THE COLOURING NUMBER OF INFINITE GRAPHS

NATHAN BOWLER, JOHANNES CARMESIN, AND CHRISTIAN REIHER

ABSTRACT. We show that, given an infinite cardinal μ, a graph has colouring number at most μ if and only if it contains neither of two types of subgraph. We also show that every graph with infinite colouring number has a well-ordering of its vertices that simultaneously witnesses its colouring number and its cardinality.

§1. INTRODUCTION

Our point of departure is a recent article by Péter Komjáth [4] one of whose results addresses infinite graphs with infinite colouring number. Recall

Definition 1.1. The *colouring number* $\mathrm{col}(G)$ of a graph $G = (V, E)$ is the smallest cardinal κ such that there exists a well-ordering $<^*$ of V with

$$|N(v) \cap \{w \mid w <^* v\}| < \kappa \qquad \text{for all } v \in V,$$

where $N(v)$ is the set of neighbours of v. We call such well-orderings *good*.

This notion was introduced by Erdős and Hajnal in [2].

What Komjáth proved in [4] is that if the colouring number of a graph G is bigger than some infinite cardinal μ, then G contains either a K_μ, i.e., μ mutually adjacent vertices, or G contains for each positive integer k an induced copy of the complete bipartite graph $K_{k,k}$. This condition is not a characterisation: there are graphs, such as $K_{\omega,\omega}$, which have small colouring number but nevertheless include an induced $K_{k,k}$ for each k.

Since having colouring number $\leqslant \mu$ is closed not only under taking induced subgraphs but even under taking subgraphs, it seems easier to look first for a characterisation in terms of forbidden subgraphs. When playing with the ideas appearing in Komjáth's proof, we realized that they can be used to give just such a transparent characterization of "having colouring number $\leqslant \mu$" in terms of forbidden subgraphs. For some explicit graphs called *μ-obstructions*, to be introduced in Definition 2.1 below, we shall prove

Theorem 1.2. *Let G be a graph and let μ denote some infinite cardinal. Then the statement $\mathrm{col}(G) > \mu$ is equivalent to G containing some μ-obstruction as a subgraph.*

The proof we describe has an interesting consequence:

Theorem 1.3. *Every graph G whose colouring number is infinite possesses a good well-ordering of length $|V(G)|$.*

It is not hard to re-obtain the result of Komjath mentioned above from our characterisation 1.2 by inspecting whether the μ-obstructions satisfy it. In fact, one can easily obtain the following strengthening:

Theorem 1.4. *If G is a graph with $\mathrm{col}(G) > \mu$, where μ denotes some infinite cardinal, then G contains either a K_μ or, for each positive integer k, an induced $K_{k,\omega}$.*

We will also give an example in Section 2 demonstrating that the conclusion cannot be improved further to the presence of an induced $K_{\omega,\omega}$. Which complete bipartite graphs exactly one gets by this approach depends on which properties the relevant cardinals have in the partition calculus.

For standard set-theoretical background we refer to Kunen's textbook [5].

§2. Obstructions

Throughout this section, we fix an infinite cardinal μ. There are two kinds of μ-obstructions relevant for the condition $\mathrm{col}(G) > \mu$ in Theorem 1.2. They are introduced next.

Definition 2.1. (1) A *μ-obstruction of type I* is a bipartite graph H with bipartition (A, B) such that for some cardinal $\lambda \geqslant \mu$ we have

- $|A| = \lambda$, $|B| = \lambda^+$,
- every vertex of B has at least μ neighbours in A, and
- every vertex of A has λ^+ neighbours in B.

(2) Let $\kappa > \mu$ be regular, and let G be a graph with $V(G) = \kappa$. Define T_G to be the set of those $\alpha \in \kappa$ with the following properties:

- $\mathrm{cf}(\alpha) = \mathrm{cf}(\mu)$
- The order type of $N(\alpha) \cap \alpha$ is μ.
- The supremum of $N(\alpha) \cap \alpha$ is α.

If T_G is stationary in κ, then G is a *μ-obstruction of type II*. We also call graphs isomorphic to such graphs μ-obstructions of type II.

Now we can directly proceed to the easier direction of Theorem 1.2.

Proposition 2.2. *If a graph G has a μ-obstruction of either type as a subgraph*[1]*, then* $\operatorname{col}(G) > \mu$.

Proof. Suppose first that G contains a μ-obstruction of type I, say with bipartition (A, B) as in Definition 2.1 above, and $|A| = \lambda \geqslant \mu$. Assume for a contradiction that there is a good well-ordering of G. Thus every $b \in B$ has a neighbour in A above it in that well-ordering. For $a \in A$, we denote by X_a the set of those neighbours of a that are below a in the well-ordering. Hence $B = \bigcup_{a \in A} X_a$. Since all the X_a have size less than μ, we deduce that $|B| \leqslant \lambda$, which is the desired contradiction.

In the second case, we may without loss of generality assume that G itself is an obstruction of type II. Again we suppose for a contradiction that there is a good well-ordering $<^*$ of $V(G)$. Notice that each $\alpha \in T_G$ has a neighbour $\beta < \alpha$ such that $\alpha <^* \beta$. Let $f \colon T_G \longrightarrow \kappa$ be a function sending each α to some such β. By Fodor's Lemma, there must be some $\beta < \kappa$ such that

$$T = \{\alpha \in T_G \mid f(\alpha) = \beta\}$$

is stationary. Now every element of T is a neighbour of β, and β comes after T in the ordering $<^*$, which in view of $|T| = \kappa > \mu$ contradicts our assumption that this ordering is good. □

We say that a graph is *μ-unobstructed* if it has no μ-obstruction of either type as a subgraph. To complete the proof of Theorem 1.2 we still need to show that every μ-unobstructed graph G satisfies $\operatorname{col}(G) \leqslant \mu$. This will be the objective of Sections 3 and 4.
In the remainder of this section, we prove two results asserting that in order to find an obstruction in a given graph G it suffices to find something weaker.

Definition 2.3. A *μ-barricade* is bipartite graph with bipartition (A, B) such that

- $|A| < |B|$,
- and every vertex of B has at least μ neighbours in A.

Lemma 2.4. *If G has a μ-barricade as a subgraph, then it also has a μ-obstruction of type I as a subgraph.*

[1] We work with the subgraph relation rather than the minor relation in this article because the colouring number is monotonic with respect to the subgraph, but not the minor, relation. For example for any μ the complete graph K_μ has colouring number μ but the graph obtained from it by subdividing all edges has colouring number only 3.

Proof. Let H with bipartition (A, B) be a barricade which is a subgraph of G, chosen so that $\lambda = |A|$ is minimal. By deleting some vertices of B if necessary, we may assume that B has cardinality λ^+. Let A' be the set of $a \in A$ for which $N_B(a)$ is of size λ^+, and let B' be the set of elements of B with no neighbour in $A \backslash A'$. By the definition of A', there are at most λ edges ab with $a \in A \backslash A'$ and $b \in B$. So $B \backslash B'$ is of size at most λ. It follows that B' has cardinality λ^+. In particular, the subgraph H' of H on (A', B') is a barricade, so by minimality of $|A|$ we have $|A'| = \lambda$. Since by construction every vertex of A' has λ^+ neighbours in B and hence in B', the subgraph H' is a μ-obstruction of type I. □

Definition 2.5. Let $\kappa > \mu$ be regular. A graph G with set of vertices κ is said to be a *μ-ladder* if there is a stationary set T such that each $\alpha \in T$ has at least μ neighbours in α. Also, every graph isomorphic to such a graph is regarded as a μ-ladder.

Lemma 2.6. *Every graph containing a μ-ladder is μ-obstructed.*

Proof. It suffices to prove that every μ-ladder is μ-obstructed. So let G with $V(G) = \kappa$ and the stationary set T be as described in the previous definition. For each $\alpha \in T$ we let the sequence $\langle \alpha_i \mid i < \mu \rangle$ enumerate the μ smallest neighbours of α in increasing order and denote the limit point of this sequence by $f(\alpha)$. Clearly we have $f(\alpha) \leqslant \alpha$ and $\mathrm{cf}\big(f(\alpha)\big) = \mathrm{cf}(\mu)$ for all $\alpha \in T$.

Let us first suppose that the set

$$T' = \{\alpha \in T \mid f(\alpha) < \alpha\}$$

is stationary in κ. Then for some $\gamma < \kappa$ the set

$$B = \{\alpha \in T' \mid f(\alpha) = \gamma\}$$

is stationary and as $|\gamma| < \kappa = |B|$ the pair (γ, B) is a μ-barricade in G. Due to Lemma 2.4 it follows that G contains a μ-obstruction of type I.

So it remains to consider the case that

$$T'' = \{\alpha \in T \mid f(\alpha) = \alpha\}$$

is stationary in κ. In that case we have $N(\alpha) \cap \alpha = \{\alpha_i \mid i < \mu\}$ for all $\alpha \in T''$. So T_G is a superset of T'' and thus stationary, meaning that G is a μ-obstruction of type II. □

§3. Regular κ

In this and the next section we shall prove the harder part of Theorem 1.2, in such a way that Theorem 1.3 is also immediate. To this end we shall show

Theorem 3.1. *Let G denote an infinite graph of order κ and let μ be an infinite cardinal. Then at least one of the following three cases occurs:*

- *G has a subgraph H with $|V(H)| < |V(G)|$ and $\operatorname{col}(H) > \mu$.*
- *G is μ-obstructed.*
- *G has a good well-ordering of length κ exemplifying $\operatorname{col}(G) \leqslant \mu$.*

Suppose for a moment that we know this. To deduce Theorem 1.2 we consider any graph with $\operatorname{col}(G) > \mu$. Let G^* be subgraph of G with $\operatorname{col}(G^*) > \mu$ and subject to this with $|V(G^*)|$ as small as possible. Then G^* is still infinite and when we apply Theorem 3.1 to G^* the first and third outcome are impossible, so the second one most occur. Thus G^* and hence G contains a μ-obstruction, as desired. To obtain Theorem 1.3 we apply Theorem 3.1 to G with $\mu = \operatorname{col}(G)$.

The proof of Theorem 3.1 itself is divided into two cases according to whether κ is regular or singular. The former case will be treated immediately and the latter case is deferred to the next section.

Proof of Theorem 3.1 when κ is regular. Let $V(G) = \kappa$ and consider the set

$$T = \{\alpha < \kappa \,|\, \text{Some } \beta \geqslant \alpha \text{ has at least } \mu \text{ neighbours in } \alpha\}\,.$$

First Case: T is not stationary in κ.

We observe that $0 \notin T$. Let $\langle \delta_i \,|\, i < \kappa\rangle$ be a strictly increasing continuous sequence of ordinals with limit κ starting with $\delta_0 = 0$ and such that $\delta_i \notin T$ holds for all $i < \kappa$. Now if for some $i < \kappa$ the restriction G_i of G to the half-open interval $[\delta_i, \delta_{i+1})$ has colouring number $> \mu$, then the first alternative holds. Otherwise we may fix for each $i < \kappa$ a well-ordering $<_i$ of $V(G_i)$ that exemplifies $\operatorname{col}(G_i) \leqslant \mu$. The concatenation $<^*$ of all these well-orderings has length κ, so it suffices to verify that it demonstrates $\operatorname{col}(G) \leqslant \mu$.

To this end, we consider any vertex x of G. Let $i < \kappa$ be the ordinal with $x \in G_i$. The neighbours of x preceding it in the sense of $<^*$ are either in δ_i or they belong to G_i and precede x under $<_i$. Since $x \geqslant \delta_i$ and $\delta_i \notin T$, there are less than μ neighbours of x in δ_i. Also, by our choice of $<_i$, there are less than μ such neighbours in G_i.

Second Case: T is stationary in κ.

Let us fix for each $\alpha \in T$ an ordinal $\beta_\alpha \geqslant \alpha$ with $|N(\beta_\alpha) \cap \alpha| \geqslant \mu$. A standard argument shows that the set

$$E = \{\delta < \kappa \,|\, \text{If } \alpha \in T \cap \delta, \text{ then } \beta_\alpha < \delta\}$$

is club in κ. Thus $T \cap E$ is unbounded in κ. Let the sequence $\langle \eta_i \,|\, i < \kappa \rangle$ enumerate the members of this set in increasing order. Then for each $i < \kappa$ the ordinal $\xi_i = \beta_{\eta_i}$ is at least η_i and smaller than η_{i+1}, because the latter ordinal belongs to E. In particular, each of the equations $\eta_i = \xi_j$ and $\xi_i = \xi_j$ is only possible if $i = j$. Thus it makes sense to define

$$v_\alpha = \begin{cases} \alpha & \text{if } \alpha \neq \eta_i, \xi_i \text{ for all } i < \kappa\,, \\ \xi_i & \text{if } \alpha = \eta_i \text{ for some } i < \kappa\,, \\ \eta_i & \text{if } \alpha = \xi_i \text{ for some } i < \kappa\,. \end{cases}$$

The map π sending each $\alpha < \kappa$ to v_α is a permutation of κ. If α belongs to the stationary set $T \cap E$, then $v_\alpha = \xi_i$ for some $i < \kappa$ and therefore v_α has at least μ neighbours in η_i and all of these are of the form v_β with $\beta < \alpha$. So π gives an isomomorphism between G and a μ-ladder, and in the light of Lemma 2.6 we are done. □

§4. Singular κ

Next we consider the case that κ is a singular cardinal. The form of our argument will be recognisable to anyone who is familiar with Shelah's singular compactness theorem (see for instance [6]). We will not, however, assume such familiarity.

Throughout this section, sets of size at least μ will be referred to as *big* and sets of size less than μ will be said to be *small*. We will often consider $\subseteq$-increasing sequences $\langle X_i \,|\, i < \gamma \rangle$ of sets for which each $N_{X_i}(v)$ is small. In such cases we would like to conclude that also $N_{\bigcup_{i<\gamma} X_i}(v)$ is small. We can do this as long as γ and μ have different cofinalities. So we fix the notation ϖ for the rest of the argument to mean the least infinite cardinal whose cofinality is not equal to $\mathrm{cf}(\mu)$. Thus ϖ is either ω or ω_1.

Definition 4.1. A set X of vertices of a graph G is *robust* if for any $v \in V(G) \backslash X$ the neighbourhood $N_X(v)$ is small.

Remark 4.2. Let $\langle X_i \,|\, i < \varpi \rangle$ be a $\subseteq$-increasing sequence of robust sets. Then $\bigcup_{i<\varpi} X_i$ is also robust.

Lemma 4.3. *Let G be a μ-unobstructed graph and let X be an uncountable set of vertices of G. Then there is a robust set Y of vertices of G which includes X and is of the same cardinality.*

Proof. Let λ be the cardinality of X. We build a $\subseteq$-increasing sequence $\langle X_i \mid i < \varpi \rangle$ of sets recursively by letting $X_0 = X$, taking $X_{i+1} = X_i \cup \{v \in V(G) \mid N_{X_i}(v) \text{ is big}\}$ in the successor step and $X_\ell = \bigcup_{i<\ell} X_i$ for ℓ a limit ordinal. Finally we set $Y = \bigcup_{i<\varpi} X_i$. Since by construction Y is robust and includes X, it remains to prove that $|Y| = \lambda$.
To do this, we prove by induction on i that each X_i is of size λ. The cases where i is 0 or a limit are clear, so suppose $i = j + 1$. By the induction hypothesis, $|X_j| = \lambda$. If $|X_{j+1}|$ were greater than λ, then the induced bipartite subgraph on $(X_j, X_{j+1} \backslash X_j)$ would be a μ-barricade, which is impossible by Lemma 2.4. Thus $|X_{j+1}| = \lambda$, as required. □

Remark 4.4. Lemma 4.3 also holds when X is countably infinite, but the proof is more involved and so we have omitted it (unlike in the above proof, we need that there are no type II obstructions).

Proof of Theorem 3.1 when κ is singular. If G is μ-obstructed then we are done, so we suppose that it is not. Let us fix any bijective enumeration $\langle v_i \mid i < \kappa \rangle$ of the set of vertices and a continuous increasing sequence $\langle \kappa_i \mid i < \mathrm{cf}(\kappa) \rangle$ of cardinals with limit κ, where $\kappa_0 > \mathrm{cf}(\kappa)$ is uncountable.
We begin by building a family $\langle X_{i,j} \mid i < \mathrm{cf}(\kappa), j < \varpi \rangle$ of robust sets of vertices of G, with $X_{i,j}$ of size κ_i. This will be done by nested recursion on i and j. When we come to choose $X_{i,j}$, we will already have chosen all $X_{i',j'}$ with $j' < j$ or with both $j' = j$ and $i' < i$. Whenever we have just selected such a set $X_{i,j}$, we fix immediately an arbitrary enumeration $\langle x^k_{i,j} \mid k < \kappa_i \rangle$ of this set. We impose the following conditions on this construction:

(1) $\{v_k \mid k < \kappa_i\} \subseteq X_{i,0}$ for all $i < \mathrm{cf}(\kappa)$.
(2) $\bigcup_{i' \leqslant i, j' \leqslant j} X_{i',j'} \subseteq X_{i,j}$ for all $i < \mathrm{cf}(\kappa)$ and $j < \varpi$.
(3) $\{x^k_{i',j} \mid k < \kappa_i\} \subseteq X_{i,j+1}$ for all $i < i' < \mathrm{cf}(\kappa)$ and $j < \varpi$.

These three conditions specify some collection of κ_i-many vertices which must appear in $X_{i,j}$. By Lemma 4.3 we can extend this collection to a robust set of the same size and we take such a set as $X_{i,j}$. This completes the description of our recursive construction.
The purpose of condition (3) is to ensure that we have

(4) $X_{\ell,j} \subseteq \bigcup_{i<\ell} X_{i,j+1}$ whenever $\ell < \mathrm{cf}(\kappa)$ is a limit ordinal and $j < \varpi$.

Indeed, for any $x \in X_{\ell,j}$ there is some index $k < \kappa_\ell$ with $x = x^k_{\ell,j}$, owing to the continuity of the κ_i that there is some ordinal $i < \ell$ with $k < \kappa_i$, and condition (3) yields $x \in X_{i,j+1}$ for any such i.

Now for $i < \mathrm{cf}(\kappa)$ the set $X_i = \bigcup_{j<\varpi} X_{i,j}$ is robust by Remark 4.2. We claim that for any limit ordinal $\ell < \mathrm{cf}(\kappa)$ we have $X_\ell = \bigcup_{i<\ell} X_i$. That each X_i with $i < \ell$ is a subset of X_ℓ is clear by condition (2) above. The other inclusion follows by taking the union over all $j < \varpi$ in (4).

Each vertex must lie in some set X_i by condition (1) above, and it follows from what we have just shown that the least such i can never be a limit. That is, X_0 together with all the sets $X_{i+1} \backslash X_i$ gives a partition of the vertex set. If the induced subgraph of G on any of these sets has colouring number $> \mu$, then the first alternative of Theorem 3.1 holds. Otherwise all of these induced subgraphs have good well-orderings. Since each X_i is robust, the well-ordering obtained by concatenating all of these well-orderings is also good, so that the third alternative of Theorem 3.1 holds. □

§5. A NECESSARY CONDITION

In this section we show that we can now easily deduce Theorem 1.4. We shall rely on the following result of Dushnik, Erdős, and Miller from [1].

Theorem 5.1. *For each infinite cardinal λ we have $\lambda \longrightarrow (\lambda, \omega)$. This means that if the edges of a complete graph on λ vertices are coloured red and green, then there is either a red clique of size λ, or a green clique of size ω.*

By restricting ones attention to the red graph, one realises that this means that every infinite graph G either contains a clique of size $|V(G)|$ or an infinite independent set. When used in this formulation, we refer to the above theorem as DEM.

Proof of Theorem 1.4. By Theorem 1.2 it remains to show that every graph with an obstruction of type I or II has a K_μ subgraph or an induced $K_{k,\omega}$.

First we check this for obstructions (A, B) of type I. By DEM, we may assume that the neighbourhood $N(b)$ of every $b \in B$ contains an independent set Y_b of size k. Let f be the function mapping b to Y_b. There must be a k-element subset Y of A such that $|f^{-1}(Y)| = |B|$. By DEM again, we may assume that $f^{-1}(Y)$ contains an infinite independent set B'. Then $G[B' \cup Y]$ is isomorphic to $K_{k,\omega}$.

Hence it remains to show that every obstruction G of type II has a K_μ subgraph or an induced $K_{k,\omega}$. For every $\alpha \in T_G$, we may assume by DEM that $N(\alpha) \cap \alpha$ contains an independent set Y_α of size k. For each i with $1 \leqslant i \leqslant k$, let $f_i \colon T \to \kappa$ be the function mapping α to the i-th smallest element of Y_α. By Fodor's Lemma, there is some stationary $T' \subseteq T_G$ at which f_1 is constant, and some stationary $T'' \subseteq T'$ at which f_2 is constant. Proceeding like this, we find some stationary $S \subseteq T_G$ at which all the f_i are constant. Let X be the set of these k constants. By DEM, we may assume that S contains a countably infinite independent set I. Then $G[X \cup I]$ is isomorphic to $K_{k,\omega}$. □

In the following example, we show that if we replace '$K_{k,\omega}$' by '$K_{\omega,\omega}$' in Theorem 1.4, then it becomes false.

Example 5.2. Let A be the set of finite 0-1-sequences, and let B be the set of 0-1-sequences with length ω. We define a bipartite graph G with vertex set $A \cup B$ by adding for each $a \in A$ and $b \in B$ the edge ab if a is an initial segment of b. Since G is bipartite, it cannot contain a K_ω. It cannot contain a $K_{\omega,\omega}$ either, since any two vertices in B have only finitely many neighbours in common. On the other hand, $\mathrm{col}(G) > \aleph_0$, since G is an $\aleph_0$-barricade.

Remark 5.3. The proof of Theorem 1.4 actually shows something slightly stronger: in order to have $\mathrm{col}(G) \leqslant \mu$ it is enough to forbid K_μ and a K_{k,μ^+}-subgraph where the k vertices on the left are independent. If $\mu = \omega$, then DEM implies it is enough to forbid K_μ and an *induced* K_{k,μ^+}. On the other hand if $\kappa = 2^\omega$ and $\mu = \omega_1$, it may happen that the bipartite graph contains neither a K_μ nor an induced K_{k,ω_1} by Sierpiński's theorem from [7], which says that

$$2^\omega \not\longrightarrow (\omega_1)^2_2 \, .$$

Our characterisation simplifies the study of many questions about colouring numbers, since they can often be reduced to questions about the properties of our obstructions. However there are some cases where our results do not appear to be helpful. For example, Halin showed in [3] that if λ is infinite and a graph G has colouring number greater than λ, then G includes a subdivision of K_λ. But this result is more closely tied to the structure of graphs with no subdivision of K_λ than of those with colouring number less than λ, and our methods appear not to provide a simplification of the proof.

6. References

[1] B. Dushnik and E. W. Miller: *Partially ordered sets,* American Journal of Mathematics **63** (1941), 600–610.

[2] P. Erdős and A. Hajnal: *On chromatic number of graphs and set-systems,* Acta Mathematica Academiae Scientarum Hungarica **17** (1966), 61–99.

[3] R. Halin: *Graphentheorie*, Second Edition, Wissenschaftliche Buchgesellschaft, Darmstadt (1989)

[4] Péter Komjáth: *A note on uncountable chordal graphs*, Discrete Mathematics **338** Issue 9 (2015), 1565–1566.

[5] Kenneth Kunen: *Set Theory*, Studies in Logic 34. London: College Publications (2011)

[6] Saharon Shelah: *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel Journal of Mathematics **21** (1975), 319–349.

[7] W. Sierpiński: *Sur un problème de la théorie des relations*, Annali della Scuola Normale Superiore di Pisa, Ser. 2, **2** (1933), 285–287.

Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg, Germany

E-mail address: Nathan.Bowler@uni-hamburg.de

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB

E-mail address: j.carmesin@dpmms.cam.ac.uk

Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, D-20146 Hamburg, Germany

E-mail address: Christian.Reiher@uni-hamburg.de