. Thus, $x\beta \neq x$ is a common predecessor of a and z. So a lies in $\langle \mathcal{A}(xy) \rangle$ on the same side as z. Thus, a and y lie in two common reachability digraphs in contradiction to the assumption. □

Now we are able to complete the investigation if $G(\Delta(D))$ is a semiregular tree:

Lemma 7.41. *If $G(\Delta(D)) \cong T_{k,\ell}$ for some $k, \ell \in \mathbb{N}^\infty$ with $k, \ell \geq 2$, then D either is locally finite or has more than one end.*

Proof. Let us assume that D is not locally finite. By reversing the direction of each edge, we may assume $k = d^+ = \omega$. Let us suppose that D has at most one end. First, we show that

the intersection of two distinct reachability digraphs lies on the same side of each of them. (7.45)

Let us suppose that this is not the case. As D is vertex-transitive, each two reachability digraphs with non-trivial intersection are a counterexample to (7.45). Let Δ_1 and Δ_2 be two distinct reachability digraphs with non-trivial intersection. By Lemma 7.40, their intersection contains at least two vertices. Since $V(\Delta_1 \cap \Delta_2)$ does not lie on the same side of Δ_1, we find two vertices $x, y \in V(\Delta_1 \cap \Delta_2)$ of odd distance in Δ_1 such that x has no successors in Δ_1. Let z be the predecessor of x on the unique x–y path P in Δ_1. Since $d^+ = \omega$, we find a successor x' of z that does not lie on P. Then the digraph $x'zPy$ is isomorphic to P and, by C-homogeneity, we find an automorphism of D that fixes zPy and maps x to x'. So we conclude that x' lies also in the same two reachability digraphs as y. Hence, the two vertices x and x' of distance 2 lie on the same side of Δ_1 and of Δ_2. Inductively, all vertices of Δ_1 that lie on the same side of Δ_1 as x, also lie in Δ_2. In particular, this holds for some successor y' of y. Hence, Δ_1 and Δ_2 share all vertices of D. For an edge $ab \in E\Delta_2$ the a–b path in Δ_1 is an alternating walk. Thus, Q together with the edge ab is a cycle witnessing that $\mathcal{A}$ is universal. This contradiction to the assumptions shows (7.45).

For the remainder of the proof, we fix two reachability digraphs Δ_1 and Δ_2 with non-trivial intersection such that the vertices in $\Delta_1 \cap \Delta_2$ have no successor in Δ_1.

With the same argument as in the proof of (7.45), just taking a path P of even length, we obtain that

every vertex on the same side of Δ_1 as $V(\Delta_1 \cap \Delta_2)$ lies in Δ_2. (7.46)
The analogous property for Δ_2 holds as soon as $\ell \geq 3$.

For the remainder of the proof, let $x \in V\Delta_1 \setminus V\Delta_2$. Next, we show that

no vertex of $N^+(x)$ separates in Δ_2 any other two vertices of $N^+(x)$. (7.47)

To show this, we suppose that $y_1 \in N^+(x)$ separates in Δ_2 the two vertices $y_2, y_3 \in N^+(x)$. By C-homogeneity and as $N^+(x)$ is independent, we find an automorphism of D that fixes x and y_3 and switches y_1 and y_2. This automorphism fixes Δ_2 setwise and we obtain that $y_2 = y_1\alpha$ separates in Δ_2 the vertices $y_1 = y_2\alpha$ and $y_3 = y_3\alpha$ which is clearly impossible. This contradiction shows (7.47).

Let us show that

D contains some directed triangle. (7.48)

Let us suppose that D contains no directed triangle. Let $y \in N^+(x)$ and let $z_1, z_2 \in N^+(y)$ such that z_1 is the neighbour of y in that component of $\Delta_2 - y$ that contains all other successors of x. Then the two 2-arcs xyz_1 and xyz_2 are induced and we obtain an automorphism α of D that fixes x and y and maps z_1 to z_2, as D is C-homogeneous. Thus, α does not fix the unique component of

$\Delta_2 - y$ that contains all successors of x. This is impossible and hence we have shown (7.48).

Let $y \in N^+(x)$ and let $z \in N^+(y)$ such that z lies in the unique component of $\Delta_2 - y$ that contains all successors of x but y, see (7.47). By the same argument as in the proof of (7.47) we obtain that

either z is the only successor of y such that $D[x, y, z]$ is a directed triangle or z is the only successors of y such that $D[x, y, z]$ is an induced 2-arc. (7.49)

If $D[x, y, z]$ is a directed triangle, then every edge of D lies on a unique directed triangle due to (7.49). So the number of directed triangles that contain a given vertex is d^+ and it is also d^-. Hence, we obtain $d^- = d^+ = \omega$. If $D[x, y, z]$ is an induced 2-arc, then the edge xy lies on infinitely many directed triangles as $D^+ = \omega$ and by (7.49). Thus, x must have infinitely many predecessors and we obtain $d^- = d^+ = \omega$ in this case, too. Hence, we have $\ell \geq 3$ and the second part of (7.46) holds. Thus, there are two reachability digraphs distinct from Δ_2 that cover the vertices of Δ_2. So the vertices of $\Delta_2 - \Delta_1$ lie in a reachability digraph $\Delta_0 \neq \Delta_1$. Since C_3 embeds into D, we have

$$\Delta_1 - \Delta_2 = \Delta_0 \cap \Delta_1 = \Delta_0 - \Delta_2.$$

As D is connected, we conclude that Δ_0, Δ_1, and Δ_2 are the only reachability digraphs of D.

The next step is to show that $D[x, y, z]$ is not an induced 2-arc:

$$D[x, y, z] \cong C_3. \tag{7.50}$$

If (7.50) does not hold, then xyz is an induced 2-arc and, by (7.49), unique with the property that xy is its first edge. Let $x' \in VD$ such that yzx' is the unique induced 2-arc with yz as its first edge. Then we have $x' \in V(\Delta_0 \cap \Delta_1)$ and x and x' lie on the same side of Δ_1. Note that xy already determines the vertex x'. So the stabilizer of the edge xy must fix x'. Let u be the first vertex on the unique x–x' path in Δ_1 that is neither x nor y. Let v be another neighbour of x, if u is a neighbour of x, and let v be another neighbour of y otherwise. Then we find an automorphism of D that fixes the edge xy and maps u to v which is clearly impossible as this automorphism does not fix x'. This shows (7.50).

Let us now show that $D[x, y, z]$ cannot be a directed cycle, either, which will be our desired contradiction. To simplify notations, let $x_0 = z$, $x_1 = x$, $x_2 = y$. Let F_i, G_i be the component of $\Delta_i - x_i x_{i+1}$ that contains x_i, x_{i+1}, respectively (we consider the indices modulo 3). Let $u \in F_1 \cap V(\Delta_1 \cap \Delta_2)$. Then we find a second vertex v in $F_1 \cap V(\Delta_1 \cap \Delta_2)$ that has distance $d_{\Delta_1}(x_2, u)$ to each of x_2

and u because of $d^+ \neq 2 \neq d^-$, where d_{Δ_1} denotes the distance in Δ_1. Let $w \in F_1$ be the unique vertex in F_1 that has the same distance to each of x_2, u, v. By C-homogeneity, we find an automorphism that fixes the unique w–u path in Δ_1 and maps the unique w–x_2 path in Δ_1 onto the unique w–v path in Δ_1 and vice versa. As in the proof of (7.47), we obtain that x_2 does not separate u and v in Δ_2. So u and v must lie in the same component C of $\Delta_2 - x_2$. Thus, all vertices a of $F_1 \cap V(\Delta_1 \cap \Delta_2)$ with $d_{\Delta_1}(a, x_2) = d_{\Delta_1}(x_2, u)$ lie in C. Let us suppose $C \subseteq F_2$. Since there are infinitely many components of $\Delta_2 - x_2$ in F_2, we find one neighbour b_1 of x_2 in C and one neighbour b_2 in another component of $F_2 \cap V(\Delta_2 - x_2)$. Both digraphs $x_1x_2b_1$ and $x_1x_2b_2$ are induced 2-arcs as neither b_1 nor b_2 is x_0 and due to (7.49). By C-homogeneity, we find an automorphism α of D that fixes x_1x_2 and maps b_1 to b_2. Thus, α cannot fix C setwise even though it fixes $F_1 \cap V(\Delta_1 \cap \Delta_2)$ setwise. This contradiction shows $C \subseteq G_2$. Thus, we have

$$F_1 \cap V(\Delta_1 \cap \Delta_2) \subseteq G_2 \cap V(\Delta_1 \cap \Delta_2).$$

By a symmetric argument, we obtain

$$F_1 \cap V(\Delta_1 \cap \Delta_2) = G_2 \cap V(\Delta_1 \cap \Delta_2).$$

Analogously, we obtain

$$F_i \cap V(\Delta_i \cap \Delta_{i+1}) = G_{i+1} \cap V(\Delta_i \cap \Delta_{i+1})$$

for all i and hence also

$$G_i \cap V(\Delta_i \cap \Delta_{i+1}) = F_{i+1} \cap V(\Delta_i \cap \Delta_{i+1}).$$

Let $D[a, b, c]$ be a directed triangle with $a \in F_1 \cap V(\Delta_0 \cap \Delta_1)$ that is disjoint from $D[x, y, z]$. Then we have

$$b \in F_1 \cap V(\Delta_1 \cap \Delta_2) = G_2 \cap V(\Delta_1 \cap \Delta_2)$$

and hence

$$c \in G_2 \cap V(\Delta_2 \cap \Delta_0) = F_0 \cap V(\Delta_2 \cap \Delta_0)$$

and

$$a \in F_0 \cap V(\Delta_0 \cap \Delta_1) = G_1 \cap V(\Delta_0 \cap \Delta_1).$$

So ab is an edge in Δ_1 between vertices of distinct components of $\Delta - x_1x_2$, which is impossible. This contradiction shows that D has more than one end. $\square$

Due to the previous two chapters, we assume in the following $G(\Delta(D)) \ncong T_{k,\ell}$ for any $k, \ell \in \mathbb{N}^\infty$, in addition to $G(\Delta(D)) \ncong C_{2m}$ for any $m \in \mathbb{N}$.

Lemma 7.42. *For each two distinct reachability digraphs Δ_1 and Δ_2 of D, the set $V(\Delta_1 \cap \Delta_2)$ lies on the same side of Δ_1.*

Proof. We may assume that Δ_1 and Δ_2 have non-trivial intersection. Due to Lemma 7.40, we have $|V(\Delta_1 \cap \Delta_2)| \geq 2$. Let us suppose that $V(\Delta_1 \cap \Delta_2)$ does not lie on the same side of Δ_1. Since $\Delta_1 \cap \Delta_2$ contains no edge, $G(\Delta(D))$ is no complete bipartite graph.

If $G(\Delta(D))$ is the countable generic bipartite graph, then any two of its vertices have distance at most 3 in $\Delta(D)$. Since $V(\Delta_1 \cap \Delta_2)$ does not lie on the same side of Δ_1, we find $x, y \in V(\Delta_1 \cap \Delta_2)$ with $d_{\Delta_1}(x, y) = 3$. So any two vertices of distance three in Δ_1 lie in the intersection of two reachability digraphs by C-homogeneity, as we can extend them to an induced alternating path of length 3 within Δ_1. This implies that all the vertices of Δ_1 lie in Δ_2, which is impossible as we already saw in the proof of Lemma 7.41. Thus, $G(\Delta(D))$ is not the countable generic bipartite graph.

So for the remainder of the proof, we may assume that $G(\Delta(D)) \cong CP_k$ for some $k \in \mathbb{N}^\infty$ with $k \geq 4$. Since it suffices to consider the case $d^+ = \omega$, we may assume $k = \omega$. As $\Delta_1 \cap \Delta_2$ contains two vertices of distinct sides of Δ_1 but no edge, it consists of precisely two vertices that are adjacent in the bipartite complement of Δ_1. For the end vertices of any 2-arc $x_1x_2x_3$, not necessarily induced, there is no $x_2' \in VD$ such that $x_1x_2'x_3$ is also a 2-arc since otherwise x_2 and x_2' lie in two common reachability digraphs and on the same side of each of them, which is impossible. In particular, every edge y_1y_2 lies on at most one directed triangle, since two directed triangle both of which contain y_1y_2 have different 2-arcs from y_2 to y_1.

Let $xy \in E\Delta_1$ with $y \in V\Delta_2$. If C_3 embeds into D, let a be the unique vertex on a directed triangle with xy. Otherwise, let a be any successor of y. In both cases, let a' (let v) be the unique neighbour of a (of y, respectively) in the bipartite complement of Δ_2. So we have $v \in V(\Delta_1 \cap \Delta_2)$. Since $k = \omega$ and each two distinct reachability digraphs have only two common vertices, we find a common successor u of x and v that is adjacent to neither a nor a'. Similar to the existence of u, we find a vertex $b \in N^+(y)$ with $b \neq a$ such that b and its unique neighbour b' in the bipartite complement of Δ_2 are adjacent to neither x nor u.

Note that Δ_1 contains rays avoiding y and v and that the reachability digraph containing a and a' that is distinct from Δ_2 contains rays avoiding a and a'. As D has at most one end, we find a path from each successor of a and each predecessor of a' to x such that the path avoids a, a', b, b', y, and v. Let P be any such path of minimal length and let c be its first vertex. Note that if C_3 embeds into D then P is the trivial path consisting only of x. By its minimality, P contains no successor of b and no predecessor of b'. Indeed, if P

has such a vertex, then this is not c, since neither a and b nor a' and b' lie in two common reachability digraphs and since $c \notin V\Delta_2$. By C-homogeneity, we find an automorphism of D that fixes xy and maps b to a and b' to a'. This would contradict the minimality of P. Note that, if P contains either a predecessor of b or a successor of b', then this is also a predecessor of a or a successor of a', respectively, and the analogue holds if P contains either a predecessor of a or a successor of a'. Thus, if $ac \in ED$, we find an automorphism of D that fixes P and $yxuv$ and maps a' to b'. Then yac and $ybc = (yac)\alpha$ are 2-arcs with the same end vertices, which cannot exist as we already mentioned. In the situation $ca' \in ED$, we obtain a similar contradiction by an automorphism that fixes P and $yxuv$ and maps a to b, where we find the two 2-arcs $ca'v$ and $cb'v$. $\square$

Now we are able to finish the situation for the cases that $G(\Delta(D))$ is either complete bipartite, or the bipartite complement of a perfect matching, or the countable generic bipartite graph.

Lemma 7.43. *If D has at most one end and is not locally finite, then it is isomorphic to one of the following digraphs:*

(i) *$C_m[I_\omega]$ for some $m \in \mathbb{N}^\infty$ with $m \geq 3$;*

(ii) *Y_ω; or*

(iii) *$\mathcal{R}_m$ for some $m \in \mathbb{N}^\infty$ with $m \geq 3$.*

Proof. Let us assume that D has at most one end and is not locally finite. Since $V(\Delta_1 \cap \Delta_2)$ lies on the same side of Δ_1 by Lemma 7.42, we may assume that the vertices in $\Delta_1 \cap \Delta_2$ have their predecessors in Δ_1 and their successors in Δ_2. Let $\{A, B\}$ be the natural bipartition of $V\Delta_1$ such that $V(\Delta_1 \cap \Delta_2) \subseteq B$. Since any two vertices in B have a common predecessor in A, we conclude $B \subseteq V\Delta_2$ by C-homogeneity. Indeed, we can map any two vertices in $V(\Delta_1 \cap \Delta_2)$ with a common predecessor onto any two vertices in B with a common predecessor, so any two vertices in B lie in two common reachability digraphs of D and hence $B \subseteq V\Delta_2$. Thus, we have $B = V(\Delta_1 \cap \Delta_2)$. By an analogous argument, we obtain that every vertex on the same side of Δ_2 as B lies in B.

Let $\sim$ be a relation on VD defined by

$$x \sim y \quad :\Longleftrightarrow \quad x \text{ and } y \text{ lie on the same side of two reachability digraphs.} \tag{7.51}$$

As we have just shown, $\sim$ is an equivalence relation on VD, which is $\mathrm{Aut}(D)$-invariant. Since each equivalence class is an independent set and since the reachability digraphs are bipartite, we conclude that $D_\sim$ is a digraph. Since every vertex of D lies in precisely two reachability digraphs, every vertex of $D_\sim$ has precisely one successor and one predecessor. Furthermore, $D_\sim$ is connected. Thus, we have $D_\sim \cong C_m$ for some $m \in \mathbb{N}^\infty$ with $m \geq 3$. If $G(\Delta(D)) \cong K_{k,\ell}$ for some $k, \ell \in \mathbb{N}$, then we obtain $k = \ell$ because B is one side of Δ_1 and one of Δ_2. It is a direct consequence that $D \cong C_m[I_\omega]$ as D is not locally finite. Similarly, if $G(\Delta(D))$ is the countable generic bipartite graph, then we directly obtain $D \cong \mathcal{R}_m$. It remains to consider the case $G(\Delta(D)) \cong CP_k$. If $m \geq 4$, then we find two distinct types of induced 2-arcs xyz: one whose end vertices are not adjacent to the same vertex y' with $y' \sim y$ and one whose end vertices do not have this property. Even though D is C-homogeneous, we cannot map the first onto the second of these induced 2-arcs by automorphisms of D. Thus, we have $m = 3$. Let $\overline{D}$ be the tripartite complement of D. Since the bipartite complement of each reachability digraph is a perfect matching, $\overline{D}$ is a disjoint union of directed cycles. Let us suppose that the length of one of these cycles is more than 3. Then it has length at least 6 and there are two $\sim$-equivalent vertices in $\overline{D}$ that have distance 3 on that cycle. Since these two $\sim$-equivalent vertices have a common predecessor, the same is true for any two $\sim$-equivalent vertices by C-homogeneity. So each two $\sim$-equivalent vertices lie on a common directed cycle in $\overline{D}$ and have distance 3 on that cycle. Hence, $\overline{D}$ consists of precisely one cycle of length at most 9 and D is locally finite in contradiction to the assumption. Thus, $\overline{D}$ is the disjoint union of directed triangles, which shows $D \cong Y_\omega$. □

Let us summarize the results of this section. The following proposition follows directly from Proposition 2.1 together with Lemmas 5.14, 7.39, 7.41, and 7.43.

Proposition 7.44. *Let D be a countable connected C-homogeneous digraph with $D^+ \cong I_n$ for some $n \in \mathbb{N}^\infty$ whose reachability relation is not universal. If D has at most one end and is not locally finite, then it is isomorphic to one of the following digraphs:*

(i) $C_m[I_\omega]$ *for some* $m \in \mathbb{N}^\infty$ *with* $m \geq 3$*;*

(ii) Y_ω*; or*

(iii) $\mathcal{R}_m$ *for some* $m \in \mathbb{N}^\infty$ *with* $m \geq 3$. □

7.2.2 Universal reachability relation

Within this section, let D be a countable connected C-homogeneous digraph with $D^+ \cong I_n$ for some $n \in \mathbb{N}^\infty$, with $D^- \cong I_{n'}$ for some $n' \in \mathbb{N}^\infty$ and with at most one end. We assume $n, n' \geq 2$ and that $\mathcal{A}$ is universal. Due to Lemma 2.2, some cycle in D witnesses that $\mathcal{A}$ is universal. By Lemma 2.3, we may assume that this is an induced cycle.

Lemma 7.45. *If D contains an induced cycle of odd length witnessing that $\mathcal{A}$ is universal, then it contains an induced cycle of length 4 witnessing that $\mathcal{A}$ is universal.*

Proof. Let C be an induced odd cycle witnessing that $\mathcal{A}$ is universal. Then C contains a unique induced 2-arc xyz. The digraphs $C - x$ and $C - y$ are isomorphic induced alternating paths. By C-homogeneity, we find an automorphism α of D that maps $C - x$ onto $C - y$. Since $N^-(z)$ is independent and $x\alpha \in N^-(z)$, the digraph $D[x, y, z, x\alpha]$ is an induced cycle of length 4 witnessing that $\mathcal{A}$ is universal. □

In the following, we fix an induced cycle C of minimal length witnessing that $\mathcal{A}$ is universal. Due to Lemma 7.45, this cycle has even length.

Lemma 7.46. *There is an isomorphic copy of C_4 in D.*

Proof. Let xyz be a 2-arc on C. Since C has even length, $C - y$ has a non-trivial automorphism: one that maps x to z and vice versa. As C is induced, we can extend this automorphism of $C - y$ to an automorphism α of D by C-homogeneity and obtain that $D[x, y, z, y\alpha]$ is a directed cycle of length 4. Note that any directed cycle of length 4 is induced since D^+ and D^- are edgeless. □

Let $xy \in ED$, let $X := N^-(x) \setminus N^+(y)$, and let $Y := N^+(y) \setminus N^-(x)$. Obviously, X and Y are disjoint. In the following, we investigate the subdigraph $\Gamma := D[X \cup Y]$ of D.

Lemma 7.47. *The subdigraph Γ is a non-empty homogeneous 2-partite digraph.*

Proof. Let A and A' be finite subdigraphs of $D[X]$ and let B and B' be finite subdigraphs of $D[Y]$. Because $V(B + B') \cap N^-(x) = \emptyset$ and because $D^+(x)$ is edgeless, x is adjacent to no vertex of $B + B'$. Similarly, because $V(A + A') \cap N^+(y) = \emptyset$ and because $D^-(y)$ is edgeless, y is adjacent to no vertex of $A + A'$. Hence, any isomorphism φ from $A + B$ to $A' + B'$ extends to an isomorphism from $A + B + x + y$ to $A' + B' + x + y$, that fixes x and y, and thus by C-homogeneity it extends to an automorphism α of D with $X\alpha = X$ and $Y\alpha = Y$. In particular, the restriction of α to Γ is an automorphism of Γ that extends φ and fixes both of X and Y setwise. Thus, Γ is homogeneous 2-partite. As C_4 embeds into D, the subdigraph Γ is not empty. □

Having shown that Γ is homogeneous 2-partite, we can apply the classification of the countable such digraphs, Theorem 3.6. So we can investigate the possible digraphs Γ one by one, similar to the different possibilities for D^+. We start with the situation that Γ is homogeneous bipartite and show that this cannot occur:

Lemma 7.48. *The subdigraph Γ is not homogeneous bipartite.*

Proof. Let us suppose that Γ is homogeneous bipartite. Since D contains some directed cycle of length 4 by Lemma 7.46, we conclude that the edges of Γ are directed from Y to X. We consider all possibilities of Theorem 3.1 one by one. Note that due to Lemma 7.47 the digraph Γ is not empty. So there are only four remaining possibilities for Γ.

If $G(\Gamma)$ is complete bipartite, then xy cannot be the inner edge of any induced 3-arc. As $\mathrm{Aut}(D)$ acts transitively on the 1-arcs, we conclude that D contains no induced 3-arc at all. Since every induced cycle of even length at least 6 that witnesses that $\mathcal{A}$ is universalcontains an induced 3-arc, C has length 4. But as xy is the inner edge of some 3-arc in a cycle isomorphic to C, the digraph Γ must contain some edges that are directed from X to Y. This contradiction shows that $G(\Gamma)$ is not complete bipartite.

If $G(\Gamma)$ is a perfect matching, then we know that every induced 2-arc lies on a unique induced directed cycle of length 4. Due to the previous case, we may assume $|X| \geq 2$. So every edge lies on at least two directed cycles of length 4. Let $xyuv$ and $xyab$ be two distinct directed cycles of length 4 and let $yuwz$ be another directed cycle of length 4 containing yu. Then neither v nor u is adjacent to any of a, b, w, z since $G(\Gamma)$ is a perfect matching and the same holds for the subdigraph defined by the edge yu instead of xy. Note that $|C| > 4$, since $|C| = 4$ implies the existence of some edge from X to Y. Thus, the digraph $D[y, z, b, x]$ cannot be a cycle of length 4 witnessing that $\mathcal{A}$ is universal. Hence, we have $zb \notin ED$. If $bz \in ED$, then a is not adjacent to z since neither zy lie in $D^-(a)$ nor bz lies in $D^+(a)$. Thus, yab lies on two distinct induced directed cycles of length 4, once together with z and once together with x. This is impossible as we already mentioned. Thus, b and z are not adjacent. Hence, C-homogeneity implies the existence of an automorphism α of D that fixes x, y, z and interchanges b and v. Since every induced 2-arc lies on a unique induced directed cycle of length 4, we conclude $a\alpha = u$ and $u\alpha = a$. As $u = a\alpha$ and $z = z\alpha$ are not adjacent, a and z are not adjacent, too. Since w and v are not adjacent, the same is true for b and $w\alpha$. If either $bw \in ED$ or $wb \in ED$, then either $D[x, b, w, u, v]$ or $D[z, w, b, a, w\alpha]$ is a cycle of length 5 witnessing that $\mathcal{A}$ is universal. By Lemma 7.45, we conclude $|C| = 4$, a contradiction. Thus, we know that b and w are not adjacent. So due to C-homogeneity, D has an automorphism β that fixes x, y, z, w and maps v to b. Since β fixes y, z, w, it must also fix u, the unique vertex that forms with the 2-arc wzy an induced directed cycle of length 4. But we have $(uv)\beta = ub \notin ED$ as previously mentioned, even though uv is an edge of D. This contradiction shows that $G(\Gamma)$ is not a perfect matching.

If $G(\Gamma)$ is the complement of a perfect matching, then we may assume $|X| \geq 3$ as otherwise $G(\Gamma)$ is also a perfect matching, which we treated before. Let $z, u, v \in X$ and let z' be the unique vertex in Y that is not adjacent to z. Considering the edge ux instead of xy, we obtain a unique vertex $z'' \in N^+(x) \setminus N^-(u)$ that is not adjacent to z'. Let us show that z'' is adjacent to neither z nor v. By the structure of Γ applied to the edge ux instead of xy, we find a vertex $u^- \in N^-(u) \setminus N^+(x)$ that is a common successor of y and z''. Since $u^- \in Y$ abd $u^- \neq z'$, we have $u^-z \in ED$. Hence, $xz''u^-z$ is a directed cycle of length 4 and we conclude that z is not adjacent to z'' since $N^+(z)$ and $N^-(z)$ are independent sets. If $u^-v \in ED$, then the same argument applies for v and z'' and hence they are not adjacent. As Γ is bipartite, we do not have $vu^- \in ED$. So let us assume that u^- and v are not adjacent. Let us suppose that v and z'' are adjacent. Since $D^+(v)$ is edgeless, we do not have $vz'' \in ED$, so we have $z''v \in ED$. Then $D[z'', v, z', u, u^-]$ is a cycle of length 5 witnessing that $\mathcal{A}$ is universal. As above, we conclude $|C| = 4$ by Lemma 7.45 and the minimality of C, which is impossible as Γ is bipartite. Thus, v and z'' are also not adjacent if u^- and v are not adjacent. We have shown that z'' is adjacent to neither v nor z. Hence, C-homogeneity implies the existence of an automorphism α of D that fixes u, x, y, z'' and maps z to v. Since α fixes u, x, z'', it must also fix the uniquely determined vertex in $N^-(u) \setminus N^+(x)$ that is not adjacent to z'', which is z'. But then α must also fix z, the unique vertex in $X = N^-(x) \setminus N^+(y)$ that is not adjacent to z', in contradiction to the definition of α. This shows that $G(\Gamma)$ is not the complement of a perfect matching.

It remains to consider the case that $G(\Gamma)$ is the generic bipartite graph. As mentioned earlier, we have $|C| \neq 4$ as otherwise Γ must contain edges from X to Y. Let $abcd$ be the induced 3-arc in C. Then $C - b$ is an induced alternating path and hence embeds into Γ. Let P be an isomorphic copy of $C - b$ in Γ. As D is C-homogeneous, we find an automorphism α of D with $(C - b)\alpha = P$. Since both end vertices of P have successors on P, they lie in Y. As $G(\Gamma)$ is generic bipartite, the end vertices of P have a common successor z in X. Then $D[a\alpha, b\alpha, c\alpha, z]$ is a cycle of length 4 witnessing that $\mathcal{A}$ is universal. This contradiction to the minimality of C shows that Γ is not homogeneous bipartite. □

Since Γ is not homogeneous bipartite, we find an edge $uv \in E\Gamma$ with $u \in X$ and $v \in Y$. So $D[x, y, u, v]$ is a cycle witnessing that $\mathcal{A}$ is universal and the minimality of C implies $|C| = 4$. In the remainder of this section, we will concentrate on arguments that involve the diameter of D. First, we show that D is homogeneous if its diameter is 2:

Lemma 7.49. *If* $\operatorname{diam}(D) = 2$*, then D is homogeneous.*

Proof. First, let us show that

for every finite independent vertex set A, there are $u, v \in VD$ with $A \subseteq N^+(u)$ and $A \subseteq N^-(v)$. (7.52)

We show (7.52) by induction: If $|A| = 2$, then we find a vertex w with $A \subseteq N(w)$ because of $\operatorname{diam}(D) = 2$. Regardless which edges between w and the elements of A lie in D, we can use C-homogeneity and the cycle C, into which every induced path of length 2 embeds, to conclude that some induced 2-arc has the two elements of A as end vertices. By the same reasons, we find some vertex u with $A \subseteq N^+(u)$ and some vertex v with $A \subseteq N^-(v)$.

Now, let us assume $|A| > 2$. First, we show the existence of some vertex with A in its out-neighbourhood. By induction, we find some $u \in VD$ and $a \in A$ with $A \setminus \{a\} \subseteq N^+(u)$. Let $a' \in N^+(u) \setminus A$. By induction, we find $z \in VD$ with $a, a' \in N^+(z)$ and such that all but at most two elements of A lie in $N^+(z)$. For all $b \in A \setminus N(z)$, the first case $|A| = 2$ gives us some $z_b \in VD$ with $b, z \in N^-(z_b)$. Since $N^+(z)$ is independent, z_b is adjacent neither to a nor to a'. Then the digraphs

$$D_1 := D[A \setminus \{a\} \cup \{z_b \mid b \in A \setminus N(z)\} \cup \{z, a'\}]$$

and

$$D_2 := D[A \cup \{z_b \mid b \in A \setminus N(z)\} \cup \{z\}]$$

are isomorphic by an isomorphism φ that maps a' to a and fixes all other vertices. By construction, D_1 and D_2 are connected, so φ extends to an automorphism α of D. Since $(A \setminus \{a\}) \cup \{a'\} \subseteq N^+(u)$, we conclude $A \subseteq N^+(u\alpha)$. By an analogous argument, we find some $v \in VD$ with $A \subseteq N^-(v)$. Thus, we have shown (7.52).

Next, we show the following:

Let A, B, A', B' be finite independent vertex sets of D such that some isomorphism $\varphi\colon D[A' \cup B'] \to D[A \cup B]$ with $A'\varphi = A$ and $B'\varphi = B$ exists. If A is maximal independent in $A \cup B$ and if D has a vertex v with $A' \subseteq D^+(v)$ and $B' \subseteq D^-(v)$, then there exists some $u \in VD$ with $A \subseteq D^+(u)$ and $B \subseteq D^-(u)$. (7.53)

If $D[A \cup B]$ is connected, then the assertion follows directly by C-homogeneity. Since the case $B = \emptyset$ is done by (7.52), we may assume $B \neq \emptyset$. By induction on $|B|$ we find some vertex $v' \in VD$ with $A \subseteq N^+(v')$ and $B \setminus \{b\} \subseteq N^-(v')$ for some $b \in B$. Applying C-homogeneity, we may assume $A' = A$ and $B' \setminus \{b'\} = B \setminus \{b\}$. Since A is maximal independent in $D[A \cup B]$, we know that b has a neighbour c in $A \cup B$. This neighbour is also a neighbour of b' with $b \in N^+(c)$ if and only if $b' \in N^+(c)$. So b and b' are not adjacent as both lie either in $N^+(c)$ or in $N^-(c)$. Let Z be a vertex set containing precisely one vertex from each component of $D[A \cup B]$ that does not contain b. Then $Z \cup \{b, b'\}$ is an independent set and we find a vertex z with $Z \cup \{b, b'\} \subseteq N^+(z)$ by (7.52). Then the digraphs $D[A \cup B \cup \{z\}]$ and $D[A \cup (B \setminus \{b\}) \cup \{b', z\}]$ are isomorphic by an isomorphism ψ that maps b to b' and fixes all other vertices. Since both digraphs are connected, ψ extends to an automorphism α of D. Then we have $A \subseteq N^+(v\alpha)$ and $B \subseteq N^-(v\alpha)$, which shows (7.53).

To show that D is homogeneous, let F and H be finite isomorphic induced subdigraphs of D and let $\varphi\colon F \to H$ be an isomorphism. Let $A \subseteq VF$ be a maximal independent subset and let $B \subseteq VF \setminus A$ be maximal independent, too. By (7.53), we find a vertex u with $A \subseteq N^+(u)$ and $B \subseteq N^-(u)$. We have $N(u) \cap VF = A \cup B$ by maximalities of A and B. Analogously, we find v with $A\varphi \subseteq N^+(v)$ and $B\varphi \subseteq N^-(v)$. Then $F + u$ and $H + v$ are connected and isomorphic via an isomorphism φ' that extends φ. By C-homogeneity, φ' extends to an automorphism of D. This shows that D is homogeneous. □

The previous lemma enables us to prove that D is homogeneous if Γ is not the generic orientation of the countable generic bipartite graph:

Lemma 7.50. *If Γ is either the generic 2-partite digraph or CP'_k for some $k \in \mathbb{N}^\infty$, then D is homogeneous.*

Proof. Up to isomorphism and/or reversing the direction of every edge, the only paths $abcd$ of length 3 in a digraph are of the form:

(a) $ab, bc, cd \in ED$;

(b) $ab, bc, dc \in ED$;

(c) $ba, bc, dc \in ED$.

If we can show that in each of these three cases the end vertices a and d have distance at most 2, then we have $\operatorname{diam}(D) = 2$ and the assertion follows from Lemma 7.49. If in any of these three cases a is adjacent to c or b is adjacent to d, we can conclude $d(a, d) \leq 2$ directly. So we may assume that this is not the case. In case (a), we may assume $bc = xy$ as $\operatorname{Aut}(D)$ acts transitively on the 1-arcs of D. Since a and c are not adjacent, we have $a \in X$ and, since b and d are not adjacent, we have $d \in Y$. As $G(\Gamma)$ is a complete bipartite graph in both possibilities for Γ, we obtain $d(a,d) = 1$. In cases (b) and (c), we may assume $c = x$, $b \in X$, and $a \in Y$ by C-homogeneity. Then either $d \in N^-(x) \setminus N^+(y) = X$ and $d(a,d) = 1$ or $d \in N^-(x) \cap N^+(y)$ and $d(a,d) = 2$ because of $a, d \in N(y)$. This proves $\operatorname{diam}(D) = 2$ and hence that D is homogeneous. □

In the following, we assume due to Lemmas 7.48 and 7.50 and by Theorem 3.1 that Γ is the generic orientation of the countable generic bipartite graph.

Lemma 7.51. *We have* $\operatorname{diam}(D) \leq 3$.

Proof. Seeking for a contradiction, let us suppose $\operatorname{diam}(D) \geq 4$. Let $P = x_0 \dots x_4$ be a shortest (not necessarily directed) path between two vertices x_0 and x_4 with $d(x_0, x_4) = 4$. Then P embeds into Γ, as every finite 2-partite digraph embeds into Γ. Hence, we find an automorphism α of D that maps P into Γ. Then either $x_0\alpha$ and $x_4\alpha$ lie in X or they lie in Y. In both cases, they have a common neighbour, either x or y. Thus, x_0 and x_4 have a common neighbour. This contradiction to $d(x_0, x_4) = 4$ shows $\operatorname{diam}(D) \leq 3$. □

Since we already investigated the case $\operatorname{diam}(D) = 2$, the only remaining situation is $\operatorname{diam}(D) = 3$. We shall prove that in this situation D and Γ are isomorphic.

Lemma 7.52. *If* $\operatorname{diam}(D) \neq 2$*, then D is the generic orientation of the countable generic bipartite graph.*

Proof. By Lemma 7.49 and Lemma 7.51, we may assume $\operatorname{diam}(D) = 3$. Let $D_i(x)$ be the set of those vertices of D whose distance to x is i. The first observation in this proof is that

there are non-adjacent vertices $a \in D_1(x)$ and $b \in D_2(x)$. (7.54)

Indeed, if all vertices $a \in D_1(x)$ and $b \in D_2(x)$ are adjacent, then every vertex in $VD = \{x\} \cup D_1(x) \cup D_2(x) \cup D_3(x)$ has distance at most 2 to a and we obtain $\operatorname{diam}(D) = 2$, a contradiction to our assumption.

Let us show that

the end vertices of any induced path of length 3 have distance 3. (7.55)

Let P_1 be a path of length 3 whose end vertices have distance 3 and let P_2 be another induced path of length 3. By using C-homogeneity and the cycle C, we can modify P_1 and obtain a path P_3 with the same end vertices like P_1 and such that P_2 and P_3 are isomorphic. Hence, (7.55) holds.

Next, we show that

D contains no triangle. (7.56)

Let us suppose that D contains some triangle. Since $N^+(x)$ and $N^-(x)$ are independent sets, this triangle is a directed triangle. Let $a \in Y$, $b \in X$, x, and $d \in N^-(x) \cap N^+(y)$. Then $D[a, b, x, d]$ is an induced path of length 3 as $N^+(y)$ and $N^-(x)$ are independent vertex sets. Due to (7.55), we have $d(a, d) = 3$, but y is a common neighbour of a and d. This contradiction shows (7.56).

A direct consequence of (7.56) is that $D_1(x)$ is an independent set. Let us show that

$D_2(x)$ *is an independent set.* (7.57)

If this is not the case, then two vertices $a, b \in D_2(x)$ are adjacent. Let c be a common neighbour of b and x. By (7.56), we know that a and c are not adjacent. Hence $D[a, b, c, x]$ is an induced path of length 3. So its end vertices have distance 3 by (7.55) in contradiction to the choice of a.

We have almost proved that D is 2-partite. The only edges that might be an obstacle for this are those with both its incident vertices in $D_3(x)$. So let us exclude such edges:

$D_3(x)$ *is an independent set.* (7.58)

Let us suppose that some edge ab has both its incident vertices in $D_3(x)$. Let P be a path of length 3 from x to a. Due to (7.56), Pab is induced and its end vertices have distance 3. As Γ is the generic orientation of the countable generic bipartite graph, we also find an isomorphic copy P' of P in Γ. By C-homogeneity, we find an automorphism α of D that maps P to P'. Since the end vertices of P' lie either both in X or both in Y, they have a common neighbour, either x or y, respectively, and thus they have distance 2. Therefore, the distance between the end vertices of $P = P'\alpha^{-1}$ must be 2, too. This contradiction to the choice of b shows (7.58).

As mentioned earlier, we obtain from (7.56), (7.57), and (7.58) that D is a 2-partite digraph with partition sets $U := \{x\} \cup D_2(x)$ and $W := D_1(x) \cup D_3(x)$. Let A, B, and C be finite subsets of U. Then we find a finite set $F \subseteq VD$ such that

$$H := D[A \cup B \cup C \cup F]$$

is connected. As $H \subseteq D$ is 2-partite and Γ is the gerneric orientation of the countable generic bipartite graph, we find an isomorphic copy of H in Γ. By C-homogeneity, there is an automorphism α of D with $H\alpha \subseteq \Gamma$ such that either $(A \cup B \cup C)\alpha \subseteq X$ or $(A \cup B \cup C)\alpha \subseteq Y$. As Γ is the generic orientation of the countable generic bipartite graph, there is a vertex v either in Y or in X with $A\alpha \subseteq N^+(v)$ and $B\alpha \subseteq N^-(v)$ and $C\alpha \cap N(v) = \emptyset$. Then $v\alpha^{-1}$ is a vertex we are searching for. An analogous argument shows the existence of such a vertex if A, B, and C are finite subsets of W. Hence, we have shown that D is the generic orientation of the countable generic bipartite graph. □

Let us summarize the results of this section:

Proposition 7.53. *Let D be a countable connected C-homogeneous digraph whose reachability relation is universal. If $D^+ \cong I_n$ for some $n \in \mathbb{N}^\infty$, then D is either homogeneous or the generic orientation of the countable generic bipartite graph.* □

7.2.3 The last partial result

By summarizing the propositions with Section 7.2, we obtain the following theorem:

Theorem 7.54. *Let D be a countable connected C-homogeneous digraph such that $D^+ \cong I_n$ for some $n \in \mathbb{N}^\infty$. If D has at most one end and is not locally finite, then it is isomorphic to one of the following digraphs:*

(i) *a homogeneous digraph;*

(ii) $C_m[I_\omega]$ *for some* $m \in \mathbb{N}^\infty$ *with* $m \geq 3$*;*

(iii) Y_ω*;*

(iv) $\mathcal{R}_m$ *for some* $m \in \mathbb{N}^\infty$ *with* $m \geq 3$*; or*

(v) *the generic orientation of the countable generic bipartite graph.* □

This finishes the proof of Theorem 1.1.

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