Infinite Matroids

December 15, 2014

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Introduction

In [23], Bruhn, Diestel, Kriesell, Pendavingh and Wollan gave axiomatic foundations for infinite matroids with duality in terms of independent sets, bases, circuits, closure and rank. This opened the way for the development of a theory of infinite matroids with duality, the beginnings of which are explained in this thesis.

It had previously been traditional to define infinite matroids like finite ones (see, for example, [56]), but with the following additional axiom:

(I4) An infinite set is independent as soon as all of its finite subsets are.

From now on, we shall refer to collections of independent sets satisfying the usual independence axioms and (I4) as *finitary matroids*. It is clear that in any finitary matroid, all circuits must be finite, since any infinite dependent set must have some finite dependent subset, and so cannot be a *minimal* dependent set. Thus one could also define finitary matroids by means of the usual circuit axioms, together with the additional axiom that all circuits be finite. Similarly, finitary matroids may easily be axiomatised in terms of their sets of bases, of their closure operators or of their rank functions.

Example 0.0.1. Let $(v_e|e \in E)$ be a (possibly infinite) family of vectors in a vector space V. We say a subset I of E is *independent* if the family $(v_e|e \in I)$ is linearly independent in V. This construction gives the set of independent sets of a finitary matroid with ground set E. Such finitary matroids are called *representable*.

Example 0.0.2. Let G be a (possibly infinite) graph. Then there is a finitary matroid $M_{FC}(G)$ with ground set $E(G)$, whose circuits are precisely the edge sets of finite cycles in G. This matroid is called the finite-cycle matroid of G.

Example 0.0.3. Let E be a set and let $n \in \mathbb{N}$. Then the set $U_{n,E}$ of subsets of X of size n is the set of bases of a finitary matroid with ground set E. Such finitary matroids are called *uniform*.

There are, however, a number of examples of non-finitary objects which it would be sensible to call matroids, beginning with the duals of the matroids listed above. The dual of $U_{2,E}$, for example, cannot be finitary for any infinite set E. If it were, then since all finite subsets of E would be independent, E

itself would be forced to be independent, although it is included in no base. More worryingly, the duals of finite cycle matroids of infinite graphs are often not finitary, even when the graphs are planar and have well-defined duals in the plane. The infinite grid, for example, is self-dual as a graph, but its finite cycle matroid doesn't have a finitary dual.

If G is a finite graph, then the circuits of the dual of $M_{FC}(G)$ are precisely the bonds of G, that is, the minimal nonempty cuts. If G is infinite, the circuits of the dual of $M_{FC}(G)$ would again have to be the bonds, since the bonds are the minimal nonempty sets of edges which never meet a circuit in just one edge [27]. Since the infinite grid has many infinite bonds, it cannot have a finitary dual. More generally, if G and G^* are dual infinite graphs, then the bonds in G^* no longer correspond to finite cycles in G, but they do still correspond to infinite cycle-like objects. In fact, they are the edge sets of topological circles in a topological space obtained by adjoining certain points 'at infinity', called *ends* to the graph G. This construction will be discussed in more detail in [27]. This construction can be carried out even when G is not planar and so has no dual graph. We still obtain a well-behaved collection of *topological cycles* in G, which ought to give the circuits of a matroid.

These topological cycles were introduced in order to allow the generalisation to infinite graphs of a number of results about cycles in finite graphs which no longer hold for the finite cycles of an infinite graph. A good overview of this field is [33]. In order to apply matroidal ideas in this area, it would be necessary to be able to form a matroid whose circuits were the (edge sets of) topological cycles in G. If our collection of infinite matroids were closed under duality, this would be easily possible: we could obtain the desired matroid as the dual of the *finite bond matroid* $M_{FC}(G)$, whose circuits are the finite bonds in G.

However, as mentioned before, the collection of finitary matroids is not closed under duality. In fact, duality fails for this collection as badly as it possibly could: a finitary matroid can only have a finitary dual if it is a direct sum of finite matroids [69, 10, 26]. Duality is such a fundamental concept and tool in the theory of finite matroids that much of this theory would not make sense for the class of finitary matroids. Those ideas which do extend, tend to do so 'by compactness'. At the time when finitary matroids were first introduced, combinatorial compactness arguments were new and fashionable. Nowadays the use of compactness arguments has become so standard that results about infinite objects proved in this way may be seen as nothing but reformulations of the corresponding finite results, so that the theory of finitary matroids can be considered a reformulation, rather than an extension, of the theory of finite matroids.

The problem lies in the crude way in which the infinite independent sets are chosen, given the finite independent sets. Given the collection of finite independent sets, (I4) throws in as many infinite independent sets as possible, subject to the axiom that subsets of independent sets should be independent. Ideally, we would like to allow the collection of infinite independent sets to have a more refined structure.

Aware of the crudity of the definition and of the disastrous failure of duality, Rado asked in 1966 whether a theory of infinite matroids with duality was possible [57, Problem P531]. In response a number of definitions were proposed, amongst which we will focus on B-matroids, which were introduced by Higgs [43]. Higgs' belief that B-matroids give a natural class of infinite matroids was confirmed by Oxley, who showed that the class of B-matroids is the largest class of preindependence spaces[1] on which duality and minors are defined and which is closed under those operations [55]. Nevertheless, the class of B-matroids remained difficult to work with because no simple axiomatisation of the class was known.

This problem was resolved by the axiomatisations developed by Bruhn et al., because the objects fulfilling their axioms are precisely the B-matroids. However, Bruhn et al. had not set out to resolve this problem. They were initially unaware of the concept of B-matroids, and were simply trying to produce natural axiomatisations according to which the (edge sets of) topological cycles in an infinite graph would give the circuits of some matroid.

Since the development of these axiomatisations, most basic finite matroid theory has been extended to infinite matroids, as will be explained in the coming chapters. In contrast to the case of finitary matroids, it is no longer possible to extend the results using straightforward compactness arguments, and even the formulation of the proper definitions is often a subtle matter (as it was for infinite matroids themselves).

After laying down some foundations in chapter 1, we will turn to one of the most basic questions one could ask about infinite matroids, namely: must all bases have the same cardinality? Higgs showed that the answer is 'yes' if we assume the generalised continuum hypothesis. In chapter 2 we show that it is also consistent with ZFC that there is a matroid with bases of two different cardinalities. We explore how to generalise the notion of representability in chapter 3 and two different but closely related approaches to infinite graphic matroids in chapters 4 and 5. We conclude by discussing the most important open problem in the theory of infinite matroids, the Packing/Covering Conjecture, in chapter 6.

0.0.1 Acknowledgements and discussion of the extent to which this thesis is my own work

This thesis is based in small part on a lecture course given at the University of Hamburg in the summer semester of 2012, but in large part on 9 papers with various of the following coauthors: Hadi Afzali, Johannes Carmesin, Robin Christan and Stefen Geschke. I am grateful to all of these coauthors, but especially to Johannes Caremesin, with whom I collaborated closely, and who is a joy to work with.

I'm also grateful to the Humboldt foundation for giving me the opportunity to take a couple of years to work on this project, and to Reinhard Diestel, who

[1] a candidate set $\mathcal{I}$ of independent sets is a *preindependence space* if it satisfies the independence axioms for finite matroids

has provided consistent support and regular suggestions about possible research directions.

I contributed a fair share of the ideas to all of the papers which form the foundation for this thesis. I'll now sketch in more detail which results are my own, though because my collaboration with Johannes has been so close it is often not possible to separate his ideas cleanly from my own. In such cases, I'll simply state that the ideas were produced in close collaboration. So in cases where I make this statement, I am confirming that my own contribution was roughly half.

Chapter 1 is largely a summary of previous research, and as such is not based on my own ideas, apart from Sections 1.3, 1.4. 1.6 and 1.7. Section 1.3 is based on ideas produced together with Johannes, though the main result is my own. Sections 1.4, 1.6 and 1.7 are my own work, though Johannes also contributed to the original texts on which these Sections are based.

The construction of uniform matroids in chapter 2 is my own, but Stefan Geschke is the one who realised that this construction still works even if we only assume Martin's Axiom (as conjectured by Adrian Mathias). I recognised that this gave an answer to Higgs' main question, and Stefan was the one who answered Higgs' other problem and showed that self-dual uniform matroids give rise to non-Baire sets.

I developed much of the theory in 3.1-3.6, though Hadi Afzali also contributed to this work, especially Sections 3.1 and 3.2. Johannes's ideas are behind Section 3.7, and mine behind 3.8 and 3.9.

The concept of trees of matroids and the proofs using determinacy that Psi-matroids are matroids are my own, though Johannes also contributed to the proof in the case of larger overlap. My ideas here were also influenced by discussions with both Johannes Carmesin and Julian Pott. The extension from the locally finite to the countable case is due to Johannes, as are the applications (thought the argument for non-well-quasiorderability is mine). Sections 4.8-4.11 were produced in close collaboration with Johannes, though Section 4.9 is just a recapitulation of older ideas of Aigner-Horev, Diestell and Postle.

The concept of graph-like space used here is my own, but is closely based on ideas of Thomassen and Vella. Sections 5.2-5.4 are my own work, though both Johannes Carmesin and Robin Christian contributed ideas here. 5.5-5.7 were produced in close collaboration with Johannes, though again Robin Christian contributed some ideas.

The idea of exchange chains in Section 6.1 is due to Johannes. Sections 6.2 and 6.3 were produced in close collaboration with Johannes. Sections 6.4 and 6.5 are again due to Johannes. The details of the argument from 6.7 onwards were produced in close collaboration with Johannes, though the overall strategy is mine.

Chapter 1

Foundations

Before we begin our explorations, we will get a rough idea of the lay of the land by looking at the various equivalent perspectives from which matroids can be seen (and axiomatised). We will further equip ourselves by developing some basic concepts and tools which we will need later on.

1.1 Axiomatisations

In [23], Bruhn et. al provided axiomatisations of matroids in terms of their independent sets, bases, circuits, closure operators and rank functions. Many of these axiomatisations involve the following potential property of a set $\mathcal{I}$ of subsets of a set E:

(M) For any $I \subseteq X \subseteq E$ with $I \in \mathcal{I}$, the set $\{I' \in \mathcal{I} | I \subseteq I' \subseteq X\}$ has a maximal element.

The axiomatisations and conversion rules they provided are listed below. It may be wise to skip this section at first, and return to it as a reference later.

1.1.1 Independence axioms

Suppose $\mathcal{I}$ is a set of subsets of E. Then $\mathcal{I}$ is the set of *independent sets* of a matroid with ground set E if and only if:

(I1) $\emptyset \in \mathcal{I}$.

(I2) For $I \in \mathcal{I}$ and $I' \subseteq I$ we have $I' \in \mathcal{I}$.

(I3) For any non-maximal element I of $\mathcal{I}$ and any maximal element I' of $\mathcal{I}$ there is an $e \in I' \setminus I$ such that $I + e \in \mathcal{I}$.

(IM) $\mathcal{I}$ satisfies (M).

Sets not in $\mathcal{I}$ are called *dependent sets* of the matroid. The axiom (I3) is called the *exchange* axiom for independent sets.

1.1.2 Basis axioms

Suppose $\mathcal{B}$ is a set of subsets of E. Then $\mathcal{B}$ is the set of *bases* of a matroid with ground set E if and only if:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) For $B, B' \in \mathcal{B}$ and $e \in B \setminus B'$, there is some $e' \in B \setminus B'$ with $B - e + e' \in \mathcal{B}$.

(BM) The set $\mathcal{I}$ of all subsets of elements of $\mathcal{B}$ satisfies (M).

The axiom (B2) is called the *base exchange* axiom.

1.1.3 Circuit axioms

Suppose $\mathcal{C}$ is a set of subsets of E. Then $\mathcal{C}$ is the set of *circuits* of a matroid with ground set E if and only if:

(C1) $\emptyset \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) Let $e \in C \in \mathcal{C}$, let $X \subseteq C - e$ and let $(C_x \in \mathcal{C} | x \in X)$ satisfy $e \notin C_x$ and $C_x \cap X = \{x\}$ for all $x \in X$. Then there is some $C' \in \mathcal{C}$ with

$$e \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x \right) \setminus X .$$

(CM) The set $\mathcal{I}$ of subsets of E which don't include any element of $\mathcal{C}$ satisfies (M).

The axiom (C3) is called the *circuit elimination* axiom. If $\{e\}$ is a circuit then e is called a *loop*.

1.1.4 Closure axioms

An operator $\mathrm{Cl}\colon \mathcal{P}E \to \mathcal{P}E$ is the *closure operator* of a matroid with ground set E if and only if:

(CL1) For all $X \subseteq E$ we have $X \subseteq \mathrm{Cl}(X)$.

(CL2) For all $X \subseteq Y \subseteq E$ we have $\mathrm{Cl}(X) \subseteq \mathrm{Cl}(Y)$.

(CL3) For all $X \subseteq E$ we have $\mathrm{Cl}(\mathrm{Cl}(X)) = \mathrm{Cl}(X)$.

(CL4) For all $X \subseteq E$ and all $e, e' \in E$ with $e' \in \mathrm{Cl}(X + e) \setminus \mathrm{Cl}(X)$ we have $e \in \mathrm{Cl}(X + e')$.

(CLM) The set $\mathcal{I} = \{I \subseteq E | (\forall e \in I) e \notin \mathrm{Cl}(I - e)\}$ satisfies (M).

Sets of the form $\mathrm{Cl}(X)$ are called *closed* sets. Thus by (CL3) a subset X of E is closed if and only if $X = \mathrm{Cl}(X)$. A subset X of E is said to be *spanning* if $\mathrm{Cl}(X) = E$.

1.1.5 Rank axioms

Let $[\subseteq]_E$ be the set of pairs (A, B) of sets with $A \subseteq B \subseteq E$. Then a function $r\colon [\subseteq]_E \to \mathbb{N}_0 + \infty$ is the *relative rank function* of a matroid with ground set E if and only if:

(R1) For all $A \subseteq B \subseteq E$ we have $r(A, B) \leq |B \setminus A|$.

(R2) For all $A, B \subseteq E$ we have $r(A \cap B, A) \geq r(B, A \cup B)$.

(R3) For all $A \subseteq B \subseteq C \subseteq E$ we have $r(A, C) = r(A, B) + r(B, C)$.

(R4) For any subset A of E and any family $(A \subseteq B_k \subseteq E | k \in K)$ with $r(A, B_k) = 0$ for all $k \in K$ we have $r(A, \bigcup_{k \in K} B_k) = 0$

(RM) The set $\mathcal{I} = \{I \subseteq E | (\forall x \in I) r(I - x, I) > 0\}$ satisfies (M).

1.1.6 Conversions

If $\mathcal{I}$ is the set of independent sets of a matroid then the set of maximal independent sets is the set of bases of the same matroid, the set $\mathcal{C}$ of minimal dependent sets is the set of circuits of the same matroid, the operator $\mathrm{Cl}\colon \mathcal{P}E \to \mathcal{P}E; X \mapsto X \cup \{e \notin X | (\exists I \in \mathcal{I}) I \subseteq X \text{ and } I + e \notin \mathcal{I}\}$ is the closure operator of the same matroid and the function

$$r\colon [\subseteq]_E \to \mathbb{N}_0 + \infty; (A, B) \mapsto \min_{\substack{I \in \mathcal{I} \\ I \subseteq A}} \max_{\substack{I' \in \mathcal{I} \\ I \subseteq I' \subseteq B}} |I' \setminus I|$$

is the relative rank function of the same matroid on E.

In the other direction, if $\mathcal{B}$ is the set of bases of a matroid then the set of subsets of elements of $\mathcal{B}$ is the set of independent sets of the same matroid. For any family $\mathcal{C}$ of subsets of E, we say a set I is $\mathcal{C}$*-independent* if it includes no element of $\mathcal{C}$. If $\mathcal{C}$ is the set of circuits of a matroid then the set of $\mathcal{C}$-independent sets is the set of independent sets of the same matroid. If $\mathrm{Cl} : \mathcal{P}E \to \mathcal{P}E$ is an operator, we say a set $I \subseteq E$ is Cl-*independent* if there is no $e \in I$ with $e \in \mathrm{Cl}(I - e)$. If Cl is the closure operator of a matroid then the set of Cl-independent sets is the set of independent sets of the same matroid. If r is the relative rank function of a matroid with ground set E then $\{I \subseteq E | (\forall x \in I) r(I - x, I) > 0\}$ is the set of independent sets of the same matroid. We can also define the closure operator in terms of the circuits: if $\mathcal{C}$ is the set of circuits of a matroid, then $\mathrm{Cl}_{\mathcal{C}}\colon \mathcal{P}E \to \mathcal{P}E; X \mapsto X \cup \{e \notin X | (\exists C \in \mathcal{C}) e \in C \subseteq X + e\}$ is the closure operator of the same matroid.

The main result of [23] is that all of the above makes sense: the conversions really do convert between objects satisfying the appropriate axiomatisations, and furthermore applying any two of the above conversions brings you back to the object that you started with. Thus it makes sense to see all of these axiomatisations as different descriptions of the same sort of mathematical object. If M is a matroid then we will refer to the set of independent sets of M as $\mathcal{I}(M)$, the set of bases of M as $\mathcal{B}(M)$, the set of circuits of M as $\mathcal{C}(M)$, the closure operator of M as Cl_M and the relative rank function of M as r_M. We will abbreviate $r_M(\emptyset, A)$ as $r_M(A)$, and call the resulting unary function the *rank function* of M. If M is a matroid with ground set E then the *rank* of M is $r(M) = r_M(E)$.

1.1.7 Spanning sets

By the definitions of the closure operator and bases in terms of independent sets in the last subsection, it is clear that an independent set is a base if and only if it is spanning. Thus any set including a base is spanning. Conversely, if X is a spanning set and B is a maximal independent subset of X then $X \subseteq \mathrm{Cl}(B)$ and so $\mathrm{Cl}(B) \supseteq \mathrm{Cl}(\mathrm{Cl}(B)) \supseteq \mathrm{Cl}(X) = E$, so that B is both independent and spanning, and hence is a base. Thus the bases are precisely the minimal spanning sets.

1.1.8 The circuit elimination axiom

The circuit elimination axiom (C3) is an extension of the usual circuit elimination axiom for finite matroids, which it extends:

(C3') For any $C, C' \in \mathcal{C}$ and any $e \in C \setminus C'$ and $x \in C \cap C'$ there is some $C'' \in \mathcal{C}$ with $e \in C'' \subseteq (C \cup C') - x$.

This is simply the special case of (C3) with $|X| = 1$. This axiom can be seen as saying that an independent set can span an edge in at most one way.

Lemma 1.1.1. *Let $\mathcal{C}$ be a set of subsets of E satisfying (C3'), x an edge of E and I a $\mathcal{C}$-independent set. Then there is at most one nonempty $C \in \mathcal{C}$ with $C \subseteq I + x$.*

Proof. Suppose for a contradiction that there are two such subsets, say C and C'. Then $x \in C \cap C'$ and without loss of generality there is some $e \in C \setminus C'$. Thus by circuit elimination there is some $C'' \in \mathcal{C}$ with $e \in C'' \subseteq (C \cup C') - x \subseteq I$, contradicting $\mathcal{C}$-independence of I. $\square$

In particular, if B is a base and $e \notin B$ then there is a unique circuit C_e^B with $e \in C_e^B \subseteq B + e$. This circuit is called the *fundamental circuit* of e with respect to B.

If $\mathcal{C}$ is a set of subsets of E, then $\mathcal{C}_{\min}$ is the set of minimal nonempty elements of $\mathcal{C}$.

Lemma 1.1.2. *Let $\mathcal{C}$ be a system of sets satisfying (C3) and (CM). Then for any $C \in \mathcal{C}$ and any $e \in C$ there is some $C' \in \mathcal{C}_{\min}$ with $e \in C' \subseteq C$.*

Proof. Let I be a maximal $\mathcal{C}$-independent subset of $C - e$, and let $X = (C - e) \setminus I$. Then for each $x \in X$ the set $I + x$ is not $\mathcal{C}$-independent, so there is some $C_x \in \mathcal{C}$ with $x \in C_x \subseteq I + x$. Then $C_x \cap (X + e) = \{x\}$. Thus by circuit elimination there is some $C' \in \mathcal{C}$ with $e \in C' \subseteq \left(C \cup \bigcup_{x \in X} C_x\right) \setminus X \subseteq I + e$. This C' is minimal among the nonempty elements of $\mathcal{C}$ by Lemma 1.1.1. $\square$

Corollary 1.1.3. *If $\mathcal{C}$ satisfies both (C3) and (CM) then so does $\mathcal{C}_{\min}$.*

The axiom (C3) also has a close relation to the axiom (O2), which will appear again later in Section 1.3. The axiom (O2) is a condition on two sets $\mathcal{C}$ and $\mathcal{D}$ of subsets of a set E:

(O2) For any partition of E as $P \dot\cup Q \dot\cup e$ one of the following holds:

- There is some $C \in \mathcal{C}$ with $e \in C \subseteq P + e$
- There is some $D \in \mathcal{D}$ with $e \in D \subseteq Q + e$.

For any set $\mathcal{C}$ of subsets of E, we define $\mathcal{C}^\perp$ to be the set of subsets of E which never meet an element of $\mathcal{C}$ in just one point.

Lemma 1.1.4. *$\mathcal{C}$ satisfies (C3) if and only if $\mathcal{C}$ and $\mathcal{C}^\perp$ satisfy (O2).*

Proof. For the "only if" implication, suppose we are given a partition $E = P \dot\cup Q \dot\cup \{e\}$ such that $P + e$ does not include an element of $\mathcal{C}$ containing e.

Let D consist of those elements x of $Q + e$ such that $P + x$ does not include an element of $\mathcal{C}$ containing x.

Suppose for a contradiction that $D \notin \mathcal{C}^\perp$. Then there is some $C \in \mathcal{C}$ meeting D only in a single element e'. Let $X = C \cap ((Q + e) \setminus D)$. For any $x \in X$ pick an element C_x of $\mathcal{C}$ such that $x \in C_x \subseteq P + x$. Applying (C3) to e', C, X and the C_x yields an element of $\mathcal{C}$ meeting $Q + e$ exactly in e', which contradicts the fact that $e' \in D$.

It remains to show the "if"-implication. Suppose we are given e, C, X and $(C_x | x \in X)$ as in (C3). Put $P = (C \cup \bigcup_{x \in X} C_x) \setminus (X + e)$ and $Q = (E \setminus P) - e$.

To prove circuit elimination, it remains to show that there is no element $D \subseteq Q + e$ of $\mathcal{C}^\perp$ containing e. Since $e \in C$ and $C \cap D \subseteq X + e$, any such set D would contain some $x \in X$ since $D \in \mathcal{C}^\perp$. But then $C_x \cap D = \{x\}$, which is impossible since $D \in \mathcal{C}^\perp$. This completes the proof. $\square$

Corollary 1.1.5. *If a set $\mathcal{C}$ of subsets of E satisfies (C3), then so does $\mathcal{C}^\perp$.* $\square$

If $\mathcal{C}$ is a set of subsets of E then the *closure* $\langle \mathcal{C} \rangle$ of $\mathcal{C}$ is the set of unions of elements of $\mathcal{C}$. Evidently, $\langle \mathcal{C} \rangle^\perp = \mathcal{C}^\perp$.

Corollary 1.1.6. *If a set $\mathcal{C}$ of subsets of E satisfies (C3), then so does $\langle \mathcal{C} \rangle$.* $\square$

1.1.9 The scrawl axioms

A subset of the ground set of a matroid is a *scrawl* if and only if it is a union of circuits. The set of scrawls of a matroid M is denoted $\mathcal{S}(M)$. Using the results of the last subsection, we can axiomatise matroids in terms of their scrawls. More precisely, if $\mathcal{S}$ is a set of subsets of E then $\mathcal{S}$ is the set of scrawls of a matroid with ground set E if and only if:

(S1) $\mathcal{S}$ is closed under taking unions.

(S2) $\mathcal{S}$ satisfies (C3).

(SM) $\mathcal{S}$ satisfies (CM).

The set of circuits of this matroid is $\mathcal{S}_{\min}$. In fact, something a little stronger is true. If $\mathcal{C}$ is a set of subsets of E satisfying $(C3)$ and (CM) then there is a matroid whose set of circuits is $\mathcal{C}_{\min}$ and whose set of scrawls is $\langle \mathcal{C} \rangle$.

1.2 Minors and duality

1.2.1 Restriction

If $\mathcal{I}$ is a set of subsets of a set E and $X \subseteq E$ then we write $\mathcal{I}\restriction X$ for $\mathcal{I} \cap \mathcal{P}(X)$. If $\mathcal{I}$ is the set of independent subsets of a matroid M with ground set E and $X \subseteq E$ then $\mathcal{I}\restriction X$ is the set of independent subsets of a matroid with ground set X, called the *restriction* of M to X and denoted $M\restriction X$ [23]. Bases of $M\restriction X$ are just maximal independent subsets of X, and are often called simply bases of X. For any set Q, the matroid $M\restriction_{E \setminus Q}$ is denoted $M\backslash Q$ and is said to be obtained from M by *deleting* Q. The following identities are easily verified:

- $\mathcal{C}(M\restriction X) = \mathcal{C}(M)\restriction X$
- $\mathrm{Cl}_{M\restriction X}(Y) = \mathrm{Cl}_M(Y) \cap X$
- $r_{M\restriction_X}(A, B) = r_M(A, B)$
- $\mathcal{S}(M\restriction_X) = \mathcal{S}(M)\restriction X$
- $M\backslash Q_1 \backslash Q_2 = M\backslash Q_2 \backslash Q_1 = M \setminus (Q_1 \cup Q_2)$.

1.2.2 Duality

If $\mathcal{B}$ is the set of bases of a matroid M with ground set E, then the set of complements of elements of $\mathcal{B}$ is the set of bases of a matroid M^* with ground set E, called the *dual* of M [23]. Clearly $M^{**} = M$. Independent sets of M^* are called *coindependent* subsets of M. Similarly, bases, circuits, loops, the closure operator, closed sets, spanning sets, the (relative) rank function and scrawls of M^* are called respectively cobases, cocircuits, coloops, the coclosure operator, coclosed sets, cospanning sets, the (relative) corank function and coscrawls of M. It is immediate from the definition of duality and the remarks in subsection 1.1.7 that the coindependent sets are precisely the complements of the spanning sets, and that $r(M) + r(M^*) = |E|$.

If B is a base of M and $e \in B$ then the fundamental circuit of e with respect to the complement of B in M^* is denoted D_e^B, and called the *fundamental cocircuit* of e with respect to B. No circuit and cocircuit of a matroid can ever meet in just a single element [23]. It follows that if B is a base of M with $e \notin B$ and $f \in B$ then $f \in C_e^B$ if and only if $e \in D_f^B$. We denote the relation this defines between edges outside of B and those in B by R_B. Note that eR_Bf if and only if $B - f + e$ is again a base of M.

Lemma 1.2.1. $\mathcal{S}(M^*) = \mathcal{C}(M)^\perp$.

Proof. We know that $\mathcal{C}(M^*) \subseteq \mathcal{C}(M)^\perp$. Thus $\mathcal{S}(M^*) = \langle \mathcal{C}(M^*) \rangle \subseteq \mathcal{C}(M)^\perp$.

For the other direction, suppose that $S \in \mathcal{C}(M)^\perp$, and let $e \in S$. In particular, e cannot be a loop of M, for then S would meet this loop in only e. So $\{e\}$ is independent. Let I be a maximal independent set with $e \in I \subseteq (E \setminus S) + e$. Let B be a base of M extending I. Suppose for a contradiction that $D_e^B \not\subseteq S$, and let $f \in D_e^B \setminus S$. Then since $f \in (E \setminus S) \setminus I$, by maximality of I there must be some circuit C with $f \in C \subseteq I + f$. Since $C \subseteq B + f$, we have $C = C_f^B$, and so $e \in C$ because $f \in D_e^B$. Thus $C \cap S = \{e\}$, which gives the desired contradiction. Thus $D_e^B \subseteq S$. Since this was true for all $e \in S$, we have $S = \bigcup_{e \in S} D_e^B \in \mathcal{S}(M^*)$. □

Corollary 1.2.2. $\mathcal{C}(M^*) = (\mathcal{C}(M)^\perp)_{\min}$. □

Corollary 1.2.3. *An edge is a coloop if and only if it is not contained in any circuits.*

Corollary 1.2.4. *A set of elements of E is a coscrawl if and only if its complement is closed.*

1.2.3 Contraction

Contraction is the dual operation to restriction: if M is a matroid with ground set E and $X \subseteq E$ then the *contraction* $M.X$ of M to X is the matroid $(M^* \restriction X)^*$. If P is any set then the matroid $M/P = M.(E \setminus P)$ is said to be obtained from M by *contracting* P. A matroid N is a *minor* of a matroid M if it is (isomorphic to) a matroid which can be obtained from M by contracting some elements and deleting some others.

The following characterisations of the contraction are taken from [23]:

Lemma 1.2.5. *Let M be a matroid with ground set E, and let $X \subseteq E$ and $I \subseteq E \setminus X$. Then the following are equivalent:*

- *I is independent in $M.X$.*
- *There is a base B of $M \backslash X$ such that $B \cup I$ is independent in M.*
- *For any independent set I' of $M \backslash X$, the set $I \cup I'$ is independent in M.*

Lemma 1.2.6. *Let M be a matroid with ground set E, and let $X \subseteq E$ and $B \subseteq E \setminus X$. Then the following are equivalent:*

- *B is a base of $M.X$.*
- *There is a base B' of $M \backslash X$ such that $B \cup B'$ is a base of M.*
- *For any base B' of $M \backslash X$, the set $B \cup B'$ is a base of M.*

The circuits of $M.X$ cannot be so easily characterised, but we can get a handle on them as follows: first, for any set $\mathcal{C}$ of subsets of a set E and any subset X of E, we write $\mathcal{C}.X$ for $\{C \cap X | C \in \mathcal{C}\}$.

Lemma 1.2.7. *$\mathcal{C}(M.X) \subseteq \mathcal{C}(M).X \subseteq \mathcal{S}(M.X)$.*

Proof. First, let C be any circuit of $M.X$, and let B be any base of $M \setminus X$. Let e be any edge of C. Then $B \cup C - e$ is independent in M, but $B \cup C$ is not, so there is some circuit C' of M with $e \in C' \subseteq B \cup C$. For any other f in C, we know that $B \cup C - f$ is independent, so $C' \not\subseteq B \cup C - f$, so that $f \in C'$. Thus $C = C' \cap X \in \mathcal{C}(M).X$. For the second part, note that no element $C \cap X$ of $\mathcal{C}.X$ can meet any element D of $\mathcal{C}(M^*)\restriction X$ just once, for then we would also have $|C \cap D| = 1$, which is impossible. Thus $\mathcal{C}(M).X \subseteq (\mathcal{C}(M^*)\restriction X)^\perp = \mathcal{C}((M.X)^*)^\perp = \mathcal{S}(M.X)$. $\square$

Corollary 1.2.8. *$\mathcal{C}(M.X) = (\mathcal{C}(M).X)_{\min}$ and $\mathcal{S}(M.X) = \langle \mathcal{C}(M).X \rangle = \mathcal{S}(M).X$.* $\square$

Corollary 1.2.9. *Let M be a matroid, and let P and Q be disjoint sets. Then $M/P \backslash Q = M \backslash Q/P$.*

Corollary 1.2.10. *Let M' be a minor of M. Further let C' be an M'-circuit and D' be an M'-cocircuit. Then there is an M-circuit $C \subseteq C' \cup (E(M) \setminus E(M'))$ and an M-cocircuit $D \subseteq D' \cup (E(M) \setminus E(M'))$ such that $o \cap b = o' \cap b'$.*

Proof. Extend C' using contracted edges only and D' using deleted edges only as in Lemma 1.2.7 and its dual. $\square$

We have seen that a set is closed in M if and only if it is the complement of a scrawl in M^*. Thus $\mathrm{Cl}_M(Y) = E \setminus \bigcup\{C \in \mathcal{C}(M^*) | C \cap Y = \emptyset\}$, from which it follows that $\mathrm{Cl}_{M.X}(Y) = \mathrm{Cl}_M(X \cup Y) \setminus X$.

From the above definitions, it is clear that $r_M(A, B) = r(M/A \backslash (E \setminus B))$, so that $r_{M/P \backslash Q}(A, B) = r_M(A \cup P, B \cup P)$ and $r_{M^*}(A, B) + r_M(E \setminus B, E \setminus A) = |M/A \backslash (E \setminus B)| = |B \setminus A|$.

1.3 The orthogonality axioms

The axiomatisations we have seen so far had few axioms, but they all relied on the condition (M). If we restrict ourselves to a countable ground set, then it is possible to give an axiomatisation involving more axioms, each of which is simpler to check. This makes the process of testing whether a given system is a matroid more straightforward.

The *orthogonality axioms* are as follows, where $\mathcal{C}$ and $\mathcal{D}$ are sets of subsets of a set E (intended to be the sets of circuits of some matroid and of its dual, respectively).

(C1) $\emptyset \notin \mathcal{C}$

(C2) No element of $\mathcal{C}$ is a subset of another.

(C1*) $\emptyset \notin \mathcal{D}$

(C2*) No element of $\mathcal{D}$ is a subset of another.

(O1) $|C \cap D| \neq 1$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(O2) For all partitions $E = P \dot\cup Q \dot\cup \{e\}$ either $P + e$ includes an element of $\mathcal{C}$ through e or $Q + e$ includes an element of $\mathcal{D}$ through e.

(O3) For every $C \in \mathcal{C}$, $e \in C$ and $X \subseteq E$, there is some $C_{\min} \in \mathcal{C}$ with $e \in C_{\min} \subseteq X \cup C$ such that $C_{\min} \setminus X$ is minimal.

(O3*) For every $D \in \mathcal{D}$, $e \in D$ and $X \subseteq E$, there is some $D_{min} \in \mathcal{D}$ with $e \in D_{min} \subseteq X \cup D$ such that $D_{min} \setminus X$ is minimal.

The axiom (IM) can be seen as saying that there are bases (that is, maximal independent sets) in all minors. Similarly, the axiom (O3) can be seen as saying that there are circuits (that is, minimal dependent sets) in all minors.

In (O3), which will usually be applied in cases with $e \notin X$, the set $C_{\min} \setminus X$ is chosen to be minimal subject to the condition that it contains e. But when $\mathcal{C}$ satisfies (C3'), this set is necessarily also minimal in a stronger sense:

Lemma 1.3.1. *Let $\mathcal{C}$ be a set of subsets of E satisfying $(C3')$, let X be a subset of E and let $e \in E \setminus X$. Let $C \in \mathcal{C}$ be such that $C \setminus X$ is minimal subject to $e \in C$. Then $C \setminus X$ is also minimal subject to $C \setminus X \neq \emptyset$.*

Proof. Suppose for a contradiction that there is some $C' \in \mathcal{C}$ with $\emptyset \subsetneq C' \setminus X \subsetneq C \setminus X$. By minimality of C, $e \notin C'$. Let $x \in C' \setminus X$. So $x \in C \cap C'$ and we may apply (C3') to obtain $C'' \in \mathcal{C}$ with $e \in C'' \subseteq C \cup C' - x$. Then $C'' \setminus X \subseteq (C \setminus X) - x$, contradicting the minimality of C. □

The first aim of this section will be to show the following:

Theorem 1.3.2. *Let E be a countable set and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$.*

Then $\mathcal{C}$ is the set of circuits of a matroid and $\mathcal{D}$ is the set of cocircuits of the same matroid if and only if $\mathcal{C}$ and $\mathcal{D}$ satisfy the orthogonality axioms.

The second aim will be to show that the axioms (O3) and (O3*) follow from a much simpler condition:

(T) For any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, the set $|C \cap D|$ is finite.

This condition is called *tameness*, and matroids M for which $\mathcal{C}(M)$ and $\mathcal{C}(M^*)$ satisfy (T) are called *tame*. Thus a matroid is tame if and only if all of its circuit-cocircuit intersections are finite.

To determine, rather than define, a matroid, the last four of the orthogonality axioms suffice. What we mean by this slightly subtle distinction is captured by the following strengthening of the theorem above:

Theorem 1.3.3. *Let E be a countable set and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$.*

Then there is a matroid M such that $\mathcal{C}(M) \subseteq \mathcal{C} \subseteq \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subseteq \mathcal{D} \subseteq \mathcal{S}(M^*)$ if and only if $\mathcal{C}$ and $\mathcal{D}$ satisfy the last four orthogonality axioms.

Proof. First we show the "only if"-implication. The axiom (O1) is just the fact that no circuit can meet a cocircuit in a single element. To show (O2) consider the matroid M_e on $\{e\}$ obtained from M by contracting P and deleting Q. If M_e is a loop, then by Lemma 1.2.7 $P+e$ includes a circuit through e, and if M_e is a co-loop, then by the dual of Lemma 1.2.7 $Q+e$ includes a cocircuit through e.

By duality, it remains to show (O3). For this we consider the matroid M_X obtained from M by contracting $X - e$. Note that $(C \setminus X) + e$ is an M_X-scrawl by Lemma 1.2.7. Hence we may pick any M_X-circuit through e included in $(C \setminus X) + e$. By Lemma 1.2.7 again, this circuit extends to an M-circuit $C_{\min}$, which has the desired properties. This completes the proof of the "only if"-implication.

For the "if"-implication, our aim is to show that the set $\mathcal{C}_{\min}$ of minimal non-empty elements of $\mathcal{C}$ is the set of circuits of a matroid M. Note that circuit elimination (C3) for $\mathcal{C}$ follows from Lemma 1.1.4, and this implies circuit elimination for $\mathcal{C}_{\min}$ using (O3) and Lemma 1.3.1.

Next, we prove (CM) for $\mathcal{C}_{\min}$. Suppose we are given a set I not including an element of $\mathcal{C}_{\min}$ and a set X with $I \subseteq X \subseteq E$. Put $I_0 = I$ and $J_0 = E \setminus X$.

Let $e_1, e_2, \ldots$ be an enumeration of X. We shall construct a partition of E into I_∞ and J_∞ such that I_∞ is a maximal $\mathcal{C}_{\min}$-independent subset of X. The construction will be recursive. So we take $I_\infty = \bigcup_{n \in \mathbb{N}} I_n$ and $J_\infty = \bigcup_{n \in \mathbb{N}} J_n$ where we construct the I_n and J_n both at step n to satisfy the following conditions:

1. I_n and J_n are disjoint.
2. $I_j \subseteq I_n$ for all $j \leq n$.
3. $J_j \subseteq J_n$ for all $j \leq n$.
4. $e_n \in I_n \cup J_n$.
5. If $e_n \in I_n$, then there is some $D \in \mathcal{D}$ with $D \subseteq J_n + e_n$ through e_n.
6. If $e_n \in J_n$, then there is some $C \in \mathcal{C}$ with $C \subseteq I_n + e_n$ through e_n.
7. If J_n includes any $D \in \mathcal{D}$, then $D \subseteq J_0$.
8. If I_n includes any $C \in \mathcal{C}$, then $C \subseteq I_0$. (That is, $C = \emptyset$: this condition says that I_n is $\mathcal{C}$-independent.)

What we do at step n depends on whether there is any $C \in \mathcal{C}$ with $e_n \in C \subseteq I_{n-1} + e_n$. If there is such a C, we let $I_n = I_{n-1}$ and $J_n = J_{n-1} + e_n$. The only nontrivial condition in this case is (7). By the induction hypothesis, any D violating this condition would contain e_n and so would meet C just once, contradicting (O1).

Next, we consider the case that $e_n \notin J_{n-1}$. In this case, we let $I_n = I_{n-1}+e_n$, but the construction of J_n is more complex. First, we note that by (O2) applied to e_n, I_{n-1} and $E \setminus I_{n-1} - e_n$ we can obtain some $D \in \mathcal{D}$ with $e_n \in D \subseteq E \setminus I_{n-1}$. Then using (O3*) we may assume that D is chosen with these properties so that $D \setminus J_{n-1}$ is minimal. We take $J_n = (J_{n-1} \cup D) - e_n$.

Once more, the only nontrivial condition is (7). Suppose for a contradiction that there is some D' violating this condition. Then D' must meet $D \setminus J_{n-1}$ in some element x. We showed above that $\mathcal{C}$ satisfies (C3), and by symmetry we may also show that $\mathcal{D}$ satisfies (C3). We apply this with $X = \{x\}$ to D and D' to obtain $D'' \in \mathcal{D}$ with $e_n \in D'' \subseteq (D \cup J_{n-1}) - x$, contradicting the minimality of $D \setminus J_{n-1}$.

The remaining case is that $e_n \in J_{n-1}$. In this case, we let $J_n = J_{n-1}$ and, dualising the construction from the last case, we choose $C \in \mathcal{C}$ such that $e_n \in C \subseteq E \setminus (J_{n-1} - e_n)$ and $C \setminus I_{n-1}$ is minimal subject to these conditions. This construction succeeds for a reason dual to that given in the last case.

This completes the recursive construction. As promised, we take $I_\infty = \bigcup_{n \in \mathbb{N}} I_n$ and $J_\infty = \bigcup_{n \in \mathbb{N}} J_n$. It is clear that this is a partition of E. Next, we show that I_∞ includes no element C of $\mathcal{C}_{\min}$. Suppose for a contradiction that there is such a C. Then there is some n with e_n in C. Then by (5) there is some $D \in \mathcal{D}$ with $e_n \in D \subseteq J_n + e_n \subseteq J_\infty + e_n$, so that $C \cap D = \{e_n\}$, violating (O1).

We can also show that I_∞ is maximal amongst the $\mathcal{C}_{\min}$-independent subsets of X. Suppose for a contradiction that there is a bigger $\mathcal{C}_{\min}$-independent set I', and pick some n with $e_n \in I' \setminus I$. Then by (6) there is $C \in \mathcal{C}$ with $e_n \in C \subseteq I_n + e_n \subseteq I'$, contradicting the $\mathcal{C}_{\min}$-independence of I' as, by (O3), C is a union of elements of $\mathcal{C}_{\min}$. This completes the proof that $\mathcal{C}_{\min}$ is the set of circuits of some matroid M.

By (O3), every element of $\mathcal{C}$ is a union of circuits of M. Hence $\mathcal{C}(M) \subseteq \mathcal{C} \subseteq \mathcal{S}(M)$. (O1) and Lemma 1.2.1 imply that $\mathcal{D} \subseteq \mathcal{S}(M^*)$. It remains only to show that $\mathcal{C}(M^*) \subseteq \mathcal{D}$. So let D be any cocircuit of M. Let $e \in D$, and apply (O2) to e, $E \setminus D$ and $D - e$. There can't be $C \in \mathcal{C}$ with $e \in C \subseteq (E \setminus D) + e$, as then we would have $C \cap D = \{e\}$, which is impossible with C a scrawl and D a cocircuit. So there is some $D' \in \mathcal{D}$ with $e \in D' \subseteq D$, and we must have $D' = D$ since no nonempty proper subset of D can be a scrawl of M^*. $\square$

We are left with the open questions of whether the restriction that E should be countable can be removed from Theorems 1.3.2 and 1.3.3, or if not whether there is a simple axiom which can be added to fix this defect.

We now show that for tame matroids we do not need (O3) or (O3*). More precisely:

Theorem 1.3.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be sets of subsets of a countable set E satisfying (O1), (O2) and (T). Then $\mathcal{C}$ and $\mathcal{D}$ also satisfy (O3) and (O3*), so that they induce a matroid as above.*

Proof. By symmetry, it is enough to show (O3). Let $C \in \mathcal{C}$, $e \in C$ and $X \subseteq E$. Let $\mathcal{Y}$ be the set of subsets Y of $C \setminus X$ such that $e \in Y$ and for every $D \in \mathcal{D}$

with $D \cap X = \emptyset$ we have $|Y \cap D| \neq 1$. We will use Zorn's Lemma to show that $\mathcal{Y}$ has a minimal element. $\mathcal{Y}$ is nonempty because it contains $C \setminus X$ by ($O1$). Let $\mathcal{Z}$ be a nonempty chain of elements of $\mathcal{Y}$. We shall show that $\bigcap \mathcal{Z}$ is in $\mathcal{Y}$ and so forms a lower bound for $\mathcal{Z}$ there. Evidently $e \in \bigcap \mathcal{Z}$. For any $D \in \mathcal{D}$ with $D \cap X = \emptyset$ we know that $D \cap (C \setminus X)$ is finite and so we can find a finite subset $\mathcal{Z}'$ of $\mathcal{Z}$ such that for any $f \in D \cap C \setminus \bigcap \mathcal{Z}$ there is $Z \in \mathcal{Z}'$ such that $f \notin Z$. Let Z be the least element of $\mathcal{Z}'$. Then $|\bigcap \mathcal{Z} \cap D| = |Z \cap D| \neq 1$.

Let Y be a minimal element of $\mathcal{Y}$. We apply ($O2$) to the partition $E = (X \cup Y - e)\dot{\cup}(E \setminus X \setminus Y)\dot{\cup}\{e\}$. By the definition of $\mathcal{Y}$ there is no $D \in \mathcal{D}$ with $e \in D \subseteq E \setminus X \setminus Y$, so there is some $C_{\min} \in \mathcal{C}$ with $e \in C_{\min} \subseteq X \cup Y$. For any other $C' \in \mathcal{C}$ with $e \in C' \subseteq X \cup C$, we have $C' \setminus X \in \mathcal{Y}$ by ($O1$) and so $C_{\min} \setminus X \subseteq C' \setminus X$. $\square$

Note that the above theorem does not require E to be countable.

We conclude with a helpful consequence of these axioms, which also holds even when E is uncountable.

Lemma 1.3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be sets of subsets of E satisfying (O1), (O2) and (O3*). Let $C \in \mathcal{C}_{\min}$ and let $e, f \in C$ be distinct. Then there is $D \in \mathcal{D}_{\min}$ with $C \cap D = \{e, f\}$.*

Proof. Let $P = C \setminus e, f$ and $Q = E \setminus P - e$. By minimality of C, there is no $C' \in \mathcal{C}$ with $e \in C' \subseteq P + e$. Thus by (O2) there is some $D \in \mathcal{D}$ with $e \in D \subseteq Q + e$. By (O3*) and Lemma 1.3.1 we may assume that $D \in \mathcal{D}_{\min}$. Then $e \in C \cap D \subseteq \{e, f\}$ so by (O1) we have $f \in D$ $\square$

This lemma has a partial converse:

Lemma 1.3.6. *Let W be a dependent set in a matroid. Then W is a circuit if and only if for any edges e and f of W there is a cocircuit D with $W \cap D = \{e, f\}$.*

Proof. The 'only if' direction is immediate from Lemma 1.3.5. For the 'if' direction, pick a circuit $C \subseteq W$. If $C \neq W$ then we can find $e \in C$ and $f \in W \setminus C$, and choosing D a cocircuit with $D \cap W = \{e, f\}$, we get $D \cap C = \{e\}$, which is impossible. $\square$

Lemma 1.3.7. *Let M be a matroid and $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$ such that every M-circuit is a union of elements of $\mathcal{C}$, every M-cocircuit is a union of elements of $\mathcal{D}$ and $|C \cap D| \neq 1$ for every $C \in \mathcal{C}$ and every $D \in \mathcal{D}$.*

Then $\mathcal{C}(M) \subseteq \mathcal{C} \subseteq \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subseteq \mathcal{D} \subseteq \mathcal{S}(M^*)$, so that $\mathcal{C}_{\min} = \mathcal{C}(M)$ and $\mathcal{D}_{\min} = \mathcal{C}(M^*)$

Proof. We begin by showing that $\mathcal{C}(M) \subseteq \mathcal{C}$. For any circuit C of M, pick an element e of C. Since C is a union of elements of $\mathcal{C}$ there is $C' \in \mathcal{C}$ with $e \in C' \subseteq C$. Suppose for a contradiction that C' isn't the whole of C, so that there is $f \in C \setminus C'$. By Lemma 1.3.5 there is some cocircuit D of M with $C' \cap D = \{e\}$. Then we can find $D' \in \mathcal{D}$ with $e \in D' \subseteq D$, and so $C' \cap D' = \{e\}$, giving the desired contradiction. Similarly we obtain that $\mathcal{C}(M^*) \subseteq \mathcal{D}$.

The fact that $\mathcal{C} \subseteq \mathcal{S}(M)$ is immediate from the dual of Lemma 1.2.1 since $\mathcal{C}(M^*) \subseteq \mathcal{D}$, and the proof that $\mathcal{D} \subseteq \mathcal{S}(M^*)$ is similar. $\square$

1.4 IE-operators and scrawl systems

1.4.1 Spaces

In [72], a *space* is defined to consist of a set E together with an operator $\mathcal{P}(E) \xrightarrow{K} \mathcal{P}(E)$ such that K preserves the order $\subseteq$ and satisfies $X \subseteq KX$ for any $X \subseteq E$. For example, for any set $\mathcal{C}$ of subsets of E the associated *closure operator* $\mathrm{Cl}_{\mathcal{C}}$, which sends X to the set

$$X \cup \{x \in E | (\exists C \in \mathcal{C}) x \in C \subseteq X + x\}$$

gives a space on the set E. If $\mathcal{C}$ is the set of circuits or the set of scrawls of a matroid M, this is just the usual closure operator associated to M, and we shall also denote this space by Cl_M in such cases.

If (E, K) is a space, the *dual space* is given by (E, K^*), where K^* is the *dual operator* to K, sending X to $X \cup \{x \in E | x \notin K(E \setminus (X + x))\}$. Thus for sets X and Y with $X \dot\cup Y \dot\cup \{x\} = E$, we have

$$x \in KX \iff x \notin K^*Y\,, \tag{†}$$

and this completely determines K^* in terms of K. Thus $K^{**} = K$. Also, by (O2), for any matroid M we have $\mathrm{Cl}_{M^*} = \mathrm{Cl}^*_M$.

1.4.2 Idempotence

For a space (K, E), we say K is *idempotent* if $K^2 = K$, and *exchange* if K^* is idempotent. If K is both idempotent and exchange, we call it an idempotent-exchange operator, or an *IE-operator* on E. Note that if K is an IE-operator then so is K^*. For any set $\mathcal{C}$ of subsets of E and any $X \subseteq E$ and $x \in E \setminus X$ we have $x \in \mathrm{Cl}^*_{\mathcal{C}}(X)$ if and only if there is no $C \in \mathcal{C}$ with $x \in C \subseteq E \setminus X$, so that $\mathrm{Cl}^*_{\mathcal{C}}(X) = E \setminus \bigcup\{C \in \mathcal{C} | C \cap X = \emptyset\}$. Thus $\mathrm{Cl}^*_{\mathcal{C}}$ is idempotent since for any $C \in \mathcal{C}$ we have $C \subseteq E \setminus X$ if and only if $C \subseteq \bigcup\{C' \in \mathcal{C} | C' \subseteq E \setminus X\}$. So for any set $\mathcal{C}$ of subsets of E the space $\mathrm{Cl}_{\mathcal{C}}$ is exchange. Thus for any matroid M the operator Cl_M is an IE-operator.

1.4.3 Scrawl systems

A *scrawl system* on a set E is a collection $\mathcal{S}$ of subsets of E satisfying (S1) and (S2). The *dual* of a scrawl system $\mathcal{S}$ is $\mathcal{S}^\perp$. By Lemma 1.1.4, $\mathcal{S}$ and $\mathcal{S}^\perp$ satisfy (O2). Thus $\mathrm{Cl}^*_{\mathcal{S}} = \mathrm{Cl}_{\mathcal{S}^\perp}$. Thus for any scrawl system $\mathcal{S}$, the operator $\mathrm{Cl}_{\mathcal{S}}$ is an IE-operator.

Conversely, let K be any IE-operator on E, and let $\mathcal{S} = \{E \setminus K^*Y | Y \subseteq E\}$. We shall show that $\mathcal{S}$ is a scrawl system with $\mathrm{Cl}_{\mathcal{S}} = K$. This clarifies how IE-operators, which were one of the suggested answers to Rado's question, are related to the notion of matroid with which we are working. The key extra property possessed by matroids but not scrawl systems (or IE-operators) is (SM).

First we must show that $\mathcal{S}$, as defined above, is closed under taking unions. So let $(Y_a|a \in A)$ be a family of subsets of E. We will show that $\bigcup_{a\in A} E\setminus K^*Y_a = E \setminus K^*(\bigcap_{a\in A} K^*Y_a)$. This is equivalent to $\bigcap_{a\in A} K^*Y_a = K^*(\bigcap_{a\in A} K^*Y_a)$, for which it suffices to show that $K^*(\bigcap_{a\in A} K^*Y_a) \subseteq \bigcap_{a\in A} K^*Y_a$. This is true since for each $a \in A$ we have $K(\bigcap_{a\in A} K^*Y_a) \subseteq K^*K^*Y_a = K^*Y_a$.

Next, we must show that $\mathcal{S}$ satisfies (S2), for which by Lemma 1.1.4 it suffices to show that $\mathcal{S}$ and $\mathcal{S}^\perp$ satisfy (O2). First we note that every set of the form $E\setminus KX$ is in $\mathcal{S}^\perp$, since if there were sets X and Y with $|(E\setminus K^*Y)\cap(E\setminus KX)| = \{x\}$, we would have $x \notin KX = KKX$ but also $x \notin K^*(E \setminus KX - x)$ since $E \setminus KX - x \subseteq K^*Y$, so $K^*(E \setminus KX - x) \subseteq K^*K^*Y = K^*Y$. This would contradict the definition of duality of spaces.

Now suppose $E = P\dot\cup Q\dot\cup\{e\}$. If $e \notin KP$ then we have $E \setminus KP \in \mathcal{S}^\perp$ as above and $e \in E \setminus KP \subseteq Q + e$. But if $e \in KP$ then $E \notin K^*Q$, so that $E \setminus K^*Q \in \mathcal{S}$ and $e \in E \setminus K^*Q \subseteq P + e$. Thus $\mathcal{S}$ and $\mathcal{S}^\perp$ satisfy (O2). This completes the proof that $\mathcal{S}$ is a scrawl system.

If $X \subseteq E$ and $x \notin X$ then we have $x \in KX$ if and only if $x \notin K^*(E\setminus X - x)$, which happens if and only if there is some Y with $x \in E \setminus K^*Y \subseteq X + x$, since for any such Y we have $E \setminus X - x \subseteq K^*Y$. Thus $\mathrm{Cl}_{\mathcal{S}} = K$.

1.5 Connectivity

In [26], Bruhn and Wollan developed the basic theory of connectivity in infinite matroids. In this section we shall summarise some of their results.

1.5.1 The connectivity of partitions and separations

The connectivity of a separation can be defined in multiple equivalent ways:

Lemma 1.5.1. *Let M be a matroid with ground set $E = X\dot\cup Y$, and let k be a natural number. Then the following are equivalent:*

- *There are bases B_X of $M\setminus Y$ and B'_X of M/Y with $B'_X \subseteq B_X$ and $|B_X \setminus B'_X| = k$.*
- *For any bases B_X of $M\setminus Y$ and B'_X of M/Y with $B'_X \subseteq B_X$ we have $|B_X \setminus B'_X| = k$.*
- *There are bases B_X of $M\setminus Y$, B_Y of $M\setminus X$ and B of M with $B \subseteq B_X \cup B_Y$ and $|B_X \cup B_Y \setminus B| = k$.*
- *For any bases B_X of $M\setminus Y$, B_Y of $M\setminus X$ and B of M with $B \subseteq B_X \cup B_Y$ we have $|B_X \cup B_Y \setminus B| = k$.*

If there is a k with these properties, then we call it the *connectivity* $\kappa_M(X)$ of the partition. Otherwise we say that the partition has connectivity ∞. We call a partition (X, Y) a *k-separation* if $\kappa_M(X) \leq k - 1$ but both X and Y have at least k elements. We say the matroid M is *k-connected* if there is no l-separation with $l < k$.

For the next few lemmas we fix a matroid M with ground set E.

Lemma 1.5.2. *For any subset X of E, we have $\kappa_M(X) = \kappa_M(E \setminus X) = \kappa_{M^*}(X)$.*

Lemma 1.5.3. *For any subsets X, X' of E, we have $\kappa_M(X) + \kappa_M(X') \geq \kappa_M(X \cap X') + \kappa_M(X \cup X')$.*

Lemma 1.5.4. *For any chain $\mathcal{X}$ of subsets of E such that $\kappa_M(X) \leq k$ for all $X \in \mathcal{X}$, we have $\kappa_M(\bigcup \mathcal{X}) \leq k$.*[1]

Lemma 1.5.5. *For any subset X of E and any disjoint subsets P and Q of E we have $\kappa_{M/P\setminus Q}(X \setminus (P \cup Q)) \leq \kappa_M(X)$.*

1.5.2 2-connectivity and direct sums

We begin by treating the notion of connectivity in slightly more generality than [26], since we will need this extra generality later.

Let $\mathcal{C}$ be a set of subsets of a set E satisfying (C3') and (O3). We define a relation $\sim_{\mathcal{C}}$ on E by $e \sim_{\mathcal{C}} f$ if $e = f$ or there is some $C \in \mathcal{C}_{\min}$ with $e, f \in C$.

Lemma 1.5.6. *$\sim_{\mathcal{C}}$ is an equivalence relation.*

Proof. Suppose $e \sim_{\mathcal{C}} f$ and $f \sim_{\mathcal{C}} g$. We must show that $e \sim_{\mathcal{C}} g$. If $e = f$ or $f = g$ then we are done. Otherwise, there are $C, C' \in \mathcal{C}_{\min}$ with $e, f \in C$ and $f, g \in C'$. If $e \in C'$ then we are done. Otherwise, choose $C_{\min} \in \mathcal{C}$ with $e \in C_{\min} \subseteq C \cup C'$ such that $C_{\min} \setminus C'$ is minimal (this is possible by (O3)). Since C is minimal nonempty in $\mathcal{C}$, we cannot have $C_{\min} \subseteq C \setminus C' \subseteq C - f$, and so there is some $f' \in C_{\min} \cap C'$. If $g \in \mathcal{C}_{\min}$ then we are done. Otherwise apply (C3') with respect to C', $C_{\min}$, g and f' to obtain some $C'' \in \mathcal{C}$ with $g \in C'' \subseteq C' \cup C_{\min} - f'$. Using (O3) again and Lemma 1.3.1, we may assume that $C'' \in \mathcal{C}_{\min}$. By minimality of C', we can't have $C'' \subseteq C'$, so $C'' \setminus C' \neq \emptyset$. Thus by Lemma 1.3.1 and our choice of $C_{\min}$ we have $C'' \setminus C' = C_{\min} \setminus C'$ and in particular $e \in C''$. Thus C'' witnesses $e \sim_{\mathcal{C}} g$. □

If $\mathcal{C}$ and $\mathcal{D}$ are as above and also satisfy (O2), then by Lemma 1.3.5 we have $\sim_{\mathcal{C}} = \sim_{\mathcal{D}}$. The equivalence classes of $\sim_{\mathcal{C}}$ are called the *connected components* with respect to $\mathcal{C}$.

In particular, if M is a matroid with ground set E then the equivalence classes with respect to $\sim_{\mathcal{C}(M)}$ are called the connected components of M. The remark in the last paragraph shows that the connected components of M are the same as those of M^*. We say M is *connected* if it has at most one connected component.

Lemma 1.5.7. *Let M be a matroid and let B be a base of M. Then the connected components of the relation R_B (see subsection 1.2.2) are the same as the connected components of M.*

[1] In [26] this is only proved for countable chains, but the same proof works for arbitrary chains.

Proof. It is clear that any e and f with eR_Bf are in the same connected component of M. So it remains to show that there cannot be a circuit C of M which meets two different connected components of R_B.

Suppose for a contradiction that there is such a circuit C. Let $e \in C$ and let Y be a connected component of R_B which meets C but does not contain e. Let $X = (Y \cap C) \setminus B$. For each $x \in X$ we have $C_x^B \subseteq Y$ and so $C_x^B \cap X + e = \{x\}$. Applying (C3), we get a circuit C' with $e \in C' \subseteq (C \cup \bigcup_{x \in X} C_x) \setminus X$. Thus $C' \setminus C \subseteq Y$ and $C' \cap Y \subseteq B$. By (C2), C' cannot be a subset of $C \setminus Y$, so let $f \in C' \cap Y$. Then $f \in B$ and $D_f^B \subseteq Y$ so that $D_f^B \cap C' = \{f\}$, which is the desired contradiction. □

The remaining results of this subsection are taken from [26].

Lemma 1.5.8. *Let X be a subset of E. Then $\kappa(X) = 0$ if and only if X is a union of equivalence classes.*

Thus a matroid is connected if and only if it is 2-connected.

Let $(M_k | k \in K)$ be a family of matroids, where M_k has ground set E_k. Then the *direct sum* $\bigoplus_{k \in K} M_k$ of this family is the matroid with ground set $\bigsqcup_{k \in K} E_k$ and independent sets of the form $\bigcup_{k \in K} I_k$ where each I_k is independent in M_k. It is not hard to check that the bases have the form $\bigcup_{k \in K} B_k$ where each B_k is a base in M_k and that the circuits are given by the circuits of the individual matroids M_k.

Lemma 1.5.9. *Any matroid is the direct sum of its restrictions to its connected components.*

1.6 Finitarisation and twinned pairs of matroids

1.6.1 Finitarisation

We say a matroid M is *finitary* if all of its circuits are finite. The *finitarisation* M^{fin} of a matroid M is the matroid whose circuits are the finite circuits of M. Thus a matroid is finitary if and only if it is equal to its own finitarisation. The class of finitary matroids introduced here is the same as that discussed in the introduction, since a set is independent in M^{fin} if and only if all of its finite subsets are independent in M.

A matroid is *cofinitary* if and only if all of its cocircuits are finite. The *cofinitarisation* M^{cofin} of a matroid M is $((M^*)^{\text{fin}})^*$. This is the dual operation to finitarisation.

1.6.2 Twinned pairs of matroids

We say that matroids M_f and M_c are *twinned* if $M_f = M_c^{\text{fin}}$ and $M_c = M_f^{\text{cofin}}$. Then we say that N *lies between* M_f and M_c if all of its circuits are M_c-circuits and all of its cocircuits are M_f-cocircuits.

Examples of twinned pairs (M_f, M_c) of matroids abound.

Proposition 1.6.1. *Let M be any finitary matroid. Then $((M^{\text{cofin}})^{\text{fin}})^{\text{cofin}} = M^{\text{cofin}}$, so that $(M^{\text{cofin}})^{\text{fin}}$ and M^{cofin} are twinned.*

Proof. No circuit of $(M^{\text{cofin}})^{\text{fin}}$ ever meets a cocircuit of M^{cofin} in just one element, so every cocircuit D of M^{cofin} is a coscrawl of $(M^{\text{cofin}})^{\text{fin}}$ by Lemma 1.2.1. The cocircuits in the union must all be cocircuits of $((M^{\text{cofin}})^{\text{fin}})^{\text{cofin}}$ since D is finite. Similarly, every circuit of M is a scrawl of $(M^{\text{cofin}})^{\text{fin}}$, so that no cocircuit of $((M^{\text{cofin}})^{\text{fin}})^{\text{cofin}}$ ever meets a circuit of M in just one element, and so every cocircuit b of $((M^{\text{cofin}})^{\text{fin}})^{\text{cofin}}$ is a coscrawl of M^{cofin} by Lemma 1.2.1 applied to M and finiteness of b. Thus $((M^{\text{cofin}})^{\text{fin}})^{\text{cofin}}$ and M^{cofin} have the same cocircuits. $\square$

Similarly, M^{fin} and $(M^{\text{fin}})^{\text{cofin}}$ are twinned.

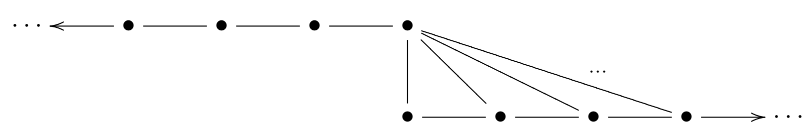
This means that if we start with any matroid M and alternately finitarise and cofinitarise then we will quickly end up going back and forth between two twinned matroids.

The *dual* of a twinned pair (M_f, M_c) is the twinned pair (M_c^*, M_f^*).

1.6.3 Basic examples associated to graphs

In the following discussion and in later chapters we shall make free use of standard graph-theoretic terminology (see, for example, [32]). However, for us a graph need not be simple, and so may have loops or multiple edges. Thus what we are calling graphs are called multigraphs in [32].

If G is a graph then Higgs showed [41] that there is a matroid with ground set $E(G)$ whose circuits are the edge sets of finite cycles or double rays in G precisely when G includes no subdivision of the Bean graph:



We shall give a simpler proof of this result in Section 3.3.

In such cases, this matroid is called the *algebraic cycle matroid* of the graph and is denoted $M_{AC}(G)$. A set I of edges of G is independent in this matroid if and only if it is a forest in which each component includes at most one ray (up to extension and truncation). Thus a set B of edges of G is a base if and only if it is either a spanning tree or a forest in which each component includes precisely one ray. Finally, a set D of edges of G is a cocircuit if and only if is a cut which is minimal amongst nonempty cuts of which at least one side is rayless. Such cuts are called *skew*. The dual of $M_{AC}(G)$ is called the *skew-cut matroid* of G and denoted $M_{SC}(G)$.

Let us consider the result of applying the process described in the last subsection to $M_{AC}(G)$. It is clear that the finitarisation of $M_{AC}(G)$ has as its circuits the edge sets of finite cycles in G. More generally, it is clear that for any graph G, whether or not G includes a subdivision of the Bean graph, the edge sets of finite cycles in G give the circuits of a matroid with ground set $E(G)$. This matroid is called the *finite-cycle matroid* of G and denoted $M_{FC}(G)$. The independent sets are forests, the bases are spanning trees and the cocircits are bonds of G. The dual of $M_{FC}(G)$ is called the *bond matroid* of G and denoted $M_B(G)$.

To continue our process we must cofinitarise M_{FC}, that is, we must finitarise $M_B(G)$ and then dualise again. The circuits of the finitarisation of $M_B(G)$ are simply the finite bonds of G, and accordingly this finitarisation is called the *finite-bond matroid* of G and denoted $M_{FB}(G)$. The circuits of the dual of $M_{FB}(G)$ can be described as edge sets of topological circles in a topological space $|G|$ obtained by adjoining some formal points at infinity, called ends, to G. If G is locally finite, then the space in question is simply the Freudenthal compactification of G. This construction is discussed in more detail in [27]. The matroid constructed here is therefore called the *topological-cycle matroid* of G, and denoted $M_{TC}(G)$.

Returning to our alternate finitarisations and cofinitarisations, we will now consider the finitarisation of $M_{TC}(G)$. Surprisingly, it is possible that $M_{TC}(G)^{\text{fin}} \neq M_{FC}(G)$. For example, let H be the graph with just two vertices but with infinitely many edges joining those vertices. Clearly H has no finite bonds, so the set of circuits of $M_{FB}(H)$ is empty. It follows that the circuits of $M_{TC}(H)$ are precisely the singletons of edges of H. These singletons are therefore also circuits of $M_{TC}(H)^{\text{fin}}$, even though they are not edge sets of finite cycles in H.

The problem is that it is possible for vertices to become topologically identified in $|G|$. More precisely, the vertices x and y are topologically identified in $|G|$ when there is no finite cut of G with x and y on opposite sides. In such cases, we say that x and y cannot be *finitely separated* in G. If any two vertices can be finitely separated in G then we say G is *finitely separable*. The graph obtained from G by identifying any two vertices which cannot be finitely separated in G is called the *finitely separable quotient* of G, and denoted G_{fs}. It is clear that $M_{FB}(G_{fs}) = M_{FB}(G)$, so that $M_{TC}(G_{fs}) = M_{TC}(G)$. Thus the following lemma is the best we could hope for:

Lemma 1.6.2. $M_{TC}(G)^{\text{fin}} = M_{FC}(G_{fs})$

This lemma is equivalent to the statement that if G is finitely separable then $M_{TC}(G)^{\text{fin}} = M_{FC}(G)$, which follows from, for example, [27, Lemma 12], together with the fact that distinct vertices are not topologically identified if G is finitely separable.

Continuing further, we now consider the cofinitarisation of $M_{FC}(G_{fs})$. This is $M_{TC}(G_{fs}) = M_{TC}(G)$, so the sequence has become periodic, as Proposition 1.6.1 implied that it must.

We have arrived at a large collection of twinned pairs of matroids in this way: if G is a finitely separable graph, then $(M_{FC}(G), M_{TC}(G))$ is a twinned pair of matroids. In [27], it is shown that the duality of infinite finitely separable planar graphs corresponds to the duality of these twinned pairs. We shall study such twinned pairs and analyse the matroids lying between them in chapter 4.

1.7 Truncation and wild matroids

1.7.1 Tame matroids again

Recall that a matroid is *tame* if any intersection of a circuit with a cocircuit is finite. Evidently, any finitary matroid is tame, as are all of the examples mentioned so far. This, together with the following lemma, demonstrates that the collection of tame matroids provides an alternative answer to Rado's question.

Lemma 1.7.1. *The class of tame matroids is closed under duality and taking minors.*

Proof. Closure under duality follows from the symmetry of the definition. Closure under taking minors is immediate from Corollary 1.2.10. □

Matroids which are not tame are called *wild*. A fertile source of wild matroids is a construction called *truncation*

1.7.2 Truncation

Definition 1.7.2. Let M be a matroid, in which $\emptyset$ isn't a base. Then the *truncation* of M is the matroid M^-, on the same groundset whose bases are those sets which can be obtained by removing a point from a base of M. That is, $\mathcal{B}(M^-) = \{B - e | B \in \mathcal{B}(M), e \in B\}$. Dually, if M is a matroid whose ground set E isn't a base, we define M^+ by $\mathcal{B}(M^+) = \{B + e | B \in \mathcal{B}(M), e \in E \setminus B\}$.

Thus $(M^+)^* = (M^*)^-$.

Since M^- is obtained from M by making the bases of M into dependent sets, we may expect that $\mathcal{C}(M^-) = \mathcal{C}(M) \cup \mathcal{B}(M)$: that is, the set of circuits of M^- contains exactly the circuits and the bases of M. This is essentially true, but there is one complication: an M-circuit might include an M-base, which would prevent it from being an M^--circuit. Let C be a circuit of M^-. If C is M-independent, it is clear that C must be an M-base. Conversely, any M-base is a circuit of M^-. If C is M-dependent, then since all proper subsets of C are M^--independent and so M-independent, C must be an M-circuit. Conversely, an M-circuit not including an M-base is an M^--circuit.

On the other hand, none of the circuits of M is a circuit of M^+: for any circuit C of M, pick any $e \in C$ and extend $C - e$ to a base B of M. Then $C \subseteq B + e$, so $C \in \mathcal{I}(M^+)$. In fact, a circuit of M^+ is a set minimal with the property that at least two elements must be removed before it becomes M-independent. To see this note that the independent sets of M^+ are those sets from which an M-independent set can be obtained by removing at most one element.

Now we are in a position to construct a wild matroid: let M be the algebraic cycle matroid of the graph in Figure 1.1. Then the dashed edges form a circuit in M^+, and the bold edges form a circuit in $(M^+)^* = (M^*)^-$ (they form a base in M^* since their complement forms a base in M). The intersection, consisting of the dotted bold edges, is evidently infinite.

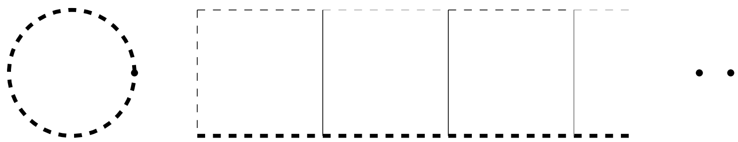


Figure 1.1: A circuit and a cocircuit with infinite intersection

For the remainder of this section, we will generalize this example to construct a large class of wild matroids. To do so, we first have a closer look at the circuits of M^+. It is clear that if M is the finite cycle matroid of a graph G, then we get as circuits of M^+ any subgraphs which are subdivisions of those in Figure 1.2.

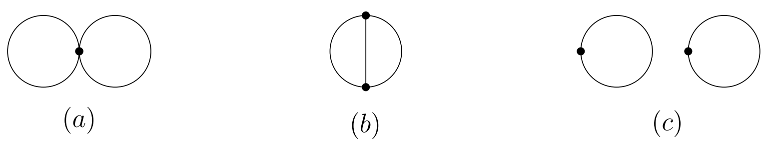


Figure 1.2: Shapes of circuits in M^+, with M a finite cycle matroid

More generally, we can make precise a sense in which every circuit of M^+ is obtained by sticking together two circuits.

Lemma 1.7.3. *Let C be a circuit of M, and $I \subseteq E(M) \setminus C$. Then $C \cup I$ is M^+-independent iff I is M/C-independent.*

Proof. If: Extend I to a base B of M/C. Pick any $e \in C$. Then $B' = B \cup C - e$ is a basis of M and $C \cup I \subseteq B' + e$.

Only if: Pick B a base of M and $e \in E \setminus B$ such that $C \cup I \subseteq B \cup e$. Since C is dependent, we must have $e \in C$, and so $I \subseteq B \setminus C$. Finally, $B \setminus C$ is a base of M/C, since $B \cap C = C - e$ is a base of C. □

Lemma 1.7.4. *Let C_1 be a circuit of M, and C_2 a circuit of M/C_1. Then $C_1 \cup C_2$ is a circuit of M^+. Every circuit of M^+ arises in this way.*

Proof. $C_1 \cup C_2$ is M^+-dependent by Lemma 1.7.3. Next, we shall show that any set $C_1 \cup C_2 - e$ obtained by removing a single element from $C_1 \cup C_2$ is M^+-independent, and so that $C_1 \cup C_2$ is a *minimal* dependent set (a circuit) in M^+. The case $e \in C_2$ is immediate by Lemma 1.7.3. If $e \in C_1$, then we pick any $e' \in C_2$. Now extend $C_2 - e'$ to a base B of M/C_1. Then $B' = B \cup C_1 - e$ is a base of M and $C_1 \cup C_2 - e \subseteq B' + e'$.

Finally, we need to show that any circuit C of M^+ arises in this way. C must be M-dependent, and so we can find a circuit $C_1 \subseteq C$ of M. Let $C_2 = C \setminus C_1$: C_2 is a circuit of M/C_1 by Lemma 1.7.3. $\square$

Corollary 1.7.5. *Any union of two distinct circuits of M is dependent in M^+.*

It follows from Lemma 1.7.4 that the subgraphs of the types illustrated in Figure 1.2 give all of the circuits of M^+ for M a finite cycle matroid. Similarly, subdivisions of the graphs in Figure 1.2 and Figure 1.3 give circuits in the algebraic cycle matroid of a graph.

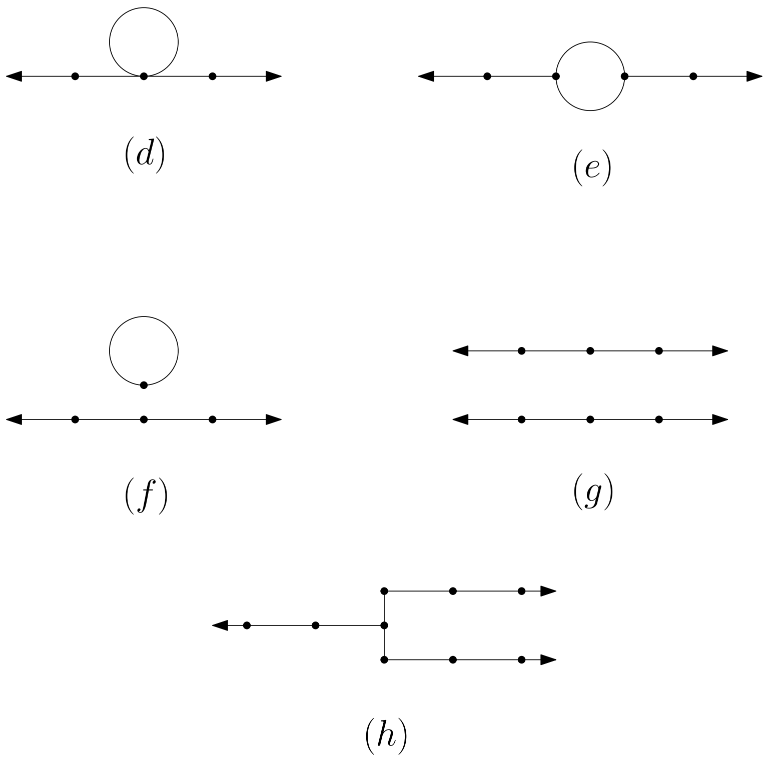


Figure 1.3: Shapes of circuits in M^+, with M an algebraic cycle matroid

Now that we have a good understanding of the circuits of matroids constructed this way, we can find many matroids M such that M^+ is wild.

Theorem 1.7.6. *Let M be a matroid such that*

1. *M contains at least two circuits;*
2. *M has a base B and a circuit C such that $C \setminus B$ is infinite.*

Then M^+ is wild.

Proof. Let C' be any circuit other than C. As C' is dependent in M/C, there is an M/C-circuit C'' included in C'. By Lemma 1.7.4, $C \cup C''$ is an M^+-circuit.

Since $E \backslash B$ is an M^*-base, it is a circuit of $(M^*)^- = (M^+)^*$. Now $(C \cup C'') \cap (E \backslash B)$ includes $C \setminus B$ and so it is infinite. $\square$

Chapter 2

Uniform matroids and equicardinality of bases

Recall from the introduction that if n is a natural number and E is a set then the *uniform* matroid $U_{n,E}$ has as its bases all subsets of E of size n.

The following is a natural infinitary generalization of uniformity.

Definition 2.0.7. Let $\mathcal{B}$ be the set of bases of a matroid $\mathcal{M}$ on a set E. Then $\mathcal{M}$ is *uniform* if the following strengthening of (B2) holds:

(U) Whenever $B \in \mathcal{B}$, $x \in B$, and $y \in E \setminus B$, then $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Clearly, a matroid on a set E is uniform iff its set $\mathcal{B}$ of bases is closed under the equivalence relation $\sim$ where for $A, B \subseteq E$ we let $A \sim B$ iff $A \setminus B$ and $B \setminus A$ are both finite and of the same size.

Under some set-theoretic assumptions we will construct self-dual, uniform matroids that are neither finitary nor cofinitary. The Continuum Hypothesis (CH) already implies the existence of a self-dual, uniform matroid on a countably infinite set.

A fundamental fact about finite matroids is that all bases of a given matroid are the same size. Higgs showed in [42] that this is still true for infinite matroids if we assume the Generalized Continuum Hypothesis. Using a fragment of Martin's Axiom together with the negation of CH we obtain a self-dual, uniform matroid on an uncountable set that has two bases of different size. This shows that Higgs' result cannot be proved without any additional assumption beyond the usual axioms of set theory, ZFC.

We also answer a question from [43] and show that the statement "all bases of a fixed matroid have the same size" does not imply GCH. Finally, we show that the existence of a self-dual, uniform matroid on a countably infinite set implies the existence of a set of reals without the Baire property. By a result of Shelah [61], the existence of a set of reals without the Baire property cannot be proved in ZF alone, i.e., without the Axiom of Choice.

This chapter is closely based on a joint paper with Stefan Geschke [22].

2.1 Martin's Axiom

We introduce the fragment of Martin's Axiom that we will use in the construction of self-dual, uniform matroids. The set-theoretic background and in particular the proof of the consistency of full Martin's Axiom with $\neg$CH can be found in either [44] or [48].

Let $(\mathbb{P}, \leq)$ be a partial order. For $p, q \in \mathbb{P}$ we say that p *extends* q if $p \leq q$. $F \subseteq \mathbb{P}$ is a *filter* if any two elements of F have a common extension in F and for all $p \in F$ and $q \in \mathbb{P}$ with $p \leq q$, $q \in F$. A set $D \subseteq \mathbb{P}$ is *dense* in $\mathbb{P}$ if every $p \in \mathbb{P}$ has an extension in D. A filter $F \subseteq \mathbb{P}$ is *generic* for a family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ if F has a nonempty intersection with every $D \in \mathcal{D}$.

Given a partial order $\mathbb{P}$, MA($\mathbb{P}$) is the statement that for every family $\mathcal{D}$ of size $< 2^{\aleph_0}$ of dense subsets of $\mathbb{P}$ there is a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$. For every partial order $\mathbb{P}$ and every countable family $\mathcal{D}$ of dense subsets of $\mathbb{P}$ there is a $\mathcal{D}$-generic filter $F \subseteq \mathbb{P}$. This is the Rasiowa-Sikorski Theorem. Hence for every partial order $\mathbb{P}$, MA($\mathbb{P}$) follows from the Continuum Hypothesis (CH, $2^{\aleph_0} = \aleph_1$).

We will be interested in partial orders of the following form:

For a cardinal κ let $\mathrm{Fn}(\kappa, 2)$ denote the set

$$\{p : \text{there is a finite set } A \subseteq \kappa \text{ such that } p : A \to \{0,1\}\}$$

ordered by reverse inclusion. For all infinite cardinals κ and λ with $\kappa \leq \lambda$, MA($\mathrm{Fn}(\kappa, 2)$) follows from MA($\mathrm{Fn}(\lambda, 2)$). The statement MA($\mathrm{Fn}(\aleph_0, 2)$) is usually denoted by MA(`countable`). *Martin's Axiom* is the statement that MA($\mathbb{P}$) holds for all partial orders $\mathbb{P}$ that satisfy the so called *countable chain condition* (c.c.c.). For all infinite cardinals κ, $\mathrm{Fn}(\kappa, 2)$ satisfies the c.c.c.

Gödel showed that if the usual system of axioms for set theory, ZFC, is consistent, then so is ZFC together with CH. Of course, we have no reason to doubt the consistency of ZFC and, following usual practice, assume it throughout the whole article. Solovay and Tennenbaum constructed a model of ZFC that satisfies both MA and $2^{\aleph_0} = \aleph_2$. Also, it is known that MA implies $2^{\aleph_0} = 2^{\aleph_1}$. It follows that the statement

$$\mathrm{MA}(\mathrm{Fn}(\aleph_1, 2)) \wedge 2^{\aleph_1} = 2^{\aleph_0} = \aleph_2$$

is consistent with ZFC.

2.2 The construction

Fix an infinite set E. Since we want to construct a self-dual matroid, we want to talk about subsets of E and their complements at the same time. In other words, we consider partitions of E into two classes A^0 and A^1. The following lemma isolates the combinatorial effect of MA($\mathrm{Fn}(\kappa, 2)$) that we will use in our construction of a uniform matroid.

Lemma 2.2.1. *Let κ be an infinite cardinal $< 2^{\aleph_0}$ and let $\mathcal{S} \subseteq \mathcal{P}(\kappa)$ be a family of infinite sets such that $|\mathcal{S}| < 2^{\aleph_0}$. Then MA*($\mathrm{Fn}(\kappa, 2)$) *implies that there is a* *partition $\{A^0, A^1\}$ of κ such that for all $S \in \mathcal{S}$ the sets $S \cap A^0$ and $S \cap A^1$ are infinite.*

Proof. For all infinite sets $S \subseteq \kappa$, all finite sets $F \subseteq \kappa$, and all $i \in \{0,1\}$ let

$$D^i_F(S) = \{p \in \mathrm{Fn}(\kappa, 1) : \exists m \in S \setminus F(p(m) = i)\}.$$

It is easily checked that the sets $D^i_F(S)$ are dense subsets of $\mathrm{Fn}(\kappa)$.

We may assume that $\kappa \in \mathcal{S}$. Let

$$\mathcal{D} = \{D^0_F(S) : S \in \mathcal{S} \wedge F \in [\kappa]^{<\aleph_0}\} \cup \{D^1_F(S) : S \in \mathcal{S} \wedge F \in [\kappa]^{\aleph_0}\}.$$

Since $\kappa < 2^{\aleph_0}$, $|[\kappa]^{<\aleph_0}| < 2^{\aleph_0}$. Hence, by MA(Fn($\kappa$, 2)) there is a $\mathcal{D}$-generic filter $G \subseteq \mathrm{Fn}(\kappa, 2)$. Let $x = \bigcup G$. Since G is a filter, x is a function. For $i \in \{0,1\}$ let $A = x^{-1}(i)$. By the choice of G and $\mathcal{D}$, A^0 and A^1 have an infinite intersection with all $S \in \mathcal{S}$. $\square$

We call two partitions $\{A^0, A^1\}$ and $\{B^0, B^1\}$ of E *independent* if the sets $A^i \cap B^j$, $i, j \in \{0,1\}$, are all nonempty. We define the equivalence relation $\sim$ on partitions of E into two classes in the natural way:

$$\{A^0, A^1\} \sim \{B^0, B^1\} \text{ iff for some } i \in \{0,1\},\ A^0 \sim B^i.$$

Note that two partitions P and P' with $P \sim P'$ are independent unless they are equal.

Lemma 2.2.2. *Suppose $2^{|E|} = 2^{\aleph_0}$. Then MA(Fn($|E|$, 2)) implies that there is a set $\mathcal{P}$ of partitions of E into two infinite classes with the following properties:*

1. *$\mathcal{P}$ is closed under the equivalence relation $\sim$.*
2. *The elements of $\mathcal{P}$ are pairwise independent.*
3. *Whenever $I^0, I^1 \subseteq E$ are disjoint with $E \setminus (I^0 \cup I^1)$ infinite, then there is a partition $\{B^0, B^1\} \in \mathcal{P}$ such that one of the following holds:*
   - *(i)* $B^0 \subseteq I^0$
   - *(ii)* $I^0 \subseteq B^0$ *and* $I^1 \subseteq B^1$.
   - *(iii)* $B^1 \subseteq I^1$

Proof. Let $((I^0_\alpha, I^1_\alpha))_{\alpha < 2^{\aleph_0}}$ be an enumeration of all pairs (I^0, I^1) of subsets of E with $I^0 \cap I^1 = \emptyset$ and $E \setminus (I^0 \cup I^1)$ infinite. We recursively choose partitions $P_\alpha = \{B^0_\alpha, B^1_\alpha\}$, $\alpha < 2^{\aleph_0}$, of E into infinite sets.

Suppose that for some $\alpha < 2^{\aleph_0}$ for all $\beta < \alpha$, P_β has been chosen. Let $\mathcal{P}_\alpha$ denote the closure of the family $\{P_\beta : \beta < \alpha\}$ under $\sim$ and let

$$\mathcal{B}_\alpha = \{B \subseteq E : \exists P \in \mathcal{P}_\alpha (B \in P)\}.$$

We distinguish two cases:

Case 1. There is a partition $\{B^0, B^1\} \in \mathcal{P}_\alpha$ such that one of the following holds:

(i) $B^0 \subseteq I^0_\alpha$

(ii) $I^0_\alpha \subseteq B^0$ and $I^1_\alpha \subseteq B^1$.

(iii) $B^1 \subseteq I^1_\alpha$

In this case let

$$\mathcal{S} = \{B^i : i \in \{0,1\} \wedge \{B^0, B^1\} \in \mathcal{P}_\alpha\} \cup \{E\}.$$

By Lemma 2.2.1, there is a partition $\{B^0_\alpha, B^1_\alpha\}$ of E such that for all $S \in \mathcal{S}$ the sets $S \cap B^0_\alpha$ and $S \cap B^1_\alpha$ are infinite. Since $\mathcal{S}$ is nonempty, B^0_α and B^1_α are both infinite. By the choice of $\mathcal{S}$, $\{B^0_\alpha, B^1_\alpha\}$ is independent of every partition $\{B^0, B^1\} \in \mathcal{P}_\alpha$.

Case 2. There is no partition $\{B^0, B^1\} \in \mathcal{P}_\alpha$ as in Case 1.

We construct a partition $B_\alpha = \{B^0_\alpha, B^1_\alpha\}$ of E such that $I^0_\alpha \subseteq B^0_\alpha$, $I^1_\alpha \subseteq B^1_\alpha$, and B_α is independent of all $B \in \mathcal{P}_\alpha$.

Let $\{A^0, A^1\}$ be a partition of E such that $I^0_\alpha \subseteq A^0$ and $I^1_\alpha \subseteq A^1$ and let $\{B^0, B^1\} \in \mathcal{B}_\alpha$. If for some $i, j \in \{0,1\}$, B^i intersects I^j_α, then $B^i \cap A^j \neq \emptyset$. It follows that for $\{B^0_\alpha, B^1_\alpha\}$ to be independent of all $P \in \mathcal{P}_\alpha$, we have to make sure that for all $i \in \{0,1\}$ and all $B \in \mathcal{B}_\alpha$, if $B \cap I^i_\alpha = \emptyset$, then $B \cap B^i_\alpha \neq \emptyset$.

Claim 2.2.3. *Suppose for some $i \in \{0,1\}$, $B \in \mathcal{B}_\alpha$ is disjoint from I^i_α. Then $B \setminus I^{1-i}_\alpha$ is infinite.*

For the proof of the claim assume that $B \setminus I^{1-i}_\alpha$ is finite. If $|I^{1-i}_\alpha \setminus B|$ is finite, then, since $E \setminus (I^0_\alpha \cup I^1_\alpha)$ is infinite, there is $B' \sim B$ such that $I^{1-i}_\alpha \subseteq B'$ and $B' \cap I^i_\alpha = \emptyset$. Since $\mathcal{B}_\alpha$ is closed under $\sim$, $B \in \mathcal{B}_\alpha$. Now the partition $\{B', E \setminus B'\} \in \mathcal{P}_\alpha$ contradicts the fact that we are in Case 2.

It follows that $|I^{1-i}_\alpha \setminus B|$ is infinite. Hence there is $B' \sim B$ such that $B' \subseteq I^{1-i}_\alpha$. As before, $B' \in \mathcal{B}_\alpha$. Again the partition $\{B', E \setminus B'\}$ contradicts the fact that we are in Case 2. This finishes the proof of the claim.

Let

$$\mathcal{S} = \{B \setminus I^{1-i}_\alpha : B \in \mathcal{B}_\alpha \wedge i \in \{0,1\} \wedge B \cap I^i_\alpha = \emptyset\}.$$

Then by the claim, all elements of $\mathcal{S}$ are infinite subsets of $E \setminus (I^0_\alpha \cup I^1_\alpha)$. Also, $|\mathcal{S}| < 2^{\aleph_0}$.

By Lemma 2.2.1, there is a partition $\{A^0, A^1\}$ of $E \setminus (I^0_\alpha \cup I^1_\alpha)$ such that A^0 and A^1 have an infinite intersection with all elements of $\mathcal{S}$. For $i \in \{0,1\}$ let $B^i_\alpha = I^i_\alpha \cup A_i$. By the previous discussion and by the choice of $\mathcal{S}$, the partition $\{B^0_\alpha, B^1_\alpha\}$ of E is independent of all the partitions in $\mathcal{P}_\alpha$.

This finishes the recursive construction of the partitions $\{B^0_\alpha, B^1_\alpha\}$. Observe that since the $\mathcal{P}_\alpha$ are closed under $\sim$ and $\{B^0_\alpha, B^1_\alpha\}$ is independent of all partitions in $\mathcal{P}_\alpha$, also every partition $\{B^0, B^1\} \sim \{B^0_\alpha, B^1_\alpha\}$ is independent of all partitions in $\mathcal{P}_\alpha$. It follows that with our choice of $\{B^0_\alpha, B^1_\alpha\}$, $\mathcal{P}_{\alpha+1}$ consists of pairwise independent partitions if $\mathcal{P}_\alpha$ does.

Finally let $\mathcal{P} = \bigcup_{\alpha < 2^{\aleph_0}} \mathcal{P}_\alpha$. It is clear that $\mathcal{P}$ is closed under $\sim$. By the previous discussion, the elements of $\mathcal{P}$ are pairwise independent. If $I^0, I^1 \subseteq E$ are disjoint and such that $E \setminus (I^0 \cup I^1)$ is infinite, then there is $\alpha < 2^{\aleph_0}$ such that $I^0 = I^0_\alpha$ and $I^1 = I^1_\alpha$. If we were in Case 1 in step α of the construction, then there was a partition $\{B^0, B^1\} \in \mathcal{P}_\alpha$ witnessing (3) for I^0_α and I^1_α. If we were in Case 2, then $\{B^0_\alpha, B^1_\alpha\}$ witnesses (3) for I^0_α and I^1_α. It follows that $\mathcal{P}$ satisfies the conditions (1)–(3). $\square$

Lemma 2.2.4. *If $\mathcal{P}$ is a set of partitions of E into two infinite classes such that (1)–(3) of Lemma 2.2.2 are satisfied, then $\mathcal{B} = \{B \subseteq E : \{B, E \setminus B\} \in \mathcal{P}\}$ is the set of bases of a uniform, self-dual matroid on E that is not finitary.*

Proof. By (3), there is a Partition $P \in \mathcal{P}$. It follows that $\mathcal{B}$ is nonempty. Since $\mathcal{P}$ is closed under $\sim$, $\mathcal{B}$ is closed under $\sim$. This shows that $\mathcal{B}$ satisfies (U) and hence (B2). We show that $\mathcal{B}$ satisfies (BM).

Let $I, X \subseteq E$ be such that $I \subseteq X$. If $X \setminus I$ is finite, then $\{B \cap X : B \in \mathcal{B} \wedge I \subseteq B\}$ is finite and therefore has a maximal element. If $X \setminus I$ is infinite, then let $I^0 = I$ and $I^1 = E \setminus X$. By (3) there is a partition $\{B^0, B^1\} \in \mathcal{P}$ such that one of (i)–(iii) holds. If $B^0 \subseteq I^0$, then $\{B \cap X : B \in \mathcal{B} \wedge I \subseteq B\} = \{\emptyset\}$ has the maximal element $\emptyset$. If $B^1 \subseteq I^1$, then $X \subseteq B^0$ and hence $\{B \cap X : B \in \mathcal{B} \wedge I \subseteq B\}$ has the maximal element X. Now assume that $I^0 \subseteq B^0$ and $I^1 \subseteq B^1$. In this case $I \subseteq B^0 \subseteq X$. Since the partitions in $\mathcal{P}$ are pairwise independent there is no $B \in \mathcal{B}$ such that $B^0 \subsetneq B$. It follows that B^0 is a maximal element of $\{B \cap X : B \in \mathcal{B} \wedge I \subseteq B\}$.

This this finishes the proof that $\mathcal{B}$ is the set of bases of a uniform matroid. Since $\mathcal{B}$ is closed under complementation, this matroid is self-dual. Now let $B \in \mathcal{B}$. Then B is infinite and has an infinite complement. For $x \in E \setminus B$ the set $B + x$ is dependent. However, since $\mathcal{B}$ is closed under $\sim$, removing any element of $B + x$ yields an element of $\mathcal{B}$. It follows that no finite subset of $B + x$ is dependent. Hence $\mathcal{B}$ is the set of bases of a matroid that is not finitary. $\square$

Theorem 2.2.5. *a) CH implies the existence of a uniform, self-dual matroid on a countable set that is not finitary.*

b) The existence of a uniform, self-dual matroid on a countable set that is not finitary is consistent with an arbitrarily large value of $2^{\aleph_0}$.

c) It is consistent that there is a uniform, self-dual matroid on an uncountable set that has one basis of size $\aleph_0$ and another basis of size $\aleph_1$.

Proof. By Lemma 2.2.2 together with Lemma 2.2.4, MA(`countable`) implies the existence of a uniform, self-dual matroid on a countable set that is neither finitary nor cofinitary. But MA(`countable`) follows from CH. This shows a). Also, MA(`countable`) is consistent with arbitrarily large values of $2^{\aleph_0}$. This implies b).

For c) let E be a set of size $\aleph_1$. We modify the construction in Lemma 2.2.2 a little bit. We may assume that the enumeration $((I^0_\alpha, I^1_\alpha))_{\alpha < 2^{\aleph_0}}$ is chosen so that $(I^0_0, I^1_0) = (\emptyset, \emptyset)$. Now choose a partition of E into a countably infinite set B^0_0 and a set B^1_0 of size $\aleph_1$. We continue the construction as in the proof of Lemma 2.2.2 and obtain a set $\mathcal{P}$ of partitions of E into two infinite classes satisfying (1)–(3). Now the set $\mathcal{B} = \{B \subseteq E : \exists P \in \mathcal{P}(B \in \mathcal{P})\}$ is the set of bases of a uniform, self-dual matroid on E and one basis, B^0_0, is countable, while another basis, B^1_0, is of size $\aleph_1$. $\square$

In [42], Higgs showed that the Generalized Continuum Hypothesis (GCH) implies that any two bases of a matroid have the same size. Together with Theorem 2.2.5 we get the following corollary:

Corollary 2.2.6. *Whether or not any two bases of a matroid have the same size cannot be decided in ZFC alone.*

2.3 Two questions of Higgs

We continue our discussion of Higgs' result about the equicardinality of bases under GCH. Higgs actually proved the following stronger statement:

Theorem 2.3.1 ([42]). *Assume GCH. Let E be a set and $\mathcal{B} \subseteq \mathcal{P}(E)$ be a family of sets such that*

(i) no one member of $\mathcal{B}$ is properly contained in another, and

(ii) if $B_1, B_2 \in \mathcal{B}$ and $I, X \subseteq E$ are such that $I \subseteq X$, $I \subseteq B_1$, and $B_2 \subseteq X$, then there is $B \in \mathcal{B}$ such that $I \subseteq B \subseteq X$.

Then the members of $\mathcal{B}$ all have the same cardinality.

It is easily checked that matroids satisfy (i) and (ii). Hence the equicardinality of bases of matroids under GCH follows from Theorem 2.3.1. Higgs asked whether the conclusion of Theorem 2.3.1 implies GCH.

The proof of Theorem 2.3.1 in [42] uses two different consequences of GCH:

1. The continuum function $\kappa \mapsto 2^\kappa$ is 1-1 on infinite cardinals.

2. For every infinite cardinal κ, the partial order $(\mathcal{P}(\kappa), \subseteq)$ has a chain of size 2^κ.

Theorem 2.3.2. *If ZFC is consistent then so is ZFC together with the statements (1) and (2) above and the negation of CH.*

Proof. We use Easton forcing (see [44, Theorem 15.18]) over a model of GCH to obtain a model of ZFC in which for each $n \in \mathbb{N}$, $2^{\aleph_n} = \aleph_{n+2}$. This can be done by a forcing of size $2^{\aleph_\omega} = \aleph_\omega^+$. This forcing does not change the size of 2^κ for any $\kappa \geq \aleph_\omega$. It follows that the continuum function is 1-1 in the resulting model.

We now work inside this forcing extension. Baumgartner and Mitchell independently showed that $\mathcal{P}(\kappa)$ contains a chain of length 2^κ iff there is a linear order of size 2^κ that has a dense subset of size κ [9, Theorem 2.1]. From [9, Theorem 3.5] together with [9, Theorem 2.2] it follows that if $2^{\aleph_n} = \aleph_{n+2}$ for all $n \in \mathbb{N}$, then for all $n \in \omega$ there is a linear order of size $2^{\aleph_n}$ with a dense subset of size $\aleph_n$. In our model GCH holds from $\aleph_\omega$ on. Moreover, $\aleph_\omega$ is the least cardinal μ such that $\aleph_\omega < 2^{\aleph_\omega}$. Now [9, Corollary 2.4] shows that for all $\kappa \geq \aleph_\omega$ there is a linear order of size 2^κ and density κ. It follows that for all infinite κ, $\mathcal{P}(\kappa)$ contains a chain of length 2^κ. $\square$

Corollary 2.3.3. *If ZFC is consistent, then the conclusion of Theorem 2.3.1 does not imply GCH.*

In [42], Higgs also asked whether every nonempty family $\mathcal{B} \subseteq \mathcal{P}(E)$ that satisfies (i) and (ii) in Theorem 2.3.1 is the set of bases of a matroid on E. We show that this is not the case in general, but it is true if E is finite.

Theorem 2.3.4. *a) There is a nonempty family $\mathcal{B}$ of subsets of a countably infinite set E satisfying (i) and (ii) in Theorem 2.3.1 such that $\mathcal{B}$ is not the set of bases of a matroid.*

b) If E is finite and $\mathcal{B} \subseteq \mathcal{P}(E)$ is nonempty and satisfies (i) and (ii) in Theorem 2.3.1, then $\mathcal{B}$ is the set of bases of a matroid on E.

Proof. a) Let E be a countably infinite set. Let $A \subseteq E$ be an infinite, co-infinite set and let $\mathcal{B}$ be the $\sim$-class of A. Then $\mathcal{B}$ is not the set of bases of a matroid.

Namely, let $I = \emptyset$ and let $X \subseteq E$ be such that both $X \cap B$ and $B \setminus X$ are infinite. Then for all $B' \in \mathcal{B}$, $(B' \setminus B) \cap X$ is finite and for all finite sets $F \subseteq X \setminus B$ there is $B' \in \mathcal{B}$ with $B' \cap X = (X \cap B) \cup F$. In particular, the set $\{X \cap B' : B' \in \mathcal{B}\}$ does not have a maximal element and hence $\mathcal{B}$ does not satisfy (BM).

On the other hand, $\mathcal{B}$ satisfies (i) and (ii) in Theorem 2.3.1. This can be seen as follows:

Let $I \subseteq X \subseteq E$ be such that for some $B_1, B_2 \in \mathcal{B}$ we have $I \subseteq B_1$ and $B_2 \subseteq X$. Since $B_1 \sim B_2$, the sets $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are finite and of the same size. We have $B_1 \setminus X \subseteq B_1 \setminus B_2$ and $B_2 \setminus B_1 \subseteq X \setminus I$. Let $C = B_1 \setminus X$ and let $D \subseteq B_2 \setminus B_1$ be a set of size $|B_1 \setminus X|$. Now let $B = (B_1 \setminus C) \cup D$. We have $B \sim B_1$ and therefore $B \in \mathcal{B}$. Also, $I \subseteq B \subseteq X$.

b) We have to show that $\mathcal{B}$ satisfies (B2). Let $B_1, B_2 \in \mathcal{B}$ and suppose $x \in B_1 \setminus B_2$. We first show that there is a set $Y \subseteq B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup Y$. Let $I = B_1 \setminus \{x\}$ and $X = I \cup B_2$. By our assumptions on $\mathcal{B}$ there is $B \in \mathcal{B}$ such that $I \subseteq B \subseteq X$. Let $Y = B \setminus B_1$. Now $B = (B_1 \setminus \{x\}) \cup Y$ and $Y \subseteq B_2$.

We show that Y is a singleton. Since no two elements of $\mathcal{B}$ are properly contained in another, Y is nonempty. Let $y \in Y$. By the same argument as before, there is a set $X \subseteq B_1 \setminus B$ such that $(B \setminus \{y\}) \cup X \in \mathcal{B}$. Since $B_1 \setminus B = \{x\}$, we must have $X = \{x\}$. Now $B \setminus \{y\} \cup \{x\}$ and $B_1 = B \setminus Y \cup \{x\}$ are both in $\mathcal{B}$. Since no two elements of $\mathcal{B}$ are properly contained in another, $Y = \{y\}$. This finishes the proof of the theorem. $\square$

2.4 The complexity of self-dual uniform matroids

In this section we work in ZF. The background in descriptive set theory used in this section can be found in either [44] or [46].

Flutters were introduced and studied by Delhommé, Mathias, and Morillon [51]. They can be constructed in ZFC, but their existence does not follow from ZF alone. We consider a notion that is formally slightly weaker than that of a 2-flutter.

Definition 2.4.1. A $(\sim, 2)$-*flutter* is a set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ that is closed under $\sim$ and has the property that for each $A \subseteq \mathbb{N}$, exactly one of the sets A and $\mathbb{N} \setminus A$ is a member of $\mathcal{A}$.

We translate the notion of a $(\sim, 2)$-flutter into a more topological setting. Instead of $\mathcal{P}(\mathbb{N})$ we consider the Cantor space $\mathcal{C} = \{0,1\}^{\mathbb{N}}$. $\mathcal{C}$ Each set $A \in \mathcal{P}(\mathbb{N})$ corresponds to its characteristic function in $\mathcal{C}$. The relation $\sim$ translates to an equivalence relation on $\mathcal{C}$, also denoted by $\sim$, where for all $x, y \in \mathcal{C}$ we have $x \sim y$ iff the sets $\{n \in \mathbb{N} : x(n) \neq y(n) \wedge x(n) = 0\}$ and $\{n \in \mathbb{N} : x(n) \neq y(n) \wedge x(n) = 1\}$ are finite and of the same size.

We also consider the equivalence relation Comp on $\mathcal{C}$ that identifies every function $x \in \mathcal{C}$ with the function $\overline{x} : \mathbb{N} \to \{0,1\}; n \mapsto 1 - x(n)$. In this setting, a $(\sim, 2)$-flutter is a subset of $\mathcal{C}$ that intersects each Comp-class in exactly one element and is closed under $\sim$.

Definition 2.4.2. Recall that the topology on $\mathcal{C}$ is generated by the sets

$$[s] = \{x \in \mathcal{C} : s \subseteq x\},$$

where $s : S \to \{0,1\}$ for some finite set $S \subseteq \mathbb{N}$. This topology is compatible with a complete metric.

If X is any complete metric space, A subset N of X is *nowhere dense* if its closure has empty interior. A subset of M of X is *meager* if it is a countable union of nowhere dense sets. Finally, a subset A of X has the *Baire property* if there is an open set $O \subseteq X$ such that the symmetric difference $A \triangle O$ is meager.

The Baire category theorem implies that no nonempty open subset of $\mathcal{C}$ is meager. In particular, no nonempty open subset of $\mathcal{C}$ is the union of two meager sets. In other words, if $O \subseteq \mathcal{C}$ is open and nonempty and $A_0, A_1 \subseteq O$ are *comeager* in O, i.e., have a meager complement relative to O, then $A_0 \cap A_1 \neq \emptyset$. Also, the family of sets with the Baire property is closed under complementation.

Theorem 2.4.3. *A $(\sim, 2)$-flutter on the Cantor space $\mathcal{C}$ does not have the Baire property.*

Proof. Let $X \subseteq \mathcal{C}$ be a $(\sim, 2)$-flutter and suppose that X has the Baire property. Now $\mathcal{C} \setminus X$ is a $(\sim, 2)$-flutter as well and has the Baire property. At most one of X and $\mathcal{C} \setminus X$ is meager. Hence we may assume that X is not meager.

Since X has the Baire property, there is an open set $O \subseteq \mathcal{C}$ such that $X \triangle O$ is meager. Since X is not meager, O is nonempty. Hence there is a finite set $S \subseteq \mathbb{N}$ and a function $s : S \to \{0,1\}$ such that $[s] \subseteq O$.

Choose an extension t of s to some finite subset T of $\mathbb{N}$ such that $t^{-1}(0)$ and $t^{-1}(1)$ have the same size. Let n be the minimal element of $\mathbb{N} \setminus \mathrm{dom}(t)$. Let $t_0 = t \cup \{(n, 0)\}$ and $t_1 = t \cup \{(n, 1)\}$. For each $x \in [t_0]$ let $h(x) \in [t_1]$ be defined by letting $h(x) \restriction \mathrm{dom}(t) = x \restriction t$ and $h(x) \restriction (\mathbb{N} \setminus \mathrm{dom}(t)) = \overline{x} \restriction (\mathbb{N} \setminus \mathrm{dom}(t))$. The map $h : [t_0] \to [t_1]$ is a homeomorphism. Since the set $[t_0] \cap X$ is comeager in $[t_0]$, $h[[t_0] \cap X]$ is comeager in $[t_1]$. Also, $[t_1] \cap X$ is comeager in $[t_1]$. Hence there is $x \in [t_0] \cap X$ such that $h(x) \in X$. Since $t^{-1}(0)$ and $t^{-1}(1)$ are of the same size, $h(x) \sim \overline{x}$.

Since X is closed under $\sim$, $\overline{x} \in X$. Hence X contains both x and $\overline{x}$. Therefore X is not a $(\sim, 2)$-flutter, a contradiction. $\square$

We will show that every uniform, self-dual matroid on a countable set gives rise to a $(\sim, 2)$-flutter. First we observe the following:

Lemma 2.4.4. *Let $\mathcal{M}$ be a uniform matroid on a set E. Then every dependent subset of E contains a basis.*

Proof. Let $X \subseteq E$ be dependent and let $\mathcal{B}$ be the set of bases of $\mathcal{M}$. By (BM), the family $\{X \cap B : B \in \mathcal{B}\}$ has a maximal element A. If $X = A$, then X is independent, contradicting our assumption on X. Hence $X \neq A$. Let $B \in \mathcal{B}$ be such that $A = B \cap X$.

We have to show that $B \subseteq X$. Suppose not. Then there are $x \in B \setminus X$ and $y \in X \setminus B$. By (U), $(B \setminus \{x\}) \cup \{y\} \in \mathcal{B}$. But now $B \cap X \subsetneq ((B \setminus \{x\}) \cup \{y\}) \cap X$, contradicting the fact that $A = B \cap X$ is maximal in $\{X \cap B : B \in \mathcal{B}\}$.

This shows that X contains a basis and finishes the proof of the lemma. $\square$

Theorem 2.4.5. *If there is a uniform, self-dual matroid on a countable set, then there is a $(\sim, 2)$-flutter.*

Proof. Let $\mathcal{M}$ be a uniform, self-dual matroid on $E = \mathbb{N} \cup \{\infty\}$. Let

$$\mathcal{A} = \{A \subseteq \mathbb{N} : A \text{ contains a basis of } \mathcal{M}\}.$$

Since the set of bases of $\mathcal{M}$ is closed under $\sim$, so is $\mathcal{A}$.

Now let $\{A_0, A_1\}$ be a partition of $\mathbb{N}$. If A_0 is dependent, then there is a basis $B \subseteq A_0$. Now $E \setminus B$ is a basis as well. Hence there is no basis contained in A_1 as A_1 is a proper subset of $E \setminus B$.

If A_0 is independent, then there is a basis B with $A_0 \subseteq B$. Now $E \setminus B$ is a basis. If $A_0 = B$, then A_1 is properly contained in the basis $E \setminus B$ and thus A_1 does not contain a basis. If $A_0 \neq B$, then $E \setminus B$ is a proper subset of $A_1 \cup \{\infty\}$. There is $B' \sim E \setminus B$ such that $\infty \notin B'$ and $B' \subseteq A_1 \cup \{\infty\}$. Now B' is a basis that is contained in A_1.

It follows that exactly one of A_0 and A_1 contains a basis. Hence $\mathcal{A}$ is a $(\sim, 2)$-flutter. $\square$

Corollary 2.4.6. *The existence of a uniform, self-dual matroid on a countable set is not provable in ZF+DC.*

Proof. If there is a uniform, self-dual matroid on a countable set then there is a subset of $\mathcal{C}$ that does not have the Baire property. However, in [61] Shelah proved that if ZF is consistent, then there is a model of ZF+DC where every subset of $\mathcal{C}$ has the Baire property. In this model there is no uniform, self-dual matroid on a countable set. $\square$

A subset A of a Hausdorff space X is *analytic* if it is the continuous image of a Borel subset of a complete metric space. Analytic subsets of complete metric spaces have the Baire property. For any countably infinite set E we can identify $\mathcal{P}(E)$ with $\mathcal{C}$ and then we know when a set $\mathcal{A} \subseteq \mathcal{P}(E)$ is analytic.

Corollary 2.4.7. *The set of bases of a uniform, self-dual matroid on a countably infinite set E is not analytic.*

Proof. Let $\mathcal{B}$ be the set of bases of a uniform, self-dual matroid on E. As in Theorem 2.4.5 we can assume $E = \mathbb{N} \cup \{\infty\}$. In the proof of Theorem 2.4.5 we defined

$$\mathcal{A} = \{A \subseteq \mathbb{N} : \exists B \in \mathcal{B}(B \subseteq A)\}.$$

Now assume that $\mathcal{B}$ is analytic in $\mathcal{P}(E)$. Then the set

$$\{(A, B) : A \subseteq \mathbb{N} \wedge B \in \mathcal{B} \wedge B \subseteq A\}$$

is an analytic subset of $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(E)$. The set $\mathcal{A}$ is the projection of this set to the first coordinate and hence analytic. But by Theorem 2.4.5 and Theorem 2.4.3, $\mathcal{A}$ does not have the Baire property and hence is not analytic, a contradiction. It follows that $\mathcal{B}$ is not analytic. $\square$

Chapter 3

Representability

The question addressed by this chapter is that of how to extend the notion of representability over a field from finitary to non-finitary matroids. Recall from the introduction that if we have a (possibly infinite) family of vectors in a vector space over some field k, we get a finitary matroid structure on that family whose independent sets are given by the linearly independent subsets of the family. Matroids arising in this way are called *finitary representable* matroids over k.

Although many interesting finite matroids (eg. all graphic matroids) are representable, many interesting examples of infinite matroids cannot be of this type, because they are not finitary. Another problem is that in restricting attention to finitary matroids we would once more lose the power of duality: if a finite matroid is representable over the field k then so is its dual, but the dual of an infinite matroid representable over k need not be finitary. So there is a question here akin to Rado's question:

Question 3.0.8. *Is there a good theory of infinite matroids representable over k with duality?*

Bruhn and Diestel explored one approach to this question in [27]. They tried extending the notion of linear combinations to allow for infinite combinations in certain constrained circumstances.

The construction relies on taking the vector space to be of the form k^A for some set A. We allow linear combinations of infinitely many vectors. However, we require these linear combinations to be well defined pointwise. This means that for each $a \in A$ there are only finitely many nonzero coefficients at vectors with nonzero component at a. Further details are given in Section 3.1. Sadly, it turns out that there are examples of systems of independent sets definable in this way which are not matroids. Accordingly, we refer to such systems in general as *thin sums systems*, and only call them thin sums matroids if they really are matroids. Thin sums matroids need not be finitary.

Because thin sums systems are often not matroids, Bruhn and Diestel focused on a class of thin sums systems which they were able to show are matroids [27], namely those generated from families of vectors in which for each $a \in A$ there are only finitely many vectors in the family whose component at a is nonzero. Such families are called *thin*. We give a characterisation of the matroids arising in this way.

Theorem 3.0.9. *A matroid arises as a thin sums matroid over a thin family for the field k if and only if its dual is finitary and representable over k.*

This theorem can also be seen as characterising which matroids can arise as duals of representable matroids.

It follows that the union of the class of finitary representable matroids with the class of thin sums matroids over thin families is closed under duality and under taking minors. However, since many of the motivating examples are not finitary and do not have finitary duals, this union is not as comprehensive as one might hope. Allowing all thin sums systems is too broad, though, as the class of thin sums matroids over $\mathbb{Q}$ is not closed under duality (this is shown in Section 3.5).

In this chapter, we show that Question 3.0.8 can be resolved by restricting to the class of tame thin sums matroids. In contrast to the bad behaviour of thin sums matroids in general, we establish the following foundational result:

Theorem 3.0.10. *The class of tame thin sums matroids over any given field is closed under duality and under taking minors.*

We prove this result by establishing a simple, self-dual characterisation of the class of tame thin sums matroids.

We are also able to show that tame thin sums matroids have close links to another approach to the construction of infinite representable matroids: matroids with coefficients, introduced by Dress [37]. Dress worked with a notion of representability with respect to more general objects, called *fuzzy rings*. The advantage of this approach is that representability over a certain minimal fuzzy ring K^o is just like no representability restrictions at all, and so the theory of infinite matroids with coefficients in K^o just gives a theory of infinite matroids. However, this theory is difficult to work with due to its wealth of necessary technical detail. These technical problems surface, for example, in the fact that contraction and deletion for matroids with coefficients have only been shown to commute in constrained circumstances.

Another feature of this approach is that duality is built into it: every matroid with coefficients over a field k has a dual matroid with coefficients over the same field k. For finite matroids, as one might expect, the duality of matroids with coefficients coincides with the duality of the underlying matroids. But for infinite matroids this is no longer true, as we shall show in Section 3.6. This phenomenon may be seen as a reflection of the greater generality of matroids with coefficients. Tameness is also built in to matroids with coefficients: indeed, every tame matroid gives rise to a matroid with coefficients over K^o. We are able to show that every tame thin sums matroid induces a matroid with coefficients over the same field which has the additional property that the duality with coefficients corresponds to the duality as a matroid.

Any finite graphic matroid is representable over every field. The situation for infinite graphs is a little more complex, in that there is more than one natural way to build a matroid from an infinite graph. We show that all six types of matroid associated to a graph which were introduced in subsection 1.6.3 are thin sums matroids over any field.

As a result of these considerations, we consider that the class of tame thin sums matroids over k gives a good answer to Question 3.0.8, and we shall take this to be our class of *representable* matroids over k.

Many results about representability for finite matroids continue to hold for this class. For example, a classical theorem of Tutte [66] states that a finite matroid is binary (that is, representable over $\mathbb{F}_2$) if and only if it does not have $U_{2,4}$ as a minor. In the same spirit, a key aim of finite matroid theory has been to determine such 'forbidden minor' characterisations for the classes of matroids representable over other finite fields. For example Bixby and Seymour [11, 60] characterized the finite ternary matroids (those representable over $\mathbb{F}_3$) by forbidden minors, and more recently there is a forbidden minors characterisation for the finite matroids representable over $\mathbb{F}_4$, due to Geelen, Gerards and Kapoor [38]. This remains an open problem for all other finite fields.

Minor closed classes of infinite matroids may have infinite 'minimal' forbidden minors. For example the class of finitary matroids has the infinite circuit $U^*_{1,\mathbb{N}}$ as a forbidden minor. Similarly, the class of tame thin sums matroids over $\mathbb{R}$ has $U_{2,\mathcal{P}(\mathbb{R})}$ as a forbidden minor. However we shall show that the class of matroids representable over a fixed *finite* field has only finite minimal forbidden minors.

Theorem 3.0.11. *Let M be a tame matroid and k be a finite field. Then M is a representable over k if and only if all of its finite minors are.*

Theorem 3.0.11 implies that each of the excluded minor characterisations for finite representable matroids mentioned in the first paragraph extends to tame matroids. Thus, for example, a tame matroid is a thin sums matroid over $\mathbb{F}_2$ if and only if it has no $U_{2,4}$ minor. Any future excluded minor characterisations for finite matroids representable over a fixed finite field will also immediately extend to tame matroids by this theorem.

As for finite matroids , we obtain nine equivalent simple characterizations of binary matroids (those representable over $\mathbb{F}_2$). We show that a tame matroid is regular (that is, representable over every field) if and only if all its finite minors are, and that regularity is equivalent to signability for tame matroids (see [70] or [56] for a definition). We also introduce an extension of the notion of uniqueness of representations for finite matroids to the infinite setting, and prove that tame thin sums matroids over $\mathbb{F}_3$ have unique representations in this sense. A consequence of this is that the finite, algebraic and topological cycle matroids of a given graph each have a unique signing. We illustrate how these signings encode some of the structure of the graph. Finally, we show that the characterisations of representability over various sets of finite fields in terms of representability over partial fields extend in a uniform way to infinite tame matroids.

This chapter is closely based on joint papers with Hadi Afzali [12] and Johannes Carmesin [13, 16].

3.1 Thin sums systems

We always use k to denote an arbitrary field. The capital letter V always stands for a vector space over k. For any set A, we write k^A to denote the set of all functions from A to k. For any function $E \xrightarrow{c} k$ the *support* $\mathrm{supp}(c)$ of c is the set of all elements $e \in E$ such that $c(e) \neq 0$. *A linear dependence* of $E \xrightarrow{\phi} V$ is a map $E \xrightarrow{c} k$ of finite support such that

$$\sum_{e\in E} c(e)\phi(e) = 0\,.$$

For a subset E' of E, we say such a c is a *linear dependence* of E' if it is zero outside E'. Recall that representability is traditionally defined for finitary matroids as follows.

Definition 3.1.1. Let V be a vector space. Then for any function $E \xrightarrow{\phi} V$ we get a finitary matroid $M(\phi)$ on the ground set E, where we take a subset E' of E to be independent if there is no nonzero linear dependence of E'. Such a finitary matroid is called *representable*.

Note that this is essentially the same as taking a family of vectors as the ground set and saying that a subfamily of this family is independent if it is linearly independent.

In [27], there is an extension of these ideas to a slightly different context. Suppose now that we have a function $E \xrightarrow{f} k^A$. A *thin dependence* of f is a map $E \xrightarrow{d} k$, not necessarily of finite support, but such that for each $a \in A$,

$$\sum_{e\in E} d(e)f(e)(a) = 0$$

(here, as in the rest of this chapter, we take this statement as including the claim that the sum is well-defined, i.e. that only finitely many summands are nonzero). This is subtly different from the concept of a linear dependence (in k^A considered as a vector space over k), since it is possible that the sum above might be well defined for each particular a in A, but the sum

$$\sum_{e\in E} d(e)f(e)$$

might still not be well defined. To put it another way, there might be infinitely many $e \in E$ such that there is *some* $a \in A$ with $d(e)f(e)(a) \neq 0$, even if there are only finitely many such e for each *particular* $a \in A$. We may also say d is a thin dependence of a subset E' of E if it is zero outside of E'.

The word *thin* above originated in the notion of a *thin family* - this is an f as above such that sums of the type given above are always defined; that is, for each a in A, there are only finitely many $e \in E$ so that $f(e)(a) \neq 0$. Notice that, for any $E \xrightarrow{f} k^A$, and any thin dependence c of f, the restriction of f to the support of c is thin.

Now we may define thin sums systems.

Definition 3.1.2. Consider a family $E \xrightarrow{f} k^A$ of functions and declare a subset of E as independent if there is no nonzero thin dependence of that subset. Let $M_{ts}(f)$ be the set system with ground set E and the set of all independent sets given in this way. We call $M_{ts}(f)$ the *thin sums system* corresponding to f. Whenever $M_{ts}(f)$ is a matroid it is called a thin sums matroid.

Since a set is dependent in a representable finitary matroid or thin sums system if and only if it has a nonzero linear or thin dependence, we normally talk about such dependences instead of dependent sets.

Not every thin sums system is a matroid[1] but it is known that if f is thin then $M_{ts}(f)$ always is a matroid. The existing proof for this is technical and we shall not review it here. However, this fact will follow from the results in Section 3.2. Next we explore the connection between representable and thin sums matroids.

Proposition 3.1.3. *For any thin sums matroid $M_{ts}(f)$, the finitarisation of $M_{ts}(f)$ is a representable finitary matroid.*

Proof. For any family $E \xrightarrow{f} k^A$ of functions, a thin dependence of f with finite support is also a linear dependence of f as a family of vectors, and conversely any linear dependence of f as a family of vectors is a thin dependence of f. $\square$

Now let's try to answer to the question: Which matroids arising from graphs are representable or thin sums matroids? It is easy to see that any algebraic cycle matroid is a thin sums matroid (in fact, this was one motivation for the definition of thin sums matroids). Recall that for any graph G which does not contain a subdivision of the Bean graph, the edge sets of cycles and double rays of G give the circuits of a matroid. Even if G does contain a subdivision of the Bean graph we shall still denote this system of sets by $M_{AC}(G)$, and call it the *algebraic cycle system* of G.

Proposition 3.1.4. *For any graph G the algebraic cycle system of G is a thin sums system over every field.*

Proof. First we give an arbitrary orientation to every edge of G, making G a digraph. For any edge e of G define a function $V(G) \xrightarrow{f(e)} k$ where for any $v \in V(G)$ $f(e)(v)$ is 1 if e originates from v, -1 if it terminates in v, and 0 if e and v are not incident. We show that D is dependent in $M_{AC}(G)$ if and only if it is dependent in $M_{ts}(f)$. If D is dependent in $M_A(G)$, then it contains a

[1]See Section 3.3 for a couple of examples.

cycle or a double ray. Let $D' \subseteq D$ be the edge set of this cycle or double ray. Give a direction to D'. For any edge $e \in D$, define $c(e)$ to be 1 if e is an edge of D' and they have the same directions, -1 if e is in D' and they have different directions, and 0 if $e \notin D'$. Now clearly we have $\sum_{e \in D'} c(e)f(e)(v) = 0$ for any vertex v of G, so c is a thin dependence of D. Conversely if D is dependent in $M_{ts}(f)$, then whenever a vertex v is an end of an edge in D, it has to be the end of at least two edges in D. Now it is not difficult to see that D has to contain a cycle or a double ray. □

An almost identical argument shows that $M_{FC}(G)$ is representable over every field. This is also true of $M_{FB}(G)$:

Proposition 3.1.5. *For any graph G, the matroid $M_{FB}(G)$ is representable over every field k.*

Proof. We start by giving fixed directions to every edge, cycle and finite bond. Let O be the set of all cycles of G and for any edge $e \in E(G)$ define a function $O \xrightarrow{\phi(e)} k$ such that for any $o \in O$, $\phi(e)(o)$ is 1 if $e \in o$ and they have the same directions, -1 if $e \in o$ and they have different directions, and 0 if e isn't an edge of o. This defines a map $E(G) \xrightarrow{\phi} k^O$. We will show $M(\phi) = M_{FB}(G)$.

We need to show that $D \subseteq E(G)$ is dependent in $M_{FB}(G)$ if and only if it is dependent in $M(\phi)$. If D is dependent in $M_{FB}(G)$ then it contains a finite bond D'. For any edge $e \in D'$ define $c(e)$ to be 1 if D' and e have the same directions, and -1 if they have different directions, and 0 if $e \notin D'$. Now consider a fixed cycle o which meets D'. Clearly D' has two sides and this cycle has to traverse D' from the first side to the second side as many times as it traverses D' from the second side to the first. As a result, for any $o \in O$ we have $\sum_{e \in E} c(e)\phi(e)(o) = 0$ and so c is a linear dependence of D.

Conversely, suppose that D is dependent in $M(\phi)$, and let D' be the support of any thin dependence of D. Whenever the edge set of a cycle meets D', they have to meet in at least two edges, which means D' (and so also D) meets every spanning tree. Thus D includes a bond and so it is a dependent set in $M_{FB}(G)$. □

In the above proof, we could exchange the role of finite bonds and arbitrary bonds to show that $M_B(G)$ is a thin sums matroid over any field. We could also exchange the role of finite cycles and arbitrary bonds, and finite bonds and finite cycles, to get another proof of the fact that $M_{FC}(G)$ is representable.

In the next section, we shall prove that duals of representable finitary matroids are alway thin sums matroids, which in particular implies that $M_{TC}(G)$ is a thin sums matroid.

3.2 Representable finitary matroids and thin sums

In this section we elucidate the connections between representable finitary matroids and thin sums matroids. First we show that any representable finitary matroid is a thin sums matroid, so thin sums matroids are a generalisation of representable finitary matroids. After that we will characterise the dual of an arbitrary representable finitary matroid and show that not only is every representable finitary matroid a thin sums matroid but every matroid whose dual is representable and finitary is also a thin sums matroid. In fact, our last result is even stronger; we show that the duals of representable finitary matroids are precisely the thin sums matroids for thin families. Since the finite bond matroid of any graph is both representable and finitary, this implies in particular that its dual, the topological cycle matroid, is a thin sums matroid.

As usual, let V^* be the dual of the vector space V (that is, the vector space consisting of all linear maps from V to k).

Theorem 3.2.1. *Consider a map $E \xrightarrow{\phi} V$ and the representable finitary matroid $M(\phi)$. For any $e \in E$ and $\alpha \in V^*$ define $E \xrightarrow{f} k^{V^*}$ by $f(e)(\alpha) := \alpha(\phi(e))$. Then,*

$$M(\phi) = M_{ts}(f).$$

In particular, $M(\phi)$ is a thin sums matroid.

Proof. We show that I is independent in $M_{ts}(f)$ if and only if I is independent in $M(\phi)$. Suppose that I is independent in $M_{ts}(f)$. Suppose that $E \xrightarrow{c} k$ is any linear dependence of ϕ that is 0 outside I. For any $\alpha \in V^*$ we have,

$$\sum_{e\in E} c(e)f(e)(\alpha) = \sum_{e\in E} c(e)\alpha(\phi(e)) = \alpha\left(\sum_{e\in E} c(e)\phi(e)\right) = 0.$$

Thus c is a thin dependence of f, and since I is independent in $M_{ts}(f)$ we get that c must be the 0 map. So I is also independent in $M(\phi)$.

Conversely, suppose that I is independent in $M(\phi)$. Suppose $E \xrightarrow{c} k$ is any thin dependence of f that is 0 outside I. Let $I' = \mathrm{supp}(c)$. Since $I' \subseteq I$, I' is also independent in $M(\phi)$, so (by extending the image of I' by ϕ to a basis of V) we can define a linear map $V \xrightarrow{\alpha_{I'}} k$ such that for any $i \in I'$, $\alpha_{I'}(\phi(i)) = 1$. As the restriction of f to $I' = \mathrm{supp}(c)$ is thin and for any $i \in I'$ $f(i)(\alpha_{I'}) = \alpha_{I'}(\phi(i)) = 1$, I' has to be finite. So for every $\alpha \in V^*$,

$$\alpha\left(\sum_{e\in E} c(e)\phi(e)\right) = \sum_{e\in E} c(e)\alpha(\phi(e)) = \sum_{e\in E} c(e)f(e)(\alpha) = 0.$$

Since this is true for every $\alpha \in V^*$, we get that $\sum_{e\in I'} c(e)\phi(e) = 0$ which means c must be a linear dependence and so must be 0. Therefore I is also independent in $M_{ts}(f)$. □

Now let's see how we can move from a representable finitary matroid to its dual. Let's start with a family $E \xrightarrow{\phi} V$. Let C_ϕ be the set of all linear dependences of ϕ. We now define a map $E \xrightarrow{\widehat{\phi}} k^{C_\phi}$ by setting $\widehat{\phi}(e)(c) := c(e)$

for any $e \in E$ and $c \in C_\phi$. Clearly $\widehat{\phi}$ is a thin family of functions. On the other hand, if we let D_f be the set of thin dependences of a thin family $E \xrightarrow{f} k^A$, we get a map $E \xrightarrow{\overline{f}} k^{D_f}$ by setting $\overline{f}(e)(d) := d(e)$ for $e \in E$ and $d \in D_f$. These processes are, in a sense, inverse to each other.

Lemma 3.2.2. *For any thin family $E \xrightarrow{f} k^A$, a map $E \xrightarrow{d} k$ is a thin dependence of f if and only if it is a thin dependence of $\widehat{\overline{f}}$.*

Proof. First, suppose that d is a thin dependence of f. Then for any $c \in C_{\overline{f}}$ we have

$$\sum_{e\in E} d(e)\widehat{\overline{f}}(e)(c) = \sum_{e\in E} d(e)c(e) = \sum_{e\in E} c(e)\overline{f}(e)(d) = 0,$$

so d is also a thin dependence of $\widehat{\overline{f}}$.

Now suppose that d is a thin dependence of $\widehat{\overline{f}}$. For any $a \in A$, let $E \xrightarrow{c_a} k$ be defined by the equation $c_a(e) := f(e)(a)$. Since f is thin, $c_a(e)$ is nonzero for only finitely many values of e. Now for any thin dependence d' of f we have

$$\sum_{e\in E} c_a(e)\overline{f}(e)(d') = \sum_{e\in E} c_a(e)d'(e) = \sum_{e\in E} d'(e)f(e)(a) = 0,$$

and so $c_a \in C_{\overline{f}}$. Now, since d is a thin dependence of $\widehat{\overline{f}}$, we have

$$\sum_{e\in E} d(e)f(e)(a) = \sum_{e\in E} d(e)c_a(e) = \sum_{e\in E} d(e)\widehat{\overline{f}}(e)(c_a) = 0.$$

Since a was arbitrary, this says exactly that d is a thin dependence of f. $\square$

An analogous argument shows that for any map $E \xrightarrow{\phi} V$, the linear dependences of $\overline{\widehat{\phi}}$ are exactly those of ϕ. We can also show that these inverse processes correspond to duality of matroids.

Theorem 3.2.3. *For any map $E \xrightarrow{\phi} V$ we have,*

$$M^*(\phi) = M_{ts}(\widehat{\phi}).$$

Proof. Suppose we have a set E_1 which is dependent in the dual of $M(\phi)$: that is, it meets every base of $M(\phi)$. Let $E_2 = E \setminus E_1$, so E_2 doesn't include any base of E - that is, E_2 doesn't span this matroid. Thus we can pick $e_1 \in E_1$ such that $\phi(e_1)$ isn't in the linear span of the family $(\phi(e)|e \in E_2)$. Consider a basis B_2 for this linear span, and extend $B_2 + \phi(e_1)$ to a basis B for V, and define a map $B \xrightarrow{h_0} k$ such that $h_0(\phi(e_1)) := 1$, and otherwise 0. Finally, extend h_0 to a linear map $V \xrightarrow{h} k$. Now, for any linear dependence c of ϕ we have

$$\sum_{e\in E} (h \cdot \phi)(e)\widehat{\phi}(e)(c) = h\left(\sum_{e\in E} c(e)\phi(e)\right) = 0$$

So $h \cdot \phi$ is a thin dependence of $\widehat{\phi}$, and since it is 0 outside E_1, E_1 is dependent with respect to $\widehat{\phi}$.

Conversely, suppose that E_1 is dependent in $M_{ts}(\widehat{\phi})$, so that there is a nonzero thin dependence d of $\widehat{\phi}$ which is 0 outside E_1. We want to show that E_1 meets every base of $M(\phi)$, so suppose for a contradiction that there is such a base B which it doesn't meet. Pick $e_1 \in E_1$ so that d is nonzero at e_1. We can express $\phi(e_1)$ as a linear combination of vectors from the family $(\phi(e)|e \in B)$ - that is, there is a linear dependence c of ϕ which is nonzero only on B and at e_1, with $c(e_1) = 1$. But then

$$d(e_1) = \sum_{e \in E} d(e)c(e) = \sum_{e \in E} d(e)\widehat{\phi}(e)(c) = 0,$$

which is the desired contradiction. Thus E_1 does meet every basis of $M(\phi)$, so it is dependent in the dual of $M(\phi)$. □

Corollary 3.2.4. *For any thin family $E \xrightarrow{f} k^A$ we have,*

$$M_{ts}(f) = M^*(\overline{f}).$$

In particular $M_{ts}(f)$ is a cofinitary matroid.

Proof. This is immediate from Theorem 3.2.3, since by Lemma 3.2.2 we have $M_{ts}(f) = M_{ts}(\widehat{\overline{f}})$. □

3.3 A sufficient condition for M_{ts} to be a matroid

Throughout this section, f will denote a map $E \xrightarrow{f} k^A$ for some sets A and E and field k. Since so many examples of matroids are of the form $M_{ts}(f)$ for some such f, it would be good to be able to characterise when the set system $M_{ts}(f)$ is a matroid. Although this set system clearly satisfies the axioms (I1) and (I2), it need not satisfy either (I3) or (IM). As we shall soon see, the algebraic cycle system of the Bean graph satisfies (IM) but not (I3).

On the other hand, we can also define a thin sums system which fails to satisfy (IM). Let $E = \mathbb{N} \times \{0, 1\}$, and define a function $E \xrightarrow{f} \mathbb{Q}^{\mathbb{N}}$ by $f((n, 0))(i) = i^n$ and $f((n, 1))(i) = -1$ if $n = i$ and 0 otherwise. Thus for any thin dependence c of f, there can only be finitely many $n \in \mathbb{N}$ with $c((n, 0))$ nonzero, and the remaining values of c are determined by the polynomial expression

$$c((i, 1)) = \sum_{n \in \mathbb{N}} c((n, 0))i^n \,. \qquad (3.1)$$

In particular, if c is 0 outside of $\mathbb{N} \times \{0\}$, then this polynomial has infinitely many roots. Hence it must be the 0 polynomial and so c must be the 0 function. This shows that $\mathbb{N} \times \{0\}$ is thinly independent in $M_{ts}(f)$. To show that $M_{ts}(f)$ doesn't satisfy (IM), we shall show that there is no maximal thinly independent superset of $\mathbb{N}\times\{0\}$. More precisely, we shall show that, for a subset X of $\mathbb{N}$, the set $\mathbb{N}\times\{0\}\cup X\times\{1\}$ is thinly independent if and only if $\mathbb{N}\setminus X$ is infinite. In fact, the same argument as that above shows that this set is thinly independent whenever $\mathbb{N}\setminus X$ is infinite, since the only polynomial which is zero in infinitely many places is the zero polynomial. Conversely, if $\mathbb{N}\setminus X$ is finite, then pick some nonzero polynomial $\sum_{n=0}^{N} a_n x^n$ with roots at all elements of $\mathbb{N}\setminus X$, and define $c((n,0))$ to be a_n for $n \leq N$ and 0 otherwise. Define $c((i,1))$ by the polynomial formula (3.1). Then c is a nontrivial thin dependence which is 0 outside $\mathbb{N}\times\{0\}\cup X\times\{1\}$, so that set is thinly dependent.

We shall argue in the next section that some condition like (IM) is unavoidable, but we can at least get rid of the condition (I3). We do this by defining for each f a different set system $M_{ts}^{\mathrm{cofin}}(f)$, which satisfies (I3) in addition to (I1) and (I2), and such that if it satisfies (IM) then $M_{ts}(f)$ is a matroid (in fact, in such cases $M_{ts}^{\mathrm{cofin}}(f) = M_{ts}^{*}(f)$).

We will make use of a compactness lemma, corresponding to the compactness of a topological space which (so far as we know) has not been introduced in the literature. We therefore introduce it here.

Definition 3.3.1. An *affine equation* over a set I with coefficients in k consists of a family $(\lambda_i \in k | i \in I)$ such that only finitely many of the λ_i are nonzero and an element κ of k.

A family $x = (x_i | i \in I)$ is a *solution* of the equation (λ, κ) if $\sum_{i\in I} \lambda_i x_i = \kappa$. Accordingly, we shall use the expression $\ulcorner \sum_{i\in I} \lambda_i x_i = \kappa \urcorner$ to denote the equation (λ, κ). x is a solution of a set Q of equations if it is a solution of every equation in Q.

The following lemma is based on ideas of Bruhn and Georgakopoulos [24], though the proof we give is a little simpler.

Lemma 3.3.2. *If every finite subset of a set Q of affine equations over I with coefficients in k has a solution then so does Q.*

Proof. The set V of affine equations over I can be given the structure of a vector space over k, with $\mu((\lambda_i|i \in I),\kappa) = ((\mu\lambda_i|i \in I),\mu\kappa)$ and $((\lambda_i|i \in I),\kappa) + ((\lambda'_i|i \in I),\kappa') = ((\lambda_i + \lambda'_i|i \in I),\kappa+\kappa')$. Let W be the subspace generated by the equations in Q, and let q_0 be the affine equation $0 = 1$ (that is, $((0|i \in I), 1)$), which has no solutions. It is clear that if $(a_i|i \in I)$ is a solution of all the equations in some finite set Q' then it is also a solution of everything in their linear span in V. So q_0 can't be in W. By choosing a basis of W and extending it to a basis of V that contains q_0, we can construct a linear map $V \xrightarrow{\alpha} k$ which is 0 on W but with $\alpha(q_0) = 1$. For each $i \in I$ let $a_i = -\alpha(\ulcorner x_i = 0\urcorner)$. Then for each equation $q \in Q$, given as $\ulcorner \sum_{i\in I} \lambda_i x_i = \kappa\urcorner$, we have $\sum_{i\in I} \lambda_i a_i = \alpha(\kappa q_0 - q) = \kappa$, so a is a solution of every equation in Q. □

This lemma is all we will really need. However, it looks like it ought to correspond to some sort of compactness, and indeed it does.

Definition 3.3.3. For any affine equation q over I with coefficients in k, let C_q be the set of solutions of q. For any finite set Q of affine equations, let $C_Q = \bigcup_{q \in Q} C_q$. The *affine Zariski topology* on k^I is that with the C_Q as its basic closed sets.

The reason for this name is the analogy between this definition and the Zariski topology on $k[X]$ for a finite set X.

Theorem 3.3.4. *The affine Zariski topology is compact.*

Proof. Let $\mathcal{Q}$ be a set of finite sets of affine equations, such that for any finite subset K of $\mathcal{Q}$ the set $\bigcap_{Q \in K} C_Q$ is nonempty. What we need to show is that $\bigcap_{Q \in \mathcal{Q}} C_Q$ is also nonempty. Let X be $\prod_{Q \in \mathcal{Q}} Q$, with the product topology. For each finite $K \subseteq \mathcal{Q}$, let X_K be the subset of X consisting of all $(q_Q | Q \in \mathcal{Q})$ such that $\{q_Q | Q \in K\}$ has a solution. X_K is closed and nonempty since K is finite. For any finite family $(K_j | j \in J)$ of such K we have that $\bigcap_{j \in J} X_{K_j} \supseteq X_{\bigcup_{j \in J} K_j}$, so it is nonempty. Since X is compact, the intersection of all the X_K is also nonempty, so we can pick an element q. Then we know that every finite subset of $\{q_Q | Q \in \mathcal{Q}\}$ has a solution, so by Lemma 3.3.2 there is a solution x of the whole set of equations. But then x lies in $\bigcap_{Q \in \mathcal{Q}} C_Q$, which is therefore nonempty. $\square$

Lemma 3.3.5. *Let d be a thin dependence of f. Then* $\mathrm{supp}(d)$ *is a union of minimal dependent sets of $M_{ts}(f)$.*

Proof. Let $I = \mathrm{supp}(d)$. It suffices to show that for any $e_0 \in I$ there is a minimal dependent set which contains e_0 and is a subset of I. We begin by fixing such an e_0.

For any $a \in A$ there are only finitely many $e \in I$ with $f(e)(a) \neq 0$, so for any $a \in A$ we get an affine equation $\ulcorner \sum_{e \in I} f(e)(a) x_e = 0 \urcorner$ over I. Let $\mathcal{Q}$ be the set of all affine equations arising in this way. Let $\mathcal{E}$ be the set of all subsets I' of I such that every finite subset of $\mathcal{Q} \cup \{\ulcorner x_e = 0 \urcorner | e \in I'\} \cup \{\ulcorner x_{e_0} = 1 \urcorner\}$ has a solution. Since $d\restriction_I$ is a solution of all equations in $\mathcal{Q}$, $(d\restriction_I)/d(e_0)$ is a solution of all equations in $\mathcal{Q} \cup \{\ulcorner x_{e_0} = 1 \urcorner\}$, so $\emptyset \in \mathcal{Q}$. $\mathcal{E}$ is also closed under unions of chains, so by Zorn's lemma it has a maximal element E_m. Now by Lemma 3.3.2 there is some solution d' of all the equations in $\mathcal{Q} \cup \{\ulcorner x_e = 0 \urcorner | e \in E_m\} \cup \{\ulcorner x_{e_0} = 1 \urcorner\}$. Since d' solves all the equations in $\mathcal{Q}$, its extension to E taking the value 0 outside I is a thin dependence of f.

We shall show that $D := \mathrm{supp}(d') = E \backslash E_m$ is the desired minimal dependent set. If it were not, there would have to be a nonzero thin dependence d'' with $\mathrm{supp}(d'') \subseteq \mathrm{supp}(d') - e_0$. But then for any $e_1 \in \mathrm{supp}(d'')$, we have that $d' - \frac{d'(e_1)}{d''(e_1)} d''\restriction_I$ is a solution of $\ulcorner x_{e_1} = 0 \urcorner$ in addition to the equations solved by d', which contradicts the maximality of E_m. $\square$

Corollary 3.3.6. *If $M_{ts}(f)$ is a matroid, and $E' \subseteq E$, then $e \notin E'$ is in the closure of E' if and only if there is a thin dependence d with* $\mathrm{supp}(d) \subseteq E' \cup \{e\}$ *and $d(e) = 1$.*

Proof. If there is such a d, by Lemma 3.3.5 we can find a minimal dependent set D with $e \in D \subseteq \text{supp}(d)$. As $D - e \subseteq E'$ is independent, $e \in \text{Cl}(E')$. If $e \in \text{Cl}(E')$ then there is a circuit D with $e \in D \subseteq E' \cup \{e\}$. Let d be a thin dependence with $\text{supp}(d) = D$. Then $d(e) \neq 0$ since $D - e_0$ is independent: scaling if necessary, we can take $d(e) = 1$. $\square$

Corollary 3.3.7. *Let $M_{ts}(f)$ be a matroid. Then a subset I is independent in $M^*_{ts}(f)$ if and only if for every $i \in I$ there is a thin dependence d_i of f such that $d_i(i) = 1$ and d_i is 0 on the rest of I.*

Proof. We recall that I is independent in $M^*_{ts}(f)$ if and only if $\text{Cl}(I^c) = E$. Now apply Corollary 3.3.6. $\square$

This motivates the definition we promised at the start of this section, of the set system M^{cofin}_{ts}.

Definition 3.3.8. A subset I of E is *coindependent* if for every $i \in I$ there is a thin dependence d_i of f such that $d_i(i) = 1$ and d_i is 0 on the rest of I. The set system $M^{\text{cofin}}_{ts}(f)$ has ground set E and consists of the coindependent subsets of E.

Thus by Corollary 3.3.7, when $M_{ts}(f)$ is a matroid, $M^{\text{cofin}}_{ts}(f) = M^*_{ts}(f)$.

Lemma 3.3.9. *Let I be coindependent and $i_0 \notin I$. If there is a thin dependence d which is nonzero at i_0 and 0 on I, then $I + i_0$ is coindependent.*

Proof. Suppose that $(d_i | i \in I)$ witnesses the coindependence of I. Let $d'_{i_0} = d/d(i_0)$, and for $i \in I$ let $d'_i = d_i - d_i(i_0)d'_{i_0}$. Then $(d'_i | i \in I + i_0)$ witnesses the coindependence of $I + i_0$. $\square$

We can now show that $M^{co}_{ts}(f)$ is always dual, in a sense, to $M_{ts}(f)$.

Lemma 3.3.10. *Let $I \subseteq E$. I is a maximal independent set with respect to f if and only if $E \setminus I$ is a maximal coindependent set with respect to f.*

Proof. Suppose first of all that I is a maximal independent set, and let $i \in E \setminus I$. Let d_i witnesses the dependence of $I \cup \{i\}$. We must have $d_i(i) \neq 0$, so without loss of generality $d_i(i) = 1$. But then the d_i witness the coindependence of $E \setminus I$. We can't have $(E \setminus I) + i$ coindependent for any $i \in I$, since the corresponding d_i would witness dependence of I.

So suppose instead for a contradiction that $E \setminus I$ is a maximal coindependent set but I is dependent, as witnessed by some thin dependence d of I. There must be $i_0 \in I$ with $d(i_0) \neq 0$ so, by Lemma 3.3.9, $(E \setminus I) + i_0$ is coindependent, contradicting the maximality of $E \setminus I$. Thus I is independent. For each $i \in E \setminus I$, $I \cup \{i\}$ is dependent, as witnessed by d_i, and so I is also maximal. $\square$

$M^{\text{cofin}}_{ts}(f)$ evidently satisfies (I1) and (I2).

Lemma 3.3.11. *$M^{\text{cofin}}_{ts}(f)$ satisfies (I3).*

Proof. Suppose we have a maximal coindependent set J, and a nonmaximal coindependent set I. We have to show that we may extend I with a point from J. Since I is nonmaximal, we can choose $i_0 \notin I$ with $I + i_0$ still coindependent. Since by Lemma 3.3.10 $E \setminus J$ is independent, there is $i_1 \in J$ with $d_{i_0}(i_1) \neq 0$. Then by Lemma 3.3.9 $I + i_1$ is coindependent. □

We can now give our slightly simplified criterion for when a thin sums system is a matroid.

Theorem 3.3.12. *If $M_{ts}^{\mathrm{cofin}}(f)$ satisfies (IM), then*

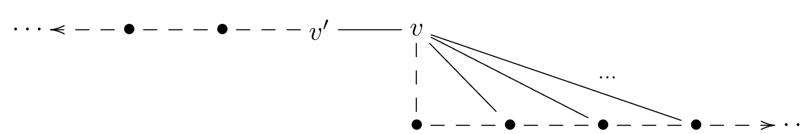
$$(M_{ts}^{co}(f))^* = M_{ts}(f).$$

In particular, $M_{ts}(f)$ is a matroid.

Proof. $M_{ts}^{co}(f)$ evidently satisfies (I1) and (I2), and satisfies (I3) by Lemma 3.3.11, so it is a matroid. It is clear from Lemma 3.3.10, that every independent set of $(M_{ts}^{co}(f))^*$ is also independent in $M_{ts}(f)$. Conversely, let I be an independent set of $M_{ts}(f)$. Then let J be a maximal independent set of $M_{ts}^{\mathrm{cofin}}(f)$ not meeting I. It suffices to show that J is a base of $M_{ts}^{\mathrm{cofin}}(f)$. Suppose not, for a contradiction: then there is some $i \in I$ with $J + i$ coindependent. But then since I is independent, the corresponding d_i is nonzero at some $j \notin I$, and by Lemma 3.3.9 we deduce that $J + j$ is coindependent, contradicting the maximality of J. □

We now return to the question of when the algebraic cycle system $M_A(G)$ of a graph G is a matroid. It evidently satisfies (I1) and (I2). A little trickery shows that $M_A(G)$ has a maximal independent set B. First, we pick a maximal collection A of disjoint rays in G, then we can take B to be any maximal set of edges including all the rays in A but not including any cycle and not connecting any 2 of the rays in A (both these steps are possible by Zorn's Lemma). B can't include a double ray, by maximality of A. A slight refinement of this argument shows that $M_A(G)$ always satisfies (IM). So we just need to determine whether $M_A(G)$ satisfies (I3).

In fact, as we mentioned in Section 3.1, it was shown by Higgs in [41] that $M_A(G)$ is a matroid if and only if G doesn't contain any subdivision of the Bean graph:



The algebraic cycle system of this graph doesn't satisfy (I3) - the dashed edges above form a maximal independent set, but there is no way to extend the nonmaximal independent set consisting of the edges meeting v (except vv') and those to the left of v' by an edge from this set. It is, however, not at all easy to see that if G doesn't contain a subdivision of the Bean graph then $M_A(G)$

satisfies (I3). In fact, Higgs didn't follow this route - the interested reader can check that his claim (3) (which is the combinatorial heart of the paper) is exactly the criterion obtained from Theorem 3.3.12 in this case. We are now in a position to give a more direct argument.

Theorem 3.3.13 (Higgs)**.** *Suppose that G includes no subdivision of the Bean graph. Then $M_A(G)$ is a matroid.*

Proof. We say a cut b of G is a *nibble* if one side (called the *small side*: the other side is the *large side*) of b is connected and includes no rays. Suppose, for a contradiction, that there are a nibble b and an algebraic cycle a meeting b infinitely often. Then a must be a double ray. Let T be a spanning tree of the small side of b. We can pick any vertex v_0 in this tree to serve as its root, and consider the subtree T' consisting of the paths from v_0 to a in T. Since T' is rayless and has infinitely many leaves there must (by König's Lemma) be a vertex v in this tree of infinite degree. Then there is an infinite set $\mathcal{P}$ of paths from v to a in T' meeting only at v, so a has some subray r containing infinitely many of the endpoints of the paths in $\mathcal{P}$. Now a together with the paths in $\mathcal{P}$ from v to r will give a subdivision of the Bean graph, contrary to our supposition. So we can conclude that a nibble and an algebraic cycle can only meet finitely often.

In fact we can say more, using the ideas of Section 3.1. Pick directions for every edge, algebraic cycle and nibble of G. Let A be the set of all algebraic cycles of G, and for any edge $e \in E(G)$ define a function $A \xrightarrow{f(e)} k$ such that for any $a \in A$, $f(e)(a)$ is 1 if $e \in a$ and they have the same directions, -1 if $e \in a$ and they have different directions, and 0 if e isn't an edge of a. This gives a map $E(G) \xrightarrow{f} k^A$. We shall show that $M_{ts}^{\text{cofin}}(f) = M_A(G)$.

First, we show that any coindependent set I for f is $M_A(G)$-independent. Suppose for a contradiction that I includes an algebraic cycle a, and pick any $i \in a$. Then $\sum_{e \in E} d_i(e) f(e)(a) = f(i)(a) \neq 0$, which is the desired contradiction.

For any nibble b of G, define the map $E(G) \xrightarrow{d_b} k$ such that $d_b(e)$ is 1 if $e \in b$ and they have the same directions, -1 if $e \in b$ and they have different directions, and 0 if e isn't an edge of b. For any algebraic cycle a, a must traverse b the same number of times in each direction (if it is a double ray, the rays in both directions must eventually end up in the large side of b). Traversals one way contribute a $+1$ term to $\sum_{e \in E} d_b(e) f(e)(a)$, and traversals the other way contribute a -1 term, so this sum is always 0. That is, each d_b is a thin dependence of f.

Now if a set I isn't coindependent then there is some $i \in I$ such that no thin dependence is nonzero at i and 0 on the rest of I. In particular, considering the thin dependences d_b above, there is no nibble b with $b \cap I = \{i\}$. Thus if the connected components of $I - i$ containing the endpoints of i are distinct then each contains a ray, so I contains a double ray. Otherwise, both ends of i are in the same component, so I contains a cycle. In either case, I contains an algebraic cycle.

We have shown that the $M_A(G)$-independent sets are exactly the coindependent sets, so they satisfy (I3) by Lemma 3.3.11. We have already checked the remaining axioms. $\square$

Remark 3.3.14. *This argument also shows a little more - namely that the dual of $M_A(G)$ is the thin sums matroid $M_{ts}(f)$. We have shown that every nibble is thinly dependent. On the other hand, if a set I contains no nibble, so that every connected component of the complement of I contains a ray, then for each i in I there is an algebraic cycle meeting I only in i, so I is thinly independent. Thus the cycles of the dual of $M_A(G)$ are exactly the minimal nonempty nibbles, as mentioned in subsection 1.6.3.*

3.4 Galois Connections

In this section, we will present a new perspective on the definition of thin sums set systems, which we believe shows that it is unlikely that any criterion much simpler than (IM) will allow us to distinguish which such systems are matroids. To do this, we shall show that thin sums systems are determined by closed classes for a particular Galois connection. We shall note that each IE-operator gives a closed class for a very similar Galois connection. Since in that case (IM) seems to be necessary to pick out the class of matroids, we think something similar will be needed for thin sums systems also. We will also show that Dress's matroids with coefficients [37] can be naturally related to the framework developed here.

Since Galois connections are not widely known, we shall review here the small portion of the theory that we shall require.

Definition 3.4.1. Let A be a set, and R a symmetric relation on A. The *Galois connection* induced by R is the function ($\mathcal{P}A \xrightarrow{p} \mathcal{P}A$) given by

$$p(A') = \{a \in A | (\forall a' \in A') aRa'\} .$$

For the remainder of this section we shall always take A, R and p to refer in this way to the constituents of a general Galois connection.

Example 3.4.2. Let V be a vector space with an inner product $\langle -, - \rangle$. We say 2 vectors v and w are *orthogonal* if $\langle v, w \rangle = 0$. This gives a relation from V to itself, and so induces a Galois connection as above. p is given by the function $\mathcal{P}V \to \mathcal{P}V$ that sends a subset of V to its *orthogonal complement*, which is always a subspace of V.

Lemma 3.4.3.

- *For $A' \subseteq A'' \subseteq A$, $p(A'') \subseteq p(A')$*
- *For $A' \subseteq A$, $A' \subseteq p^2(A')$*

Proof. To prove the first property, note that for any $a \in p(A'')$, for any $a' \in A' \subseteq A''$ we have aRa', so that $a \in p(A')$. To prove the second property, note that for any $a' \in A'$, for any $a \in p(A')$ we have $a'Ra$, so that $a' \in p^2(A')$. $\square$

Lemma 3.4.4. *For $A' \subseteq A$, the following are equivalent:*

- $A' = p^2(A')$.
- *A' is in the image of p.*

Proof. The first statement clearly implies the second. Suppose the second is true, and let $A' = p(A'')$. Then we have $A'' \subseteq p^2(A'')$, so $p(A'') \supseteq p^3(A'')$, that is $A' \supseteq p^2(A')$. Since we also know $A' \subseteq p^2(A')$, we have $A' = p^2(A')$ as required. □

In such cases, we say A' is a *closed* subset of A (with respect to this Galois connection). It is immediate from Lemma 3.4.4 that p restricts to an order reversing involutory automorphism of the poset of closed subsets of A. For any closed set A', $p(A')$ is called the *dual* closed set to A'.

Example 3.4.5. If, in Example 3.4.2, V is finite dimensional, then the closed sets for this Galois connection are precisely the subspaces of V.

Example 3.4.6. Let G be a finite graph, and let V be the free vector space $\mathbb{F}_2^E$ over $\mathbb{F}_2$ on the set E of edges of G. We can identify subsets of E with vectors in V: each subset gets identified with its characteristic function. There is a standard inner product on this space, with $\langle v, w\rangle = \sum_{e\in E} v(e)w(e)$. Then the cuts of G generate a subspace of V, which is the orthogonal complement of the subspace of V generated by the cycles of G. Thus the cycles and the bonds of G generate dual closed classes in the associated Galois connection.

Let E be any set, and define a relation R_1 from $\mathcal{P}E$ to itself by letting XR_1Y when $|X \cap Y| \neq 1$. This slightly odd relation is motivated by the fact that it holds between any circuit and any cocircuit in a matroid. We shall show that each matroid with ground set E induces a closed subset of $\mathcal{P}E$ in the associated Galois connection, and that the dual matroid induces the dual closed subset. In fact, we can go further and get such a result for idempotent-exchange operators (see Section 3.1 for a definition of this concept).

Definition 3.4.7. Let S be an IE-operator on a set S. A set $X \subseteq E$ is S-*closed* if $SX = X$. A subset X is an S-scrawl if for each $x \in X$ it is true that $x \in S(X - x)$. The set of S-scrawls is denoted $\mathcal{S}(S)$.

Thus if M is a matroid then a set is Cl_M-closed if and only if it is M-closed and is a Cl_M-scrawl if and only if it is a union of M-circuits.

Lemma 3.4.8. *Let S be an IE-operator on E, and let $X \subseteq E$. Then X is S-closed if and only if $E \setminus X$ is an S^*-scrawl.*

Proof. Note that by (†) of Section 3.1 for any $x \in E \setminus X$ we have $x \notin SX$ if and only if $x \in S^*((E \setminus X) - x)$. □

Corollary 3.4.9. *Let M be a matroid with ground set E, and let $s \subseteq E$ be a set which never meets an M-cocircuit in just one point. Then s is a union of M-circuits.* □

Theorem 3.4.10. *Let S be an IE-operator on a set E, and let $p = p_1$ be given as above by the Galois connection associated to R_1. Then $\mathcal{S}(S) = p(\mathcal{S}(S^*))$.*

Proof. We must show that a subset X of E is in $\mathcal{S}(S)$ if and only if it is in $p(\mathcal{S}(S^*))$.

First of all, suppose that $X \in \mathcal{S}(S)$, and pick any $X' \in \mathcal{S}(S^*)$. Suppose for a contradiction that $|X \cap X'| = 1$, and call the unique element of this set x. Then $x \in S(X - x)$ and so $x \in S(E \setminus X')$, which contradicts the fact that by Lemma 3.4.8, $E \setminus X'$ is S-closed. Since X' was arbitrary we get that $X \in p(\mathcal{S}(S^*))$.

Now suppose instead that $X \in p(\mathcal{S}(S^*))$. For any $x \in X$, $S(X - x)$ is S-closed, since S is idempotent, so $E \setminus S(X - x) \in \mathcal{S}(S^*)$ by Lemma 3.4.8. So $X \cap (E \setminus S(X - x))$, which is a subset of $\{x\}$, can't have just one element. So $x \notin E \setminus S(X - x)$ and so $x \in S(X - x)$. Since x was arbitrary, $X \in \mathcal{S}(S)$. $\square$

Thus although every matroid corresponds to a closed class for such a Galois connection, not every such closed class corresponds to a matroid: the far more general collection of IE-operators gives rise to many such closed classes which don't come from matroids. Thus, in order to determine which closed classes for these Galois connections correspond to matroids, some condition akin to (IM) is essential.

However, there is a similar Galois connection whose closed classes capture the information behind thin sums systems. Let E be a set, and k a field. We have a relation R_2 from k^E to itself with cR_2d when

$$\sum_{e \in E} c(e)d(e) = 0\,.$$

Here, as usual, we take this to include the statement that the sum is well defined, i.e. that only finitely many of the summands are nonzero.

Just as in Example 3.4.2, any closed set is necessarily a subspace of the vector space k^E. The link between this relation and the relation R_1 defined above is that, since no sum evaluating to zero can have precisely one nonzero term, if cR_2d then there can't be just one $e \in E$ at which both are nonzero. Explicitly, $cR_2d \Rightarrow \mathrm{supp}(c) R_1 \mathrm{supp}(d)$.

From any closed class, we can define a corresponding set system.

Definition 3.4.11. For any closed set C with respect to R_2, we say a subset I of E is *C-independent* if the only $c \in C$ which is zero outside I is the 0 function. Otherwise, I is *C-dependent*. The *thin sums system M_C* corresponding to C is the system of C-independent subsets of the ground set E.

We shall now show that this notion corresponds to the usual notion of a thin sums system. Let p_2 be given as above by the Galois connection associated to R_2.

Proposition 3.4.12. *Suppose we have a function $E \xrightarrow{f} k^A$. Let D be the set of functions $d_a \colon e \mapsto f(e)(a)$ with $a \in A$. Then $M_{ts}(f) = M_{p_2(D)}$.*

Proof. It is enough to show that the elements of $p_2(D)$ are exactly the thin dependences for f. But using the substitution given above, the condition that $c \in p_2(D)$, namely that for each $a \in A$

$$\sum_{e \in E} c(e) d_a(e) = 0\,,$$

becomes the condition that for each $a \in A$

$$\sum_{e \in E} c(e) f(e)(a) = 0$$

which is the condition for c to be a thin dependence for f. $\square$

Because thin sums systems correspond in this way to closed classes for the Galois connection correponding to R_2, and a condition like (IM) seems necessary to pick out the matroids amongst the closed classes for R_1, it is likely that some condition akin to (IM) will also be needed to distinguish which thin sums systems are matroids. On the other hand, the evident similarity of this connection to the sort employed in example 3.4.6 provides another indication of why the various types of cycle and bond matroids corresponding to a graph are all thin sums systems.

3.4.1 Dress's matroids with coefficients

We are now almost ready to explain the links between the notion of thin sums matroids explored by this chapter and the notion of matroids with coefficients introduced by Dress [37]. Before doing this, we just need to introduce a couple more Galois connections, very similar to those introduced above. First of all, for any set E, we get a relation R_3 from $\mathcal{P}(E)$ to itself by letting $X R_3 Y$ whenever $X \cap Y$ is finite. Let p_3 be given by the Galois connection associated to R_3.

Secondly, for any set E, and any field k let $\mathcal{P}_k E$ be the set of pairs (X, c) with $X \subseteq E$ and $c \in k^E$ taking the value 0 outside X. We get a relation R_4 from $\mathcal{P}_k E$ to itself by letting $(X, c) R_4 (Y, d)$ whenever $X \cap Y$ is finite and $\sum_{e \in X \cap Y} c(e) d(e) = 0$. Equivalently, $(X, c) R_4 (Y, d)$ whenever $X R_3 Y$ and $c R_2 d$. Let p_4 be given by the Galois connection associated to R_4.

Note that for any closed class $\mathcal{M}$ of p_4 the set

$$\mathcal{X}(\mathcal{M}) = \{X | (\exists c \in k^E)(X, c) \in \mathcal{M}\} = \{X | (X, 0) \in \mathcal{M}\}$$

is a closed class of p_3. Also, $\mathcal{M}$ may be recovered from $\mathcal{X}(\mathcal{M})$ together with

$$\mathcal{R}(\mathcal{M}) = \{c | (\exists X \in \mathcal{P}E)(X, c) \in \mathcal{M}\}$$

as $\mathcal{M} = (\mathcal{X}(\mathcal{M}) \times \mathcal{R}(\mathcal{M})) \cap \mathcal{P}_k E$.

Now we can explain how Dress's matroids with coefficients fit into this framework. Sadly, it will be necessary to ignore many of the more interesting features of Dress's account. For example, in this chapter we always work over a field k, whereas Dress works over more fragile objects which he calls fuzzy rings. Every field induces a fuzzy ring, and we shall only deal with matroids with coefficients over fuzzy rings induced in this way from fields. Dress does not give a direct definition of a matroid with coefficients over a field, preferring to work with presentations. First, in [37, §2], he introduces closed classes for p_3, which he calls *matroid support systems.* Then in [37, 3.5], he says that, if $\mathcal{X}$ is a matroid support system and $\mathcal{R} \subseteq k^E$ with the support of every element of $\mathcal{R}$ in $\mathcal{X}$, then $\mathcal{R}$ *presents a matroid* relative to $\mathcal{X}$ if and only if it satisfies a certain technical condition which he calls (M). He then defines an equivalence relation $\sim_M$, saying that $\mathcal{R}$ and $\mathcal{R}'$ *present the same matroid* if and only if $\mathcal{R} \sim_M \mathcal{R}'$.

The details of (M) and $\sim_M$ need not concern us, since it follows from [37, Theorem 5.4] that if $\mathcal{R}$ presents a matroid relative to $\mathcal{X}$ then there is a closed class $\mathcal{M}$ of p_4 such that $\mathcal{X}(\mathcal{M}) = \mathcal{X}$ and $\mathcal{R}(\mathcal{M}) \sim_M \mathcal{R}$: the construction is $\mathcal{M} = p_4(p_4((\mathcal{X} \times \mathcal{R}) \cap \mathcal{P}_k E))$. Furthermore, it is not hard to check that there do not exist two distinct closed classes $\mathcal{M}$ and $\mathcal{M}'$ with $\mathcal{X}(\mathcal{M}) = \mathcal{X}(\mathcal{M}')$ and $\mathcal{R}(\mathcal{M}) \sim_M \mathcal{R}(\mathcal{M}')$. Thus, since we are not concerned in this chapter with issues of how matroids with coefficients may be presented, we shall take a *matroid with coefficients* in a field k to be simply a closed class of p_4. The *dual* $\mathcal{M}^*$ of a matroid with coefficients is defined to be $p_4(\mathcal{M})$. For any thin sums system there is a corresponding matroid with coefficients.

Lemma 3.4.13. *For any closed class C of p_2 there is a matroid with coefficients $\mathcal{M}$ such that $C = \mathcal{R}(\mathcal{M})$ and $p_2(C) = \mathcal{R}(\mathcal{M}^*)$.*

Proof. We may take $\mathcal{M} = (\mathcal{X} \times C) \cap \mathcal{P}_k E$, where $\mathcal{X}$ is given by the expression $p_3(p_3(\{\mathrm{supp}(c) | c \in C\}))$. □

Dress goes on in §4 to define for each matroid with coefficients $\mathcal{M}$ a closure operator, which he denotes $\langle - \rangle_{\mathcal{M}}$. Modulo the rearrangements of his definition given above, this is as follows.

Definition 3.4.14. For any $F \subseteq E$ we set

$$\langle F \rangle_{\mathcal{M}} = F \cup \{e \in E | (\forall Y \in \mathcal{X}(\mathcal{M}^*))(\exists c \in \mathcal{R}(\mathcal{M})) e \in \mathrm{supp}(c) \cap Y \subseteq F \cup \{e\}\}.$$

For a matroid with coefficients $\mathcal{M}$, let $\mathcal{C}(\mathcal{M}) = \{\mathrm{supp}(c) | c \in \mathcal{R}(\mathcal{M})\}$.

Lemma 3.4.15. *For any matroid with coefficients $\mathcal{M}$ on E and any set $F \subseteq E$ we have $\langle F \rangle_{\mathcal{M}} = \mathrm{Cl}^*_{\mathcal{C}(\mathcal{M}^*)}(F)$ (see Section 2 for the definition of $\mathrm{Cl}_{\mathcal{C}}$).*

Proof. For $e \in \langle F \rangle_{\mathcal{M}} \setminus F$ there cannot be any $d \in \mathcal{R}(\mathcal{M}^*)$ with $e \in \mathrm{supp}(d) \subseteq E \setminus F$ because if there were we would have $\mathrm{supp}(d) \in \mathcal{X}(\mathcal{M}^*)$ so that there would be $c \in \mathcal{R}(\mathcal{M})$ with $e \in \mathrm{supp}(c) \cap \mathrm{supp}(d) \subseteq F \cup \{e\}$, meaning $\mathrm{supp}(c) \cap \mathrm{supp}(d) = \{e\}$ and so $\sum_{f \in E} c(f)d(f) = c(e)d(e) \neq 0$, which is impossible. Thus for any such e we have $e \notin \mathrm{Cl}_{\mathcal{C}(\mathcal{M}^*)}((E \setminus F) - e)$, so $e \in \mathrm{Cl}^*_{\mathcal{C}(\mathcal{M}^*)}(F)$. This shows that $\langle F \rangle_{\mathcal{M}} \subseteq \mathrm{Cl}^*_{\mathcal{C}(\mathcal{M}^*)}(F)$.

For the reverse implication, suppose that $e \in \mathrm{Cl}^*_{\mathcal{C}(\mathcal{M}^*)}(F) \setminus F$. We must show that $e \in \langle F \rangle_{\mathcal{M}}$. So let $Y \in \mathcal{X}(\mathcal{M}^*)$. Without loss of generality, $Y \cap F = \emptyset$. For each $f \in Y$, let $Y \xrightarrow{\delta_f} k$ be the function sending f to 1 and everything else to 0. Since $\mathcal{M}$ is closed in p_4, the set $\{c\restriction_Y | c \in \mathcal{R}(\mathcal{M})\}$ gives a linear subspace V of k^Y. Suppose for a contradiction that this subspace does not contain δ_e. Then there is a linear map : $k^Y \xrightarrow{\alpha} k$ which is 0 on V but with $\alpha(\delta_e) = 1$. Define $d \colon E \to k$ by $d(f) = \alpha(\delta_f)$ for $f \in Y$ and 0 otherwise. Then for any $(X, c) \in \mathcal{M}$ we have that $X \cap Y$ is finite (since $Y \in \mathcal{X}(\mathcal{M}^*)$) and

$$\sum_{f \in X \cap Y} c(f)d(f) = \sum_{f \in X \cap Y} c(f)\alpha(\delta_f) = \alpha\left(\sum_{f \in X \cap Y} c(f)\delta_f\right) = \alpha(c\restriction_Y) = 0\,,$$

so that $(Y, d) \in \mathcal{M}^*$. But then d witnesses that $e \in \mathrm{Cl}_{\mathcal{C}(\mathcal{M}^*)}(E \setminus F - e)$ since $d(e) = \alpha(\delta_e) = 1$, which is the desired contradiction. Thus there is some $c \in \mathcal{R}(\mathcal{M})$ with $c\restriction_Y = \delta_e$, which implies that $\mathrm{supp}(c) \cap Y = \{e\}$. The required condition, that $e \in \mathrm{supp}(c) \cap Y \subseteq F \cup \{e\}$, then follows. $\square$

Thus $\langle - \rangle_{\mathcal{M}}$ is an idempotent space. However, it is not always the closure operator of a matroid, as we will see in the next section. For a matroid with coefficients $\mathcal{M}$ derived from a closed class C for p_2 as in Lemma 3.4.13, the above Lemma shows that $\langle - \rangle_{\mathcal{M}}$ is the closure operator of a matroid if and only if the thin sums system $M_{p_2(C)}$ is a matroid.

3.5 A thin sums matroid over $\mathbb{Q}$ whose dual is not a thin sums matroid

Our counterexample will be built from the algebraic cycle matroid for the graph G in Figure 3.1, in which we have assigned directions to all the edges and labelled them for future reference. We will refer to this matroid as M for the rest of this section.

We showed in Section 1.7 that M^+ is wild.

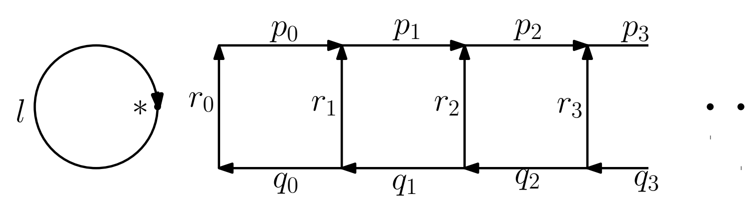


Figure 3.1: The graph G

As usual, we denote the vertex set of G by V and the edge set by E. We call the unique vertex lying on the loop at the left $*$.

Theorem 3.5.1. *M^+ is a thin sums matroid over the field $\mathbb{Q}$.*

Proof. We begin by specifying the family $(f(e)|e \in E)$ of functions from V to $\mathbb{Q}$ for which $M^+ = M_f$. We take $f(e)$ as in Proposition 3.1.3 if e is one of the p_i or q_i, to be χ_* if $e = l$, and to be $f(e) + i \cdot \chi_*$ if $e = r_i$.

First, we have to show that every circuit of M^+ is dependent in M_f. There are a variety of possible circuit types: in fact, types (b), (c), (e) and (f) from Figures 1.2 and 1.3 can arise. We shall only consider type (f): the proofs for the other types are very similar. Figure 3.2 shows the two ways a circuit of type (f) can arise.

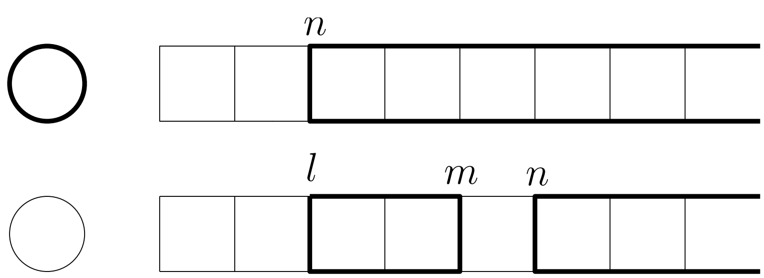


Figure 3.2: The two ways of obtaining a circuit of type (f)

The first includes the edge l, together with r_n for some n and all those p_i and q_i with $i \geq n$. We seek a thin dependence λ such that λ is nonzero on precisely these edges.

We shall take $\lambda_{r_n} = 1$. We can satisfy the equations $\sum_{e \in E} \lambda_e f(e)(v)$ with $v \neq *$ by taking $\lambda_{p_i} = \lambda_{q_i} = 1$ for all $i \geq n$. The equation $\sum_{e \in E} \lambda_e f(e)(*) = 0$ reduces to $\lambda_* + n\lambda_{r_n} = 0$, which we can satisfy by taking $\lambda_* = -n$. It is immediate that this gives a thin dependence of f.

The second way a circuit of type (f) can arise includes the edges r_l, r_m and r_n, together with those p_i and q_i with either $l \leq i < m$ or $n \leq i$. We seek a thin dependence λ such that λ is nonzero on precisely these edges.

The equations $\sum_{e \in E} \lambda_e f(e)(v)$ with $v \neq *$ may be satisfied by taking $\lambda_{p_i} = \lambda_{q_i} = \lambda_{r_l} = -\lambda_{r_m}$ for $l \leq i \leq m$ and $\lambda_{p_i} = \lambda_{q_i} = \lambda_{r_n}$ for $i \geq n$. The equation $\sum_{e \in E} \lambda_e f(e)(*) = 0$ reduces to $l\lambda_{r_l} + m\lambda_{r_m} + n\lambda_{r_n} = 0$, which since $\lambda_{r_m} = -\lambda_{r_l}$ reduces further to $(m - l)\lambda_{r_m} = n\lambda_{r_n}$. We can satisfy this equation by taking $\lambda_{r_m} = n$ and $\lambda_{r_n} = m - l$. Taking the remaining λ_e to be given as above then gives a thin dependence of f. Note that $\lambda \neq 0$ since $m \neq l$ and thus $\lambda_{r_n} \neq 0$.

Next, we need to show that every dependent set of M_f is also dependent in M^+, completing the proof. Let D be such a dependent set, as witnessed by a nonzero thin dependence λ of f which is 0 outside D. Let $D' = \{e|\lambda_e \neq 0\}$, the *support* of D. Using the equations $\sum_{e \in E} \lambda_e f(e)(v)$ with $v \neq *$, we may deduce that the degree of D' at each vertex (except possibly $*$) is either 0 or at least 2. Therefore any edge (except possibly l) contained in D' is contained in some circuit of M included in D'. Since $\{l\}$ is already a circuit of M, we can even drop the qualification 'except possibly l'.

Since D' is nonempty, it must include some circuit O of M. Suppose first of all for a contradiction that $D' = O$. The intersection of D' with the set $\{l\} \cup \{r_i | i \in \mathbb{N}_0\}$ is nonempty, so by the equation $\sum_{e\in E} \lambda_e f(e)(*) = 0$ this intersection must have at least 2 elements. The only way this can happen with D' a circuit is if there are $m < n$ such that D' consists of r_m, r_n, and the p_i and q_i with $m \le i < n$. We now deduce, since λ is a thin dependence, that $\lambda_{p_i} = \lambda_{q_i} = \lambda_{r_m} = -\lambda_{r_n}$ for $m \le i \le n$. In particular, the equation $\sum_{e\in E} \lambda_e f(e)(*) = 0$ reduces to $(m-n)\lambda_{r_m} = 0$, which is the desired contradiction as by assumption $\lambda_{r_m} \neq 0$ and $m < n$. Thus $D' \neq O$, and we can pick some $e \in D \setminus O$. As above, D' includes some M-circuit O' containing e. Then the union $O \cup O' \subseteq D$ is M^+-dependent by Corollary 1.7.5. □

Theorem 3.5.2. *$(M^+)^*$ is not a thin sums matroid over any field.*

Proof. Suppose for a contradiction that it is a thin sums matroid M_f, with $f : E \to k^A$. For each circuit O of $(M^+)^*$, we can find a nonzero thin dependence λ of f which is nonzero only on O - it must be nonzero on the whole of O by minimality of O.

The circuits of $(M^+)^* = (M^*)^-$ are precisely the circuits and the bases of M^*, the dual of the algebraic cycle matroid of G, since no circuit in M^* includes a base. This dual M^*, called the *skew cuts* matroid of G, is known to have as its circuits those cuts of G which are minimal subject to the condition that one side contains no rays.

Thus since $\{r_0, p_0\}$ is a skew cut, we can find a thin dependence λ^0 which is nonzero precisely at r_0 and q_0. Similarly, for each $i > 0$ we can find a thin dependence λ^i which is nonzero precisely at q_{i-1}, r_i and q_i. Since the set of bold edges in Figure 1.1 is also a circuit of $(M^+)^*$, there is a thin dependence λ which is nonzero on precisely those edges.

To obtain a contradiction, we will show that $\{r_i | i \in \mathbb{N}\}$ is dependent in M_f. The idea behind the following calculations is to consider $\{r_i | i \in \mathbb{N}\}$ as the limit of the M_f-circuits $\{r_i | 0 \le i \le n\} \cup \{p_n\}$ and then to use the properties of thin sum representations to show that the "limit" $\{r_i | i \in \mathbb{N}\}$ inherits the dependence.

Now define the sequences $(\mu_i | i \in \mathbb{N})$ and $(\nu_i | i \in \mathbb{N})$ inductively by $\nu_0 = 1$, $\nu_i = -(\lambda^i_{p_i}/\lambda^i_{p_{i-1}})\nu_{i-1}$ for $i > 0$ and $\mu_i = -(\lambda^i_{r_i}/\lambda^i_{p_i})\nu_i$. Pick any $a \in A$. Then we have $0 = \sum_{e\in E} \lambda^0_e f(e)(a) = \lambda^0_{r_0} f(r_0)(a) + \lambda^0_{p_0} f(p_0)(a)$, and rearranging gives

$$\nu_0 f(p_0)(a) = \mu_0 f(r_0)(a) .$$

Similarly, $0 = \sum_{e\in E} \lambda^i_e f(e)(a) = \lambda^i_{p_{i-1}} f(p_{i-1})(a) + \lambda^i_{r_i} f(r_i)(a) + \lambda^i_{p_i} f(p_i)(a)$, and rearranging gives

$$\nu_i f(p_i)(a) = \nu_{i-1} f(p_{i-1})(a) + \mu_i f(r_i)(a) .$$

So by induction on i we get the formula

$$\nu_i f(p_i)(a) = \sum_{j=0}^{i} \mu_j f(r_j)(a) .$$

The formula $\sum_{e\in E}\lambda_e f(e)(a) = 0$ implicitly includes the statement that the sum is well defined, so only finitely many summands can be nonzero. In particular, there can only be finitely many i for which $f(p_i)(a) \neq 0$. It then follows by the formula above that there are only finitely many i such that $f(r_i)(a)$ is nonzero, since if $f(r_i) \neq 0$, then as $\mu_i \neq 0$ we have $\nu_i f(p_i)(a) \neq \nu_{i-1} f(p_{i-1})(a)$. So as $\nu_i \neq 0$ and $\nu_{i-1} \neq 0$, one of $f(p_i)(a)$ or $f(p_{i-1})(a)$ is not equal to zero. Therefore all but finitely many $f(r_i)(a)$ are zero since all but finitely many $f(p_i)(a)$ are zero. So the following sum is well defined and evaluates to zero.

$$\sum_{i=0}^{\infty} \mu_i f(r_i)(a) = 0\,.$$

Therefore, if we define a family $(\lambda'_e | e \in E)$ by $\lambda'_{r_i} = \mu_i$ and $\lambda'_e = 0$ for other values of e, then we have

$$\sum_{e\in E} \lambda'_e f(e)(a) = 0\,.$$

Since $a \in A$ was arbitrary, this implies that λ' is a thin dependence of f. Note that $\lambda' \neq 0$ since $\lambda'_{r_0} \neq 0$. Thus the set $\{r_i | i \in \mathbb{N}\}$ is dependent in $M_f = (M^*)^-$. But it is also an $(M^*)^-$basis, since adding l gives a basis of M^*. This is the desired contradiction. $\square$

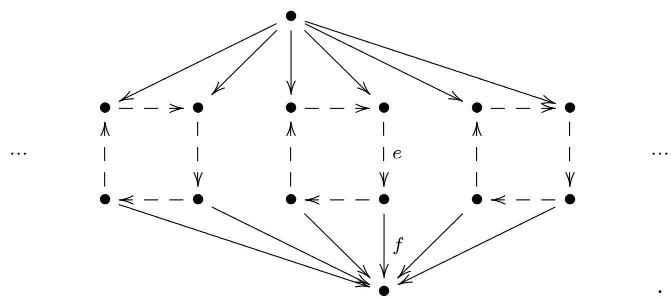
3.6 Tameness and duality

The problem with the matroid in the last section was that it was wild: the main result of this section will be that the class of tame thin sums matroids is closed under duality. It will then quickly follow that it is also closed under taking minors.

The class of tame thin sums matroids includes all the interesting examples arising from graphs: any finitary or cofinitary matroid must be tame, and this includes the finite and topological cycle matroids as well as the bond and finite bond matroids of a given graph. We showed in the proof of Theorem 3.3.13 that the algebraic cycle and skew cuts matroids are also tame.

A natural strategy for showing that the dual of a thin sums matroid is again a thin sums matroid is suggested by the results of Section 3.2. These results suggest that in attempting to construct the representation $E \xrightarrow{\overline{f}} k^{\overline{A}}$ of $M^*_{ts}(f)$ we should take $\overline{A}$ to be the set of all thin dependencies of f, and define $\overline{f}(e)(c)$ to be $c(e)$. Under the correspondence given in Proposition 3.4.12, this would correspond to the hope that the dual of M_C would be $M_{p_2(C)}$. However, this natural attack fails to work, even if $M_{ts}(f)$ is tame, as our next example shows.

Example 3.6.1. Let G be the graph



We may represent the algebraic cycle matroid of G as $M_{ts}(f)$ as in the proof of Proposition 3.1.4. Recall that for any edge e of G the function $V(G) \xrightarrow{f(e)} k$ is given by taking $f(e)(v)$ to be 1 if e originates from v, -1 if it terminates in v, and 0 if e and v are not incident. Thus the function which takes the value 1 on the dotted edges and 0 elsewhere is a thin dependence of f. So no function with support given by the skew cut consisting of the vertical dotted edges can be a thin dependence of $\overline{f}$ as given above. That is, for this matroid and this definition of $\overline{f}$, we have $M^* \neq M_{ts}(\overline{f})$.

This example also allows us to answer a couple of open problems of Dress [37, 4.4, (i-ii)]. First we must translate into the language of matroids with coefficients, as in Section 3.4. We know from Proposition 3.4.12 and Lemma 3.4.13 that there is a matroid with coefficients $\mathcal{M}$ with $\mathcal{R}(\mathcal{M})$ the set of thin dependences of f and $\mathcal{R}(\mathcal{M}^*)$ the set of thin dependences of $\overline{f}$. Using the constructions given there, we get that $\mathcal{X}(\mathcal{M})$ is the set of subsets of $E = E(G)$ that contain only finitely many non-dotted edges, and $\mathcal{X}(\mathcal{M}^*)$ is the set of subsets of E that contain only finitely many dotted edges.

Dress asked the following:

(i) For X, F subsets of the ground set E of a matroid with coefficients $\mathcal{M}$ and $e \in \langle X \cup F \rangle_{\mathcal{M}}$, is there always a minimal subset X' of X with $e \in \langle X' \cup F \rangle$?

(ii) For X, Y, F subsets of the ground set E of a matroid with coefficients $\mathcal{M}$ and such that for any $K \subseteq X$ we have $K = X \cap \langle K \cup F \rangle_{\mathcal{M}}$ if and only if $K = X \cap \langle K \cup F \cup Y \rangle_{\mathcal{M}}$, does it follow that for any $L \subseteq Y$ we have $L = Y \cap \langle L \cup F \rangle_{\mathcal{M}}$ if and only if $L = Y \cap \langle L \cup F \cup X \rangle_{\mathcal{M}}$?

If we take $\mathcal{M}$ as above, let F be the set of all edges adjacent with the top vertex of the graph together with $\{f\}$, X be the set of remaining non-dotted edges, e be as marked in the picture and $Y = \{e\}$ it is not hard to check using Definition 3.4.14 that we get a negative answer to both questions. The fact that we get a negative answer to (i) also implies that $\langle - \rangle_{\mathcal{M}}$ is not the closure operator of a matroid, so that $M_{\overline{f}}$ is not a matroid either by Lemma 3.4.15.

Our approach will be a little different in character, although our results will imply that the restriction of the $\overline{f}$ defined above to the set of thin dependences whose supports are circuits *does* give a representation of the dual of $M_{ts}(f)$. We shall proceed by giving a self-dual characterisation of the class of tame thin sums matroids.

Theorem 3.6.2. *Let M be a tame matroid with ground set E. Then M is a thin sums matroid over the field k if and only if there is for each circuit o of M a function $o \xrightarrow{c_o} k^*$ (here k^* is the set of nonzero elements of k) and for each cocircuit b of M a function $b \xrightarrow{d_b} k^*$ such that for any circuit o and cocircuit b we have*

$$\sum_{e \in o \cap b} c_o(e) d_b(e) = 0\,. \tag{3.2}$$

Definition 3.6.3. Such a family of functions is called a *k-painting* of the matroid M, and the property of having a k-painting is called *k-paintability.* So the theorem shows that a tame matroid is a thin sums matroid over k if and only if it is k-paintable.

Proof. Suppose first of all that we have such c_o and d_b. Let A be the set of cocircuits of M, and let $E \xrightarrow{f} k^A$ be defined by $f(e)(b) = d_b(e)$ if $e \in b$ and 0 otherwise. We shall show that $M = M_{ts}(f)$, by showing that a set $I \subseteq E$ is M-dependent if and only if it is $M_{ts}(f)$-dependent. If I is M-dependent, it includes some circuit o, and then the function extending c_o to E and taking the value 0 everywhere outside o is a nontrivial thin dependence of f which is 0 outside of I. If I is $M_{ts}(f)$-dependent, then let c be a nontrivial thin dependence of f which is 0 outside of I, and let $s = \mathrm{supp}(c)$. Then for any M-cocircuit b we have

$$\sum_{e \in E} c(e) d_b(e) = 0\,.$$

The collection of those e such that $c(e)d_b(e) \neq 0$ is $s \cap b$, which therefore can't have just one element. So by Corollary 3.4.9 s is a union of M-circuits. Since s is nonempty, it is therefore M-dependent, and therefore so is I.

Conversely, let M be given as $M_{ts}(f)$ for some $E \xrightarrow{f} k^A$. For each circuit o of M, pick some thin dependence $\hat{c}_o$ of f with support o, and let $c_o = \hat{c}_o\restriction_o$. Now let b be any cocircuit of M, and fix some $e_b \in b$. By Lemma 1.3.5, we can find for each $e \in b - e_b$ some circuit $o(e)$ of M such that $o(e) \cap b = \{e_b, e\}$. We define the map $b \xrightarrow{d_b} k^*$ to be 1 at e_b and $-\frac{c_{o(e)}(e_b)}{c_{o(e)}(e)}$ for $e \in b - e_b$ (note that this choice ensures that (3.4) holds for b and each $o(e)$).

Let o be any circuit of E. It remains to show that $\sum_{e \in o \cap b} c_o(e) d_b(e) = 0$. Plugging in the values for $d_b(e)$, this means that we need to show

$$\hat{c}_o(e_b) - \sum_{e \in o \cap (b - e_b)} \frac{c_o(e) c_{o(e)}(e_b)}{c_{o(e)}(e)} = 0\,.$$

That is, we need $c(e_b) = 0$, where

$$c = \hat{c}_o - \sum_{e \in o \cap (b - e_b)} \frac{c_o(e)}{c_{o(e)}(e)} \hat{c}_{o(e)}\,.$$

As c is a finite linear combination of thin dependences, it is again a thin dependence. But for any $e \in b - e_b$, we have $c(e) = \hat{c}_o(e) - \frac{\hat{c}_o(e)}{c_{o(e)}(e)} c_{o(e)}(e) = 0$. If $c(e_b) \neq 0$, then by Lemma 3.3.5, there is a circuit o such that $e_b \in o \subseteq \mathrm{supp}(c)$, which gives $o \cap b = \{e_b\}$, a contradiction. Thus $c(e_b) = 0$, as desired.

□

Theorem 3.6.4. *The class of tame thin sums matroids is closed under duality and under taking minors.*

Proof. The closure under duality follows from the fact that the characterisation given in Theorem 3.6.2 is self-dual. For the closure under taking minors, let M be a tame thin sums matroid with functions c_o, d_b given as in Theorem 3.6.2, and let $N = M/C\backslash D$ be a minor of M. For each circuit o of N, let $\hat{o}$ be a circuit of M with $o \subseteq \hat{o} \subseteq o \cup C$ (such a circuit exists by Lemma 1.2.7), and take c_o to be $c_{\hat{o}}\restriction_o$. Similarly, for each cocircuit b of N let $\hat{b}$ be a cocircuit of M with $b \subseteq \hat{b} \subseteq b \cup D$ and let $d_b = d_{\hat{b}}\restriction_b$. These c_o and d_b satisfy the conditions of Theorem 3.6.2, so that N is also a thin sums matroid over k. □

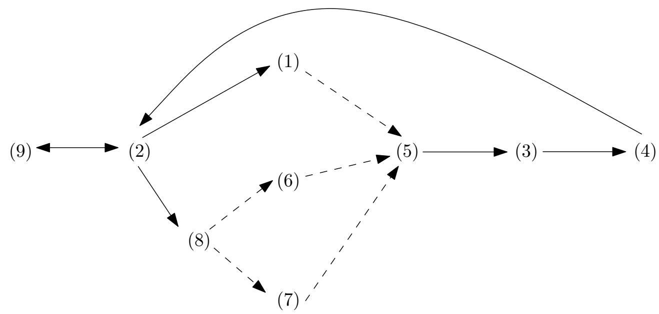
Because this notion is so well behaved, we call a tame matroid *representable over* k if and only if it has a thin sums representation over k. In the succeeding chapters, we will explore some basic properties and generalisations of representability.

3.7 Binary matroids

Theorem 3.7.1. *Let M be a tame matroid. Then the following are equivalent:*

1. *M is binary.*
2. *For any circuit o and cocircuit b of M, $|o \cap b|$ is even.*
3. *For any circuit o and cocircuit b of M, $|o \cap b| \neq 3$*
4. *M has no minor isomorphic to $U_{2,4}$.*
5. *If o_1, o_2 are circuits then $o_1 \triangle o_2$ is empty or includes a circuit.*
6. *If o_1, o_2 are circuits then $o_1 \triangle o_2$ is a disjoint union of circuits.*
7. *If $(o_i | i \in I)$ is a finite family of circuits then $\triangle_{i \in I} o_i$ is empty or includes a circuit.*
8. *If $(o_i | i \in I)$ is a finite family of circuits then $\triangle_{i \in I} o_i$ is a disjoint union of circuits.*
9. *For any base s of M, and any circuit o of M, $o = \triangle_{e \in o \backslash s} o_e$, where o_e is the fundamental circuit of e with respect to s.*

Proof. We shall prove the following implications:



Those implications indicated by dotted arrows are clear. We shall prove the remaining implications.

(2) implies (1): We need to find a suitable thin sums system. Let A be the set of cocircuits of M, and let $E \xrightarrow{f} \mathbb{F}_2^A$ be the map sending e to the function which sends $b \in A$ to 1 if $e \in b$ and 0 otherwise.

We are to show that the thin sums matroid M_{ts} defined by f is M. Since the characteristic function of any M-circuit is a thin dependence for f with support equal to that circuit by (2), any M-dependent set is also $M_{ts}(f)$-dependent.

It remains to show that the support of every non-zero thin dependence is M-dependent. By the dual of Lemma 1.2.1 the support of every non-zero thin dependence is a nonempty scrawl and so includes a circuit, as desired.

(2) implies (8): Let $(o_i|i \in I)$ be a finite family of circuits. By Zorn's Lemma, we can choose a maximal family $(o_j|j \in J)$ of disjoint circuits such that $\bigcup_{j\in J} o_j \subseteq \triangle_{i\in I} o_i$, and let $w = \triangle_{i\in I} o_i \setminus \bigcup_{j\in J} o_j$. Let b be any cocircuit of M, so that $|b \cap o_i|$ is even for each $i \in I$. Then $|b \cap \triangle_{i\in I} o_i|$ is also even, and in particular finite. Since the o_j are disjoint, there can only be finitely many of them that meet $b \cap \triangle_{i\in I} o_i$, and since for each such j we have that $|b \cap o_j|$ is even, it follows that $|b \cap w|$ is even. In particular, $b \cap w$ doesn't have just one element. Since b was arbitrary, by the dual of Lemma 1.2.1 w is a scrawl of M and so if it is nonempty it includes a circuit. But in that case, we could add that circuit to the family $(o_j|j \in J)$, contradicting the maximality of that family. Thus w is empty, and $\triangle_{i\in I} o_i = \bigcup_{j\in J} o_j$ is a disjoint union of circuits.

(5) implies (3): Suppose, for a contradiction, that (5) holds but (3) fails, and choose a circuit o and a cocircuit b with $o \cap b = \{x, y, z\}$ of size 3. Pick a base s of $(E \setminus b) + x$ including $o - y - z$, which exists by (IM). As b is a cocircuit, $b - x$ avoids some M-base, thus $(E \setminus b) + x$ is spanning and thus s is spanning, as well. Let o_y and o_z be the fundamental circuits of y and z with respect to s.

It suffices to show that $o_y \triangle o_z \subseteq o - x$. Indeed, since $y, z \in o_y \triangle o_z$, (5) then yields a circuit properly included in o, which is impossible. We can't have $o_y \cap b = \{y\}$ so we must have $x \in o_y$. Similarly, $x \in o_z$, and so $x \notin o_y \triangle o_z$. So it is sufficient to show that o_y and o_z agree outside o, in other words: $o_y \subseteq o_z \cup o$ and $o_z \subseteq o_y \cup o$.

To see this, first note that by uniqueness of the fundamental circuit of y it suffices to show that y is spanned by $(o_z - z) \cup (o - y - z)$. As z is spanned by $(o_z - z)$, $o - y$ is spanned by $(o_z - z) \cup (o - y - z)$. Since o is a circuit, y

is also spanned by $(o_z - z) \cup (o - y - z)$, as desired. A similar argument yields $o_z \subseteq o_y \cup o$, completing the proof.

(3) implies (4): Since any subset of the ground set of $U_{2,4}$ of size 3 is both a circuit and a cocircuit, it is easy to find a circuit and cocircuit in $U_{2,4}$ whose intersection has size 3. So we simply apply Corollary 1.2.10.

(4) implies (2): Suppose for a contradiction that (4) holds but (2) does not. Then let o be a circuit and b a cocircuit such that $|o \cap b| = k$ is odd. By contracting $o \setminus b$ and deleting $b \setminus o$, we obtain a minor M' of M in which $o \cap b$ is both a circuit and a cocircuit. Let s be a minimal spanning set containing $o \cap b$, which exists by (IM^*). Then in the minor M'' of M' obtained by contracting $s \setminus (o \cap b)$, $(o \cap b)$ is spanning, and is still both a circuit and a cocircuit. By a similar removal, we can find a minor M''' of M'' in which $o \cap b$ is a circuit and a cocircuit and is both spanning and cospanning. Let $x \in o \cap b$. Then $o \cap b - x$ is both a base and a cobase of M''', and it is finite (it has size $k-1$). As $o \cap b - x$ is a base and a cobase, the complement of $o \cap b - x$ is also a base and a cobase. Thus the ground set of M''' is also finite (it has size $2k-2$). Applying the finite version of the theorem, then, M''' contains a $U_{2,4}$ minor, which is also a minor of M, giving the desired contradiction.

(9) implies (2): first we will show that the following implies (2):

> For any base s of M, any circuit o meets every fundamental cocircuit of s in an even number of edges. $(\diamond)$

To see that $(\diamond)$ implies (2), it suffices to show that every cocircuit b is fundamental cocircuit of some base s. Let $e \in b$. Then as b is a cocircuit, $E \setminus (b - e)$ is spanning. Thus by (IM) there is a base s of $E \setminus (b - e)$, which clearly has b as fundamental cocircuit.

So it remains to see that (9) implies $(\diamond)$. By (9), $o = \triangle_{e \in o \setminus s} o_e$. Let b_f be some fundamental cocircuit of s for some $f \in s$. Thus $o_e \cap b_f$ is empty or $o_e \cap b_f = \{e, f\}$. So it suffices to show that every f is in only finitely many o_e, which follows from the fact that $o = \triangle_{e \in o \setminus s} o_e$ is well defined at f. This completes the proof.

(2) implies (9): we have to show for every edge f that it is contained in only finitely many o_e and that $f \in o \iff f \in \triangle_{e \in o \setminus s} o_e(f)$. If $f \notin s$, this is easy, so let $f \in s$. Now $f \in o_e$ iff $e \in b_f$. As M is tame $|o \cap b_f|$ is finite, so there are only finitely many such e. By (2), $|o \cap b_f|$ is even. If $f \notin o$, all such e are not contained in s, so $f \notin \triangle_{e \in o \setminus s} o_e$. If $f \in o$, all such e but f are not contained in s, so $f \in \triangle_{e \in o \setminus s} o_e$. This completes the proof. $\square$

We remark that we might also put the duals of the statements in the list onto the list. It might be worth noting that (7) becomes false if we also allow I to be infinite. To see this, consider the finite cycle matroid of the graph obtained from a ray by adding a vertex that is adjacent to every vertex on the ray. Indeed, the symmetric difference of all 3-cycles is a ray starting at this new vertex. This set is not empty, and nor does it include a circuit, so the infinite version of (7) fails.

However, the following can be shown with the same method used to prove (7) from (2):

Lemma 3.7.2. *Let M be a binary matroid and $X \subseteq E(M)$ with the property that it meets every cocircuit finitely and evenly. Then X is a disjoint union of circuits.*

We offer the following related open questions. Let (10) be the statement like (9) but for only one base of M. For finite matroids, (10) is equivalent to (9). Is the same true for tame matroids?

The following simple question also remains open:

In Theorem 3.7.1, we assumed that M is tame. Without this assumption, the theorem is no longer true. For example, in [21] there is an example of a wild matroid satisfying (2-6) and (10), but not (1) or (7-9). However, this matroid is not a binary thin sums matroid. In fact, we still do not know the answer to the following:

Open question 3.7.3. *Is every binary thin sums matroid tame?*

In a binary tame matroid, it is easy to see that any set meeting every cocircuit not in an odd number of edges is a disjoint union of circuits provided that the set is either countable or does not meet any cocircuit infinitely. A well-known result of Nash-Williams says that the above is also true if the matroid is the finite cycle matroid of some graph. Does this extend to all binary tame matroids?

Open question 3.7.4. *Let M be a binary tame matroid and let X be a set that meets no cocircuit in an odd number of edges. Must X be a disjoint union of circuits?*

3.8 Representable matroids

The aim of this section is to provide an excluded-minors characterisation of thin sums matroids in the class of tame matroids. The following definition will be essential.

Let k be a field and let k^* denote the set of nonzero elements of k. A *k-painting for the matroid M* is a choice of a function $c_o\colon o \to k^*$ for each circuit o of M and a function $d_b\colon b \to k^*$ for each cocircuit b of M such that for any circuit o and cocircuit b we have

$$\sum_{e \in o \cap b} c_o(e) d_b(e) = 0\,. \tag{3.3}$$

A matroid is *k-paintable* if it has a k-painting. The method we will use is motivated by Theorem 3.6.2, which may now be reformulated as follows:

Theorem 3.8.1. *Let M be a tame matroid. Then M is a thin sums matroid over the field k iff M is k-paintable.*

Note that the painting of the cocircuits of M is determined (up to scaling) by that of the circuits and vice versa. This fact also allows us to observe that a painting of M uniquely induces paintings of all minors of M.

Definition 3.8.2. Let M be a matroid, and let $((c_o|o \in \mathcal{C}(M)), (d_b|b \in \mathcal{C}(M^*))$ be a k-painting of M. Let the ground set E of M be partitioned as $X\dot{\cup}C\dot{\cup}D$, and let $M' = M/C\backslash D$. We say that a painting $((c'_o|o \in \mathcal{C}(M')), (d'_b|b \in \mathcal{C}(M'^*))$ of M' is *induced* by that of M if and only if for each $o' \in \mathcal{C}(M')$ there is $o \in \mathcal{C}(M)$ with $o' \subseteq o \subseteq o' \cup C$ and such that $c'_{o'} = c_o\restriction_{o'}$ and for each $b' \in \mathcal{C}(M'^*)$ there is $b \in \mathcal{C}(M^*)$ with $b' \subseteq b \subseteq b' \cup D$ and such that $d'_{b'} = d_b\restriction_{b'}$.

It is clear from Lemma 1.2.7 that any painting of M induces at least one painting of each minor M'. We can use the fact that the paintings of the circuits and cocircuits determine each other to show that these induced paintings are unique up to scalar factors on the c'_o and d'_b.

Lemma 3.8.3. *Let M, M' and their paintings be as in Definition 3.8.2. Let o' be any circuit of M' and o any circuit of M with $o' \subseteq o \subseteq o' \cup C$. Then there is $\lambda \in k^*$ with $c_o\restriction_{o'} = \lambda c'_{o'}$.*

Proof. Pick any $e \in o'$, and let $\lambda = \frac{c_o(e)}{c'_{o'}(e)}$. For any other $f \in o'$, by Lemma 1.3.5 there is a cocircuit b' of M' with $o' \cap b' = \{e, f\}$. Since the painting of M' is induced from that of M, there is a cocircuit b of M such that $d_b(g) = d'_{b'}(g)$ for all $g \in E(M')$ and $b' \subseteq b \subseteq b' \cup D$, and so $o \cap b = \{e, f\}$. Using the identities in the definition of painting, we deduce that

$$c_o(e)d_b(e) + c_o(f)d_b(f) = 0 \quad \text{ and } \quad c'_{o'}(e)d'_{b'}(e) + c'_{o'}(f)d'_{b'}(f) = 0$$

and so

$$c_o(f) = -\frac{c_o(e)d_b(e)}{d_b(f)} = -\frac{(\lambda c'_{o'}(e))d'_{b'}(e)}{d'_{b'}(f)} = \lambda c'_{o'}(f)$$

which gives the desired result, since f was arbitrary. □

Our main result is the following.

Theorem 3.8.4. *Let M be a tame matroid and k be a finite field. Then the following are equivalent.*

1. *M is a thin sums matroid over k.*
2. *M is k-paintable.*
3. *Every finite minor of M is k-representable.*

In Theorem 3.7.1 we already proved this theorem if $k = \mathbb{F}_2$. The general case uses similar ideas but is more complex.

Proof. To see that (1) implies (3), we use that the class of tame thin sums matroids is closed under taking minors.

That (2) implies (1) is immediate from Theorem 3.8.1.

Conversely, for any M-circuit o the function from E to k sending e to $c_o(e)$ if $e \in o$ and to 0 otherwise is a thin dependence for f with support equal to that circuit, and so any M-dependent set is also $M_{ts}(f)$-dependent.

It remains to show that (3) implies (2). We will use a compactness argument, so we begin by defining the topological space we will use. We would like an element of of this space to correspond to a choice of functions as in Lemma 3.6.2 (though not necessarily satisfying the restrictions of that Lemma), so we take

$$H = \left(\bigcup_{o \in \mathcal{C}(M)} \{o\} \times o \right) \amalg \left(\bigcup_{b \in \ \mathcal{C}(M^*)} \{b\} \times b \right)$$

and take the underlying set of our space to be $X = (k^*)^H$ - the compact topology on X that we will use is given by the product of H copies of the discrete topology on k^*.

For each circuit o and cocircuit b of M, the set

$$C_{o,b} = \left\{ c \in (k^*)^H \,\middle|\, \sum_{e \in o \cap b} c(o,e)c(b,e) = 0 \right\}$$

is closed because $o \cap b$ is finite. We shall now show that any finite intersection of such sets is nonempty.

Let $K \subseteq \mathcal{C}(M) \times \mathcal{C}(M^*)$ be finite. Let O be the set of circuits appearing as first components of elements of K, and let B be the set of cocircuits appearing as second components of elements of K. Let $F = \bigcup O \cap \bigcup B$. Note that F is finite, since M is tame.

Next, we shall construct a finite minor M' of M that will help us to prove that the finite intersection is nonempty.

Lemma 3.8.5. *There exists a finite minor M' of M such that for every $o \in O$ there is an M'-circuit $o' \subseteq o$ such that $o' \cap F = o \cap F$ and for every $b \in B$ there is an M'-cocircuit $b' \subseteq b$ such that $b' \cap F = b \cap F$.*

Proof of the lemma. We may assume that for each $o \in O$ and $b \in B$ the sets $o \cap F$ and $b \cap F$ are nonempty by adding an edge from o or b to F if necessary. We pick an element $e_o \in o \cap F$ for each $o \in O$. Next, for each $o \in O$ and each $e \in o \cap F - e_o$ we pick a cocircuit $b_{o,e}$ with $o \cap b_{o,e} = \{e_o, e\}$ (this is possible by Lemma 1.3.5). Let B' be the set of all cocircuits picked in this way or contained in B. Note that B' is finite. Similarly, we pick for each $b \in B$ an element $e_b \in F \cap b$ and then pick a circuit $o_{b,e}$ with $o_{b,e} \cap b = \{e_b, e\}$ for each $e \in F \cap b - e_b$, and we collect all of these, together with all circuits contained in O, in a finite set O'.

Let $F' = \bigcup O' \cap \bigcup B'$. Note that F' is also finite since M is tame. Let $C = \bigcup O' \setminus F'$, and let $D = E \setminus \bigcup O'$. Thus $E = C \dot\cup F' \dot\cup D$. Let M' be the finite minor of M with ground set F' that is given by $M/C\backslash D$. For each $o \in O$, $o \setminus F' \subseteq C$ and so $o \cap F'$ is a scrawl of M' by Lemma 1.2.7. Let o' be a circuit of M' with $e_o \in o' \subseteq o \cap F'$. Then for each $e \in o \cap F - e_o$ we know that $F' \cap b_{o,e}$ is a coscrawl of M', again by Lemma 1.2.7, so it can't meet o' in just one point. But $e_o \in o' \cap F' \cap b_{o,e} \subseteq \{e_o, e\}$ so we must have $o' \cap F' \cap b_{o,e} = \{e_o, e\}$ and we conclude that $e \in o'$. Since e was arbitrary, this implies that $o \cap F \subseteq o'$. Moreover, $o \cap F = o' \cap F$.

Similarly, for each $b \in B$, we find a cocircuit b' of M' such that $e_b \in b' \subseteq F' \cap b$, and it follows by a dual argument that $F \cap b = b' \cap F$, completing the proof of the lemma. □

Since M' is finite, it is representable over k. So we can find functions c_o and d_b giving a k-painting of this matroid. Let $c \in (k^*)^H$ be chosen so that, for each $o \in O$ and each $e \in o \cap F$ we have $c(o,e) = c_{o'}(e)$, and also so that for each $b \in B$ and each $e \in F \cap b$ we have $c(b,e) = c_{b'}(e)$. These choices ensure that $c \in \bigcap_{(o,b)\in K} C_{o,b}$.

Since $(k^*)^H$ is compact, and any finite intersection of the $C_{o,b}$ is nonempty, we have that $\bigcap_{(o,b)\in \mathcal{C}(M)\times\mathcal{C}(M^*)} C_{o,b}$ is nonempty. As any element in the intersection is a k-painting, this completes the proof. □

We note that this gives a uniform way to extend excluded minor characterisations of representability from finite to infinite matroids. For example, we may immediately extend the result of [11, 60] as follows:

Corollary 3.8.6. *A tame matroid M is a thin sums matroid over $GF(3)$ if and only if it has no minor isomorphic to $U_{2,5}$, $U_{3,5}$, F_7 or F_7^*.*

3.9 Other applications of the method

3.9.1 Regular matroids

A key definition to prove Theorem 3.8.4 was that of a k-painting. The corresponding notion for regular matroids is as follows.

A *signing* for a matroid M is a choice of a function $c_o\colon o \to \{1, -1\}$ for each circuit o of M and a function $d_b\colon b \to \{1, -1\}$ for each cocircuit b of M such that for any circuit o and cocircuit b we have

$$\sum_{e\in o\cap b} c_o(e)d_b(e) = 0\,,$$

where the sum is evaluated over $\mathbb{Z}$. A matroid is *signable* if it has a signing.

Lemma 3.9.1. *[[56, Proposition 13.4.5],[70]] Let M be a finite matroid. Then M is regular if and only if M is signable.*

Using similar ideas to those in the proof of Theorem 3.8.4, we obtain the following.

Theorem 3.9.2. *Let M be a tame matroid. Then the following are equivalent.*

1. *M is a thin sums matroid over every field.*
2. *M is signable*
3. *Every finite minor of M is regular.*

Proof. (2) implies that M is k-paintable for every field k, and so implies (1). (1) implies that every finite minor of M is representable over every field, and so is regular, which gives (3). (3) implies that every finite minor of M is signable, by Lemma 3.9.1. We may then deduce (2) by a compactness argument like that in the proof of Theorem 3.8.4. □

Motivated by this theorem, we call a tame matroid *regular* if any of these equivalent conditions hold.

3.9.2 Partial fields

Theorem 3.9.2 is a special case of a more general result extending characterisations of simultaneous representations over multiple fields using partial fields to tame infinite matroids. For some background on partial fields, see [68].

A *partial field* consists of a pair (R, S), where R is a ring and S is a subgroup of the group of units of R under multiplication, such that $-1 \in S$. In this context, an (R, S)*-painting* for a matroid M is a choice of a function $c_o \colon o \to S$ for each circuit o of M and a function $d_b \colon b \to S$ for each cocircuit b of M such that for any circuit o and cocircuit b we have

$$\sum_{e \in o \cap b} c_o(e) d_b(e) = 0 \,. \tag{3.4}$$

For example, for any field k a matroid M is k-paintable if and only if it is (k, k^*)-paintable, and M is signable if and only if it is $(\mathbb{Z}, \{-1, 1\})$-paintable. It is clear that the class of (R, S)-paintable matroids is closed under duality and under taking minors. In particular, any finite minor of an (R, S)-paintable matroid is (R, S)-paintable. The converse follows from an almost identical compactness argument to that used for Theorem 3.8.4, giving:

Theorem 3.9.3. *Let (R, S) be a partial field with S finite. A tame matroid is (R, S)-paintable if and only if all its finite minors are.*

It follows from the results of [68, Section 2.7] that a finite matroid is (R, S)-paintable if and only if it is (R, S)-representable. For finite matroids it is known that simultaneous representability over sets of fields corresponds to representability over partial fields, and we are now in a position to lift many such results to all tame matroids. For example, we can lift [71, Theorem 1.2] as follows:

Corollary 3.9.4. *A tame matroid M is a thin sums matroid over both $\mathbb{F}_3$ and $\mathbb{F}_4$ if and only if it is $(\mathbb{C}, \{\zeta^i | i \leq 6\})$-paintable for ζ a primitive sixth root of unity.*

3.9.3 Ternary matroids

For finite matroids, a useful property of $\mathbb{F}_3$-representable matroids is the uniqueness of the representations. In this section, we shall prove the corresponding property for tame ternary matroids.

Let M be a k-paintable matroid for some field k. We say that two k-paintings $((c_o|o \in \mathcal{C}(M)), (d_b|b \in \mathcal{C}(M^*)))$ and $((\tilde{c}_o|o \in \mathcal{C}(M)), (\tilde{d}_b|b \in \mathcal{C}(M^*)))$ are *equivalent* if and only if there are constants $x(o)$ for every $o \in \mathcal{C}(M)$, constants $x(b)$ for every $b \in \mathcal{C}(M^*)$, constants $x(e)$ for every edge e and a field automorphism φ such that the following are true:

1. $\tilde{c}_o(e) = \varphi(x(o)x(e)c_o(e))$ for any $e \in o \in \mathcal{C}(M)$.
2. $\tilde{d}_b(e) = \varphi\left(\frac{x(b)d_b(e)}{x(e)}\right)$ for any $e \in b \in \mathcal{C}(M^*)$.

Two signings of the same matroid M are *equivalent* if and only if they induce equivalent $\mathbb{F}_3$-paintings of M.

Via Theorem 3.6.2 for any tame matroid any thin sums representation over k corresponds to a k-painting. For finite matroids, the notions of equivalence for representations and paintings coincide: it is straightforward to check that two representations are equivalent iff the corresponding paintings are. As for finite matroids, we obtain the following.

Theorem 3.9.5. *Any two $\mathbb{F}_3$-paintings of the same matroid M are equivalent.*

Proof. M, being $\mathbb{F}_3$-paintable, must be tame. Without loss of generality we may also assume that M is connected and has more than one edge. Thus any edges e and f of M lie on a common circuit[2]. We nominate a particular edge g_1, and for each other edge g we nominate a circuit $o(g)$ containing both g_1 and g. We also nominate for each circuit o of M an edge $e(o) \in o$ and for each cocircuit b of M an edge $e(b) \in b$.

We denote the two $\mathbb{F}_3$-paintings $((c_o|o \in \mathcal{C}(M)), (d_b|b \in \mathcal{C}(M^*)))$ and $((\tilde{c}_o|o \in \mathcal{C}(M)), (\tilde{d}_b|b \in \mathcal{C}(M^*)))$. We shall construct witnesses to the equivalence as in the definition above. Since every automorphism of $\mathbb{F}_3$ is trivial, we shall take φ to be the identity.

We now set $x(g) = \frac{\tilde{c}_{o(g)}(g)c_{o(g)}(g_1)}{\tilde{c}_{o(g)}(g_1)c_{o(g)}(g)}$ for each $g \in E$, $x(o) = \frac{\tilde{c}_o(e(o))}{x(e(o))c_o(e(o))}$ for each circuit o of M and $x(b) = \frac{x(e(b))\tilde{d}_b(e(b))}{d_b(e(b))}$ for each cocircuit b of M.

In order to prove that these values satisfy (1) at a particular circuit o and $g \in o$, let $O = \{o, o(g), o(e(o))\}$ and $F = \{g, g_1, e(o)\}$ and use the construction from the proof of Lemma 3.8.5 to obtain a finite minor $M' = M/C\backslash D$ such that for every $o \in O$ there is an M'-circuit $o' \subseteq o$ such that $o' \cap F = o \cap F$ and for every $b \in B$ there is an M'-cocircuit $b' \subseteq b$ such that $b' \cap F = b \cap F$.

Let $((c'_o|o \in \mathcal{C}(M')), (d'_b|b \in \mathcal{C}(M'^*))$ be the $\mathbb{F}_3$-painting of M' induced by $((c_o|o \in \mathcal{C}(M)), (d_b|b \in \mathcal{C}(M^*))$, and $((\tilde{c}'_o|o \in \mathcal{C}(M')), (\tilde{d}'_b|b \in \mathcal{C}(M'^*))$ that induced by $((\tilde{c}_o|o \in \mathcal{C}(M)), (\tilde{d}_b|b \in \mathcal{C}(M^*)))$.

[2] In Section 3 of [26], it is shown that the relation 'e is in a common circuit with f' is indeed an equivalence relation for infinite matroids.

By uniqueness of representation for finite matroids, we can find constants $x'(o')$ for every $o' \in \mathcal{C}(M')$, constants $x'(b')$ for every $b' \in \mathcal{C}(M'^*)$ and constants $x'(g)$ for every $g \in X$ such that

3. $\tilde{c}'_{o'}(g) = x'(o')x'(g)c'_{o'}(g)$ for any $g \in o' \in \mathcal{C}(M')$.
4. $\tilde{d}'_{b'}(g) = \frac{x'(b')d'_{b'}(g)}{x'(g)}$ for any $g \in b' \in \mathcal{C}(M'^*)$.

Lemma 3.9.6. *For each $o \in O$ there is $\lambda_o \in k^*$ such that*

5. $c_o\restriction_F = \lambda_o c'_{o'}\restriction_F$

Proof. As part of the construction of M', we picked a canonical element e_o of o'. Let $\lambda = \frac{c_o(e_o)}{c'_{o'}(e_o)}$. For any other $e \in o' \cap F$, there is by construction a cocircuit $b_{o,e}$ of M with $o \cap b_{o,e} = \{e_o, e\}$. Then by the dual of Lemma 1.2.7 $b_{o,e} \cap E(M')$ is a coscrawl of M', and so there is a cocircuit b' of M' with $e_o \in b' \subseteq b_{o,e}$, and so $e_o \in o' \cap b' \subseteq \{e_o, e\}$. Since o' and b' can't meet in only one element, $e \in b'$. Since the painting of M' is induced from that of M, there is a cocircuit b of M such that $d_b(e) = d'_{b'}(e)$ for all $e \in E(M')$ and $b' \subseteq b \subseteq b' \cup D$, and so $o \cap b = \{e_o, e\}$. Using the identities in the definition of painting, we deduce that

$$c_o(e_o)d_b(e_o) + c_o(e)d_b(e) = 0 \quad \text{and} \quad c'_{o'}(e_o)d'_{b'}(e_o) + c'_{o'}(e)d'_{b'}(e) = 0$$

and so

$$c_o(e) = -\frac{c_o(e_o)d_b(e_o)}{d_b(e)} = -\frac{\lambda c'_{o'}(e_o)d'_{b'}(e_o)}{d'_{b'}(e)} = \lambda c'_{o'}(e)$$

which gives the desired result, since $e \in o' \cap F$ was arbitrary. □

Similarly, we can find constants $\tilde{\lambda}_o$ for each $o \in O$ such that

6. $\tilde{c}_o\restriction_F = \tilde{\lambda}_o \tilde{c}'_{o'}\restriction_F$

Now we must simply unwind all the algebraic relationships to obtain the desired result.

$$x(g) = \frac{\tilde{c}_{o(g)}(g)c_{o(g)}(g_1)}{\tilde{c}_{o(g)}(g_1)c_{o(g)}(g)} = \frac{\tilde{c}'_{o(g)'}(g)c'_{o(g)'}(g_1)}{\tilde{c}'_{o(g)'}(g_1)c'_{o(g)'}(g)} = \frac{x'(o(g)')x'(g)}{x'(o(g)')x'(g_1)} = \frac{x'(g)}{x'(g_1)}$$

where the first equation follows from the definitions, the second from (5) and (6) and the third from (3). Similarly, we get:

$$x(o) = \frac{\tilde{c}_o(e(o))}{x(e(o))c_o(e(o))} = \frac{\tilde{\lambda}_o}{\lambda_o}\frac{\tilde{c}'_{o'}(e(o))}{x(e(o))c'_{o'}(e(o))} = \frac{\tilde{\lambda}_o}{\lambda_o}\frac{x'(o')x'(e(o))}{x(e(o))}$$

And finally:

$$x(o)x(g)c_o(g) = \frac{\tilde{\lambda}_o}{\lambda_o}\frac{x'(o')x'(e(o))x'(g_1)}{x'(e(o))}\frac{x'(g)}{x'(g_1)}c_o(g) = \frac{\tilde{\lambda}_o}{\lambda_o}x'(o')x'(g)c_o(g)$$

Now the last term is just $\tilde{c}_o(g)$ by first applying (5) and then (3). This completes the proof of the above assignment satisfies (1). The proof that it also satisfies (2) is similar.

□

As every tame regular matroid is a thin sums matroid over $\mathbb{F}_3$, it also has a unique representation. In particular the finite cycle matroid, the algebraic cycle matroid and the topological cycle matroid of a given graph (and their duals) have a unique signing.

In what follows, we will describe this signing of the finite cycle matroid of a given graph G — the other cases are similar. First direct the edges of G in an arbitrary way. To define the functions c_o, let o be some cycle of G. Pick a cyclic order of o. For $e \in o$, let $c_o(e) = 1$ if e is directed according to the cyclic order of o and -1 otherwise.

Next, let b be some cocircuit. By minimality of the cocircuit, it is contained in a single component of G and its removal separates this component into two components, say $C_1(b)$ and $C_2(b)$. Note that every edge in b has precisely one endvertex in each of these components. For $e \in b$, let $d_b(e) = 1$ if e points to a vertex in C_1 and -1 otherwise.

It remains to check that $\sum_{e\in o\cap b} c_o(e)d_b(e) = 0$ for all circuits o and cocircuits b. As every circuit is finite, the above sum is finite. Since the directions we gave to the edges of G do not influence the values of the products $c_o(e)d_b(e)$, we may assume without loss of generality that in the bond b all edges are directed from $C_1(b)$ to $C_2(b)$. So we get a summand of $+1$ for each edge along which o traverses b from $C_1(b)$ to $C_2(b)$ and a summand of -1 for each edge along which o traverses b from $C_2(b)$ to $C_1(b)$. Since o must traverse b the same number of times in each direction, the sum evaluates to 0.

Let us look at how to modify the above construction to make it work for the algebraic cycle matroid and the topological cycle matroid instead. Finite circuits in the algebraic cycle matroid may be dealt with as before. To define c_o for a double ray o, we pick an orientation of o and let $c_o(e)$ be 1 if e is directed in agreement with this orientation and -1 otherwise. The above argument still applies: using the tameness of the algebraic cycle matroid, we obtain that a double ray can cross a skew cut only finitely many times, and both tails of the double ray must lie on the same side (as one side is rayless), so the double ray must cross the skew cut the same number of times in each direction.

Using the fact that topological circles are homeomorphic to the unit circle, we get a cyclic order on each circuit of the topological cycle matroid and the above construction again gives us a signing.

Chapter 4

Ψ-matroids

We will now discuss a large class of infinite matroids, motivated by recent work of Diestel and Pott. Before we talk about the class itself, we shall first explain a little bit about what they were doing.

They looked at the question how one could extend the following theorem to infinite graphs: Two finite graphs are dual if and only if their cycle matroids are dual to each other. In the infinite case, the situation is no longer that easy since there are at least two different cycle matroids associated to an infinite locally finite graph G: the finite cycle matroid $M_{FC}(G)$, whose circuits are the finite circuits of G, and the topological cycle matroid $M_C(G)$, whose circuits are edge sets of topological circles in the end-compactification $|G|$ of G [23]. Note that $M_{FC}(G)$ is finitary and $M_C(G)$ is cofinitary. In fact, if G and G^* are dual in a suitable sense, then $M_{FC}(G)$ and $M_C(G^*)$ are dual to each other.

Motivated by the slight asymmetry of this fact, Diestel and Pott [35] introduced a more general context in which a stronger result is true. Given a partition of the ends of G into Ψ and $\Psi^{\complement}$, a Ψ-circuit is a topological circuit using only ends from Ψ, and a Ψ-tree is a set of edges maximal with the property that it does not include a Ψ-circuit. If $\Psi = \Omega(G)$, then the Ψ-circuits and Ψ-trees are the $M_C(G)$-circuits and $M_C(G)$-bases, whereas if $\Psi = \varnothing$, then the Ψ-circuits and Ψ-trees are the $M_{FC}(G)$-circuits and $M_{FC}(G)$-bases.

Let $G = (V, E, \Omega)$ and $G^* = (V^*, E, \Omega)$ be two finitely separable[1] 2-connected graphs with the same set of edges E and the same set of ends Ω. Diestel and Pott showed that if G and G^* are duals, then for every Ψ the complements of Ψ-trees in G are precisely the $\Psi^{\complement}$-trees in G^*. This means that if the set of Ψ-trees were the set of bases of some matroid, then the set of $\Psi^{\complement}$-trees in G^* would also be the set of bases of a matroid, namely its dual. This tempted Diestel and Pott to ask[2] the following.

[1] A graph is *finitely separable* if any two vertices can be separated by removing only finitely many vertices

[2] personal communication

Question 4.0.7. *Let G be a locally finite graph and $\Psi \subseteq \Omega(G)$. Is the set of Ψ-trees the set of bases of a matroid?*

Unfortunately, the answer to this question is no. Indeed, with some effort the question can be reduced to the question about path-connectedness in certain connected subspaces of $|G| \setminus \Psi^{\complement}$. Questions of this type have been considered by Georgakopoulos in [40], and his main counterexample from there also gives a counterexample here. However, the construction of the set Ψ in this case heavily relies on the Axiom of Choice (we will return to this point later).

The purpose of this chapter is to show that if the set Ψ is pleasant enough, in a sense we will now explain, then the set of Ψ-trees is the set of bases of a matroid.

It will turn out that the way pleasantness is measured has to with Determinacy of Sets (See Section 4.3 for an explanation why this is a good way to measure pleasantness here). Determinacy of sets is usually defined using games. Let $\Psi \subseteq A^{\mathbb{N}}$ for some set A, then the Ψ-game $\mathcal{G}(\Psi)$ is the following game between two players which has one move for every natural number. In each odd move the first player chooses an element of A whereas in each even move the second player chooses such an element. The first player wins if and only if the sequence they generate between them is in Ψ. The set Ψ is *determined* if one player has a winning strategy. The question which sets are determined has been investigated a lot in set theory [45]: The statement that all subsets $\Psi \subseteq A^{\mathbb{N}}$ with A countable are determined is called *the Axiom of Determinacy*, and is sometimes taken as an alternative to the Axiom of Choice. Indeed, if one assumes the Axiom of Determinacy instead of the Axiom of Choice, every set of real numbers becomes Lebesgue measurable [52]. A deep result in this area says that if Ψ is Borel (in the product topology), then it is determined [50].

We will want to consider slightly more general games in which the set of moves available to a player may vary depending on the moves made so far in the game, and may even sometimes be empty. Any game like this can be coded up by an equivalent game of the above type, so we will not worry too much about this issue. A game is *determined* if at least one of the players has a winning strategy.

Next we sketch how we transform Question 4.0.7 into an equivalent statement about determinacy of games. First we build from a given locally finite graph G what we call a *tree of matroids* which is a tree T whose ends are the ends of G, where for each node we store a finite matroid, and for each edge we store information about how to glue together the matroids for the two incident nodes. We do this in such a way that if we do all the gluing at once we get back all the relevant information about G.

Then we introduce the *circuit games* which are games of the above type in which each possible play defines a (possibly infinite) path in T starting at a fixed node of T. If play continues forever, then the path is infinite and the first player wins if and only if that path belongs to some end in Ψ (for a precise Definition of the game see Section 4.3 or 4.5). Having done this, we then are able to reduce Question 4.0.7 to a question about the determinacy of circuit games:

Theorem 4.0.8. *The set of Ψ-trees is the set of bases of a matroid if and only if certain circuit games are all determined.*

Applying the determinacy of Borel sets mentioned above, we obtain the following.

Corollary 4.0.9. *Let G be a locally finite graph and $\Psi \subseteq \Omega(G)$ a Borel set. Then the set of Ψ-trees is the set of bases of a matroid.*

So far we have talked only about locally finite graphs. Our proof in that case heavily relies on the assumption that the graph is locally finite (Once the definition of a tree of matroids is made precise, it is clear that this requires the graph to be locally finite). However, we are able to extend our results to all countable graphs. The argument takes the whole of Section 4.6 and uses a new technique; we expect that this technique can also be used in other contexts to extend results from locally finite to countable graphs.

The new matroids constructed here can be used to find counterexamples to various conjectures about infinite matroids. We shall illustrate this with three examples.

A class $\mathcal{F}$ of matroids is *well-quasi-ordered* if for every sequence $(M_n | n \in \mathbb{N})$ with $M_n \in \mathcal{F}$ there are $i < j$ such that $M_i \preceq M_j$. Robertson and Seymour proved [58] that the class of finite graphs is well-quasi-ordered. In 1965, Nash-Williams [53] proved that infinite trees are well-quasi-ordered. This was extended by Thomas to the class of graphs of bounded branch width [64]. On the other hand, he provided a sequence of uncountable graphs showing that the class of all graphs is not well-quasi-ordered [63]. It is not known if the class of countable graphs is well-quasi-ordered. For finite matroids, it is at the moment an important project to prove that the class of matroids representable over a fixed finite field is well-quasi-ordered. Geelen, Gerards and Whittle [39] proved that this is true if the matroids have bounded branch width. For infinite matroids almost nothing is known. Azzato and Jeffrey [8] made a first step towards proving that the class of finitary matroids of bounded branch width representable over a fixed finite field is well-quasi-ordered. In this chapter we consider the corresponding question for infinite matroids, not just for the finitary ones. The new matroids we construct can be used to show that the answer to this question is no, even in a very special case.

Corollary 4.0.10. *The countable binary matroids of branch-width at most* 2 *are not well-quasi-ordered (under the minor relation).*

The next conjecture concerns the number of possible non-isomorphic matroids on a countable ground set. Clearly, there cannot be more than $2^{2^{\aleph_0}}$. We show that this bound is actually attained.

Corollary 4.0.11. *There are $2^{2^{\aleph_0}}$ non-isomorphic tame matroids with no $M(K_4)$-minor and no $U_{2,4}$-minor on a countable ground set.*

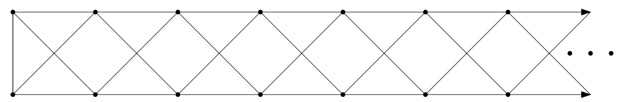


Figure 4.1: The graph Q

Diestel and Kühn [36] proved that there is a countable planar graph that has all other countable planar graphs as minors. Such a graph is called a *universal countable planar graph (with respect to the minor relation)*. In the same spirit, we call a matroid *universal for a class $\mathcal{F}$ of matroids (with respect to the minor relation)* if it is in $\mathcal{F}$ and it has every member of $\mathcal{F}$ as a minor. A matroid is *planar* if it is tame and all its finite minors are planar [19]. The result of Diestel and Kühn does not extend to infinite matroids:

Corollary 4.0.12. *There is no universal matroid for the class of countable planar matroids.*

If G is a locally finite graph then the *G-matroids* are those lying between the twinned matroids $M_{FC}(G)$ and $M_{TC}(G)$. Thus every Ψ-matroid is a G-matroid, but the converse does not hold.

Example 4.0.13. Let Q be the graph depicted in Figure 4.1. We say that two topological circuits of Q are *equivalent* if their symmetric difference is finite. Let $\mathcal{C}$ be any union of equivalence classes which includes all finite circuits. Then it can be shown that $\mathcal{C}$ is the set of circuits of a Q-matroid [21].

In cases like the above example, we can get a large set of badly behaved[3] matroids, each specified by a great deal of information 'at infinity'. In order to characterise the G-matroids of a general graph G we would require a specification of complex information distributed somehow over the ends of G, and this is currently intractable. However, this extra information is only necessary because the matroids in question are wild.

Theorem 4.0.14. *The tame G-matroids are exactly the Ψ-matroids for G.*

However the set Ψ need not be Borel. The question of which sets Ψ of ends give rise to matroids is tied to subtle set theoretic questions about determinacy of games.

Restricting our attention to tame matroids also allows us to resolve a problem due to Aigner-Horev, Diestel and Postle from [6] about the reconstruction of connected matroids from their 3-connected pieces. Any finite connected matroid M can be decomposed canonically into a tree of pieces, each of which is 3-connected, a circuit or a cocircuit [31, 59]. Any two adjacent pieces share only a single edge, and M can be reconstructed from this tree by taking 2-sums along all these edges.

[3] for example, such matroids can be non-binary in the sense that there are 3 circuits whose symmetric difference does not include a circuit

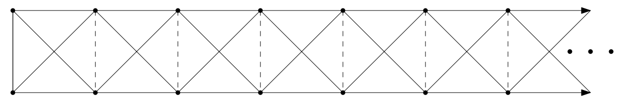


Figure 4.2: The tree decomposition of the graph Q

Recall that for any two matroids M_1 and M_2 that share only one edge, the *2-sum* $M_1 \oplus_2 M_2$ of M_1 and M_2 is the matroid whose edge set is the symmetric difference of those of M_1 and M_2 and whose circuits are of the following 3 types: circuits of M_1 avoiding e, circuits of M_2 avoiding e, and symmetric differences of an M_1-circuit containing e with an M_2-circuit containing e. This construction is associative, in the sense that if M_1 and M_2 meet in only one edge and M_2 and M_3 meet in only one edge, and M_1 has no edge in common with M_3, then $(M_1 \oplus_2 M_2) \oplus_2 M_3 = M_1 \oplus_2 (M_2 \oplus_2 M_3)$. Because of this associativity, it doesn't matter in what order we take the 2-sums at the edges of the tree: we always get back the original matroid M.

Aigner-Horev, Diestel and Postle partially extended this result to infinite matroids: they were able to show that there is such a canonical tree decomposition of any connected matroid [6]. It is a little surprising that the structure obtained is a genuine graph-theoretic tree, rather than one of the more order-theoretic or topological notions of infinite tree discussed in [54]. However, reconstruction of the original matroid from this tree is not so straightforward if the tree is infinite. For example, every Q-matroid decomposes into a ray of pieces, each of which is isomorphic to $M(K_4)$, as in Figure 4.2.

This example shows that the tree decomposition alone does not provide enough information to reconstruct the matroid: more information is needed. In [21], we answer the question of which extra information is needed to carry out this reconstruction. The answer is complicated and beyond the scope of this thesis. However, if we once more restrict our attention to tame matroids then there is a much more natural solution. In this chapter, we give a self-contained account of this more natural solution, for which the necessary arguments are much simpler than in [21].

The tree alone is still not enough information - the finite and topological cycle matroids of Q are both tame, and they give rise to the same tree of matroids. Just as in the graphic case, we may think of the topological cycle matroid as the matroid we get by allowing the end of the ray to be used by circuits and the finite cycle matroid as the matroid we get by forbidding circuits to use the end. This suggests what is in fact the right answer: the extra information we need is simply the set Ψ of ends of the tree which may be used by circuits.

More precisely, in Section 4.9 we give a construction which can be thought of as taking infinitely many 2-sums simultaneously. Given a suitable tree $\mathcal{T}$ of matroids and a suitable set Ψ of ends of $\mathcal{T}$, we showed in [14] that this construction allows us to build a matroid $M_\Psi(\mathcal{T})$ by (roughly speaking) sticking together the matroids along the edges of the tree and only allowing circuits to use the ends in Ψ. We can show that this construction suffices to rebuild any tame matroid from its canonical decomposition into circuits, cocircuits, and 3-connected pieces, together with information about which ends are used by circuits:

Theorem 4.0.15. *Let N be a tame matroid and let $\mathcal{T}$ be the tree of matroids $\mathcal{T}$ arising from the canonical tree decomposition of N.*

Then there is some $\Psi \subseteq \Omega(\mathcal{T})$ such that $N = M_\Psi(\mathcal{T})$.

The proof that the Ψ-matroids of a locally finite graph really are matroids also relied on gluing together an infinite tree of finite pieces. In this case, the tree structure arose from a tree decomposition of the graph. So both of the results mentioned above say that, for some particular tree structure, if we have any tame matroid whose circuits and cocircuits all fit, in some sense, with that tree structure, then this constrains the matroid to be of a very special type, which we call a Ψ-matroid. We give a general result of this type for trees of matroids in which any two adjacent matroids share at most one edge. See Theorem 4.10.10.

This chapter is closely based on two joint papers with Johannes Carmesin [14, 18]

4.1 What are the Ψ-matroids of a graph?

In this section we shall review the definitions of Ψ-circuits and $\Psi^{\complement}$-bonds for a graph G with a specified set Ψ of ends. Much of what we say will be a review of the early parts of [35], though we shall work in a slightly more general context: in [35], only finitely separable graphs are considered (a graph is *finitely separable* if any two vertices lie on opposite sides of some finite cut). We shall rely on [35] for the results we need about finitely separable graphs.

We say that two rays in a graph G are *equivalent* if they cannot be separated by removing finitely many vertices from G. An *end* of G is an equivalence class of rays under this relation, and the set of ends of G is denoted $\Omega(G)$.

Let d be the distance function on $V(G) \sqcup (0,1) \times E(G)$ considered as the ground set of the simplicial 1-complex formed from the vertices and edges of G. We define a topology VTOP on the set $V(G) \sqcup \Omega(G) \sqcup (0,1) \times E(G)$ by taking basic open neighbourhoods as follows:

- For $v \in V(G)$, the basic open neighbourhoods of v are the ϵ-balls $B_\epsilon(v) = \{x | d(v,x) < \epsilon\}$ for $\epsilon \leq 1$.
- For $(x,e) \in (0,1) \times E$ we say (x,e) is an *interior point* of e, and take the basic open neighbourhoods to be the ϵ-balls about (x,e) with $\epsilon \leq \min(x, 1-x)$.
- For $\omega \in \Omega(G)$, the basic open neighbourhoods of ω will be parametrised by the finite subsets S of $V(G)$. Given such a subset, we let $C(S,\omega)$ be the unique component of $G - S$ that contains a ray from ω, and let $\hat{C}(S,\omega)$ be the set of all vertices and inner points of edges contained in or incident

with $C(S,\omega)$, and of all ends represented by a ray in $C(S,\omega)$. We take the basic open neighbourhoods of ω to be the sets $\hat{C}(S,\omega)$.

We call the topological space obtained in this way $|G|$. We will need a fundamental lemma about this topology. A *comb* in G consists of a ray R together with infinitely many vertex-disjoint finite paths having precisely their first vertex on R. R is called the *spine* of the comb, and the final vertices of the paths are called the *teeth* of the comb.

Lemma 4.1.1. *Let G be a graph. Let X be a set of vertices of G and ω an end of G. Let R_ω be some ray in ω.*

Then ω is in the closure of X if and only if there is a comb with spine R_ω all of whose teeth are in X.

Proof. For the 'if' direction, let $\hat{C}(S,\omega)$ be a basic open neighbourhood of ω. Then only finitely many of the paths in the comb can meet S, so without loss of generality none of them do. Some tail of R must lie in $C(S,\omega)$, so without loss of generality the whole of R does. Then all teeth of the comb lie in $\hat{C}(S,\omega)$.

For the 'only if' direction, we apply Menger's Theorem to get either infinitely many vertex-disjoint R_ω-X-paths or a finite vertex set S whose removal separates X from R_ω. In the first case we are done and in the second we get a contradiction to the assumption that ω lies in the closure of X. □

For any set Ψ of ends of G, we set $\Psi^{\complement} = \Omega(G) \setminus \Psi$ and $|G|_\Psi = |G| \setminus \Psi^{\complement}$. This topological space, derived from a graph, seems almost to fit the notion of graph-like space explored in [19] (and closely related to the earlier work of [65]). We can make this precise as follows:

Definition 4.1.2. An *almost graph-like space* G is a topological space (also denoted G) together with a *vertex set* $V = V(G)$, an *edge set* $E = E(G)$ and for each $e \in E$ a continuous map $\iota_e \colon [0,1] \to G$ such that:

- The underlying set of G is $V \sqcup (0,1) \times E$
- For any $x \in (0,1)$ we have $\iota_e(x) = (x,e)$.
- $\iota_e(0)$ and $\iota_e(1)$ are vertices (called the *endvertices* of e).
- $\iota_e\restriction_{(0,1)}$ is an open map.

Such an almost graph-like space is a *graph-like space* if in addition for any $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$. This ensures that $V(G)$, considered as a subspace of G, is totally disconnected, and that G is Hausdorff.

Thus we can give $|G|_\Psi$ the structure of an almost graph-like space, with edge set $E(G)$ and vertex set $V(G) \cup \Psi$.

Let e be an edge in a graph-like space with $\iota_e(0) \neq \iota_e(1)$. Then ι_e is a continuous injective map from a compact to a Hausdorff space and so it is a homeomorphism onto its image. The image is compact and so is closed, and so is the closure of $(0,1) \times \{e\}$ in G. So in this case ι_e is determined by the properties above and the topology of G. The same is true if $\iota_e(0) = \iota_e(1)$: in this case we can lift ι_e to a continuous map from $S^1 = [0,1]/(0=1)$ to G, and argue as above that this map is a homeomorphism onto the closure of $(0,1) \times \{e\}$ in G.

Definition 4.1.3. We say that two vertices v and v' of an almost graph-like space G are *equivalent* (denoted $v \sim v'$) if for any disjoint open subsets U, U' of G partitioning $V(G)$, v and v' lie on the same side of the partition. The *graph-like quotient* $\widetilde{G}$ of G is the space obtained from G by identifying equivalent vertices. $\widetilde{G}$ has the structure of a graph-like space with the same edge set as G and with vertex set $V(G)/\sim$.

Lemma 4.1.4. *If G is an almost graph-like space, then $\widetilde{G}$ is a graph-like space.*

Proof. It is clear that $\widetilde{G}$ is an almost graph-like space. Let $[v]_\sim$ and $[v']_\sim$ be distinct vertices of $\widetilde{G}$. Then $v \not\sim v'$, and so there are disjoint open sets U and U' in G which partition $V(G)$ and with $v \in U$ and $v' \in U'$. Then any pair of equivalent vertices of G are either both in U or both in U', so U and U' induce disjoint open subsets $U/\sim$ and $U'/\sim$ of $\widetilde{G}$ which partition the vertices of $\widetilde{G}$ and such that $[v]_\sim \in U/\sim$ and $[v']_\sim \in U'/\sim$. □

We say that a cut b in a graph G is Ψ-*bounded* if the closure of b in $|G|_\Psi$ contains no ends. Thus if b is Ψ-bounded and ω is an end in Ψ then any ray to ω in G lies eventually on one side of b - we then say that ω is on that side of b.

Lemma 4.1.5. *Two vertices of $|G|_\Psi$ are equivalent if and only if they lie on the same side of every Ψ-bounded cut.*

Proof. For the 'if' direction, let v and v' be inequivalent vertices of $|G|_\Psi$, and let U and U' be disjoint open subsets partitioning $V(|G|_\Psi)$ with $v \in U$ and $v' \in U'$. Let b be the cut of G consisting of those edges with one endvertex in U and the other in U'. We shall show that b is Ψ-bounded. Let $\omega \in \Psi$. Without loss of generality $\omega \in U$ and so there is some S with $\hat{C}(S,\omega) \subseteq U$. Let $e \in b$, so one endvertex is in U'. Then since U' is open some interior point of e is in U', so that interior point of e isn't in $\hat{C}(S,\omega)$, so e doesn't meet $\hat{C}(S,\omega)$. Since e was arbitrary, $\hat{C}(S,\omega) \cap b = \varnothing$ and so ω isn't in the closure of b, as required.

For the 'only if' direction, let v and v' be equivalent vertices of $|G|_\Psi$ and let b be a Ψ-bounded cut of G. For each end $\omega \in \Psi$ there is by the definition of Ψ-boundedness a basic open set $U_\omega = \hat{C}(S_\omega, \omega)$ that doesn't meet b. Each set $C(S_\omega, b)$ is connected and so lies entirely on one side of b. Letting the sides of b be X and X', we may take $U = \bigcup_{v \in V(G) \cap X} B_{\frac{1}{2}}(v) \cup \bigcup_{\omega \in \Psi \cap X} U_\omega$ and $U' = \bigcup_{v \in V(G) \cap X'} B_{\frac{1}{2}}(v) \cup \bigcup_{\omega \in \Psi \cap X'} U_\omega$. Now since v and v' are equivalent they must either be both in U or both in U', so they lie on the same side of b. Since b was arbitrary, we are done. □

For a vertex v of G and a ray R of G, we say that v *dominates* R if there are infinitely many paths from v to R, vertex-disjoint except at v. We say that v *dominates* some end ω if it dominates some ray (or equivalently all rays) in ω.

Lemma 4.1.6. *Let $v \in V(G)$ dominate some end $\omega \in \Psi$. Then v and ω are equivalent as vertices of $|G|_\Psi$.*

Proof. Let R be a ray in ω and let $(P_i | i \in \mathbb{N})$ be a sequence of paths from v to ω meeting only at v. Suppose for a contradiction that there is a Ψ-bounded cut with v and ω on opposite sides. Then R must eventually lie on the same side of b as ω, so without loss of generality it lies entirely on that side. For each P_i, let v_i be the first vertex of P_i on the same side of b as ω. Then R together with the paths $v_i P_i$ forms a comb, so by Lemma 4.1.1 the end ω is in the closure of the set of teeth v_i, so it is in the closure of b, which is the desired contradiction. □

We let $\simeq$ be the smallest equivalence relation identifying any vertex with any end that it dominates. If G is finitely separable, then by [[35], Lemma 6], no two vertices will be equivalent under $\simeq$. In [35], the topological space $\widetilde{G}_\Psi$ is defined, for G a finitely separable graph, to be the quotient of $|G|_\Psi$ by $\simeq$. By the above lemma, $\sim$ refines $\simeq$ and so there is a continuous quotient map $f_G \colon \widetilde{G}_\Psi \to \widetilde{|G|_\Psi}$.

Lemma 4.1.7. *If G is finitely separable, then f_G is an homeomorphism.*

Proof. Since f is a quotient map, it suffices to show that it is injective.

Let v and v' be vertices of $\widetilde{G}_\Psi$. By [[35], Lemma 6], there is a finite set F of edges such that v and v' lie in disjoint open subsets of $\widetilde{G}_\Psi \setminus (0,1) \times F$ whose union is $\widetilde{G}_\Psi \setminus (0,1) \times F$. Let C be the connected component of $G \setminus F$ containing v (or a ray to v if v is an end), and let $b \subseteq F$ be the cut consisting of edges with one endvertex in C and the other not. Since b is finite, it is a Ψ-bounded cut, and so $v \not\sim v'$, as required. □

We therefore extend the definition in [35] by taking $\widetilde{G}_\Psi$ for G an arbitrary graph to be the graph-like quotient of $|G|_\Psi$.

In [19], topological circuits and topological bonds are defined in any graph-like space. A *circuit of* $\widetilde{G}_\Psi$, or just *a* Ψ*-circuit,* is an edge set whose $\widetilde{G}_\Psi$-closure is homeomorphic to the unit circle. A *bond of* $\widetilde{G}_\Psi$, or just *a* $\Psi^{\complement}$*-bond*-is an edge set of a minimal nonempty Ψ-bounded cut. In the following sense the Ψ-circuits and $\Psi^{\complement}$-bonds behave like the circuits and cocircuits of some matroid.

Lemma 4.1.8. *No Ψ-circuit meets any $\Psi^{\complement}$-bond in a single edge.*

Proof. Suppose for a contradiction that some Ψ-circuit o meets some $\Psi^{\complement}$-bond b in a single edge f

Then $\widetilde{G}_\Psi$ with all the interior points of edges of b removed has two connected components, namely the two sides of the bond. This contradicts the fact that $o - f$ is connected and contains both endvertices of f. □

We say that (G, Ψ) *induces a matroid* M if $E(M) = E(G)$ and the M-circuits are the Ψ-circuits and the M-cocircuits are the $\Psi^{\complement}$-bonds. In this case, we call M the Ψ*-matroid* of G. Even if we don't get a matroid, we call $(\mathcal{C}, \mathcal{D})$, where $\mathcal{C}$ is the set of Ψ-circuits of G and $\mathcal{D}$ is the set of $\Psi^{\complement}$-bonds of G, the Ψ-*system* of G.

A Ψ-tree is an edge set maximal with the property that it includes no Ψ-circuit. The main results of Diestel and Pott [35] are phrased in terms of Ψ-trees. These results let them to suspect that the Ψ-trees are the bases of some matroid. Although we shall mostly work with Ψ-circuits and $\Psi^{\complement}$-cocircuits instead, the fact that our results do confirm this suspicion in many cases follows from the following lemma.

Lemma 4.1.9. *If (G, Ψ) induces a matroid, then the bases of this matroid are the Ψ-trees.* □

We say that *G and G^* are plane duals* if there is an isomorphism ι from $E(G)$ to $E(G^*)$ that maps the G-cycles to the G^*-bonds. In [28], it is proved that ι induces a bijection ι_Ω between the ends of G and the ends of G^*. Then Lemma 4.1.9 yields the following.

Corollary 4.1.10. *Let G and G^* be two finitely separable graphs that are plane duals, as witnessed by some map ι. If (G, Ψ) induces a matroid, then its dual is induced by $(G^*, \iota_\Omega(\Psi^{\complement}))$.* □

Let's look at how tameness, $(O1)$ and $(O2)$ look for $\mathcal{C}$ the set of Ψ-circuits and $\mathcal{D}$ the set of $\Psi^{\complement}$-bonds for a locally finite graph G and $\Psi \subseteq \Omega(G)$. We abbreviate $G_\Psi = |G| \setminus (\Omega(G) \setminus \Psi)$.

First, we look at tameness. If a Ψ-circuit o and a $\Psi^{\complement}$-bond b meet infinitely, this gives rise to a minimal cover of o with infinitely many open sets, contradicting the compactness of o. Hence tameness is implied by the fact that every circuit is compact.

Next, we look at $(O1)$. If we have a Ψ-circuit o, then for every $e \in o$, the closure of $o - e$ in G_Ψ is still connected and hence there cannot be a $\Psi^{\complement}$-bond b meeting o only in e. Thus $(O1)$ is implied by the fact that for every Ψ-circuit o and every $e \in o$, the closure of $o - e$ in G_Ψ is connected.

Finally, we look at $(O2)$, so we are given a partition $E = P \dot\cup Q \dot\cup \{e\}$. Let $\bar{P}$ be the closure of the edge set P in G_Ψ. Let us consider the topological space $G_\Psi \cap \bar{P}$. Then $(O2)$ says that we either find an arc joining the two endvertices of e or we find a $\Psi^{\complement}$-bond whose induced topological bond separates the two endvertices. The first is equivalent to the statement that the two endvertices of e are in the same arc-component since the topological circles in $|G|_\Psi$ are precisely the Ψ-circuits.

The second is equivalent to the statement that the two endvertices of e are not in the same connected component. Indeed, if there is a such a bond, then the two endvertices are clearly not in the same connected component. For the converse, we assume that there is an open partition of $|G|_\Psi$ into two sets C_1 and C_2 with the two endvertices of e on different sides. Let V_1 be the set of those vertices in C_1. Then the set of edges crossing the G-separation $(V_1, V(G) \setminus V_1)$ is a (possibly infinite) cut of G. This cut is a disjoint union of bonds, which are all $\Psi^{\complement}$-bonds. From these, the bond including e is the desired one.

Hence $(O2)$ is equivalent to the following: The two endvertices of e lie in the same connected component of $G_\Psi \cap \bar{P}$ if and only if they lie in the same arc-component of $G_\Psi \cap \bar{P}$.

The question whether any connected subspace of $|G|$ is path-connected was solved by Georgakopoulos in [40]. Idneed, he constructed a locally finite graph G such that $|G|$ has a subset S that is connected but not path-connected. Note that since $|G|$ is a Hausdorff space, path-connectedness is equivalent to arc-connectedness. It is straightforward to show that $(G, S \cap \Omega(G))$ does not induce a matroid since it does not satisfy $(O2)$ for $P = S \cap E(G)$. We note for future reference that Georgakopoulos's argument heavily relies on the Axiom of Choice. We will soon be in a position to examine for which G and Ψ the pair (G, Ψ) induces a matroid.

4.2 Trees of matroids I

We wish to paste together infinite collections of matroids to obtain interesting new infinite matroids. Before we can be more explicit about this construction, we must give a precise account of the configurations of matroids we will seek to paste together. These will be given by tree-like structures.

Definition 4.2.1. A *tree $\mathcal{T}$ of matroids* consists of a tree T, together with a function M assigning to each node t of T a matroid $M(t)$ on ground set $E(t)$, such that for any two nodes t and t' of T, if $E(t) \cap E(t')$ is nonempty then tt' is an edge of T.

For any edge tt' of T we set $E(tt') = E(t) \cap E(t')$. We also define the *ground set* of $\mathcal{T}$ to be $E = E(\mathcal{T}) = \left(\bigcup_{t \in V(T)} E(t)\right) \setminus \left(\bigcup_{tt' \in E(T)} E(tt')\right)$.

We shall refer to the edges which appear in some $E(t)$ but not in E as *dummy edges* of $M(t)$: thus the set of such dummy edges is $\bigcup_{tt' \in E(T)} E(tt')$.

The idea is that the dummy edges are to be used only to give information about how the matroids are to be pasted together, but they will not be present in the final pasted matroid, which will have ground set $E(\mathcal{T})$.

Definition 4.2.2. If T is a tree, and tu is a (directed) edge of T, we take $T_{t \to u}$ to be the connected component of $T - t$ that contains u. If $\mathcal{T} = (T, M)$ is a tree of matroids, we take $\mathcal{T}_{t \to u}$ to be the tree of matroids $(T_{t \to u}, M\restriction_{T_{t \to u}})$.

We shall consider a couple of different sorts of pasting. First, in this section, we will consider a type of pasting corresponding to 2-sums. Later, in Section 4.4, we will define a type of pasting along larger separators. In each case, we will make use of some additional information to control the behaviour at infinity: a set Ψ of ends of T. The first type of pasting is only possible for a restricted class of trees of matroids.

Definition 4.2.3. A tree $\mathcal{T} = (T, M)$ of matroids is *of overlap 1* if, for every edge tt' of T, $|E(tt')| = 1$. In this case, we denote the unique element of $E(tt')$ by $e(tt')$.

Given a tree of matroids of overlap 1 as above and a set Ψ of ends of T, a *Ψ-pre-circuit* of $\mathcal{T}$ consists of a connected subtree C of T together with a function o assigning to each vertex t of C a circuit of $M(t)$, such that all ends of C are in Ψ and for any vertex t of C and any vertex t' adjacent to t in T, $e(tt') \in o(t)$ if and only if $t' \in C$. The set of Ψ-pre-circuits is denoted $\overline{\mathcal{C}}(\mathcal{T}, \Psi)$.

Any Ψ-pre-circuit (C, o) has an *underlying set* $\underline{(C, o)} = E \cap \bigcup_{t \in V(C)} o(t)$. Nonempty subsets of E arising in this way are called *Ψ-circuits* of $\mathcal{T}$. The set of Ψ-circuits of $\mathcal{T}$ is denoted $\mathcal{C}(\mathcal{T}, \Psi)$.

We shall show in Section 4.3 that $\mathcal{C}(\mathcal{T}, \Psi)$ very often gives the set of circuits of a matroid on $E_{\mathcal{T}}$. To do this, we will make use of the orthogonality axioms, and so we will also need a specified collection of putative cocircuits. These will be given by the $\Psi^{\complement}$-circuits of a tree of matroids dual to $\mathcal{T}$. Not only is there a natural notion of duality for trees of matroids, there are also natural notions of contraction and deletion.

Definition 4.2.4. Let $\mathcal{T} = (T, M)$ be a tree of matroids. Then the *dual* $\mathcal{T}^*$ of $\mathcal{T}$ is given by (T, M^*), where M^* is the function sending t to $(M(t))^*$. For a subset C of the ground set, the tree of matroids $\mathcal{T}/C$ obtained from $\mathcal{T}$ by *contracting* C is given by $(T, M/C)$, where M/C is the function sending t to $M(t)/(C \cap E(t))$. For a subset D of the ground set, the tree of matroids $\mathcal{T} \backslash D$ obtained from $\mathcal{T}$ by *deleting* D is given by $(T, M \backslash D)$, where $M \backslash D$ is the function sending t to $M(t) \backslash (C \cap E(t))$. We say that a tree of matroids $\mathcal{T}$ of overlap 1 together with a set Ψ of its ends *induce a matroid* $M = M(\mathcal{T}, \Psi)$ if $\mathcal{C}(M) \subseteq \mathcal{C}(\mathcal{T}, \Psi) \subseteq \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subseteq \mathcal{C}(\mathcal{T}^*, \Psi^{\complement}) \subseteq \mathcal{S}(M^*)$.

Lemma 4.2.5. *For any tree $\mathcal{T}$ of matroids, $\mathcal{T} = \mathcal{T}^{**}$. For any disjoint subsets C and D of the ground set of $\mathcal{T}$ we have $(\mathcal{T}/C)^* = \mathcal{T}^* \backslash C$, $(\mathcal{T} \backslash D)^* = \mathcal{T}^*/D$ and $\mathcal{T}/C \backslash D = \mathcal{T} \backslash D/C$. If $\mathcal{T}$ has overlap 1 and $(\mathcal{T}, \Psi)$ induces a matroid M, then $(\mathcal{T}/C \backslash D, \Psi)$ induces the matroid $M/C \backslash D$ and $(\mathcal{T}^*, \Psi^{\complement})$ induces the matroid M^*.* $\square$

We will sometimes use the expression *$\Psi^{\complement}$-cocircuits of $\mathcal{T}$* for the $\Psi^{\complement}$-circuits of $\mathcal{T}^*$.

We will examine the question of when $(\mathcal{T}, \Psi)$ induces a matroid using the orthogonality axioms. The question of whether $(O2)$ holds for these systems is tricky and will be addressed in Section 4.3. However, we are already in a position to give simple proofs of $(O1)$, and of tameness if all the $M(t)$ are tame.

Lemma 4.2.6 (($O1$) for trees of matroids of overlap 1)**.** *Let $\mathcal{T} = (T, M)$ be a tree of matroids, Ψ a set of ends of T, and let (C, o) and (D, b) be respectively a Ψ-pre-circuit of $\mathcal{T}$ and a $\Psi^{\complement}$-pre-circuit of $\mathcal{T}^*$. Then $|\underline{(C, o)} \cap \underline{(D, b)}| \neq 1$.*

Proof. Suppose for a contradiction that $|\underline{(C, o)} \cap \underline{(D, b)}| = \{e\}$, with $e \in t_0$. We recursively construct a sequence of nodes $t_n \in C \cap D$ forming a ray from t_0. To construct t_n, we note that $o(t_{n-1})$ meets $b(t_{n-1})$ (in e if $n = 1$, and in $e(t_{n-2}t_{n-1})$ if $n > 1$), so since they are respectively a circuit and a cocircuit of $M(t_{n-1})$ they must meet at least twice. Since they cannot meet in any edge of

E, they must meet in some edge $e(t_{n-1}t_n)$ with t_n adjacent to t_{n-1} in T and $t_n \neq t_{n-2}$ (for $n > 1$). It follows that $t_n \in C \cap D$. Then the end of this ray is in Ψ by the definition of (C, o) and is in $\Psi^{\complement}$ by the definition of (D, b), which is the desired contradiction. $\square$

Lemma 4.2.7 (Tameness for trees of tame matroids of overlap 1)**.** *Let $\mathcal{T} = (T, M)$ be a tree of tame matroids, Ψ a set of ends of T, and let (C, o) and (D, b) be respectively a Ψ-pre-circuit of $\mathcal{T}$ and a $\Psi^{\complement}$-pre-circuit of $\mathcal{T}^*$. Then $\underline{(C, o)} \cap \underline{(D, b)}$ is finite.*

Proof. Otherwise $C \cap D$ is infinite, and is locally finite since the $M(t)$ are all tame, and so it has an end ω of T in its closure. Then ω is in Ψ by the definition of (C, o) and is in $\Psi^{\complement}$ by the definition of (D, b), which is a contradiction. $\square$

We shall later show that any Ψ-system for a locally finite graph can be recovered by a more complex version of the construction above from a tree of finite matroids. We illustrate this by showing that many interesting Ψ-systems can already be recovered from the construction given above.

Definition 4.2.8. Let G be a graph. A *tree structure* on G is a tree T whose nodes form a partition of the vertices of G, such that distinct nodes are adjacent in T if and only if they contain adjacent vertices of G and the induced subgraph on each partition class is finite and connected. A tree structure *has width 2* if and only if for any pair of adjacent partition classes in T there are precisely 2 edges of G with one endvertex in each class.

Remark 4.2.9. *Any tree structure T on G induces a tree decomposition of G, in which the parts are the sets $E(t, t')$ of edges of G with one endvertex in t and the other in t', for t and t' (not necessarily distinct) nodes of T.*

Example 4.2.10. The wild cycle graph (so called because it includes a wild cycle in the sense of [34]), depicted in Figure 4.3, has a tree structure of width 2. The grey blobs represent the nodes of the tree.

Lemma 4.2.11. *Let T be a tree structure on a locally finite graph G. Then there is a canonical homeomorphism from the ends of G to the ends of T, sending an end ω of G to the unique end in the closure of the set of vertices of T that meet some ray R to ω.*

Remark 4.2.12. *We shall use this homeomorphism to identify the ends of T with those of G.*

Proof. First, we show that for any ray R in G there is a unique end $\varphi(R)$ of T in the closure of the set of vertices of T that meet R. There is certainly at least one such end, since R is infinite and so must meet infinitely many of the (finite) partition classes. If there were 2, say ω and ω', then for any vertex t of T whose removal separated ω and ω', R would have to meet t infinitely often, which would be a contradiction.

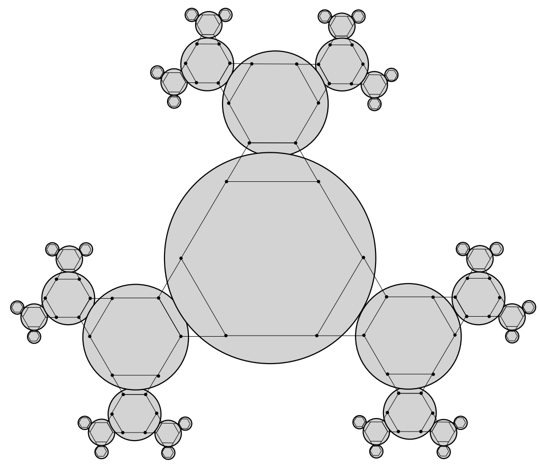


Figure 4.3: A tree structure on the Wild Cycle Graph

A similar argument shows that $\varphi(R)$ only depends on the end of G containing R: if there were 2 equivalent rays R and R' in G, with $\varphi(R) \neq \varphi(R')$, then for any vertex t of T whose removal separated $\varphi(R)$ from $\varphi(R')$, R and R' would eventually have to lie in the same component of $G \setminus t$, and so the components of $T - t$ meeting R and R' infinitely often would be the same, which would be a contradiction.

Thus φ induces a map $\tilde{\varphi}$ taking ends of G to ends of T. This map is injective, because for any distinct ends ω and ω' of G there is a finite set X of vertices of G separating ω from ω' in G: the (finite) set of vertices of T containing elements of X then separates $\tilde{\varphi}(\omega)$ from $\tilde{\varphi}(\omega')$ in T. It is surjective, because for any ray R in T there is a ray in G meeting exactly the nodes of T on R (here we use the fact that each node t of T is connected in G). It is continuous because for any node t of T the components of $G \setminus t$ are precisely the unions of the components of $T - t$, and it is open by the same fact together with the fact that any finite set X of vertices of G is a subset of a union of finitely many nodes of T. $\square$

Definition 4.2.13. Given a graph G together with a tree structure T on G and a node t of T, the *torso* $\tau(t)$ of G at a node t is the graph constructed as follows: the vertices are the elements of t, together with a new dummy vertex v_e for each edge e of G with one endpoint in t and the other not in t. The edges are of three types: edges of G with both ends in t, an edge vv_e for each edge $e = vv'$ of G with $v \in t$ and $v' \in t'$ with t' adjacent to t, and an edge joining any two dummy vertices corresponding to edges of G from vertices in t to vertices in the same adjacent node t' of T.

For a graph G with a tree structure T this gives a corresponding tree of finite matroids $\mathcal{T}(G,T) = (T, t \mapsto M(\tau(t)))$.

Observe that if T has width 2, then $\mathcal{T}(G,T)$ has overlap 1.

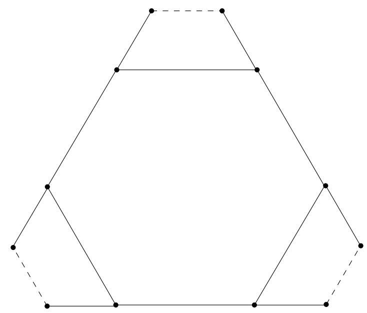


Figure 4.4: A typical torso of the Wild Cycle Graph

Example 4.2.14. Each torso arising from the tree structure in Example 4.2.10 is isomorphic to the graph in Figure 4.4.

We shall see later that this is a particularly simple example of a tree structure of width 2, but it illustrates that the topological space $\Omega(G)$ may still be rich enough in such cases to support very complicated subsets Ψ. We end this section by showing that the construction outlined above does capture the Ψ-systems of graphs in the width 2 case.

Lemma 4.2.15. *Let G be a graph, and let T be a tree structure on G of width 2. Let Ψ be a set of ends of G. Let G' be the graph obtained from G by subdividing each edge which has endpoints in different nodes of T.*[4] *Then the Ψ-circuits of G' are exactly the Ψ-circuits of $\mathcal{T}(G,T)$ and the Ψ^{C}-bonds of G' are exactly the Ψ^{C}-cocircuits of $\mathcal{T}(G,T)$.*

Proof. First we show that every Ψ^{C}-bond of G' is a Ψ^{C}-cocircuit of $\mathcal{T}(G,T)$. Let $\underline{b}$ be such a Ψ^{C}-bond. Let X be the set of vertices of G' on one side of $\underline{b}$. Let D be the set of vertices t of T such that $\tau(t)$ contains a vertex from X and a vertex not from X. Since both X and $V(G') \setminus X$ are connected, D is an intersection of 2 connected subsets of the tree T, and so is also connected. D doesn't include a ray to any end in Ψ, because $\underline{b}$ is a Ψ^{C}-bond.

For each $t \in D$, let $b(t)$ be the $\tau(t)$-cut of edges of $\tau(t)$ with one endpoint in X and the other not in X. Both sides of $b(t)$ are connected, since both X and $V(G') \setminus X$ are, so $b(t)$ is a circuit of $(M(\tau(t)))^*$. For any t' adjacent to t in T, let the shared dummy vertices of $\tau(t)$ and $\tau(t')$ be v_e and v_f. If $t' \notin D$ then v_e and v_f are on the same side of $\underline{b}$, so $e(tt') \notin b(t)$. If $t' \in D$, then since both X and $V(G') \setminus X$ are connected exactly one of v_e or v_f is in X, so $e(tt') \in b(t)$. Thus we obtain that $\underline{b} = \underline{(D,b)}$ is a Ψ^{C}-cocircuit of $\mathcal{T}(G,T)$.

Next, we show that every Ψ-circuit of G' is a Ψ-circuit of $\mathcal{T}(G,T)$. Let $\underline{o}$ be such a Ψ-circuit, and let C be the set of vertices t of T such that $\underline{o}$ meets $\tau(t)$. For any $t \notin C$, there can only be one component of $T - t$ meeting C, since the

[4] formally, we add a new vertex v_e corresponding to each such edge $e = vv'$, and replace e in the set of edges by the two new edges vv_e and $v'v_e$.

unions of these components are separated by t in $G \setminus t$. Thus C is a subtree of T. Any end in the closure of C is also in the closure of $\underline{o}$ and so must lie in Ψ.

For any $t \in C$, let $o(t)$ be the union of $\underline{o} \cap E(\tau(t))$ with the set of all edges ee' of $\tau(t)$ where e and e' are the two edges of G with endpoints in both t and t' for some t' adjacent to t in C. Then every vertex of $\tau(t)$ has degree 0 or 2 with respect to $o(t)$: this is immediate for vertices in t, and vertices given by edges with one endpoint in t and the other in t' have degree 0 if $t' \notin C$, 2 if $t' \in C$. To show that $o(t)$ is a circuit, it remains to show that it is connected. Suppose not, for a contradiction. Then there is a cut b of $\tau(t)$ not meeting $o(t)$ but with edges of $o(t)$ on both sides, so there is such a cut that doesn't contain any dummy edges. This cut is a finite cut of G not meeting $\underline{o}$ but with edges of $\underline{o}$ on both sides, which is the desired contradiction. Thus each $o(t)$ is a circuit in $M(\tau(t))$. Thus we obtain that $\underline{o} = \overline{(C, o)}$ is a Ψ-circuit of $\mathcal{T}(G, T)$.

To show that every Ψ^{C}-cocircuit $\overline{(D, b)}$ of $\mathcal{T}(G, T)$ is a Ψ^{C}-bond of G', we pick any edge $e_0 \in \overline{(D, b)}$ and let X and Y be the sets of vertices in the same connected components of $G' \setminus \overline{(D, b)}$ as the endvertices x_0, y_0 of e_0. If $X = Y$ then there is a finite circuit in G' meeting $\overline{(D, b)}$ just once, which is impossible by the argument above and Lemma 4.2.6. Let t_0 be the vertex of T with $e_0 \in \tau(t_0)$. We prove by induction on the distance of t from t_0 that $X \cup Y$ includes all vertices of $\tau(t)$ and if $t \in D$ then $b(t)$ is the set of edges of $\tau(t)$ with one end in X and the other in Y. This is immediate if $t = t_0$, since $b(t_0)$ is a bond of $\tau(t_0)$. For any other $t' \in V(T)$, let t be the neighbour of t' in the direction of t_0. If $t' \in D$ then also $t \in D$ and so of the two dummy vertices shared by $\tau(t)$ and $\tau(t')$ one is in X and the other in Y, giving the result since $b(t')$ is a bond of $\tau(t')$. If $t' \notin D$ then the two dummy vertices shared by $\tau(t)$ and $\tau(t')$ are either both in X or both in Y, so either all vertices of $\tau(t')$ are in X or all of them are in Y. This shows that $\overline{(D, b)}$ is the bond of G' consisting of all edges with one end in X and the other in Y. It is a Ψ^{C}-bond since every end in its closure is in the closure of D and so is in Ψ^{C}.

Finally, we show that every Ψ-circuit of $\mathcal{T}(G, T)$ is a Ψ-circuit of G'. Consider such a circuit $\overline{(C, o)}$. By the above argument and Lemma 4.2.6 it never meets a finite bond of G' just once and so, by the dual of Lemma 1.2.1 applied to the topological cycle matroid of G' it is a union of topological circuits. To show that it is the edge set of a single topological circle, it is enough by Lemma 1.3.6 to show that for any $e, f \in \overline{(C, o)}$ there is a finite bond b of G with $b \cap \overline{(C, o)} = \{e, f\}$. Consider the unique finite path $t_1, ...t_n$ in T with $e \in E(\tau(t_1))$ and $f \in E(\tau(t_n))$. Let $e_0 = e, e_n = f$ and for $0 < i < n$ let $e_i = e(t_i t_{i+1})$. For each $i \leq n$ we let b_i be any bond of $\tau(t_i)$ with $b_i \cap o(t_i) = \{e_{i-1}, e_i\}$. Without loss of generality we may choose the b_i to contain no dummy edges other than the e_i. Then $\bigcup_{i=1}^{n} b_i \setminus E$ is the desired finite bond of G. Thus $\overline{(C, o)}$ is a topological circuit of G. It is a Ψ-circuit since every end in its closure is in the closure of C and so is in Ψ. $\square$

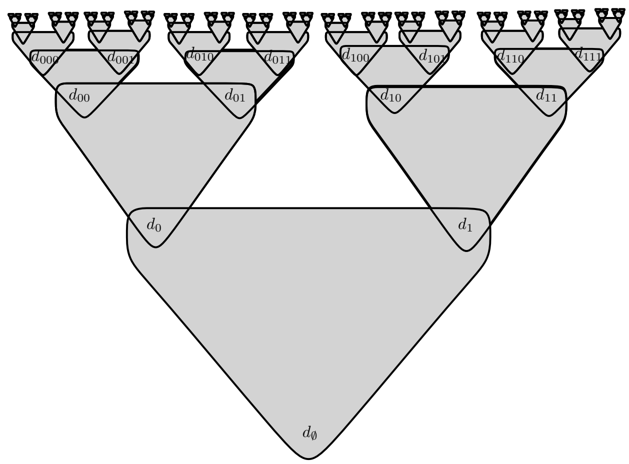


Figure 4.5: The tree of matroids $\mathcal{T}^{game}$

4.3 Determinacy and $(O2)$ for trees of matroids of overlap 1

In Section 4.11, we saw that $(O2)$ corresponds, for Ψ-systems, to a principle implying path-connectedness from connectedness. Here we will show that, for the systems arising from trees of matroids, $(O2)$ has close links with determinacy of games. We begin by analysing an illuminating example.

Let $\mathcal{T}^{game}$ be the tree of matroids given by (T_2, M^{game}), as follows: T_2 is the infinite rooted binary tree (to fix notation, we take the vertices of T_2 to be the finite sequences from $\{0, 1\}$, with s adjacent to each of $s0$ and $s1$ for any such sequence s, and we call the empty sequence $\emptyset$). For any node s of T_2, we take the ground set of $M^{game}(s)$ to be $\{d_s, d_{s0}, d_{s1}\}$ and we take $M^{game}(s)$ to be uniform, of rank 1 if the length of s is even and of rank 2 if the length of s is odd. This tree of matroids has overlap 1, with all edges except $d_\emptyset$ being dummy edges. The ground set E^{game} of $\mathcal{T}^{game}$ is simply $\{d_\emptyset\}$. The structure of this tree of matroids is displayed in Figure 4.5.

Although the ground set has only 1 element, so that the sets of Ψ-circuits of $\mathcal{T}$ or $\mathcal{T}^*$ must be very simple for any Ψ, our analysis of $(O2)$ will still be complex because of the way in which these sets arise from $\mathcal{T}$. Any instance of $(O2)$ for trees of matroids is reducible to one on which the ground set has only one element, since $(O2)$ holds for the partition $E = \{e\}\dot\cup P\dot\cup Q$ of the ground set of $\mathcal{T}$ if and only if it holds for the partition $\{e\} = \{e\}\dot\cup\varnothing\dot\cup\varnothing$ of the ground set of $\mathcal{T}/P\backslash Q$. However, as this section will illustrate, this reduction does not diminish the complexity of the problem.

Let's fix some set $\Psi \subseteq \{0, 1\}^{\mathbb{N}}$ and examine the meaning of $(O2)$ applied to the Ψ-circuits and $\Psi^{\complement}$-cocircuits of $\mathcal{T}^{game}$, with the partition $E^{game} = \{d_\emptyset\}\dot\cup\varnothing\dot\cup\varnothing$. If $(O2)$ is true, then one of the following 2 things happens:

1. There is a Ψ-circuit through $d_\emptyset$.
2. There is a $\Psi^{\complement}$-cocircuit through $d_\emptyset$.

Let's think first of all about (1). This says that we can find a Ψ-precircuit (C, o) with $\emptyset \in C$, $d_\emptyset \in o(\emptyset)$. The shape of C is now quite constrained. For any $s \in C$ we have $d_s \in o(s)$. If s has even length, then $o(s)$ can only be $\{d_s, d_{s0}\}$ or $\{d_s, d_{s1}\}$. On the other hand, if s has odd length then $o(s)$ can only be $\{d_s, d_{s0}, d_{s1}\}$. Thus C is a set of finite sequences from $\{0, 1\}$ with the following properties:

- $\emptyset \in C$.
- C is closed under taking initial segments.
- For any $s \in C$ of even length, exactly one of $s0$ and $s1$ is in C.
- For any $s \in C$ of odd length, both of $s0$ and $s1$ are in C.
- For any $s \in \{0, 1\}^{\mathbb{N}}$ such that all finite initial segments of s are in C, $s \in \Psi$.

These properties collectively state that C gives a winning strategy for the first player in the game $\mathcal{G}(\Psi)$ from the introduction, with Ψ considered as a subset of $\{0, 1\}^{\mathbb{N}}$: the first player should play so as to ensure that the finite sequence generated so far always remains in s. Conversely, given a set C with these properties, we can define a function o on C sending s to $\{d_s, d_{s0}\}$ if s has even length and $s0 \in C$, to $\{d_s, d_{s1}\}$ if s has even length and $s1 \in C$, and to $\{d_s, d_{s0}, s_{s1}\}$ if s has odd length. Then $\underline{(C, o)}$ is a Ψ-circuit of $\mathcal{T}^{game}$ with $\underline{(C, o)}$ witnessing (1).

What this shows is that (1) is equivalent to the statement that the first player has a winning strategy in the game $\mathcal{G}(\Psi)$. A similar argument shows that (2) is equivalent to the statement that the second player has a winning strategy in that game. Thus in this case $(O2)$ is equivalent to determinacy of the game $\mathcal{G}(\Psi)$. By introducing some slightly more complex games, we will now show that for any tree $\mathcal{T}$ of matroids of overlap 1 and any set Ψ of ends of $\mathcal{T}$ there is a collection of games such that $(\mathcal{T}, \Psi)$ induces a matroid if and only if all of the games in that collection are determined.

We temporarily fix such a $\mathcal{T}$ and Ψ, together with a partition $E = \{e\} \dot\cup P \dot\cup Q$ of the ground set of $\mathcal{T}$. Let t_0 be the node of T such that $e \in E(t_0)$.

Definition 4.3.1. The *circuit game* $\mathcal{G} = \mathcal{G}(\mathcal{T}, \Psi, P, Q)$ is played between two players, called Sarah and Colin[5], as follows:

Play alternates between the players, with Sarah making the first move. At any point in the game there is a *current node* $t_c \in V(t)$, and a *current edge* $e_c \in E(t_c)$. Initially we set $t_c = t_0$ and $e_c = e$ to be the node of T with $e_c \in E(t_c)$. For any n the $(2n-1)^{\text{st}}$ move is made by Sarah: she must play a

[5]The name 'Sarah' has been chosen because it sounds similar to 'circuit', and 'Colin' because it may be pronounced co-lin, to sound a bit like 'cocircuit'

circuit o_n of $M(t_c)$ such that $e_c \in o_n$ but $o_n \cap Q = \emptyset$. Then the $2n^{\text{th}}$ move is made by Colin: he must play a node t_n adjacent to t_c and further from t_0 than t_c is, such that $e(t_c t_n) \in o_n$. After he does this, the current node is updated to t_n, and the current edge to $e(t_{n-1}t_n)$. If play continues forever, then Sarah wins if the end ω of T containing $(t_n | n \in \mathbb{N})$ is in Ψ, and Colin wins if $\omega \in \Psi^{\complement}$.

The *cocircuit game* $\mathcal{G}^* = \mathcal{G}^*(\mathcal{T}, \Psi, P, Q)$ is the game like the dual circuit game $\mathcal{G}(\mathcal{T}^*, \Psi^{\complement}, Q, P)$, but with the roles of Sarah and Colin reversed. We will also use a different notation for the cocircuit game, putting stars on the notation for the circuit game. Thus for example the current edge is denoted e_c^* and Colin's n^{th} move is denoted o_n^*.

Lemma 4.3.2. *Sarah has a winning strategy in $\mathcal{G}$ if and only if there is a Ψ-circuit $\underline{(C, o)}$ of $\mathcal{T}$ with $e \in \underline{(C, o)} \subseteq \{e\} \dot\cup P$.*

Proof. Suppose first that there is such a Ψ-circuit $\underline{(C, o)}$. Then Sarah can win in $\mathcal{G}$ by always choosing $o(t_c)$ when it is her turn to play.

Suppose for the converse that Sarah has a winning strategy σ in $\mathcal{G}$. Let C be the set of nodes t of T such that there is some finite play according to σ consisting of $2n + 1$ moves for some n after which t is the current node. Then this play is unique, since Sarah's moves are determined by σ, and Colin's must be the sequence of vertices along the finite path in T from t_0 to t. We set $o(t)$ to be the final move o_n made by Sarah in that play. It is immediate that (C, o) is a Ψ-pre-circuit of $\mathcal{T}$ with the desired properties. □

Corollary 4.3.3. *Colin has a winning strategy in $\mathcal{G}^*$ if and only if there is a $\Psi^{\complement}$-cocircuit $\underline{(C, o)}$ of $\mathcal{T}$ with $e \in \underline{(C, o)} \subseteq \{e\} \dot\cup Q$.* □

In order to relate $(O2)$ to determinacy of $\mathcal{G}$, we need to show that $\mathcal{G}$ and $\mathcal{G}^*$ are closely related games.

Lemma 4.3.4. *Colin has a winning strategy in $\mathcal{G}$ if and only if he has one in $\mathcal{G}^*$.*

Proof. For the 'if' part, suppose that he has a winning strategy σ^* in $\mathcal{G}^*$. Then he can win in $\mathcal{G}$ by playing as follows:

He should imagine an auxilliary play in the game $\mathcal{G}^*$, in which he plays according to σ^*, and for which he should ensure that at any point the current edge and node agree with those in $\mathcal{G}$. When Sarah makes the move o_n, he should pick some edge in $o_n \cap o_n^*$ other than e_n. This edge t can then only be a dummy edge $e(t_c t)$ for some t adjacent to t_c. He should play t as t_n in $\mathcal{G}$ and imagine that Sarah also plays t as t_n^* in $\mathcal{G}^*$. If play continues forever, then the end ω containing $(t_n | n \in \mathbb{N})$ is in $\Psi^{\complement}$ since σ^* is winning.

For the 'only if' part, suppose that he has a winning strategy σ in $\mathcal{G}$. Then he can win in $\mathcal{G}^*$ by playing as follows:

He should imagine an auxilliary play in the game $\mathcal{G}$, in which he plays according to σ, and for which he should ensure that at any point the current edge and node agree with those in $\mathcal{G}^*$. When he has to make a move o_n^*, he should consider the set R of responses prescribed by σ to legal moves o_n that Sarah could make in $\mathcal{G}$. Then $R \cup Q$ meets every circuit o of $M(t_c)$ with $e_c \in o$. Thus since $(O2)$ holds for the matroid $M(t_c)$ there is some cocircuit o_n^* of that matroid with $e_c \in o_n^* \subseteq \{e_c\} \cup R \cup Q$, and Colin should play such a cocircuit. If Sarah responds by playing t_n^*, then we must have $t_n^* \in R$ and so there is some legal move o_n in $\mathcal{G}$ to which σ prescribes the response t_n^*. Then Colin should imagine that, in the play of $\mathcal{G}$, Sarah plays o_n and he responds by playing t_n^* as t_n. If play continues forever, then the end ω containing $(t_n^* | n \in \mathbb{N})$ is in $\Psi^{\complement}$ since σ is winning. $\square$

Corollary 4.3.5. *$(O2)$ holds for the partition $E = \{e\} \dot\cup P \dot\cup Q$ of the groundset of $\mathcal{T}$ if and only if $\mathcal{G}(\mathcal{T}, \Psi, P, Q)$ is determined.* $\square$

Since any game $\mathcal{G}(\Psi)$ with $\Psi \subseteq A^{\mathbb{N}}$ and A countable can be coded by such a game with $A = \{0, 1\}$, we also get:

Corollary 4.3.6. *The Axiom of Determinacy is equivalent to the statement that every set Ψ of ends of every tree of finite matroids of overlap 1 induces a matroid. If the Axiom of Choice holds then there is a tree of finite matroids of overlap 1 and a set Ψ of ends of that tree that doesn't induce a matroid.* $\square$

Corollary 4.3.7. *For any tree of countable tame matroids $\mathcal{T} = (T, M)$ of overlap 1 and any Borel set Ψ of ends of T, the pair $(\mathcal{T}, \Psi)$ induces a matroid.*

In the Appendix, we prove that the assumptions that $\mathcal{T}$ is countable and tame are not needed.

Proof. This is immediate from Borel determinacy, Corollary 4.3.5 and the fact that for each partition of the ground set as $\{e\} \dot\cup P \dot\cup Q$ the projection map from the set of legal infinite plays in $\mathcal{G}(\mathcal{T}, \Psi, P, Q)$ to $\Omega(T)$ sending a play to the end containing the sequence $(t_n | n \in \mathbb{N})$ for that play is continuous. $\square$

In Section 4.5 we will extend these techniques to trees of finite representable matroids and so get results applying to all locally finite graphs. However, our results so far already have implications for graphs with a tree structure of width 2.

Theorem 4.3.8. *Let G be a graph with a tree structure T of width 2, and Ψ a Borel set of ends of G. Then (G, Ψ) induces a matroid.*

Proof. Let G' be obtained from G by subdividing certain edges as in the proof of Lemma 4.2.15. Then by Corollary 4.3.7, $(\mathcal{T}(G, T), \Psi)$ induces a matroid M, which by Lemma 4.2.15 is also induced by (G', Ψ). Then the matroid obtained from M by contracting one of each pair of edges subdividing an edge of G is induced by (G, Ψ). $\square$

Assuming the Axiom of Choice holds, we can also give another example of a graph G and a set of ends Ψ of G such that (G, Ψ) doesn't induce a matroid.

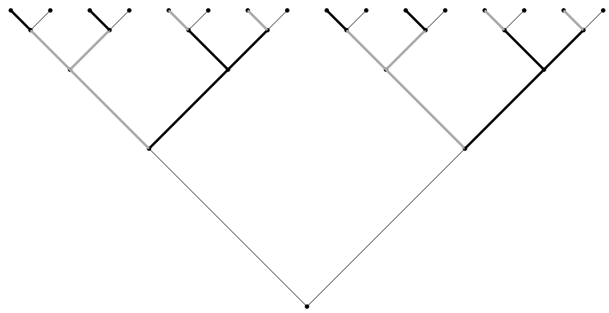


Figure 4.6: The binary tree with a particular 3-coloring of its edges.

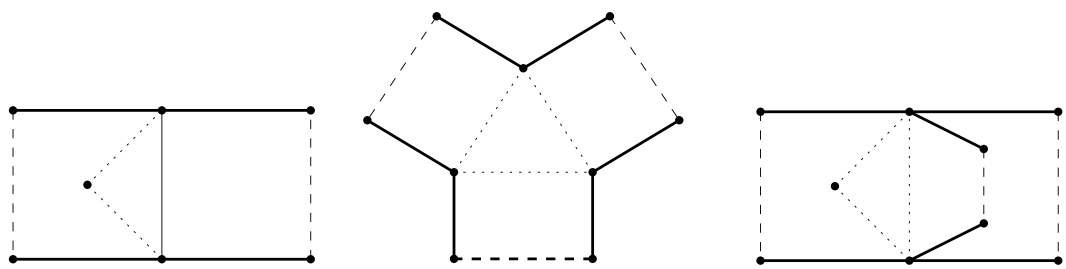


Figure 4.7: The Torsos from Example 4.3.9.

Example 4.3.9. Figure 4.6 illustrates that we may 3-colour the edges of T_2 in such a way that the edges incident with any vertex s are the same colour if s has even length considered as a finite $\{0,1\}$-sequence, but are all different colours if s has odd length.

We fix such a 3-colouring given as a function $c\colon E(T_2) \to V(K_3)$. Let G be the graph obtained from $T_2 \times K_3$ by removing all edges of the form $e \times \{c(e)\}$ with $e \in E(T_2)$. Then G has a tree structure of width 2, in which the vertices of T are the sets $\{s\} \times V(K_3)$ with s a vertex of T_2. The shapes of the torsos for this tree structure are given in Figure 4.7.

Let Ψ be a set of ends of G such that $\mathcal{G}(\Psi)$ is not determined. Then the tree of matroids obtained from $\mathcal{T}(G,T)$ by contracting the bold edges in Figure 4.7 and deleting the dotted edges is isomorphic to $\mathcal{T}^{game}$, and we know $(\mathcal{T}^{game}, \Psi)$ does not induce a matroid. Thus $(\mathcal{T}(G,T), \Psi)$ does not induce a matroid, and so (G, Ψ) cannot induce a matroid, and so $(T_2 \times K_3, \Psi)$ does not induce a matroid.

Now we can explain the sense in which we said that the wild cycle graph was relatively simple when we discussed it in Section 4.9.

Lemma 4.3.10. *For any set Ψ of ends of the wild cycle graph G_{wild}, the pair (G_{wild}, Ψ) induces a matroid.*

Proof. As in the proof of Theorem 4.3.8, it is enough to check that $(\mathcal{T}(G_{wild}, T), \Psi)$ induces a matroid, where T is the tree structure from Example 4.2.10. Now we may note that the torsos for this tree structure, depicted in Figure 4.4, have the property that no bond contains more than 2 dummy edges. Thus in the cocircuit games for this tree of matroids, all of Sarah's moves apart from her first one are forced. Thus all these games are determined, and we are done by Lemma 4.3.4 and Corollary 4.3.5. □

There are other simple examples of graphs which induce a matroid for any Ψ:

Lemma 4.3.11. *Let T be any locally finite tree, and let Ψ be any set of ends of $T \times K_2$. Then $(T \times K_2, \Psi)$ induces a matroid.*

Proof. Once more it is enough to check that $(\mathcal{T}(T \times K_2, T'), \Psi)$ induces a matroid, where T' is the tree structure whose vertices are the sets $\{t\} \times V(K_2)$ for $t \in V(T)$. The torsos are of the form $S \times K_2$, where S is a finite star. They have the property that no circuit contains more than 2 dummy edges, and so all the circuit games for this tree of matroids are determined, and we are done by Corollary 4.3.5. □

4.4 Trees of matroids II

To capture graphs which cannot be given a tree structure of width 2, we need a more general notion of pasting in a tree of matroids, for which we will work with representable matroids. Strictly speaking, we will be pasting together *represented* matroids, since the matroid structure after pasting can depend on the choices of representation before pasting.

Before returning to trees of matroids, we shall first outline how to paste together just 2 matroids in this way. We shall take a slightly unusual point of view on representations: we think of a representation of a finite matroid M over a field k as given by a subspace U of $k^{E(M)}$ such that the minimal nonempty supports of elements in U are the M-circuits (there is such a subspace if and only if M is representable in the usual sense over k). The dual of M is then represented by the orthogonal complement $U^\perp$ of U.

Now suppose that we have two finite matroids M_1 and M_2 where M_i has ground set E_i and is represented over k by a subspace U_i of k^{E_i}. Then there are canonical embeddings of U_1, U_2 and $k^{E_1 \triangle E_2}$ as subspaces of $V = k^{E_1 \cup E_2}$. We let $U_1 \triangle U_2$ be $(U_1 + U_2) \cap k^{E_1 \triangle E_2}$: the vectors in this space are those v such that there are $v_1 \in U_1$ and $v_2 \in U_2$ with $v_1\restriction_{E_1 \cap E_2} = -v_2\restriction_{E1 \cap E_2}$, $v\restriction_{E_1 \setminus E_2} = v_1\restriction_{E_1 \setminus E_2}$ and $v\restriction_{E_2 \setminus E_1} = v_2\restriction_{E_2 \setminus E_1}$.

This construction is well behaved with respect to duality. The orthogonal complement of $U_1 \triangle U_2$ in V is $(U_1^\perp \cap U_2^\perp) + \left(k^{E_1 \triangle E_2}\right)^\perp = (U_1^\perp \cap U_2^\perp) + k^{E_1 \cap E_2}$. So the orthogonal complement of $U_1 \triangle U_2$ in $k^{E_1 \triangle E_2}$ is the intersection of that space with $k^{E_1 \triangle E_2}$, which is the set of those w such there are $w_1 \in U_1^\perp$ and $w_2 \in U_2^\perp$ with $w_1\restriction_{E_1 \cap E_2} = w_2\restriction_{E_1 \cap E_2}$, $w\restriction_{E_1 \setminus E_2} = w_1\restriction_{E_1 \setminus E_2}$ and $w\restriction_{E_2 \setminus E_1} = w_2\restriction_{E_2 \setminus E_1}$.

This isn't quite the same as $U_1^\perp \triangle U_2^\perp$ - there is a missing minus sign in one of the equations - but the supports of the vectors, and so the induced matroids, are the same. Thus we have $(M_1 \triangle M_2)^* = M_1^* \triangle M_2^*$.

This construction also allows us to glue together pairs of tame thin sums matroids, provided that the overlap of their ground sets is finite. The details will not be addressed here, but the basic reason is that in proving (O2), which is potentially the trickiest of the axioms, it is possible by contracting P and deleting Q to reduce the problem to one on the finite set consisting of e and the edges in the overlap set.

If we want to use a construction like this to glue together a tree of matroids, we will need a representation of each of the (finite) matroids.

Definition 4.4.1. Let k be a finite field. A *k-representation* of a tree (T, M) of matroids is a function V assigning to each vertex t of T a subspace $V(t)$ of $k^{E(t)}$ such that $M(V(t)) = M(t)$. The *dual* $V^\perp$ of such a k-representation is the representation of the dual tree of matroids which assigns to each node t of T the space $V(t)^\perp$. We will only ever consider representations of trees of finite matroids.

In this context, a *Ψ-vector of V* consists of a function v assigning to each vertex t of T a vector $v(t) \in V(t)$, in such a way that for any edge tt' of T we have $v(t)\restriction_{E(tt')} = v(t')\restriction_{E(tt')}$ and that every end of T in the closure of $\{t \in V(T) | v(t) \neq 0\}$ is in Ψ. The set of such Ψ-vectors is denoted $\mathcal{V}(V, \Psi)$. The *support* of a Ψ-vector v is the set $\underline{v} = E \cap \bigcup_{t \in T} \underline{v(t)}$. The set of such supports is denoted $\underline{\mathcal{V}}(V, \Psi)$.

We say that (V, Ψ) *induces a matroid* $M = M(V)$ if $\mathcal{C}(M) \subseteq \underline{\mathcal{V}}(V, \Psi) \subseteq \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subseteq \underline{\mathcal{V}}(V^\perp, \Psi^\complement) \subseteq \mathcal{S}(M^*)$.

The question of when $(O2)$ holds for these systems is once more tricky, and will be addressed in Section 4.5. However, we are already in a position to give a simple proof of $(O1)$ and of tameness.

Lemma 4.4.2 (($O1$) and tameness for representable trees of finite matroids)**.** *Let $\mathcal{T} = (T, M)$ be a tree of finite matroids with a k-representation V, let Ψ be a set of ends of T, and let v and w be respectively a Ψ-vectors of V and a $\Psi^\complement$-vector of $V^\perp$. Then $|\underline{v} \cap \underline{w}|$ is finite but not equal to 1.*

Proof. If it were infinite, then there would be an end ω in the closure of $\underline{v} \cap \underline{w}$ and so in the closure of both $\{t \in V(T) | v(t) \neq 0\}$ and $\{t \in V(T) | w(t) \neq 0\}$, so that ω would have to be in both Ψ and $\Psi^\complement$, a contradiction. So it is finite.

Now fix some node t_0 of T and for any node t let $d(t)$ be the distance from t_0 to t in T (thus $d(t_0) = 0$). Let $\hat{v} \colon E \to k$ be the function sending $e \in E(t)$ to $v(t)(e)$, and $\hat{w} \colon E \to k$ be the function sending $e \in E(t)$ to $(-1)^{d(t)} w(t)(e)$.

Then we have

$$\begin{aligned}\sum_{e\in E}\hat{v}(t)\hat{w}(t) &= \sum_{t\in V(T)}(-1)^{d(t)}\left(\sum_{e\in E(t)}v(t)(e)w(t)(e)-\sum_{tt'\in E(T)}\sum_{e\in E(tt')}v(t)(e)w(t)(e)\right)\\ &= -\sum_{tt'\in E(T)}\left((-1)^{d(t)}+(-1)^{d(t')}\right)\left(\sum_{e\in E(tt')}v(t)(e)w(t)(e)\right)\\ &= 0\end{aligned}$$

and it follows that $|\underline{v}\cap\underline{w}| = |\underline{\hat{v}}\cap\underline{\hat{w}}| \neq 1$. $\square$

Remark 4.4.3. *Once we have shown that this system induces a (tame) matroid M, the proof above will also show that it is a thin sums matroid over k according to the characterisation given in [12], since we can choose the function $c_{\underline{v}}\colon \underline{v}\to k$ for a circuit $\underline{v}$ to be given by $\hat{v}\restriction_{\underline{v}}$ and similarly take $d_{\underline{w}} = \hat{w}\restriction_{\underline{w}}$.*

With this new construction, we can capture the Ψ-system of any graph with a tree decomposition.

Definition 4.4.4. For a graph G with a tree structure T, let $V(G,T)$ be the unique representation of $\mathcal{T}(G,T)$ over $\mathbb{F}_2$ (such a representation exists since for each $t\in V(t)$ the matroid $M(\tau(t))$ is graphic and so binary).

Lemma 4.4.5. *Let G be a graph, and let T be a tree structure on G. Let Ψ be a set of ends of G. Let G' be the graph obtained from G by subdividing each edge which has endpoints in different nodes of T.[6] Then every Ψ-circuit of G' is the support of a Ψ-vector of $V(G,T)$ and every $\Psi^{\complement}$-bond of G' is the support of a $\Psi^{\complement}$-vector of $(V(G,T))^\perp$.*

Proof. First we show that every $\Psi^{\complement}$-bond of G' is the support of a vector of $(V(G,T,\Psi))^*$. Let $\underline{b}$ be such a $\Psi^{\complement}$-bond. Let X be the set of vertices of G' on one side of $\underline{b}$. For each $t\in V(T)$, let $b(t)$ be the $\tau(t)$-cut of edges of $\tau(t)$ with one endpoint in X and the other not in X, and let $w(t)$ be the characteristic function of $b(t)$: thus w is a vector of $(V(G,T,\Psi))^\perp$. Then $\underline{b}=\underline{w}$.

Next, we show that every Ψ-circuit of G' is the support of a vector of $V(G,T,\Psi)$. Let $\underline{o}$ be such a Ψ-circuit, and let O be the circle in $\tilde{G}_\Psi$ inducing $\underline{o}$. Fix some vertex t_0 of T such that $\underline{o}$ meets $E(t_0)$. For any other vertex t of T let $T{\uparrow}t$ be the set of vertices t' of t on the other side of t from t_0, together with t itself. Let $E{\uparrow}t$ be $E\cap\bigcup_{t'\in T\uparrow t}E(t')$. For any $tt'\in E(T)$, with t' further from t_0 than t, let $F(tt')\subseteq E(tt')$ be the set of those edges v_ev_f such that there is an arc in O from v_e to v_f using only edges of $E{\uparrow}t'$. For any vertex t of T, let $o(t)$ be the union of $\underline{o}\cap E(t)$ with all of the $F(tt')$ for t' adjacent to t in T, and let $v(t)$ be the characteristic function of $o(t)$. Since $\underline{o}=\underline{v}$, it suffices to prove

[6] as before, we add a new vertex v_e corresponding to each such edge $e=vv'$, and replace e in the set of edges by the two new edges vv_e and $v'v_e$.

that $\underline{v}$ is a Ψ-vector. Every end in the closure of $\{t \in V(t) | v(t) \neq 0\}$ is in the closure of o and so is in Ψ. So we just need to show that for each node t of T the function $v(t)$ is in the circuit space of $\tau(t)$. In fact, we shall show something stronger: that $o(t)$ is a vertex-disjoint union of circuits of $\tau(t)$.

The circle O can be broken into finitely many arcs each of which uses either only edges in $E(t_0)$ or else only edges not in $E(t_0)$, with consecutive arcs around O being of opposite types. For each arc using only edges not in $E(t_0)$ there is some t' adjacent to t_0 such that that arc only uses edges from $E{\uparrow}t'$. Replacing each such arc with the corresponding edge in $F(t_0t')$ gives the set $o(t_0)$, which is therefore a circuit of $\tau(t_0)$.

For any $t \neq t_0$, let t_- be the neighbour of t in the direction of t_0, and let $v_e v_f$ be any edge in $F(t_- t)$. Then there is an arc A in O from v_e to v_f using only edges of $E{\uparrow}t$. A can be broken into finitely many arcs each of which uses either only edges in $E(t)$ or else only edges not in $E(t)$, with consecutive arcs along A being of opposite types. For each arc using only edges not in $E(t)$ there is some t' adjacent to t such that that arc only uses edges from $E{\uparrow}t'$. Replacing each such arc with the corresponding edge in $F(tt')$ gives a path $P(ef)$ of $\tau(t)$, which together with $v_e v_f$ itself gives a circuit $o(v_e v_f)$ of $\tau(t)$. Then $o(t)$ is the union of the vertex-disjoint circuits $o(v_e v_f)$, completing the proof. $\square$

Lemma 4.4.6. *In the context of Lemma 4.4.5, for any Ψ-vector v of $V(G,T)$, $\underline{v}$ is a union of Ψ-circuits. For any vector w of $(V(G,T,\Psi))^{\perp}$, $\underline{w}$ is a union of $\Psi^{\complement}$-bonds.*

Proof. By Lemma 4.4.5 and Lemma 4.4.2, $\underline{v}$ never meets a finite bond of G' just once and so, by the dual of Lemma 1.2.1 it is a union of topological circuits of G'. Each such circuit is a Ψ-circuit since every end in its closure is in the closure of $\{t \in V(T) | v(t) \neq 0\}$ and so is in Ψ. The proof for w is analogous. $\square$

In fact, the results above apply to all locally finite graphs.

Lemma 4.4.7. *Any connected locally finite graph G can be given a tree structure.*

Proof. Let U be a normal spanning tree of G, with root node v_0. For any down-closed set X of vertices of G we take $\delta(X)$ to be the set of minimal vertices not in X (here minimality is with respect to the tree order $\leq$ on U). For any set X of vertices of G, let $X{\downarrow}$ be the down-closure of X in U, and $N(X)$ the set of vertices adjacent to or in X. We build a sequence of finite subsets V_n of the vertices of G by setting $V_0 = \varnothing$ and $V_{n+1} = N(V_n){\downarrow} \cup \delta(V_n)$. For any n and any vertex $v \in \delta(V_n)$, we set $t(v) = \{v' \in V_{n+1} | v \leq v'\}$. Let T be the set of sets $t(v)$ arising in this way. By construction, T is a partition of the vertices of T into finite, connected sets. We order the vertices of T by $t(v) \leq t(v')$ if and only if $v \leq v'$ in the tree order on N. This gives a tree-order (with root $t(v_0)$) on T, making T a tree. It remains to show that distinct vertices of T are adjacent if and only if they contain adjacent vertices of G.

If $t(v)$ and $t(v')$ are adjacent in T, with $v < v'$, then let n be such that $v \in \delta(V_n)$. As $v < v'$, $v' \notin V_n \cup \delta(V_n)$ so $v' \notin V_{n+1}$. Let w be minimal such that

$v < w \le v'$ and $w \notin V_{n+1}$. Then $w \in \delta(V_{n+1})$ and we have $t(v) < t(w) \le t(v')$ in T, so $w = v'$. Thus the predecessor v^- of v' in U is in V_{n+1}, but it can't be in V_n since $v' > v$. So $v^- \in t(v)$ and so there is an edge from $t(v)$ to $t(v')$.

Now let $v \neq v'$ be such that there is an edge from $t(v)$ to $t(v')$ in G. Say the endpoints of this edge are $w \in t(v)$ and $w' \in t(w)$. Since U is normal we have without loss of generality that $w < w'$. Let n be such that $v \in \delta(V_n)$. Then $v \le w < w'$, so since $w' \notin t(v)$ we have $w' \notin V_{n+1}$. Since $w \in t(v)$ we have $w \in V_{n+1}$ and so $w' \in V_{n+2}$, so that $v' \in \delta(V_{n+1})$. Since both v and v' lie below w', we have $v < v'$ and so v and v' are adjacent in T. $\square$

4.5 Determinacy and $(O2)$ for representable trees of matroids

We fix a finite field k. We will rely on the following basic fact:

Fact 4.5.1. *Let X be a finite set of vectors in a finite dimensional vector space V, and let $y \in V$. $X^\perp \subseteq \{y\}^\perp$ if and only if y is in the span $\langle X \rangle$ of X.* $\square$

We fix a k-representation V of a tree $\mathcal{T} = (T, M)$ of finite matroids and a set Ψ of ends of T, together with a partition $E = \{e\} \dot\cup P \dot\cup Q$ of the ground set of $\mathcal{T}$. Let t_0 be the node of T such that $e \in E(t_0)$. For a function f whose domain is a subset of $\bigcup_{t \in V(T)} E(t)$, we obtain a function $\bar{f} \colon \bigcup_{t \in V(T)} E(t) \to k$ from f by assigning to each value in $\bigcup_{t \in V(T)} E(t)$ but not in the domain of f the value zero.

Definition 4.5.2. The *circuit game* $\mathcal{G} = \mathcal{G}(V, \Psi, P, Q)$ is played between two players, called Sarah and Colin, as follows:

Play alternates between the players, with Sarah making the first move. At any point in the game there is a *current node* $t_c \in V(T)$, a *current challenge set* $S_c \subseteq E(t_c)$ and a *current challenge function* $x_c \colon S_c \to k$. Initially we set $t_c = t_0$, $S_c = \{e\}$ and $x_c(e) = 1$. For any n the $(2n-1)^{\text{st}}$ move is made by Sarah: she must play a vector $v_n \in V(t_c)$ such that $\bar{v}_n\restriction_Q = 0$ and $\bar{v}_n \not\perp \bar{x}_c$. Then the $2n^{\text{th}}$ move is made by Colin: he must play a node t_n adjacent to t_c and further away from t_0 than t_c is and a vector $x_n \in k^{E(t_c t_n)}$ such that $\bar{v}_n \not\perp \bar{x}_n$. After he does this, the current node is updated to t_n, the current challenge set to $S_n = E(t_n t_{n-1})$ and the current challenge function to x_n. If play continues forever, then Sarah wins if the end ω of T containing $(t_n | n \in \mathbb{N})$ is in Ψ, and Colin wins if $\omega \in \Psi^{\complement}$.

The *cocircuit game* $\mathcal{G}^* = \mathcal{G}^*(V, \Psi, P, Q)$ is the game like the dual circuit game $\mathcal{G}(V^\perp, \Psi^{\complement}, Q, P)$, but with the roles of Sarah and Colin reversed. We will also use a different notation for the cocircuit game, putting stars on the notation for the circuit game. Thus for example the current challenge function is denoted x_c^* and Colin's n^{th} move is denoted v_n^*.

Lemma 4.5.3. *Sarah has a winning strategy in $\mathcal{G}$ if and only if there is a Ψ-vector v of V such that $e \in \underline{v} \subseteq \{e\} \dot\cup P$.*

Proof. Suppose first that there is such a vector v. Then Sarah can win in $\mathcal{G}$ by always choosing the vector $v(t_c)$ when it is her turn to play. Indeed, for any edge $tt' \in E(\mathcal{T})$, the vectors $v(t)$ and $v(t')$ coincide when restricted to $E(tt')$. Hence if $\bar{v}_n \not\perp \bar{x}_n$, then also $\bar{v}_{n+1} \not\perp \bar{x}_n$. So choosing $v(t_c)$ is a legal move and since v is a vector, the nodes t_n from any play that is played according to this strategy will converge to some end in Ψ.

Suppose for the converse that Sarah has a winning strategy σ in $\mathcal{G}$. For each n, let R_n be the set of sequences $(v_i | i \leq n)$ which can arise as the first n moves made by Sarah in a game played according to σ.

Sublemma 4.5.4. *Let $r \in R_n$ and let $t(r)$ be the node of T that is current when r_n is played. Let t' be a node of T that is adjacent to $t(r)$ and further away from t_0 than $t(r)$. Let $P_{t'}(r)$ be the set of those $v \in V(t')$ such that the extension $r.v$ of the sequence r by $r_{n+1} = v$ is in R_{n+1}.*

Then $r_n\restriction_{E(t(r)t')} \in \langle v\restriction_{E(t(r)t')} | v \in P_{t'}(r)\rangle$.

Before proving Sublemma 4.5.4, let us see how to derive Lemma 4.5.3 from it. By Sublemma 4.5.4, for each $r \in R_n$, $t(r)$, t' and $P_{t'}(r)$ as in that Lemma, we can choose a representation.

$$r_n\restriction_{E(t(r)t')} = \sum_{v \in P_{t'}(r)} \lambda_{r.v} v\restriction_{E(t(r)t')}$$

Let $r\restriction_i$ denote the initial sequence of r of length i. For any $t \in V(T)$ at distance $n-1$ from t_0, we set:

$$v(t) = \sum_{r \in R_n:\ t(r)=t} r_n \cdot \prod_{i=2}^{n} \lambda_{r\restriction_i}$$

Since each r_n in this expression is in $V(t)$, the vector $v(t)$ is in $V(t)$. And also $e \in \underline{v} \subseteq \{e\}\dot\cup P$. Next we check $t \mapsto v(t)$ is a Ψ-vector of V. For this, we first check that for any $tt' \in E(T)$ with t' further away from t_0 than t we have $v(t)\restriction_{E(tt')} = v(t')\restriction_{E(tt')}$:

$$\begin{aligned}
v(t)\restriction_{E(tt')} &= \sum_{r \in R_n:\ t(r)=t} r_n\restriction_{E(tt')} \cdot \prod_{i=2}^{n} \lambda_{r\restriction_i} \\
&= \sum_{r \in R_n:\ t(r)=t} \left(\sum_{v \in P_{t'}(r)} \lambda_{r.v} v\restriction_{E(tt')} \right) \cdot \prod_{i=2}^{n} \lambda_{r\restriction_i} \\
&= \sum_{r \in R_{n+1}:\ t(r)=t'} r_{n+1}\restriction_{E(tt')} \cdot \prod_{i=2}^{n+1} \lambda_{r\restriction_i} \\
&= v(t')\restriction_{E(tt')}
\end{aligned}$$

Next, suppose for a contradiction that there is a sequence t_n with the support of $v(t_n)$ nonempty such that its limit is not in Ψ. Without loss of generality, we may assume that t_n has distance at least n from t_0. Hence for each $n \in \mathbb{N}$ there is some $j \geq n$ and some $r \in R_j$ such that $r_j \neq 0$ and $t(r) = t_n$. Since $0 \perp x$ for every x, no $r\restriction_i$ can be 0 for any $i \leq j$ since the play would then be finished after the i^{th} move, which is not true. So without loss of generality, we may assume that t_n has distance precisely n from t_0.

Now we apply the Infinity Lemma where we take the V_n from that Lemma to be the sets $\{r \in R_n | t(r) = t_n, r_n \neq 0\}$. And we join $r \in R_{n+1}$ to $r' \in R_n$ if and only if $r\restriction_n = r'$. Note that each V_n is finite since k is finite. Hence we find a sequence of $a^n \in R_n$ such that $a^{n+1}\restriction_n = a^n$. This gives rise to an infinite play according to σ whose end is not in Ψ, contradicting the fact that σ is a winning strategy. Thus $t \mapsto v(t)$ is a Ψ-vector of V.

Having shown how Lemma 4.5.3 can be deduced from Sublemma 4.5.4, it remains to prove Sublemma 4.5.4. For this, we fix a particular finite play of length $2n+1$ according to σ and giving rise to r, and consider the situation just after this play. For any $w \in k^{E(t(r)t')}$ with $\bar{w} \not\perp \bar{r}_n$ Sarah has a response prescribed by σ, that is, there is some $v \in P = P_{t'}(r)$ such that $\bar{w} \not\perp \bar{v}$. In other words, any $w \in k^{E(t(r)t')}$ that is not orthogonal to x_n is also not orthogonal to some $v \in P$. Put yet another way, any $z \in k^{E(t(r)t')}$ that is orthogonal to every $v \in P$ is orthogonal to x_n. By Fact 4.5.1, $r_n\restriction_{E(t(r)t')} \in \langle v\restriction_{E(t(r)t')} | v \in P\rangle$. This completes the proof of Sublemma 4.5.4, and so of Lemma 4.5.3. $\square$

Corollary 4.5.5. *Colin has a winning strategy in $\mathcal{G}^*$ if and only if there is a $\Psi^{\complement}$-vector v^* of $V^\perp$ such that $e \in \underline{v}^* \subseteq \{e\}\dot\cup Q$.* $\square$

In order to relate $(O2)$ to determinacy of $\mathcal{G}$, we need to show that $\mathcal{G}$ and $\mathcal{G}^*$ are closely related games.

Lemma 4.5.6. *Colin has a winning strategy in $\mathcal{G}$ if and only if he has one in $\mathcal{G}^*$.*

Proof. For the 'if' part, suppose that he has a winning strategy σ^* in $\mathcal{G}^*$. Then he can win in $\mathcal{G}$ by playing as follows:

He should imagine an auxilliary play in the game $\mathcal{G}^*$, in which he plays according to σ^*, and for which he should ensure that at any point the current node and current challenge set agree with those in $\mathcal{G}$, and additionally ensure that $x_n = v_n^*\restriction_{S_n}$ and $x_n^* = v_{n+1}\restriction_{S_n}$. We shall assume, without loss of generality, that $v_1(e) = 1$ (otherwise we can just multiply v_1 by some constant to make this true).

Suppose Sarah makes some move v_n. Then $x_c^* = v_n\restriction_{S_{n-1}}$: if $n = 1$ then this is true by our assumption, and otherwise it is true by the condition that $x_n^* = v_{n+1}\restriction_{S_n}$. Let v_n^* be the move in $\mathcal{G}^*$ that is prescribed by σ^*. Then $\sum_{f \in S_{n-1}} v_n(f)v_n^*(f) = \sum_{f \in S_{n-1}} x_c^*(f)v_n^*(f) \neq 0$ but $v_n \perp v_n^*$. Since the support of the map $f \mapsto v_n(f)v_n^*(f)$ consists of dummy edges only, there is some $t_n \in V(T)$ that is adjacent to t_{n-1} and has distance n from t_0, such that $\sum_{f \in E(t_{n-1}t_n)} v_n(f)v_n^*(f) \neq 0$. Then Colin plays t_n, $S_n = E(t_{n-1}t_n)$ and $x_n = v_n^*\restriction_{S_n}$. And he plays v_n^* in the imagined cocircuit-game, and imagines that Sarah plays $x_n^* = v_{n+1}\restriction_{S_n}$ there. Note that this is a legal move since

$\sum_{f\in S_n} v_n^*(f)x_n^*(f) = \sum_{f\in S_n} x_n(f)v_{n+1}(f) \neq 0$. If the play of the circuit game continues forever, then the end ω containing $(t_n|n \in \mathbb{N})$ is in $\Psi^{\complement}$ since σ^* is winning.

For the 'only if' part, suppose that he has a winning strategy σ in $\mathcal{G}$. Then he can win in $\mathcal{G}^*$ by playing as follows:

He should imagine an auxilliary play in the game $\mathcal{G}$, in which he plays according to σ, and for which he should ensure that at any point the current node and current challenge set agree with those in $\mathcal{G}^*$.

When it is his turn to move, either it is his first move, in which case we let x_0^* be the function with support $\{e\}$ that sends e to 1 or Sarah has just played x_{n-1}^* in $\mathcal{G}^*$. Then he imagines the corresponding game of $\mathcal{G}$ where he has just played x_{n-1}, or else it is his first move, in which case we set $x_0 = x_0^*$.

Let O be the set of Sarah's legal moves in $\mathcal{G}$. For $v \in O$, let $t(v)$ and $x(v)$ be the node and challenge function prescribed by σ. Let $T_n = \{t(v)|v \in O\}$. And for each $t \in T_n$, let $P(t) = \{x(v)|v \in O : t(v) = t\}$.

Sublemma 4.5.7. *There is some $v^* \in V(t_{n-1})^\perp$ and coefficients $\lambda_{t,x} \in k$ and a vector $w \in k^{E(t_{n-1})\cap Q}$ such that*

$$\bar{x}_{n-1} = \overline{v^*} + \overline{w} + \sum_{t\in T_n}\sum_{x\in P(t)} \lambda_{t,x}\bar{x}.$$

Before proving Sublemma 4.5.7, let us complete the description of his strategy. In $\mathcal{G}^*$, he plays $v_n^* = v^*$ - by the equation above the support of this vector cannot meet the set P_{co}. Let t_n and x_n^* be the node and challenge set that Sarah plays in her next move in $\mathcal{G}^*$. Then by the choice of v_n^*, the node t_n is in T_n, and $\bar{x}_n^* \not\perp \bar{v}_n^*$. Since v_n^* restricted to $E(t_{n-1}t_n)$ is equal to $\sum_{x\in P(t_n)} \lambda_{t,x}x$, there is some $x_n \in P(t_n)$ with $x_n \not\perp x_n^*$. Then he imagines that she plays some $v \in O$ with $x(v) = x_n$, and that he then plays t_n and x_n. This completes the description of his strategy. If play continues forever, then the end ω containing $(t_n^*|n \in \mathbb{N})$ is in $\Psi^{\complement}$ since σ is winning.

Hence it remains to prove Sublemma 4.5.7. For this, by Fact 4.5.1, it remains to show that $(V^\perp \cup k^{E(t_{n-1})\cap Q} \cup \bigcup_{t\in T_n}\bigcup_{x\in P(t)} \bar{x})^\perp \subseteq \bar{\{x_{n-1}\}}^\perp$. In other words, any y that is not orthogonal to x_{n-1} is not orthogonal to some $v^* \in V^\perp$ or to some $\bar{x}$ or has support meeting Q. This follows from the fact that for every $v \in V$ with $v \not\perp x_{n-1}$ and $\underline{v} \cap Q = \varnothing$, there is some x such that $v \not\perp \bar{x}$. This completes the proof of Sublemma 4.5.7, and so also the proof of Lemma 4.5.6. □

Corollary 4.5.8. *(O2) holds for the partition $E = \{e\}\dot\cup P\dot\cup Q$ of the groundset of $\mathcal{T}$ if and only if $\mathcal{G}(V, \Psi, P, Q)$ is determined.* □

Corollary 4.5.9. *The Axiom of Determinacy is equivalent to the statment that every tree of finite matroids representable over a finite field induces a matroid.* □

Corollary 4.5.10. *For any tree of finite matroids $\mathcal{T} = (T, M)$ represented over a finite field and any Borel set Ψ of ends of T, $(\mathcal{T}, \Psi)$ induces a matroid.*

Proof. Just like the proof of Corollary 4.3.7. □

Theorem 4.5.11. *Let G be a locally finite graph, and Ψ a Borel set of ends of G. Then (G, Ψ) induces a matroid.*

Proof. Just like the proof of Theorem 4.3.8. □

4.6 From the locally finite case to the countable case

4.6.1 From the locally finite case to the case that the graph has a locally finite normal spanning tree

For any graphs G and H, we will use $G \times H$ to denote the graph with vertex set $V(G) \times V(H)$ and with edge set

$$\{e \times \{v\} | e \in E(G), v \in V(H)\} \cup \{\{v\} \times e | v \in V(G), e \in E(H)\}.$$

The edges in $\{e \times \{v\} | e \in E(G), v \in V(H)\}$ are called G-edges, and those in $\{\{v\} \times e | v \in V(G), e \in E(H)\}$ are called H-edges.

Let G be a graph having a normal spanning tree T. Then the *Undomination-graph* $U = U(G, T)$ of G is the following. Its vertex set is $V(U) = V(G) \times V(T)$. The pair $(v, t)(v', t')$ is an edge if and only if either $v = v'$ and t and t' are adjacent in T or v and v' are adjacent in G and $v = t'$ and $v' = t$. We call the edges of the first type *T-edges* and the ones of the second type *G-edges*. We will sometimes implicitly identify the G-edge $(v, v')(v', v)$ with the corresponding edge vv' of G.

The following properties of U are immediate. Any vertex of U is incident with at most one G-edge. U has G as a minor, where the branching set of the vertex v has the form $\{v\} \times V(T)$. In other words, we obtain G as a minor of U by contracting all T-edges.

Definition 4.6.1. Let $P_G = p_1(p_1, p_2)p_2 \dots (p_{n-1}, p_n)p_n$ be a walk in G. Let $t, t' \in V(T)$. Then $u_{t,t'}(P_G)$ denotes the following walk in U.

$$u_{t,t'}(P_G) = [\{p_1\} \times (tTp_2)] \circ [(p_1, p_2)(p_2, p_1)] \circ [\{p_2\} \times (p_1Tp_3)] \circ$$
$$[(p_2, p_3)(p_3, p_2)] \circ [\{p_3\} \times (p_2Tp_4)] \circ \dots \circ [\{p_n\} \times (p_{n-1}Tt')]$$

Definition 4.6.2. Let P_U be a walk in U from (p_1, t) to (p_n, t'). Then the set of its G-edges forms a walk in G from p_1 to p_n. We denote this walk by $g(P_U)$.

Lemma 4.6.3. *The operations u and g are inverse to each other for walks that traverse no edge more than once.*

Proof. It is immediate from the definitions that $g(u_{t,t'}(P)) = P$.

For the other direction, let P be a walk in U from (p_1, t) to (p_n, t'). We are to show that $u_{t,t'}(g(P)) = P$. This follows from the fact the branching set of every $v \in V(G)$ is a tree. □

Note that if P is a path in G, then $u_{t,t'}(P)$ is a path whereas if P is a path in U, the walk $g(P)$ need not be a path.

Corollary 4.6.4. *Let $R_G = p_1(p_1, p_2)p_2 \dots$ be a ray in G. Then for any $t \in T$, there is a unique ray $u_t(R_G)$ starting at (p_1, t) in U included in the T-edges together with $\{(p_1, p_2)(p_2, p_1), \dots\}$.*

More precisely:

$$u_t(R_G) = [\{p_1\} \times (tTp_2)] \circ [(p_1, p_2)(p_2, p_1)] \circ [\{p_2\} \times (p_1Tp_3)] \circ \dots$$

Remark 4.6.5. *A result similar to Corollary 4.6.4 also holds for combs since we have it for paths and rays. A little bit of care is needed when choosing the starting points t of the paths $u_{t,t'}(P)$ to ensure that these paths only meet the spine of the comb in their initial vertices.*

The following lemma allows us to turn finite separators in G into finite separators in U.

Lemma 4.6.6. *Let X be a finite set of vertices of G, and let $w = (v, t)$ and $w' = (v', t')$ be vertices of U such that v and v' are in different components of $G \setminus X$.*

Then $X \times X$ separates w from w' in U.

Proof. Let P_U be some w-w'-path in U. Let $g(P_U) = p_1(p_1, p_2)p_2 \dots (p_{n-1}, p_n)p_n$ with $p_1 = v$ and $p_n = v'$.

Let C_1 be the component of $G \setminus X$ containing p_1. Let $i \in \{1, \dots, n\}$ be maximal such that $p_i \in C_1$. Such an i exists as $p_1 \in C_1$. Note that $p_i \neq p_n$. Then p_{i+1} is in X.

Since $p_{i+1} \neq p_n, p_1$, the path P_U has $\{p_{i+1}\} \times (p_iTp_{i+2})$ as a subpath. Since $p_i \in C_1$ but $p_{i+2} \notin C_1$, the path p_iTp_{i+2} has to meet X in some point x. Then P_U meets $X \times X$ in (p_{i+1}, x), completing the proof. □

The following lemma is the reason why we call U the Undomination-graph of G.

Proposition 4.6.7. *In $U(G, T)$, no vertex dominates a ray.*

Proof. Suppose for a contradiction that U has a vertex (v, t) dominating a ray R. Then there is an infinite collection $(P_n | n \in \mathbb{N})$ of (v, t)-R-paths in U that meet only in (v, t). Since all edges except for at most one edge incident with (v, t) are T-edges, we may assume that the second vertex on each P_n has the form (v_n, t) where v_n is an neighbour of v in T. Since v has at most one lower neighbour in T, we may even assume that all the v_n are upper neighbours of v in T.

Let $\lceil v \rceil$ be the set of those vertices that are less than or equal to v in the tree order of T. As T is normal, all the v_n are in different components of $G \setminus \lceil v \rceil$. By Lemma 4.6.6, all the (v_n, t) are in different components of $U \setminus (\lceil v \rceil \times \lceil v \rceil)$.

Since $\lceil v \rceil \times \lceil v \rceil$ is finite, we can find a tail R' of R that avoids $\lceil v \rceil \times \lceil v \rceil$. Then for any two paths P_i and P_j that avoid $\lceil v \rceil \times \lceil v \rceil$ and meet R', the set

$R' \cup P_i \cup P_j$ is connected in $U \setminus (\lceil v \rceil \times \lceil v \rceil)$. Hence the vertices (v_i, t) and (v_j, t) are in the same connected component of $U \setminus (\lceil v \rceil \times \lceil v \rceil)$.

Since $R \setminus R'$ and $\lceil v \rceil \times \lceil v \rceil$ are both finite, there exist such paths P_i and P_j, which yields the desired contradiction. □

Next we shall investigate how the ends of U relate to the ends of G.

Lemma 4.6.8. *Let R_1 and R_2 be rays of G. Then R_1 and R_2 belong to the same end of G if and only if $u_t(R_1)$ and $u_{t'}(R_2)$ belong to same end of U for any $t, t' \in V(T)$.*

Proof. First suppose that $u_t(R_1)$ and $u_{t'}(R_2)$ belong to different ends of U. Then there is a finite set $S = (v_1, t_1), \ldots, (v_n, t_n)$ separating them. Without loss of generality we may assume that $u_t(R_1)$ and $u_{t'}(R_2)$ do not meet S. Let P be some R_1-R_2-path, which goes from (w, s) to (w', s'). Then $u_{s,s'}(P)$ meets S in some point, say (v_i, t_i). Hence $\{v_1, \ldots, v_n\}$ separates R_1 from R_2, yielding the first implication.

The other implication is an immediate consequence of Lemma 4.6.6. □

By Lemma 4.6.8, the map u induces an inclusion $\tilde{u}$ from the ends of G into the ends of U. Let C be the set of T-edges. The purpose of this subsection is to prove the following.

Theorem 4.6.9. *Assume that $(U, \tilde{u}(\Psi))$ induces a matroid M. Then (G, Ψ) induces the matroid M/C.*

The Undomination-graph $U(G, T)$ is locally finite whenever T is locally finite. Thus Theorem 4.6.9, reduces the case where G has a locally finite normal spanning tree to the locally finite one, which is the aim of this subsection.

The proof of Theorem 4.6.9 takes the rest of this subsection.

Lemma 4.6.10. *Assume that $(U, \tilde{u}(\Psi))$ induces a matroid M. Then the edge set b is an M/C-cocircuit b if and only if it is a $\Psi^{\complement}$-bond of G.*

Proof. First suppose that b is an M/C-cocircuit. The cocircuit b is a $\tilde{u}(\Psi)^{\complement}$-bond of U that does not meet C. Since the graphs U/C and G are equal, it remains to show that b considered as an edge set of G does not have any end of Ψ in its closure.

Suppose for a contradiction that there is such an end $\omega \in \Psi$ that is in the closure of b. Let R_ω be some ray in ω.

By Lemma 4.1.1, there is a comb K with spine R_ω all of whose teeth are endvertices of b. Then in U, the set $K \cup C$ contains a comb all of whose teeth are in endvertices of b with spine $u_t(R_\omega)$ for some t by Remark 4.6.5. Hence $\tilde{u}(\omega)$ is in the closure of b, a contradiction.

Next suppose that b is a $\Psi^{\complement}$-bond of G. As above, it is clear that b considered as an edge set of U is a bond.

Now suppose for a contradiction that there is some end $\omega \in \tilde{u}(\Psi)$ in the closure of b. We pick a ray $R_\omega \in \tilde{u}^{-1}(\omega)$. By Lemma 4.1.1 there is a comb in U with spine $u(R_\omega)$ all of whose teeth are endvertices of b. Then this comb defines a comb in G with comb R_ω, which is impossible. This completes the proof. □

Next we prove Lemma 4.6.10 for circuits, which is a little more complicated.

We define the map $p : |U|_{\tilde{u}(\Psi)} \to |G|_\Psi$ as follows. A vertex (v,t) maps to v, a G-edge $(v,t)(t,v)$ maps to the edge vt, all interior points of a T-edge $(v,t)(v,t')$ map to v, and an end $\omega \in \tilde{u}(\Psi)$ maps to $\tilde{u}^{-1}(\omega)$.

Lemma 4.6.11. *p is continuous.*

Proof. Let O be some open set in $|G|_\Psi$. Let $x \in p^{-1}(O)$. If x is an interior point of a G-edge, then $p^{-1}(O)$ clearly includes a neighbourhood around x. If x is an interior point of a T-edge, then $p^{-1}(O)$ included the whole interior of that edge.

If x is a vertex, then there is some ϵ with $B_\epsilon(p(x)) \subseteq O$: then $B_\epsilon(x) \subseteq p^{-1}(O)$.

If x is an end, then some basic open set $\hat{C}(S, p(x))$ is included in O. Let $D = D(S \times S, x)$ be the unique component of $U \setminus S \times S$ having x in its closure. We show that $\hat{D}(S \times S, x)$ is a subset of $p^{-1}(O)$. Clearly all edges and vertices of $\hat{D}(S \times S, x)$ are in $p^{-1}(O)$. So let $\omega \in \hat{D}(S \times S, x)$ be an end.

Let $(v,t) \in D$. Let R be a ray in G that is in $\tilde{u}^{-1}(\omega)$. Then (for any t) $u_t(R)$ is eventually in D as $u_t(R) \in \omega$. By Lemma 4.6.6, R is then eventually in the same component as v. So it is in $C(S, p(x))$. Hence $\omega \in \hat{C}(S, p(x))$. This completes the proof of the continuity of p. □

Since $\tilde{G}_\Psi$ has the quotient topology, the quotient map $\pi_G : |G|_\Psi \to \tilde{G}_\Psi$ is continuous. Similarly, the quotient map $\pi_U : |U|_{\tilde{u}(\Psi)} \to \tilde{U}_{\tilde{u}(\Psi)}$ is continuous. All the maps occurring here are shown in Figure 4.8.

Lemma 4.6.12. *For any two $x, y \in \tilde{U}_{\tilde{u}(\Psi)}$ with $\pi_G(p(x)) \neq \pi_G(p(y))$, we have $\pi_U(x) \neq \pi_U(y)$.*

In particular, there is a unique map $\tilde{p} : \tilde{U}_{\tilde{u}(\Psi)} \to \tilde{G}_\Psi$ satisfying $\tilde{p}(\pi_U(x)) = \pi_G(p(x))$. Moreover, $\tilde{p}$ is continuous.

It might be worth noting that since U is locally finite, the map π_U is the identity, which makes the Lemma rather trivial. However we will not use this in the proof as we rely on this Lemma later on in a slightly different context where π_U is not the identity.

Proof. Since $\pi_G(p(x)) \neq \pi_G(p(y))$, there is some Ψ-bounded cut of G with $p(x)$ and $p(y)$ on different sides by Lemma 4.1.5. Then there is also a $\Psi^{\complement}$-bond b of G with $p(x)$ and $p(y)$ on different sides. By Lemma 4.6.10, the bond b is also a $\tilde{u}(\Psi)^{\complement}$-bond in U. And this bond witnesses that $\pi_U(x) \neq \pi_U(y)$ by the other implication of Lemma 4.1.5. This proves the first part of the Lemma.

It remains to show that $\tilde{p}$ is continuous. This follows from the universal property of the quotient map π_U since the concatenation of π_G and p is continuous. □

Corollary 4.6.13. *Assume that $(U, \tilde{u}(\Psi))$ induces a matroid M. Then for any M/C-circuit o and any edge $e \in o$, the circuit o includes a Ψ-circuit of G containing e.*

$$\begin{tikzcd}
 & {|U|_{\tilde{u}(\Psi)}} \arrow[r, "p"] \arrow[d, "\pi_U"] & {|G|_\Psi} \arrow[d, "\pi_G"] \\
S^1 \arrow[r, "o'"] \arrow[rr, bend right, dashed, "o''"'] & \tilde{U}_{\tilde{u}(\Psi)} \arrow[r, dashed, "\tilde{p}"] & \tilde{G}_\Psi
\end{tikzcd}$$

Figure 4.8: The construction of the map o''.

Proof. Let o be some M/C-circuit. Then there is some M-circuit $o \subseteq o' \subseteq o \cup C$ by Lemma 1.2.7. Let $o'' = \tilde{p} \circ o'$ as in Figure 4.8.

Let e be some edge in o. Then e considered as an edge of U is mapped under $\tilde{p}$ to the edge e considered as an edge of G, which is then in the image of o''.

Then the restriction of o'' to those points that do not map to interior points of e is a path between the two endvertices of e, that is a continuous function from $[0, 1]$ to $\tilde{G}_\Psi$ mapping 0 and 1 to the endvertices of e. By a well-known Lemma of basic topology[7], there is an arc (injective path) between the two endvertices of e whose image is included in the image of that path. The concatenation of this arc with some continuous function from $[0, 1]$ to e defines the desired Ψ-circuit. $\square$

By Corollary 4.6.13, Lemma 4.6.10 and Lemma 4.1.8, we can apply Lemma 1.3.7 and deduce Theorem 4.6.9.

4.6.2 From the case that the graph has a locally finite normal spanning tree to the countable case

The aim of this subsection is to prove the following.

Proposition 4.6.14. *For every countable graph G together with $\Psi_G \subseteq \Omega(G)$ there is a graph H having a locally finite normal spanning tree together with $\Psi_H \subseteq \Omega(H)$ and $C \subseteq E(H)$ such that if (H, Ψ_H) induces a matroid M, then (G, Ψ_G) induces M/C.*

First we need the following lemma.

Lemma 4.6.15. *Let G be a countable graph together with a normal spanning tree T_G. Then there is a countable graph H together with a locally finite normal spanning tree T_H and $C \subseteq E(T_H)$ such that $G = H/C$ and $T_G = T_H/C$.*

Proof of Lemma 4.6.15. First we construct T_H. Let X be the set of those vertices of T_G that have infinitely many upper neighbours. We obtain the tree T' from T_G by adding a ray R_x starting at x for every $x \in X$.

We obtain T_H from T' by replacing each edge of the type vx where v is an upper neighbour in T' of x by the edge vx' for some $x' \in R_x - x$ in such a way that for all $x \in X$ all vertices in $R_x - x$ get degree 3. This is possible by the choice of X. Note that any $x \in X$ has degree at most 2 in T_H, and hence T_H is locally finite. Let C be the set of all those edges contained in some R_x. Then $C \subseteq E(T_H)$ and $T_G = T_H/C$.

Note that $V(G) \subseteq V(T_H)$. We obtain H from T_H by adding all edges $e \in E(G) \setminus E(T_G)$. It is straightforward to check that $G = H/C$ and T_H is normal in H. This completes the proof. $\square$

Proof of Proposition 4.6.14. First note that every countable graph has a normal spanning tree [34]. Hence we may pick a normal spanning tree T_G of G.

By Lemma 4.6.15, there is a countable graph H together with a locally finite normal spanning tree T_H and $C \subseteq E(T_H)$ such that $G = H/C$ and $T_G = T_H/C$.

Every normal ray R of G starting at some vertex $v \in V(G)$ extends to a unique normal ray $h(R)$ starting at the same vertex v and that is included in $R \cup C$.

It is straightforward to check that R and R' belong to the same end of G if and only if $h(R)$ and $h(R')$ belong to the same end of H. This defines an inclusion $\tilde{h}$ from the ends of G into the ends of H. We let $\Psi_H = \tilde{h}(\Psi_G)$.

We define the map $p : |H|_{\Psi_H} \to |G|_{\Psi_G}$ to be $\tilde{h}^{-1}$ on the ends, map R_x to x for every $x \in X$, and to be the identity everywhere else. As in the proof of Theorem 4.6.9, we show the following.

Lemma 4.6.16. *p is continuous.*

Proof. Let O be some open set in $|G|_{\Psi_G}$. Let $y \in p^{-1}(O)$. If y is a vertex or an interior point of an edge, then $p^{-1}(O)$ includes an open neighbourhood around y as in the proof of Lemma 4.6.11.

If y is an end in Ψ_H, then O includes a basic open set of the form $\hat{C}(S, \tilde{h}^{-1}(y))$. We pick $v \in V(G)$ such that in T_G it separates S from $\tilde{h}^{-1}(y)$. Note that this is possible since S is finite.

Then $\hat{C}(S_v, y) \subseteq p^{-1}(O)$ where S_v is the down-closure of v in T_H. This completes the proof of the continuity of p. $\square$

Now assume that (H, Ψ_H) induces a matroid M. The proofs of Lemmata 4.6.10, 4.6.12 and 4.6.13 extend immediately to our setting. Hence we can apply the proof of Theorem 4.6.9 from the last section to conclude that (G, Ψ_G) induces M/C. $\square$

4.7 Applications

We are now in a position to begin applying our main results, to answer some of the basic questions about matroids discussed in the introduction. We begin by showing that there are as many countable tame matroids as there could possibly be: we prove that there are $2^{2^{\aleph_0}}$ non-isomorphic countable tame matroids with no $M(K_4)$-minor and no $U_{2,4}$-minor (Corollary 4.0.11 from the Introduction).

Proof. First we outline the construction of the $2^{2^{\aleph_0}}$ non-isomorphic matroids. Let T be a tree with precisely one vertex of each finite degree ≥ 2. We will use the graph $G = T \times K_2$: that is, G is built from two disjoint copies of T by adding an edge between each vertex and its clone. We call the two copies of the vertex of degree n v_n and v'_n. So for any $\Psi \subseteq \Omega(G)$ the pair (G, Ψ) induces a matroid $M(\Psi)$ by Lemma 4.3.11.

It suffices to show for any isomorphism $f : M(\Psi) \to M(\Psi')$ that $\Psi = \Psi'$.

The edge $v_n v'_n$ is in precisely n circuits of length 4. Since all edges not of the type $v_n v'_n$ are in precisely one circuit of length 4, the map f maps $v_n v'_n$ to itself.

Let e be some edge not of the type $v_n v'_n$. Then it is contained in a unique circuit of length 4 which contains its clone and two other edges, say $v_i v'_i$ and $v_j v'_j$. The edge e cannot be distinguished from its clone and f may map it to itself or to its clone but it cannot map it to some other edge because $f(e)$ must lie in a common 4-circuit with $v_i v'_i$ and $v_j v'_j$.

For every end ω of G, $\omega \in \Psi$ if and only if the unique double ray D containing $v_2 v'_2$ and with both ends in ω is a circuit of $M(\Psi)$. But by the above argument, each such D is fixed by f. Hence $\Psi = \Psi'$.

Having shown that there are $2^{2^{\aleph_0}}$ non-isomorphic tame matroids, it remains to show that none of them has $M(K_4)$ or $U_{2,4}$ as a minor. Combining the fact that in these matroids every circuit-cocircuit intersection is even by Remark 4.4.3 with a result of [13], yields that they do not have a $U_{2,4}$-minor.

If $M(\Psi)$ had an $M(K_4)$-minor, we would be able to find a 2-separation of G with at least two of the six edges of that minor on each side. But this would induce a 2-separation of $M(\Psi)$, which in turn would induce a separation of $M(K_4)$ with at least 2 edges on each side. Since there is no such 2-separation, there can be no $M(K_4)$ minor. $\square$

A direct consequence of Corollary 4.0.11 is that there is no universal matroid for the class of countable planar matroids (Corollary 4.0.12 from the Introduction) since every countable matroid has at most 2^ω many non-isomorphic minors but the class of countable planar matroids has $2^{2^{\aleph_0}}$ many non-isomorphic members.

Finally, we prove that the countable binary matroids of branch-width at most 2 are not well-quasi-ordered (Corollary 4.0.10 from the Introduction).

Throughout the rest of this section $2^{\mathbb{N}}$ is endowed with the product topology. The next Lemma finds complicated subsets of $2^{\mathbb{N}}$.

Lemma 4.7.1. *There is a sequence of subsets $\Psi_n \subseteq 2^{\mathbb{N}}$ with the following properties.*

1. *Each Ψ_n has cardinality $2^{\aleph_0}$.*
2. *There do no not exist $i < j \in \mathbb{N}$ and an injective continuous map $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $f(\Psi_i) \subseteq \Psi_j$.*

Before proving this lemma, let us see how we can deduce Corollary 4.0.10 from it.

Proof that Lemma 4.7.1 implies Corollary 4.0.10. As in Lemma 4.3.11, we consider the graph $G = T_2 \times K_2$. Note that $\Omega(G)$ and $2^{\mathbb{N}}$ are homeomorphic. Let M_n be the Ψ-matroid $M(G_n, \Psi_n)$ with $G_n = G$, which is a matroid as shown in that Lemma. It is easy to check that G has branch-width 2, so M_n has branch-width 2 as well.

Suppose for a contradiction that there are $i < j$ such that $M_i \cong M_j/C\backslash D$. By Lemma 4.7.1, it remains to find an injective continuous map $f\colon \Omega(G_i) \to \Omega(G_j)$ such that $f(\Psi_i) \subseteq \Psi_j$.

For $\omega \in \Omega(G_i)$, we pick a double ray $D(\omega)$ having only ω in its closure. Then the edge set of $D(\omega)$ considered as an edge set of G_j has only a single end in its closure. Indeed, if there were two ends in its closure, then there is a 2-separation of M_j having infinitely many edges from $D(\omega)$ on both sides. This then would give rise to a 2-separation of M_i with infinitely many edges from $D(\omega)$ on both sides, which is impossible.

This motivates the following definition: we define $f(\omega)$ to be the unique end of G_j in the closure of $D(\omega)$. Note that this does not depend on the choice of $D(\omega)$ since any two such choices differ by finitely many edges only.

To see that $f(\Psi_i) \subseteq \Psi_j$, note that for every $\omega \in \Psi_i$ the set $D(\omega)$ extends to a circuit of M_j using additionally only edges from C. This circuit has only ends from Ψ_j in the closure. Hence the unique end in the closure of $D(\omega)$ must be in Ψ_j.

To see that f is continuous, let $\omega \in \Omega(G_i)$ and let $\hat{C}_\epsilon(S, f(\omega))$ be a basic open neighbourhood of $f(\omega)$. Then S defines a separation of M_j of finite order with the edges of $C(S, \omega)$ on one side. Then the set F of all these edges without $C \cup D$ forms the side of a separation of finite order in M_i, which gives rise to a vertex separator S' in G_i (Formally, S' consists of those vertices that are incident with one edge in F and one outside). Then $\hat{C}_\epsilon(S', \omega) \subseteq f^{-1}(\hat{C}_\epsilon(S, f(\omega)))$. Hence f is continuous.

It remains to show that f is injective. So suppose for a contradiction that there are $\omega_1 \neq \omega_2$ in $\Omega(G_i)$ that are mapped to the same end τ in $\Omega(G_j)$. We may assume that we picked $D(\omega_1)$ and $D(\omega_2)$ such that they are vertex-disjoint.

We shall construct a 2-separation (A, B) of M_j such that A and B both include an edge from each of $D(\omega_1)$ or $D(\omega_2)$. For this, we pick some $e_1 \in D(\omega_1)$ and some $e_2 \in D(\omega_2)$. Then in G_j, there are two vertices v and w such that the components of $G - v - w$ containing e_1 or e_2 do not have τ in their closure. Let B consist of those edges of G that are only incident with v, w or vertices of the component $G/\{v, w\}$ that has τ in its closure. Let $A = E(M_j) \setminus B$.

Since $A \setminus (C \cup D)$ and $B \setminus (C \cup D)$ both have at least 2 elements, $(A \setminus (C \cup D), B \setminus (C \cup D))$ is a 2-separation of $M_j/C\backslash D$. Since $M_i \cong M_j/C\backslash D$, this gives rise to a 2-separation of M_i having on each side at least one edge from each of $D(\omega_1)$ and $D(\omega_2)$.

This gives rise to a 2-separation (A', B') in G_i, and it induces a separation on the closure of $D(\omega_1)$ in M_i. Since this closure is 2-connected, this separation has order 2 and thus $D(\omega_1)$ includes the separator of (A', B'). Similarly, $D(\omega_2)$ includes this separator, contradicting the fact that $D(\omega_1)$ and $D(\omega_2)$ are vertex-disjoint. This completes the proof.

□

Proof of Lemma 4.7.1. We build the sets Ψ_n recursively. So let us suppose that $\Psi_1, \dots, \Psi_n$ are already constructed such that they satisfy (1) and (2) for all $j \leq n$.

Let K be the set of pairs (i, f) where $i \leq n$ and $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is a continuous injective function.

Since $2^{\mathbb{N}}$ has a countable basis as a topological space, the set K has size $2^{\aleph_0}$. Let κ be the least ordinal of size $2^{\aleph_0}$. We can well-order K as $((i_\alpha, f_\alpha)|\alpha < \kappa)$.

For every $\alpha < \kappa$ we pick two elements $s_\alpha, t_\alpha \in 2^{\mathbb{N}}$ such that all the s_α and t_α are distinct and $t_\alpha \in f_\alpha(\Psi_{i_\alpha})$. This is possible as $|2^{\mathbb{N}}| = 2^{\aleph_0}$.

We let $\Psi_{n+1} = \{s_\alpha | \alpha < \kappa\}$. Then $|\Psi_{n+1}| = 2^{\aleph_0}$ since all the s_α are disjoint. Let f be some continuous function $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ and $i < n+1$. Then there is some $\alpha < \kappa$ such that $f_\alpha = f$ and $i_\alpha = i$. We ensured at step α that $t_\alpha \in f_\alpha(\Psi_{i_\alpha})$ and hence $f(\Psi_i) \not\subseteq \Psi_{n+1}$, yielding (2) for all $j \leq n+1$. This completes the proof. □

4.8 Trees of matroids of overlap 1 revisited

The purpose of this section is to show the following

Theorem 4.8.1. *If $\mathcal{T} = (T, M)$ is a tree of matroids of overlap 1 and Ψ is a Borel set of ends of $\mathcal{T}$ then there is a matroid $M_\Psi(\mathcal{T})$ whose circuits are the Ψ-circuits.*

We will prove this by showing that $\mathcal{S}$ satisfies the scrawl axioms, where $\mathcal{S}$ is the set of unions of underlying sets of Ψ-precircuits. Then $\mathcal{S}$ satisfies (S1) by definition. If we let $\mathcal{D}$ be the set of underlying sets of $\Psi^{\complement}$-precocircuits then $\mathcal{S}$ and $\mathcal{D}$ satisfy (O1) by Lemma 4.2.6 and satisfy (O2) by Corollary 4.3.5, so $\mathcal{S}$ satisfies $(S2)$ by Lemma 1.1.4. In order to show (SM), we will need a preliminary lemma.

Definition 4.8.2. Let $B \subseteq E(\mathcal{T})$. We say B is Ψ-spanning if for any $x \in E(\mathcal{T}) \setminus B$ there is a Ψ-precircuit $(S_o, \hat{o})$ with $x \in \underline{(S_o, \hat{o})} \subseteq B + x$.

Lemma 4.8.3. *Let $\mathcal{T} = (T, M)$ be a tree of matroids of overlap 1, and Ψ a Borel set of ends of T. Then there is a partition of $E(\mathcal{T})$ into a Ψ-spanning set B and a $\Psi^{\complement}$-cospanning set B^*.*

Proof. Pick a root t_0 of T. For any edge tu of T directed away from t_0, and any subset K of $E(\mathcal{T})$, we say that $e(tu)$ is *loopy* if there is a Ψ-precircuit of $\mathcal{T}_{t \to u}$ with underlying set $\{e(tu)\}$. And it is *coloopy* if it is loopy for $\mathcal{T}^*$.

For every non-coloopy dummy edge $e(tu)$, we may by (O2) applied in $\mathcal{T}_{t \to u}$ pick a Ψ-precircuit $(S(t \to u), \hat{o}_{t \to u})$. We choose these precircuits recursively, choosing them for edges closer to t_0 earlier, in such a way that if $t'u'$ is an edge of $S(t \to u)$ then $S(t' \to u') = S(t \to u)_{t' \to u'}$ and $\hat{o}_{t' \to u'} = \hat{o}_{t \to u}\restriction_{S(t' \to u')}$.

Dually, for every nonloopy dummy edge $e(tu)$, we may pick a Ψ-precocircuit $(R(t \to u), \hat{b}_{t\to u})$, in such a way that if $t'u'$ is an edge of $R(t \to u)$ then $R(t' \to u') = R(t \to u)_{t'\to u'}$ and $\hat{b}_{t'\to u'} = \hat{b}_{t\to u}\restriction_{R(t'\to u')}$.

Building the partition: For any vertex t of T, let $I(t)$ consist of the loopy dummy edges $e(tu)$ of $M(t)$ with tu directed away from t_0. Similarly, let $I^*(t)$ consist of the coloopy dummy edges $e(tu)$ of $M(t)$ with tu directed away from t_0. Note that $I(t)$ and $I^*(t)$ are disjoint since no edge can be both loopy and coloopy by Lemma 4.2.6.

For each node t of T, we shall construct a partition of $E(t)$ into $B(t)$ and $B^*(t)$ such that $I(t) \subseteq B(t)$ and each $x \in B^*(t) \setminus I^*(t)$ is spanned by $B(t)$, and dually $I^*(t) \subseteq B^*(t)$ and each $x \in B(t) \setminus I(t)$ is cospanned by $B^*(t)$. We shall construct the partitions recursively, where at step n, we construct the partitions for all nodes t_n that have distance n from the root node t_0.

Now suppose that all partitions for t with distance less than n from the root t_0 are already defined. Let t_n be at distance n from the root. If $n = 0$, it is clear that there is a partition $E(t_0) = B(t_0)\dot{\cup}B^*(t_0)$ such that $I(t_0) \subseteq B(t_0)$ and each $x \in B^*(t_0) \setminus I^*(t_0)$ is spanned by $B(t_0)$, and dually $I^*(t_0) \subseteq B^*(t_0)$ and each $x \in B(t_0) \setminus I(t_0)$ is cospanned by $B^*(t_0)$. If $n > 0$, let t_{n-1} be the neighbour of t_n with distance $n-1$ from the root.

Now we distinguish 2 cases. If $e(t_{n-1}t_n)$ is in $B(t_{n-1})$, then in particular $e(t_{n-1}t_n)$ is not coloopy at the node t_{n-1}. Thus we may pick a circuit $o_{min}(t_n)$ of $M(t_n)$ with $e(t_{n-1}t_n) \in o_{min}(t_n) \subseteq I(t_n) \cup \hat{o}_{t_{n-1}\to t_n}(t_n)$, in such a way as to minimise $o_{min}(t_n)\setminus I(t_n)$ (this is possible by (O3) applied to $M(t_n)$). Now we let $B(t_n)$ be a minimal spanning set of $E(t_n) \setminus I^*(t_n)$ including $[I(t_n) \cup o_{min}(t_n)] - e(t_{n-1}t_n)$.

In the dual case where $e(t_{n-1}t_n)$ is not in $B(t_{n-1})$, so it is in $B^*(t_{n-1})$, we do the dual thing: We pick a cocircuit $b_{min}(t_n)$ of $M(t_n)$ with $e(t_{n-1}t_n) \in b_{min}(t_n) \subseteq I^*(t_n) \cup \hat{b}_{t_{n-1}\to t_n}(t_n)$, in such a way as to minimise $b_{min}(t_n) \setminus I^*(t_n)$. Now we let $B^*(t_n)$ be a minimal cospanning set of $E(t_n) \setminus I(t_n)$ including $[I^*(t_n) \cup b_{min}(t_n)] - e(t_{n-1}t_n)$.

Having defined the partitions for each node, we take B to be the union of all sets $B(t)$ intersected with the set $E(\mathcal{T})$ of real edges, and $B^* = E(\mathcal{T}) \setminus B$.

Proof that the partition is suitable: By duality, it remains to show that every $x \in B^*$ is Ψ-spanned by B. For every edge tu directed away from t_0, the dummy edge $e(tu)$ is in $B(t)$ if and only if it is not in $B(u)$. If $e(tu)$ is in $B(t)$, we will think of this edge as being 'spanned from above'. We make this more formal by showing the first of two auxilliary facts: that for any edge tu of T directed away from the root t_0, if $e(tu) \in B(t)$ then $e(tu)$ is Ψ-spanned in $\mathcal{T}_{t\to u}$ by $B \cap E(\mathcal{T}_{t\to u})$.

So suppose we have such an edge $e(tu) \in B(t)$. We first define a subtree of $S(t \to u)$. We say an edge vw, directed away from u, of $S(t \to u)$ is *helpful* if in the construction we defined a circuit $o_{min}(v)$ and $e(vw) \in \hat{o}_{min}(v)$. Let S be the subtree of $S(t \to u)$ on those vertices v such that all edges of the path from u to v are helpful. For each edge vw of $T_{t\to u}$ directed away from u

and with v but not w a vertex of S and $e(vw) \in o_{min}(v) \cap I(v)$, we choose a Ψ-precircuit $(S_w, \hat{o}_w)$ of $\mathcal{T}_{v \to w}$ with underlying set $\{e(vw)\}$. Now we obtain the desired Ψ-precircuit $(S_o, \hat{o})$ by sticking all of these precircuits together: that is, we let S_o be the union of S and all the S_w, and we take $\hat{o}(x)$ to be $o_{min}(x)$ if $x \in S$ and $\hat{o}_w(x)$ if $x \in S_w$.

Having shown this first auxiliary fact, we next show a second auxiliary fact, telling us when dummy edges are 'spanned from below': more precisely, we show that for any edge $t_n t_{n+1}$ of T with distance n from t_0 and directed away from the root t_0, if $e(t_n t_{n+1}) \in B^*(t_n) \setminus I^*(t_n)$, then $e(t_n t_{n+1})$ is Ψ-spanned in $\mathcal{T}_{t_{n+1} \to t_n}$ by $B \cap E(\mathcal{T}_{t_{n+1} \to t_n})$.

We prove this by induction on n. Let $\hat{o}(t_n)$ be a fundamental circuit of $e(t_n t_{n+1})$ into $B(t_n)$. If $n > 0$, we take t_{n-1} to be the unique neighbour of t_n such that $t_{n-1}t_n$ is directed away from t_0. Note that if $e(t_{n-1}t_n) \in I^*(t_{n-1})$ then there is an $M(t_n)$-cocircuit b with $e(t_{n-1}t_n) \in b \subseteq I^*(t_n) + e(t_{n-1}t_n)$, so by (O1) applied in $M(t_n)$ we cannot have $e(t_{n-1}t_n) \in \hat{o}(t_n)$. For each dummy edge $t_n u$ with $u \neq t_{n+1}$ and $e(t_n u) \in \hat{o}(t_n)$ there is a Ψ-precircuit $(S_u, \hat{o}_u)$ of $\mathcal{T}_{t_n \to u}$ all of whose real edges are in B: this is by the induction hypothesis if $u = t_{n-1}$ and by the first auxiliary fact otherwise. Sticking these precircuits $(S_u, \hat{o}_u)$ onto $\hat{o}(t_n)$ gives the desired precircuit.

Now suppose that we have some $x \in E(\mathcal{T}) \setminus B$. Let t be the node of T with $x \in E(t)$. Let $\hat{o}(t)$ be a fundamental circuit of x into $B(t)$. If t has a neighbour u with tu directed towards t_0 and $e(tu) \in \hat{o}(t)$ then as in the last paragraph we see that $e(tu) \notin I^*(u)$. So for each neighbour u of t with $e(tu) \in \hat{o}(t)$ we get, by one of the auxilliary facts, a Ψ-precircuit $(S_u, \hat{o}_u)$ of $\mathcal{T}_{t_n \to u}$ all of whose real edges are in B. Sticking these precircuits $(S_u, \hat{o}_u)$ onto $\hat{o}(t)$ gives the desired Ψ-precircuit. This completes the proof. $\square$

To deduce (SM), let $\mathcal{I}$ be the set of subsets of $E(\mathcal{T})$ not including a nonempty element of $\mathcal{S}$. Suppose we have $I \subseteq X \subseteq E(\mathcal{T})$ with $I \in \mathcal{I}$. Let $Y = E(\mathcal{T}) \setminus X$. We apply Lemma 4.8.3 to $\mathcal{T}/I\backslash Y$ to obtain a partition of $E(\mathcal{T}) \setminus I \setminus Y$ into a Ψ-spanning set B and a $\Psi^{\complement}$-cospanning set B^*. We will show that $I \cup B$ is maximal amongst the subsets of X that are in $\mathcal{I}$.

First, we show that every proper superset of $I \cup B$ is not in $\mathcal{I}$. Suppose not for a contradiction, so there is some $e \in X \setminus (I \cup B)$ such that $(I \cup B) + e \in \mathcal{I}$. Since B is Ψ-spanning in $\mathcal{T}/I\backslash Y$, there is a Ψ-precircuit $(S_o, \hat{o})$ for $\mathcal{T}/I\backslash Y$ whose underlying set contains e and is included in $B + e$. For each $t \in S_o$, there is an $M(t)$-circuit $\hat{o}_1(t)$ with $\hat{o}(t) \subseteq \hat{o}_1(t) \subseteq \hat{o}(t) \cup I$ by Lemma 1.2.7. Then $(S_o, \hat{o}_1)$ is a Ψ-precircuit with nonempty underlying set that is included in $(I \cup B) + e$, which is a contradiction.

It remains to show that $I \cup B \in \mathcal{I}$. Suppose not, for a contradiction, and let $(S_o, \hat{o})$ be a Ψ-circuit whose underlying set is nonempty and included in $I \cup B$. The underlying set must meet B in some edge e, since $I \in \mathcal{I}$. Let $(S_b, \hat{b})$ be a precocircuit witnessing that e is $\Psi^{\complement}$-cospanned by B^*. As above, we may find a precocircuit $(S_b, \hat{b}_1)$ of $\mathcal{T}$ with $\hat{b}(t) \subseteq \hat{b}_1(t) \subseteq \hat{b}(t) \cup I$ for each $t \in S_b$. Then the underlying sets of $(S_o, \hat{o})$ and $(S_b, \hat{b}_1)$ have only the edge e in their intersection.

This contradicts Lemma 4.2.6. Hence $I \cup B \in \mathcal{I}$. This completes the proof of (SM) and so also that of Theorem 4.8.1.

4.9 Tree decompositions of matroids

In [6], the following notion of a decomposition of a matroid into a tree of such smaller parts was introduced.

Definition 4.9.1. A *tree decomposition of adhesion 2* of a matroid N consists of a tree T and a partition $R = (R_v)_{v \in V(T)}$ of the ground set E of N such that for any edge tt' of T the partition $(\bigcup_{v \in V(T_{t \to t'})} R_v, \bigcup_{v \in V(T_{t' \to t})} R_v)$ is a 2-separation of N.

Given such a tree decomposition, and a vertex v of T, we define a matroid $M(v)$, called the *torso* of T at v, as follows: the ground set of $M(v)$ consists of R_v together with a new edge $e(vv')$ for each edge vv' of T incident with v. For any circuit o of N not included in any set $\bigcup_{t \in V(T_{v \to v'})} R_v$, we have a circuit $\hat{o}(v)$ of $M(v)$ given by $(o \cap R_v) \cup \{e(vv') \in E(v) | o \cap \bigcup_{t \in V(T_{v \to v'})} R_v \neq \emptyset\}$. These are all the circuits of $M(v)$.

In this way we get a tree of matroids $\mathcal{T}(N, T, R) = (T, v \mapsto M(v))$ of overlap 1 from any tree decomposition of adhesion 2. For any circuit o of N we get a corresponding precircuit $(S_o, \hat{o})$, where S_o is just the subtree of T consisting of those vertices v for which $\hat{o}(v)$ is defined.

Note that $\overline{(S_o, \hat{o})} = o$. Each $M(v)$ really is a matroid [6, §4, §8], isomorphic to a minor of N [21], and that $\mathcal{T}(N^*, T, R) = (\mathcal{T}(N, T, R))^*$. [6] also contains the following theorem.

Theorem 4.9.2 (Aigner-Horev, Diestel, Postle)**.** *For any matroid N there is a tree decomposition $\mathcal{D}(N)$ of adhesion 2 of N such that all torsos have size at least 3 and are either circuits, cocircuits or 3-connected, and in which no two circuits and no two cocircuits are adjacent in the tree. This decomposition is unique in the sense that any other tree decomposition with these properties must be isomorphic to it.*

The above theorem is a generalisation to infinite matroids of a standard result about finite matroids [31, 59]. If N is a finite matroid, it is possible to reconstruct N from the decomposition $\mathcal{D}(N)$. However, as noted in the introduction, it is not in general possible to reconstruct N from $\mathcal{D}(N)$ if N is infinite. Our aim in the next section will be to show that if N is tame, then not much extra information is needed to recover N. All we need is the set Ψ consisting of those ends of T that appear in the closure of some circuit of N.

4.10 Reconstruction

Let N be a tame matroid and let (T, R) be a tree decomposition of N of adhesion 2. We begin by considering the case that T is a ray $t_1, t_2, \ldots$. In this case, we can show that the tree $\mathcal{T}(N, T, R)$ is well behaved.

Definition 4.10.1. A precircuit (S, o) for a tree $\mathcal{T} = (T, M)$ of matroids of overlap 1 is called a *phantom precircuit* if there is an edge tt' of S such that $o(v) \cap E(\mathcal{T}) = \emptyset$ for $v \in V(S_{t \to t'})$.

$\mathcal{T} = (T, M)$ is *nice* if neither $\mathcal{T}$ nor $\mathcal{T}^*$ has any phantom precircuits.

Note that $\mathcal{T} = (T, M)$ is nice iff there is not $tt' \in E(T)$ such that in $\mathcal{T}_{t \to t'} = (T_{t \to t'}, M\restriction_{V(T_{t \to t'})})$ the edge $e(tt')$ is either a loop in $M_{\Omega(\mathcal{T}_{t \to t'})}(\mathcal{T}_{t \to t'})$ or a coloop $M_\emptyset(\mathcal{T}_{t \to t'})$.

Lemma 4.10.2. *Let N be a matroid with a tree decomposition (T, R) of adhesion 2.*

1. *For every N-circuit o its corresponding precircuit $(S_o, \hat{o})$ is not phantom.*
2. *If T is a ray, and there is a circuit o and a cocircuit b of N that both have edges in infinitely many of the R_v, then $\mathcal{T}(N, T, R)$ is nice.*

Proof. (1) follows from the definition of S_o.

For (2), let $T = t_1, t_2, \ldots$ be a ray. Now suppose for a contradiction that there is a phantom precircuit (S_c, c). Then for all sufficiently large n, the circuit $c(t_n)$ consists of $e(t_{n-1}t_n)$ and $e(t_n t_{n+1})$. In other words, $e(t_{n-1}t_n)$ and $e(t_n t_{n+1})$ are in parallel.

So $c(t_n) \subseteq \hat{o}(t_n)$, hence $c(t_n) = \hat{o}(t_n)$. This contradicts (1). The case that there is a phantom precocircuit (S_c, c) is similar. Hence $\mathcal{T}(N, T, R)$ is nice. $\square$

Lemma 4.10.3. *Let $\mathcal{T} = (T, M)$ be a nice tree of matroids, then every $\emptyset$-circuit is an $\Omega(T)$-circuit.* $\square$

By duality, an analogue of Lemma 4.10.3 is also true for cocircuits.

Lemma 4.10.4. *Let $\mathcal{T} = (T, M)$ be a nice tree of matroids, and N be a matroid such that $\mathcal{C}(N) \subseteq \mathcal{C}(M_{\Omega(T)}(\mathcal{T}))$ and $\mathcal{C}(N^*) \subseteq \mathcal{C}(M^*_\emptyset(\mathcal{T}))$.*

Then $\mathcal{C}(M_\emptyset(\mathcal{T})) \subseteq \mathcal{C}(N)$ and $\mathcal{C}(M^*_{\Omega(T)}(\mathcal{T})) \subseteq \mathcal{C}(N^*)$.

Proof. By duality, it suffices to prove only that $\mathcal{C}(M_\emptyset(T)) \subseteq \mathcal{C}(N)$. So let $o \in \mathcal{C}(M_\emptyset(T))$. Since o never meets an element of $\mathcal{C}(M^*_\emptyset(T))$ just once, it never meets an element of $\mathcal{C}(N^*)$ just once. Hence o includes an N-circuit o' by the dual of Lemma 1.2.1. Thus $o' \in \mathcal{C}(M_{\Omega(T)}(T))$. By Lemma 4.10.3, we must have $o' = o$. So $o \in \mathcal{C}(N)$, as desired. $\square$

Lemma 4.10.5. *Let N be a tame matroid with a tree decomposition (T, R) of adhesion 2. Assume that $T = t_1 t_2 \ldots$ is a ray.*

Then there are not a circuit o and a cocircuit b of N that both converge to the end ω of T.

Indeed, either $N = M_\emptyset(\mathcal{T}(N, T, R))$ or $N = M_{\{\omega\}}(\mathcal{T}(N, T, R))$

Proof. Suppose for a contradiction that there are such o and b. Then there are $l < m < n$ and $e_l, e_m, e_n \in E(N)$ such that $e_l \in b \cap E(t_l)$, and $e_m \in o \cap E(t_m)$, and $e_n \in b \cap E(t_n)$. Using the tameness of N, we make these choices in such a way that for any $i \geq m$, the intersection of $o \cap b$ with $E(t_i)$ is empty. We may also assume that o has an edge in some $E(t_k)$ with $k < m$, so that both dummy edges of $E(t_m)$ are in $\hat{o}(t_m)$.

Now b is a cocircuit of $M(\mathcal{T}(N,T,R),\emptyset)$ since $(S_b,\hat{b})$ is a precocircuit, and there cannot be a precocircuit whose cocircuit at any node t is a subset of $\hat{b}(t)$. By the dual of Lemma 1.3.5 there is some $M_\emptyset(\mathcal{T})$-circuit o_b meeting b only in e_l and e_n. Note that o_b is also a circuit of N by Lemma 4.10.4 since $\mathcal{T}(N,T,R)$ is nice by Lemma 4.10.2.

Now we build an $\emptyset$-precircuit $(S_C,\hat{C})$ as follows. First we set $S_C = (S_{o_b} \setminus \{t_1 \dots t_m\}) \cup (S_o \cap \{t_1 \dots t_m\})$. We take $\hat{C}(t_j) = \hat{o}_b(t_j)$ for $j > m$, and $\hat{C}(t_j) = \hat{o}(t_j)$ for $j \leq m$. Let C be the underlying circuit of $(S_C,\hat{C})$. Note that C is a circuit of $M_\emptyset(\mathcal{T})$ and so also a circuit of N by Lemma 4.10.4 and Lemma 4.10.2 as before.

We now apply circuit elimination in N to the circuits o and C, eliminating the edge e_m and keeping the edge e_n. Call the resulting circuit C'.

If $t_m \in S_{C'}$, then $\hat{C}'(t_m) \subseteq \hat{o}(t_m) - e_m$ (since both dummy edges of $E(t_m)$ are in $\hat{o}(t_m)$), which is impossible. So $S_{C'} \subseteq \{t_{m+1}, t_{m+2}, \dots\}$. Hence $C' \cap b = \{e_n\}$, which is also impossible.

We have now established that there cannot be a circuit o and a cocircuit b of N such that ω is in the closure of both o and b.

If ω is in the closure of some N-circuit, then every N-circuit is a $\{\omega\}$-circuit, and every N-cocircuit is an $\emptyset$-cocircuit. Since by Lemma 4.2.6 no Ψ-circuit ever meets a $\Psi^{\complement}$-cocircuit just once, we may apply Lemma 1.3.7 to deduce that $N = M_{\{\omega\}}(\mathcal{T})$. In the case that ω is not in the closure of any N-circuit a similar argument yields that $N = M_\emptyset(\mathcal{T})$. This completes the proof. $\square$

Having considered the case that the tree T is a ray, we now reduce the general case to this special case. Let N be a matroid with a tree decomposition (T,R) of adhesion 2. Let $Q = q_1, q_2, \dots$ be a ray in T. We define R^Q to be the following coarsening of R. We define $R^Q_{q_i}$ to be the union of all the R_v such that in T the vertices v and q_i can be joined by a path that does not contain any other q_j.

Then (Q,R^Q) is a tree decomposition of N of adhesion 2. An N-circuit o has the end ω of Q in its closure with respect to (T,R) if and only if o has ω in its closure with respect to (Q,R^Q). So by Lemma 4.10.5, we deduce that there cannot be a circuit and a cocircuit of N that have a common end in both of their closures (with respect to (T,R)).

Let Ψ be the set of ends of T that appear in the closure of some circuit of N. Thus every N-circuit is a Ψ-circuit and every N-cocircuit is a $\Psi^{\complement}$-cocircuit. Since by Lemma 4.2.6 no Ψ-circuit ever meets a $\Psi^{\complement}$-cocircuit just once, we may apply Lemma 1.3.7 to deduce that $N = M_\Psi(\mathcal{T})$. Hence we get the following theorem.

Theorem 4.10.6. *Let N be a tame matroid with a tree decomposition (T,R) of adhesion 2.*

Then there is some $\Psi \subseteq \Omega(T)$ such that $N = M(\mathcal{T}(N,T,R),\Psi)$.

Combining this theorem with Theorem 4.9.2 yields:

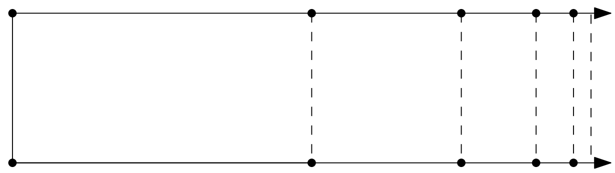


Figure 4.9: A non-nice tree of matroids

Theorem 4.10.7. *Let N be a connected tame matroid. Then $N = M_\Psi(\mathcal{T})$ where each $M(t)$ is either a circuit, a cocircuit or else is 3-connected.*

Remark 4.10.8. *In the proof of this theorem, it might look as if we would have some freedom in choosing the set Ψ, namely that we could take Ψ to be any set containing all the ends to which some circuit converges and avoiding all ends to which some cocircuit converges. However, it can be shown that for every end in Ψ, there is a Ψ-circuit having this end in the closure.*

The arguments above make use of the fact that the matroid N is severly constrained by the restriction that each of its circuits comes from some precircuit of the tree, and each of its cocircuits comes from some precocircuit. In investigating how restrictive constraints of this form might be in general, we are led to the following question. Suppose that we have a tree $\mathcal{T} = (T, M)$ of matroids. We say a matroid N is a $\mathcal{T}$-matroid if every circuit of N is an $\Omega(T)$-circuit and every cocircuit of N is an $\emptyset$-cocircuit. How constrained is N? If $\mathcal{T}$ is not nice, then N can be quite unconstrained.

Example 4.10.9. Here the tree T is a ray and each $M(t) = M(C_4)$, arranged as in Figure 4.9. Then $M_\emptyset(\mathcal{T})$ is the free matroid but $M_{\Omega(T)}(\mathcal{T})$ consists of a single infinite circuit. So any pair of edges forms an $M_{\Omega(T)}(\mathcal{T})$-cocircuit which is not an $M_\emptyset(\mathcal{T})$-cocircuit.

However, if $\mathcal{T}$ is nice and N is tame then N has to be of the form $M_\Psi(\mathcal{T})$:

Theorem 4.10.10. *Let $\mathcal{T} = (T, M)$ be a nice tree of matroids of overlap 1, and let N be a tame $\mathcal{T}$-matroid. Then there is some $\Psi \subseteq \Omega(T)$ such that $N = M_\Psi(\mathcal{T})$.*

Proof. We begin by showing that there cannot be a circuit o and a cocircuit b of N such that there is some end ω of T in the closure of both o and b. So suppose for a contradiction that there are such o, b, and ω. We fix some notation, as illustrated in Figure 4.10. Pick a ray $R = v_1, v_2, \dots$ in T to ω. By taking a suitable tail of R if necessary, we may assume that there is some edge f of b in $E(\mathcal{T}) \setminus E(\mathcal{T}_{v_1 \to v_2})$, some edge g of o in $E(\mathcal{T}_{v_1 \to v_2}) \setminus E(\mathcal{T}_{v_2 \to v_3})$ and some edge h of b in $E(\mathcal{T}_{v_2 \to v_3})$ (here we use that $\mathcal{T}$ is nice). Since $o \cap b$ is finite, we may even assume that no edge of $o \cap b$ lies in $E(\mathcal{T}_{v_1 \to v_2})$.

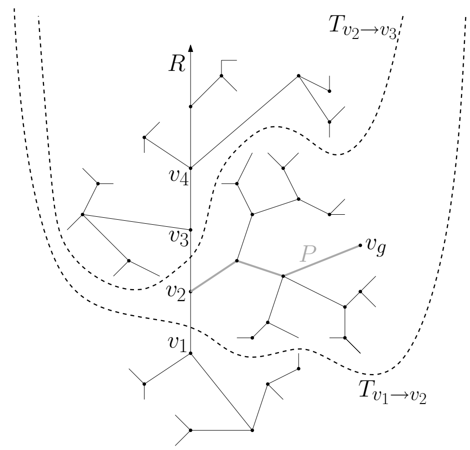


Figure 4.10: Objects appearing in the proof of Theorem 4.10.10

We may also assume that o has an edge in $E(\mathcal{T}_{v_2 \to v_1})$, so that both dummy edges of $E(v_2)$ are in $\hat{o}(v_2)$.

By Lemma 1.3.5 there is some $M_\emptyset(\mathcal{T})$-circuit o_b meeting b only in f and h. Let $(S, \hat{o})$ be an $\Omega(T)$-precircuit representing o, and $(S_b, \hat{o}_b)$ be an $\emptyset$-precircuit representing o_b. Let v_g be the node of T with $g \in E(v)$. Let P be the path joining v_2 to v_g in T. Let ∂ be the set of edges tt' of T with t in $V(P)$ but t' not in either $V(P)$ or $V(T_{v_2 \to v_3})$. For each edge $tt' \in \partial$ there is by niceness of $\mathcal{T}$ some $M_\emptyset(\mathcal{T}_{t \to t'})$-circuit $o_{t \to t'}$ through $e(tt')$. Let $(S_{t \to t'}, \hat{o}_{t \to t'})$ be an $M_\emptyset(\mathcal{T}_{t \to t'})$-precircuit representing $o_{t \to t'}$.

Now we build a $\emptyset$-precircuit $(S_C, \hat{C})$ from all this data as follows. First we set

$$S_C = (S_b \cap T_{v_2 \to v_3}) \cup P \cup \bigcup_{tt' \in \partial} S_{t \to t'}.$$

Then we take $\hat{C}(u)$ to be $\hat{o}_b(u)$ for $u \in (S_b \cap T_{v_2 \to v_3})$, $\hat{o}(u)$ for $u \in P$ and $\hat{o}_{t \to t'}(u)$ for $u \in S_{t \to t'}$. Let C be the underlying circuit of $(S_C, \hat{C})$. By Lemma 4.10.4, C is an N-circuit.

We now apply circuit elimination in N to the circuits o and C, eliminating the edge g and keeping the edge h. Call the resulting circuit C', and let $(S_{C'}, \hat{C}')$ be an $\Omega(T)$-precircuit representing C'. Let the vertices of P be, in order, $v_g = p_1, p_2, \ldots p_k = v_2$. We shall show by induction on i that $p_i \notin S_{C'}$. For the base case, we note that if v_g were in $S_{C'}$ we would have to have $\hat{C}'(v_g) \subseteq \hat{o}(v_g) \setminus \{g\}$, which is impossible. For the induction step, we similarly note that if p_{i+1} were in $S_{C'}$ we would have to have $\hat{C}'(p_{i+1}) \subseteq \hat{o}(p_{i+1}) \setminus \{e(p_i p_{i+1})\}$, by the induction hypothesis, which is impossible. In particular, we deduce that $v_2 \notin S_{C'}$. On the other hand, we know that $h \in C'$ so that $S_{C'} \subseteq T_{v_2 \to v_3}$, so that $C' \cap b = \{h\}$, a contradiction.

We have now established that there cannot be a circuit o and a cocircuit b

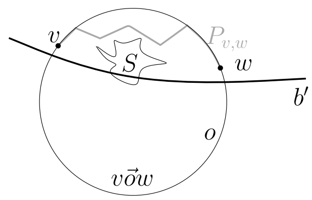


Figure 4.11: The situation of Lemma 4.11.2.

of N such that there is some end ω of T in the closure of both o and b. Let Ψ be the set of ends of T that appear in the closure of some circuit of N. Thus every N-circuit is a Ψ-circuit and every N-cocircuit is a $\Psi^{\complement}$-cocircuit. Since by Lemma 4.2.6 no Ψ-circuit ever meets a $\Psi^{\complement}$-cocircuit just once, we may apply Lemma 1.3.7 to deduce that $N = M_\Psi(\mathcal{T})$ as required. $\square$

4.11 Tame G-matroids are Ψ-matroids

Let G be a locally finite graph. Recall that a matroid N on the ground set $E(G)$ is a G-matroid if $\mathcal{C}(N) \subseteq \mathcal{C}(M_C(G))$ and $\mathcal{C}(N^*) \subseteq \mathcal{C}(M_{FC}(G)^*)$. Since $\mathcal{C}(M_{FC}(G)) \subseteq \mathcal{C}(M_C(G))$ and $\mathcal{C}(M_C^*(G)) \subseteq \mathcal{C}(M_{FC}^*(G))$ both $M_{FC}(G)$ and $M_C(G)$ are G-matroids, and an argument like that for Lemma 4.10.4 shows that for any G-matroid N we have $\mathcal{C}(M_{FC}(G)) \subseteq \mathcal{C}(N)$ and $\mathcal{C}(M_C^*(G)) \subseteq \mathcal{C}(N^*)$. The aim of this section is to prove the following.

Theorem 4.11.1. *Let G be a locally finite graph, and let N be a tame G-matroid. There there is some $\Psi \subseteq \Omega(G)$ such that $N = M_\Psi(G)$.*

For the rest of this section we fix some locally finite graph G and some tame G-matroid N.

In this section, we will have to use two different notions of path. Finite paths in graphs will simply be called *paths*, whereas paths in the topological sense, namely continuous images of the closed unit interval, will be called *topological paths*.

For a pair of points on a topological circle, there are two arcs joining them through the circle. To allow us to distinguish them, we shall make use of orientations of circles and topological paths. For distinct points x and y on an oriented circle $\vec{o}$ we use $x\vec{o}y$ to denote the (oriented) topological path from x to y through o whose orientation agrees with that of $\vec{o}$. We denote the other topological path by $x\overleftarrow{o}y$. If $x = y$, we do not take the trivial topological path but the topological path that goes all the way around the circle.

Lemma 4.11.2. *Let o be a topological circle in G. Let $v, w \in V(o)$ and let S be a finite set of vertices avoiding $V(o)$. Then for any orientation $\vec{o}$ of o, there is a finite v-w-path $P_{v,w}$ not meeting $v\vec{o}w$ in interior points and avoiding S.*

Moreover if $v\vec{o}w$ has at least two edges, then there is a bond b' of G that has $P_{v,w} \cup v\overleftarrow{o}w$ on one side and all interior vertices of $v\vec{o}w$ on the other side.

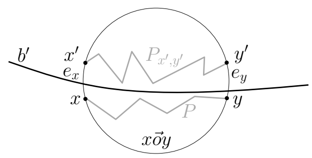


Figure 4.12: The situation of Lemma 4.11.3.

Figure 4.11 gives an overview of the terminology used in this lemma.

Proof. The proof is trivial if $v = w$. Thus we may assume that $v \neq w$. Let e_v be the first edge on $v\vec{o}w$ and e_w be the last edge on $v\vec{o}w$. Note that e_v and e_w exist since v and w are vertices. If $e_v = e_w$, we pick $P_{v,w} = e_v$. So we may assume that $e_v \neq e_w$.

Let $G' = G \setminus S$. Since o is a topological circle in $|G'|$, there is a finite bond b of G' meeting o in precisely e_v and e_w.

All edges and vertices of $v\vec{o}w - e_v - e_w - v - w$ are on the same side of b. Let C be the other side. Note that v and w are in C. Now let $P_{v,w}$ be some path in C joining v and w.

The bond b extends to a finite bond b' of G by adding finitely many deleted edges. The bond b' has the desired property. $\square$

Lemma 4.11.3. *Let o be a circuit of the G-matroid N, and $\vec{o}$ be some orientation of o. Further, let $x, y \in V(o)$ and let $\vec{P}$ be an x-y-path meeting o in precisely x and y.*

Then $x\vec{o}y\overleftarrow{P}x \in \mathcal{C}(N)$.

Proof. The proof is trivial if $x = y$. Thus we may assume that $x \neq y$. Let e_x be the first edge on $x\overleftarrow{o}y$ and e_y be the last edge on $x\overleftarrow{o}y$. Let x' be the endvertex of e_x that is not x, and y' be the endvertex of e_y that is not y, as depicted in Figure 4.12.

Applying Lemma 4.11.2 to $\vec{o}$ with $S = V(P) - x - y$, yields an x'-y'-path $P_{x',y'}$, and a bond b' as in that lemma. By assumption the finite circuit $x\vec{P}ye_yy'\overleftarrow{P}_{x',y'}x'e_xx$ is an N-circuit. See Figure 4.12 to get an overview of all the definitions.

Now we apply circuit elimination in N to this new circuit and o eliminating e_x and keeping some $z \in E(P)$. Note that z exists since $x \neq y$. We obtain an N-circuit $o' \subseteq (o - e_x) \cup P \cup P_{x',y'}$ including z.

It remains to show that $o' = x\vec{o}y \cup P$. Since each vertex of G is incident with 0 or 2 edges of o', we conclude that each edge adjacent to z on P is in o'. In fact an inductive argument yields that $P \subseteq o'$.

If $x\overleftarrow{o}y$ consists of a singe edge xy, then by the same argument xy also cannot be in o'. Thus $o' \subseteq x\vec{o}y\overleftarrow{P}x$. Since the latter is a topological cycle, we must have equality, hence $x\vec{o}y\overleftarrow{P}x \in \mathcal{C}(N)$. Thus we may assume that $x\overleftarrow{o}y$ includes at least two edges.

We know that o' is the union of P and some topological arc A from x to y. The edge set of this arc is included in $L := (o - e_x) \cup P_{x',y'}$. The set L meets b' precisely in e_y. Let K be the side of b' not containing x and y. Suppose for a contradiction that L includes an edge e_k from K. Then there are two disjoint arcs L_x and L_y from e_k to x and from e_k to y. By the Jumping-Arc Lemma [34, Lemma 8.5.3], both of these have to meet b', contradicting the fact that L meets b' just in e_y.

This means that $L \subseteq x\vec{o}y$. Since $x\vec{o}y$ is an x-y-arc, we actually get $L = x\vec{o}y$. Thus we have shown that $o' = x\vec{o}y \cup P$, which completes the proof. □

Corollary 4.11.4. *Let $o \in \mathcal{C}(N)$, and $\vec{o}$ be some orientation of o. Further, let $x, y \in V(o)$ and let $\vec{P}$ be an x-y-path meeting $x\vec{o}y$ not in interior points.*

Then $x\vec{o}y\overleftarrow{P}x \in \mathcal{C}(N)$.

Remark 4.11.5. *The only difference between Lemma 4.11.3 and Corollary 4.11.4 is that in the second the path P may meet o in some of the interior points.*

Proof. We prove this by strong induction on $|P \cap o|$. Let z be the second point in the order of P in $P \cap o$ (the first such point is x). Now we apply Lemma 4.11.3 to o and xPz and obtain a new circuit $o_z := x\vec{o}z\overleftarrow{P}x$ and a new path $P_z := zPy$. Since $|o_z \cap P_z| < |o \cap P|$, we may apply the induction hypothesis. □

Lemma 4.11.6. *Let $o \in \mathcal{C}(N)$ and let ω be an end of o. Then there is $o' \in \mathcal{C}(N)$ that has only the end ω in its closure.*

Proof. First we pick an orientation $\vec{o}$ of o. Then we pick a $\mathbb{Z}$-indexed family of distinct edges e_i such that their ordering on $\omega\vec{o}\omega$ is the same as the ordering of their indices and $(e_i|i > 0)$ and $(e_i|i < 0)$ both converge to ω.

Let s_i and t_i be the endvertices of e_i such that $s_i < t_i$ on $\omega\vec{o}\omega$. We repeatedly apply Corollary 4.11.4 to get a $\mathbb{Z}$-indexed family of vertex-disjoint t_i-s_{i+1}-paths P_i with P_i disjoint from $t_{i+1}\vec{o}s_i$.

Let D be the double ray obtained from sticking the P_i and the e_i together, formally:

$$\dots t_{-1}P_{-1}s_0e_0t_0P_0s_1e_1t_1P_1\dots$$

By construction, both tails of D belong to ω. So D is a topological cycle. It remains to show that $D \in \mathcal{C}(N)$. Suppose not, for a contradiction: Then $D \in \mathcal{I}(N)$, so there is a N-bond b with $b \cap D = \{e_0\}$.

Since N is tame, $o \cap b$ is finite, so there are only finitely many $i \in \mathbb{Z}$ such that b meets $t_i\vec{o}s_{i+1}$. Let K be the set of such i.

By applying Corollary 4.11.4 finitely often, we get a circuit o'' that meets b precisely in e_0. Formally,

$$o'' = o \setminus \left(\bigcup_{i\in K}(t_i\vec{o}s_{i+1})\right) \cup \left(\bigcup_{i\in K} P_i\right)$$

So there are a circuit and a cocircuit of N which meet just once, which is the desired contradiction. □

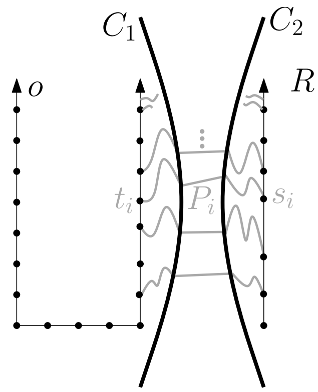


Figure 4.13: The o-R-paths P_i.

Lemma 4.11.7. *Let $b \in \mathcal{C}(N^*)$ and ω be an end in the closure of b. Assume there is a double ray $o \in \mathcal{C}(N)$ both of whose tails converge to ω.*

Then there is such an o that does not meet b.

Proof. We prove this by induction on $|o \cap b|$; the base case $|o \cap b| = 0$ is clear. The case $|o \cap b| = 1$ is impossible. So suppose for the induction step that $|o \cap b| \geq 2$. Thus we may pick $e, f \in o \cap b$. Since $b \in \mathcal{C}(M_{FC}(G))$, there is a finite circuit o' meeting b in precisely e and f. Note that o' is an N-circuit. Now pick z in the infinite component of $o \setminus o'$ containing ω.

Applying circuit elimination to o and o' eliminating e and keeping z yields an N-circuit $o'' \subseteq o \cup o' - e$ through z. By the choice of z, the subgraph with edge set $o \cup o' - e - z$ has two components one of which is a ray R from one endvertex of z converging to ω. Since each vertex is incident with 0 or 2 edges of o'' and $z \in o''$, the ray R must be included in o''. Hence o'' must be infinite, it also has only the end ω in its closure since $o'' \subseteq o \cup o' - e$ has no other end in its closure. Now $o'' \cap b \subseteq (o \cap b) - e$. This completes the induction step. $\square$

Lemma 4.11.8. *Let $b \in \mathcal{C}(N^*)$ and ω be an end in the closure of b.*

Then there is no double ray $o \in \mathcal{C}(N)$ both of whose tails converge to ω with $o \cap b = \emptyset$.

Proof. Suppose for a contradiction that there is such an N-circuit o. Let C_1 and C_2 be the two sides of b. Since $o \cap b = \emptyset$ and since the double ray o is connected as a subgraph, it lies entirely on one side, say C_1. Since G is locally finite, C_2 includes a ray R converging to ω.

Now we construct o-R-paths P_i as in Figure 4.13. Since o and R both have ω in their closure, there are infinitely many vertex disjoint R-o-paths $(P_i | i \in \mathbb{N})$. We enumerate the P_i such that in the linear order on R, the starting vertex of P_i is less than the starting vertex of P_j if and only if $i > j$. By Ramsey's theorem there is a tail R_o of o and $N \subseteq \mathbb{N}$ such that all P_i with $i \in \mathbb{N}$ have their endvertex on R_o, and in the linear order on R_o, the endvertex of P_i is less than the endvertex of P_j if and only if $i > j$. By relabeling the indices of the P_i if necessary, we may assume that $N = \mathbb{N}$. Let s_i be the starting vertex of P_i, and t_i be its endvertex.

Now we prepare to apply the infinite circuit elimination axiom. We pick some edge x_i between t_{2i-1} and t_{2i} on R_o, and pick some $z \in o \setminus R_o$. Then $C_{x_i} = t_{2i-1}R_o t_{2i}P_{2i}s_i R s_{2i-1}P_{2i-1}t_{2i-1}$ is a finite circuit. So C_{x_i} is an N-circuit. We apply circuit elimination to o and the C_{x_i} eliminating the x_i and keeping z. Thus there is an N-circuit o' through z that is included in:

$$\left(o \cup \left(\bigcup_{i \in \mathbb{N}} t_{2i}P_{2i}s_i R s_{2i-1}P_{2i-1}t_{2i-1} \right) \right) \setminus \{x_i | i \in \mathbb{N}\}$$

Since each vertex has degree 0 or 2 on o', no edge from any of the finite paths $X_i := t_{2i-1}R_o t_{2i}$ is in o'. Hence o' is included in:

$$D := (o \setminus R_o) \cup \left(\bigcup_{i \in \mathbb{N}} C_{x_i} \setminus X_i \right)$$

But D is a double ray. So $D = o'$ and is an N-circuit. But $D \cap b$ is infinite. This contradicts the tameness of N. $\square$

Proof of Theorem 4.11.1. First we show that there cannot be an N-circuit o, and an N-cocircuit b that have a common end in their closure. Suppose for a contradiction there are such o and b. By Lemma 4.11.6, we get that there is such an o with only the end ω in its closure. By Lemma 4.11.7, we get there is such an o that additionally does not meet b. By Lemma 4.11.8, we then get the desired contradiction. So no end ω is ever in the closure of both a N-circuit and a N-cocircuit.

This motivates the following definition. Let Ψ be the set of ends that are in the closure of some N-circuit. Then every N-circuit is a Ψ-circuit and every N-cocircuit is a $\Psi^{\complement}$-cocircuit. Since N is a matroid, and the intersection of any Ψ-circuit with any $\Psi^{\complement}$-cocircuit is never of size 1 by Lemma 4.1.8, we are in a position to apply Lemma 1.3.7. Hence $N = M(G, \Psi)$. $\square$

Chapter 5

Graphic matroids

There is a rich theory describing and employing the relationship between finite graphic matroids and finite graphs. In this chapter, we will show how the foundations of this theory can be extended to infinite matroids [23]. A central result in the finite context is Tutte's characterisation by finitely many excluded minors of the class of matroids which can be represented by graphs [67].

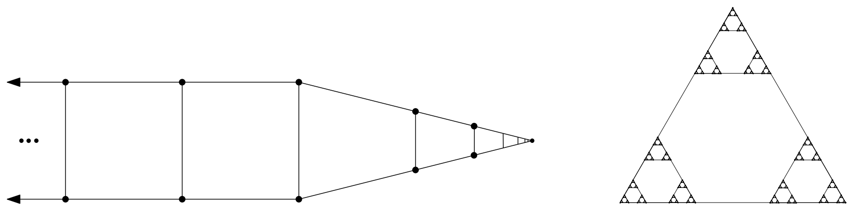
Existing work with infinite graphic matroids has focused on a few possible constructions of matroids, such as the finite-cycle, algebraic-cycle or topological-cycle matroids. Various ad-hoc extensions of these notions suggest themselves. For example, we could allow identification of ends with vertices in the definition of the topological cycle matroid [33].

Certain results about finite graphic matroids have been proved for these classes of infinite graphic matroids [25], [27], [28], [33], [62], and could also be proved about the ad-hoc extensions without too much trouble. But since all these notions fall far short of the natural boundary, namely the class of infinite matroids satisfying Tutte's excluded minor characterisation, in this chapter we instead take the approach of isolating a notion of representation for which the representable matroids are precisely those satisfying Tutte's condition. Such matroids, and their representations, provide a natural context for the extension of results from finite to infinite graphic matroids.

That the existing approaches fall far short of providing representations of all graphic matroids is shown by examples like those depicted in Figure 5.1. Here the circuits of the matroids in question are again given by the (edge sets of) homeomorphic copies of the unit circle in the subspaces of the plane given in the pictures.

What these examples show is that infinite graphic matroids should, in general, be taken to be represented not by graphs but rather by graph-like topological spaces, in a sense akin to that of Thomassen and Vella [65]. This includes the existing approaches: the finite cycle matroid of a graph would be represented by its geometric realisation, the algebraic cycle matroid by a 1-point compactification and the topological cycle matroid by the end compactification.

We restrict our attention to tame matroids because as we have seen this



(a) (b)

Figure 5.1: Subspaces of the plane inducing matroids

restriction is both natural and necessary in related representability problems. We shall introduce a notion of representability of matroids over graph-like spaces for which we can prove the following:

Theorem 5.0.9. *A tame matroid satisfies Tutte's excluded minor characterisation if and only if it is representable over a graph-like space.*

We call matroids satisfying either of these equivalent conditions *graphic*.

At least for 3-connected matroids, the notion of representability is what you would hope: the circuits are given just as usual by homeomorphic copies of the unit circle. That this hope can be fulfilled is a little strange. After all, any circuit given in this way must be countable, and there is nothing in Tutte's excluded minor characterisation which appears to restrict the cardinality of circuits. We are saved by the following miraculous fact:

Theorem 5.0.10. *In any 3-connected tame matroid satisfying Tutte's excluded minor characterisation, all circuits are countable.*

In fact, in order to prove this we first introduce a notion of representability which doesn't entail any cardinality restrictions, then play the topological structure of the representing graph-like space off against the matroidal structure.

This chapter is closely based on a joint paper with Carmesin and Christian [20].

5.1 Graph-like spaces

The key notion of this section is the following, which we briefly saw in the last chapter and which is based on a definition from [65]:

Definition 5.1.1. A *graph-like space* G is a topological space (also denoted G) together with a *vertex set* $V = V(G)$, an *edge set* $E = E(G)$ and for each $e \in E$ a continuous map $\iota_e^G \colon [0, 1] \to G$ (the superscript may be omitted if G is clear from the context) such that:

- The underlying set of G is $V \sqcup [(0, 1) \times E]$

- For any $x \in (0,1)$ and $e \in E$ we have $\iota_e(x) = (x, e)$.
- $\iota_e(0)$ and $\iota_e(1)$ are vertices (called the *endvertices* of e).
- $\iota_e\restriction_{(0,1)}$ is an open map.
- For any two distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$.

The inner points of the edge e are the elements of $(0,1) \times \{e\}$.

Note that $V(G)$, considered as a subspace of G, is totally disconnected, and that G is Hausdorff.

Let e be an edge in a graph-like space with $\iota_e(0) \neq \iota_e(1)$. Then ι_e is a continuous injective map from a compact to a Hausdorff space and so it is a homeomorphism onto its image. The image is compact and so is closed, and therefore is the closure of $(0,1) \times \{e\}$ in G. So in this case ι_e is determined by the topology of G. The same is true if $\iota_e(0) = \iota_e(1)$: in this case we can lift ι_e to a continuous map from $S^1 = [0,1]/(0=1)$ to G, and argue as above that this map is a homeomorphism onto the closure of $(0,1) \times \{e\}$ in G. In this case, we say that e is a loop of G.

Next we shall define maps of graph-like spaces. Let G and G' be graph-like spaces. Two maps $\varphi_V : V(G) \to V(G')$ and $\varphi_E : E(G) \to (E(G') \times \{+,-\}) \sqcup V(G)$ induce a function φ sending points of G to points of G' as follows: a vertex v of G is mapped to $\varphi_V(v)$. Let e be an edge, and (r, e) one of its interior points. If $\varphi_E(e)$ is a vertex, then (r, e) is mapped to $\varphi_E(e)$. If $\varphi_E(e) = (f, +)$ for some $f \in E(G')$, then (r, e) is mapped to (r, f). Similarly, if $\varphi_E(e) = (f, -)$ for some $f \in E(G')$, then (r, e) is mapped to $(1-r, f)$. If a function arising in this way is continuous we call it a *map of graph-like spaces*. From this definition, it follows that if v is an endvertex of e, then $\varphi(v)$ is either an endvertex of or equal to the image of e.

Let us consider some examples of graph-like spaces. We shall write $[0,1]$ for the unique graph-like space without loops having precisely one edge and two vertices. There are exactly seven maps of graph-like spaces from $[0,1]$ to two copies of $[0,1]$ glued together at a vertex: four of these have one of the copies of $[0,1]$ as their image and the other three map the whole interval to a vertex. However, none of these maps is bijective nor has an inverse, even though the underlying topological spaces are homeomorphic.

Figures 5.1a and 5.1b from the introduction define graph-like spaces with vertices and edges as in the figures. In each case the topology is that induced by the embedding in the plane suggested by the figures. For a locally finite graph $G = (V, E)$, the topological space $|G|$ is a graph-like space with vertex set $V \cup \Omega(G)$ and edge set E (see [34] for the definition of $|G|$). Note that if G is finite, then $|G|$ is homeomorphic to the geometric realisation of G considered as a simplicial complex.

Lemma 5.1.2. *Let G be a graph-like space with only finitely many edges and finitely many vertices. Then G is homeomorphic to $|H|$ for some finite graph H.*

Proof. G is compact, since it is a union of finitely many compact subspaces. Let H be the graph with edge set $E(G)$ and vertex set $V(G)$, and in which v is an endpoint of e if and only if this is true in G. We now construct a map $\varphi\colon G \to |H|$ as follows: taking φ_V to be the identity and φ_E to be the function sending each edge e to $(e, +)$, we build φ as in the definition of a map of graph-like spaces.

It remains to show that the function φ is continuous: since it is a bijection from a compact to a Hausdorff space, it will then be a homeomorphism. We begin by noting that for any $e \in E(G)$, the restriction of φ to the image of ι_e^G is a homeomorphism, by the remarks following Definition 5.1.1. Now we need to show for any $x \in |H|$ that the inverse image of any open neighbourhood U of $\varphi(x)$ includes an open neighbourhood of x. If x is an interior point of an edge, this is clear. Otherwise, x is a vertex of $|H|$. Then there is an open neighbourhood $U' \subseteq U$ of x which only meets edges incident with x. For each such edge e, since the restriction of φ to the image of ι_e^G is a homeomorphism, there is an open set V_e of G with $V_e \cap \mathrm{Im}(\iota_e^G) = \varphi^{-1}(U') \cap \mathrm{Im}(\iota_e^G)$. Letting V be the intersection of the V_e, we obtain that V is an open neighbourhood of x included in $\varphi^{-1}(U)$, completing the proof that φ is continuous. $\square$

All the above examples of graph-like spaces will turn out to induce matroids. Before we can make this more explicit, we must first introduce the notions of topological circuits and bonds in a graph-like space. The discussion of topological circuits will be delayed until the next section, but we will introduce topological bonds now.

Definition 5.1.3. Given a pair of disjoint open subsets of a graph-like space G partitioning the vertices, we call the set of those edges having an endvertex in both sets *a topological cut* of G. A *topological bond* of G is a minimal nonempty topological cut of G.

Given a graph-like space G and a set of edges $R \subseteq E(G)$, we define the graph-like space $G\restriction_R$, the *restriction* of G to R, to have the same vertex set as G and edge set R. Then the ground set of $G\restriction_R$ is a subset of that of G, and we give it the subspace topology. Evidently, for any topological cut b of G, $b \cap R$ is a topological cut of $G\restriction_R$. The *deletion* of D from G, denoted by $G\backslash D$, is $G\restriction_{(E\backslash D)}$. We abbreviate $G\backslash\{e\}$ by $G - e$. The inclusion map g_D from $G\backslash D$ to G is a map of graph-like spaces.

Note that $G\restriction R$ has the same vertex set as G, even though only the vertices in the closure of $(0,1) \times R$ play an important role in the new space. By analogy to the notation of [34], we also introduce a notation for the graph-like space whose edges are those in R but whose vertices are those in the closure of $(0,1) \times R$. We will call this subspace the *standard subspace with edge set* R, and denote it $\overline{R}$.

Given a graph-like space G and $C \subseteq E(G)$, we define the *contraction* G/C of G onto C as follows:

Let $\equiv_C$ be the relation on the vertices of G defined by $u \equiv_C v$ if every topological cut with u and v in different parts meets C. It is easy to check that

$\equiv_C$ is an equivalence relation. The vertex set of G/C is the set of $\equiv_C$-equivalence classes, and the edge set is $E(G) \setminus C$.

It remains to define the topology of G/C. We shall obtain this as the quotient topology derived from a function $f_C : G \to G/C$, to be defined next.

The function f_C sends each vertex to its $\equiv_C$-equivalence class and is bijective on the interior points of edges of $E \setminus C$. The two endpoints of an edge in C are in the same equivalence class, and we send all of its interior points to that equivalence class.

Taking this quotient topology ensures that G/C is a graph-like space, and makes f_C a map of graph-like spaces. In G/C, the endpoints of an edge are the equivalence classes of its endpoints in G. For any topological cut b of G with $b \cap C = \emptyset$, the two sides of b are closed under $\equiv_C$ by definition, and so b is also a topological cut in G/C.

We define $G.X := G/(E \setminus X)$ and $G/e := G/\{e\}$. It is straightforward to check for disjoint sets C and D that $(G\backslash D)/C$ and $(G/C)\backslash D$ are equal and the following diagram commutes.

$$\begin{tikzcd} G\backslash D \arrow[r, "g_D"] \arrow[d, "f_C"'] & G \arrow[d, "f_C"] \\ G/C\backslash D \arrow[r, "g_D"'] & G/C \end{tikzcd}$$

Contraction behaves especially well when applied to one side of a topological cut [20].

5.2 Pseudoarcs and Pseudocircles

When investigating a topological space, it is common to consider arcs in that space, that is, continuous injections from the unit interval to that space. We must consider maps from a slightly more general kind of domain. These domains, which we will call *pseudo-lines*, will be graph-like spaces built from total orders in the following way:

Definition 5.2.1. Let P be a totally ordered set. To construct the pseudo-line $L(P)$, we take as our vertex set V the set of initial segments of P, and as our edge set P itself. Next, we take a subbasis of the topology to consist of the sets of the type $S(p,r)^+$ or $S(p,r)^-$ defined below.

For every $p \in P$ and $r \in (0,1)$, let $S(p,r)^-$ contain precisely those vertices which do not contain p. Furthermore, let $S(p,r)^-$ contain all interior points of edges x with $x < p$ together with $(0,r) \times \{p\}$.

Similarly, let $S(p,r)^+$ contain precisely those vertices which contain p. Furthermore, let $S(p,r)^+$ contain all interior points of edges x with $x > p$ together with $(r,1) \times \{p\}$.

A *pseudo-path from v to w in a graph-like space G* is a map φ of graph-like spaces from a pseudo-line $L(P)$ to G with $\varphi(\emptyset) = v$ and $\varphi(P) = w$. The vertex v is called the *start-vertex* of the pseudo-path, and w is called the *end-vertex*.

A *pseudo-arc* is an injective pseudo-path. Any pseudo-arc is a homeomorphism onto its image since the domain is (as we shall soon show) compact, and the codomain is Hausdorff. Thus we will also refer to the images of pseudo-arcs as pseudo-arcs. In particular, a *pseudo-arc in* a graph-like space G is the image of such a map (in other words, it is a subspace of G which is also a pseudo-line).

Lemma 5.2.2. *The spaces $L(P)$ defined above are connected and compact.*

Proof. For the connectedness, let U be an open and closed set containing the start-vertex $\emptyset$. Since for any edge e the subspace topology of $\iota_e([0,1])$ is that of $[0,1]$, which is connected, the set $\iota_e([0,1])$ is either completely included in U or disjoint from U. Let $v = \{p \in P | S(p, 1/2)^- \subseteq U\}$. Then the vertex v is in U since any neighbourhood of it meets U (even if $v = \emptyset$). So since U is open, it includes an open neighbourhood O of v. Since by our earlier remarks U includes all edges $p \in v$ and so also all vertices $w \subseteq v$, we may assume without loss of generality that either $v = P$ or else O has the form $S(p,r)^-$ for some $p \notin v$. In the second case we conclude that $p \in v$, which is impossible. Hence $v = P$. Since the closure of $\bigcup_{p \in P} \iota_p((0,1))$ is the whole of $L(P)$, the closed set U is the whole of $L(P)$. Hence $L(P)$ is connected, as desired.

It remains to show that $L(P)$ is compact. By Alexander's theorem, it suffices to check that any open cover by subbasic open elements has a finite subcover. Let $L(P) = \bigcup_{i \in I^+} S(p_i, r_i)^+ \cup \bigcup_{i \in I^-} S(p_i, r_i)^-$ be an open cover by subbasic open sets. Let $v = \{p \in P | \exists i \in I^- : p < p_i\}$.

First we consider the case where there is some $i \in I^+$ with $v \in S(p_i, r_i)^+$. Then $p_i \in v$, so there is some $j \in I^-$ such that $p_i < p_j$. This means that $S(p_i, r_i)^+$ and $S(p_j, r_j)^-$ cover $L(P)$.

Otherwise there is some $i \in I^-$ with $v \in S(p_i, r_i)^-$. Then $p_i \notin v$ and so p_i is maximal amongst the p_j with $j \in I^-$. Thus $v + p_i$ is contained in some $S(p_k, r_k)^+$ with $k \in I^+$. Then $S(p_i, r_i)^-$ and $S(p_k, r_k)^+$, together with some finite collection of sets from our cover covering the compact subspace $\iota_{p_i}([0,1])$, form a finite subcover, completing the proof. □

Example 5.2.3. If $P = \omega_1$, then $L(P)$ is the *long line*, which is not homeomorphic to $[0,1]$.

Remark 5.2.4. *Any nontrivial pseudo-line is the closure of the set of interior points of its edges. Any nontrivial pseudo-arc in a graph-like space is the standard subspace corresponding to its set of edges.*

Remark 5.2.5. *Contracting a set of edges of a pseudo-line $L(P)$ corresponds to removing that set of edges from the associated poset P.*

Corollary 5.2.6. *Any contraction of a pseudo-line is a pseudo-line.* □

Lemma 5.2.7. *Any nontrivial pseudo-line $L(P)$ with only countably many edges is homeomorphic to the unit interval.*

Proof. Let $\bar{\mathbb{Q}} = \mathbb{Q} \cap (0,1)$. Consider the lexicographic linear order on $P \times \bar{\mathbb{Q}}$. This is dense, countable and has neither a largest nor a smallest element. Since the theory of such linear orders is countably categorical, this order is isomorphic to the order of $\bar{\mathbb{Q}}$. Pick an isomorphism ϕ from $P \times \bar{\mathbb{Q}}$ to $\bar{\mathbb{Q}}$.

For any $x \in [0,1]$ such that there are $p \in P$ and $q, r \in \bar{\mathbb{Q}}$ with $\phi(p,q) < x < \phi(p,r)$ we set $f(x) = (p, \sup\{q \in \bar{\mathbb{Q}} | \phi(p,q) < x\})$ (in such cases, p is clearly uniquely determined). Otherwise we set $f(x) = \{p \in P | (\forall q \in \bar{\mathbb{Q}}) \phi(p,q) < x\}$. This gives an injection f from $[0,1]$ to $L(P)$. It is continuous by the definition of the topology on $L(P)$, and so is a homeomorphism since $[0,1]$ is compact and $L(P)$ is Hausdorff. $\square$

Lemma 5.2.8. *Let $s_1 <_L \ldots <_L s_n$ be finitely many edges of a pseudo-line L. Let $S = \bigcup_{i=1}^n \iota_{s_i}((0,1))$. Then $L \setminus S$ has $n+1$ components each of which is a pseudo-line. These are $S(s_1, 1/2)^- \setminus S$, and $S(s_{i+1}, 1/2)^- \cap S(s_i, 1/2)^+) \setminus S$ for $1 \leq i \leq n-1$ and $S(s_n, 1/2)^+ \setminus S$.*

Proof. The assertion follows by induction from the following. Let $e \in L$. Then $L - e$ has two components that are both pseudo-arcs. These are $S(e, 1/2)^- \setminus ((0,1) \times \{e\})$ and $S(e, 1/2)^+ \setminus ((0,1) \times \{e\})$. $\square$

We get a total order $\leqq$ on the set of points of the space $L(P)$ as follows: if v and w are vertices, we set $v \leqq w$ when $v \subseteq w$. If v is a vertex and (p,q) an interior point of an edge, we set $v \leqq (p,q)$ when $p \notin v$ and $(p,q) \leqq v$ when $p \in v$. Finally, we order the interior points of edges by the lexicographic order on $P \times (0,1)$.

Lemma 5.2.9. *Let X be a nonempty closed subset of a pseudo-line $L(P)$. Then X contains a $\leqq$-smallest and a $\leqq$-biggest element.*

Proof. First we show that X contains a $\leqq$-biggest element.

Let $v = \{p \in P | (\exists x \in X)(\exists r \in (0,1))(p,r) \leqq x\}$. If $v \in X$ then it is evidently the $\leqq$-biggest element of X. Otherwise, since X is closed, there must be some basic open set containing v but avoiding X. Without loss of generality this set is of the form $S(e,r)^+$. Then $e \in v$, and so there must be some $r' \in (0,1)$ with $(e, r') \in X$. Since X is closed there is a maximal such r'. Then (e, r') is the maximal element of X.

The proof that X contains a $\leqq_L$-smallest element is analogous. $\square$

The *concatenation of two pseudo-lines L and M* is obtained from the disjoint union of L and M by identifying the end-vertex of L with the start-vertex of M.

Remark 5.2.10. *The concatenation of two pseudo-lines is a pseudo-line.* $\square$

Remark 5.2.11. *Taking the concatenation of 2 pseudo-lines corresponds to taking the disjoint union of the two corresponding posets, where in the new ordering we take all elements of the second poset to be greater than all elements of the first.*

Let $P : L \to G$ and $Q : M \to G$ be two pseudo-arcs such that the end-vertex t_P of P is the start-vertex s_Q of Q. Then *their concatenation* is the function $f : (L \sqcup M)/(t_P = s_Q) \to G$ which restricted to L is just P and restricted to M is just Q. For a pseudo-arc $Q : M \to G$ and vertices x and y in the image of Q, we write xQy for the restriction of Q to those points of M that are both $\leqq_L$-bigger than $Q^{-1}(x)$ and $\leqq_L$-smaller than $Q^{-1}(y)$. Note that xQy is a pseudo-arc from x to y. If Q is a pseudo-arc from v to w and x and y are vertices in the image of Q, we abbreviate xQw by xQ and vQy by Qy.

Lemma 5.2.12. *Let $P : L \to G$ be a pseudo-arc from x to y and $Q : M \to G$ be a pseudo-arc from y to z. Then the concatenation of P and Q includes a pseudo-arc from x to z*

The corresponding Lemma about arcs needs the requirement that $x \neq z$. However, we avoid this requirement because there is a pseudo-line whose start- and end-vertex are equal, namely the trivial pseudo-line.

Proof. Let I be the intersection of the image of P with the image of Q, which is closed, being the intersection of two closed sets. Then $P^{-1}(I)$ is closed as P is continuous, and contains a $\leqq_L$-minimal element w by Lemma 5.2.9.

If w is not a vertex, then $P(w)$ is not a vertex and thus is contained in $\iota_e((0,1))$ for some edge e. Since P and Q both contain the whole of $\iota_e([0,1])$ if they contain some point from $\iota_e((0,1))$, the same is true for I. But then $\iota_e([0,1]) \subseteq I$, which contradicts the choice of w. Hence w is a vertex. Let $w' = P(w)$

Thus $w'Q$ is a pseudo-arc. By Remark 5.2.10, the concatenation of Pw' and $w'Q$ is the desired pseudo-arc since their images meet precisely in w'. $\square$

A *pseudo-circle* is a graph-like space obtained by identifying the end-vertices of a nontrivial pseudo line.

We have the following relation between pseudo-lines and pseudo-circles. Every pseudo-circle C with one edge removed is a pseudo-line with endvertices the endvertices of the removed edge.

Conversely, let P and Q be pseudo-lines where P has endvertices s_P and t_P and Q has endvertices s_Q and t_Q. Then the graph-like space obtained from the disjoint union of P and Q by identifying s_P with t_Q and t_P with s_Q is a pseudo-circle or else is the trivial graph-like space.

So from Corollary 5.2.6 we obtain the following:

Corollary 5.2.13. *Any contraction of a pseudo-circle in which not all edges are contracted is a pseudo-circle.* $\square$

Using Lemma 5.2.7 we get:

Corollary 5.2.14. *Any countable pseudo-circle is homeomorphic to S^1.* $\square$

Definition 5.2.15. A *cyclic order on a set X* is a relation $R \subseteq X^3$, written $[a,b,c]_R$, that satisfies the following axioms:

1. Cyclicity: If $[a, b, c]_R$ then $[b, c, a]_R$.
2. Asymmetry: If $[a, b, c]_R$ then not $[c, b, a]_R$.
3. Transitivity: If $[a, b, c]_R$ and $[a, c, d]_R$ then $[a, b, d]_R$.
4. Totality: If a, b, and c are distinct, then either $[a, b, c]_R$ or $[c, b, a]_R$.

Remark 5.2.16. *The edge set of a pseudo-circle C has a canonical cyclic order R_C (up to choosing an orientation). Conversely, for any nonempty cyclic order there exists a pseudo-circle (unique up to isomorphism) such that its edge set has the same cyclic order.* $\square$

We also get a cyclic order R'_C on the set of all points of a pseudo-circle C, corresponding to the order $\leqq$ on the set of points of a pseudo-line. Once more there are two canonical choices of cyclic order on C, one for each orientation of C; in fact, we shall take this as our definition of an orientation of C. For us, an orientation of a pseudo-circle C is a choice of one of the two canonical cyclic orders of the points of C.

Let $s \subseteq o$ and let $R \subseteq o^3$ be a cyclic order. The *cyclic order of s inherited from R* is R restricted to s^3. We say that e, g are *clockwise adjacent* in the cyclic order R if $[e, g, f]_R$ for any other f in o. In a finite cyclic order, for each e there is a unique g clockwise adjacent to e, which we denote by $n(e)$.

From Lemma 5.2.8 we obtain the following.

Corollary 5.2.17. *Let s be a finite nonempty set of edges of a pseudo-circle C. Let $S = \bigcup_{e \in s} \iota_e((0, 1))$. Then $L \setminus S$ has $|s|$ components each of which is a pseudo-line.*

For each such component there is a unique $e \in s$ such that the component contains precisely those edges f with $[e, f, n(e)]_{R_C}$, where $n(e)$ is taken with respect to the induced cyclic order on s. $\square$

For a graph-like space G, we also use the term pseudo-circle to describe an injective map of graph-like spaces from a pseudo-circle to G, as well as the image of such a map. In particular, a *pseudo-circle in G* is the image of such a map (or, in other words, it is a subspace of G which is also a pseudo-circle). If G is a graph-like space and C is a pseudo-circle in G, the set of edges of C is called a *topological circuit* of G. Thus the pseudo-circles in G are precisely the standard subspaces of G corresponding to the topological circuits.

Lemma 5.2.18. *The intersection of a topological circuit with a topological cut is never only one edge.*

Proof. Suppose for a contradiction that there are a topological circuit o and a topological cut b that intersect in only one edge f. In the graph-like space $\overline{o}$, the set $b \cap o$ is a topological cut consisting of a single edge f. This contradicts the fact that removing any edge does not disconnect the pseudo-circle $\overline{o}$, which completes the proof. $\square$

We can also show that the intersection of topological circuits with topological cuts is finite. In fact, we can prove something a little more general.

Lemma 5.2.19. *Let o be a set of edges in a graph-like space G such that $\overline{o}$ is compact. The the intersection of o with any topological cut b is finite.*

Proof. Let b be induced by the open sets U and U'. The sets $U \cap \overline{o}$ and $U' \cap \overline{o}$, together with all the sets $(0,1) \times \{e\}$ with $e \in o$, comprise an open cover of $\overline{o}$. So there is a finite subcover, which can only contain $(0,1) \times \{e\}$ for finitely many edges e. For any other edge f of o we must have $(0,1) \times \{f\} \subseteq U \cup U'$, and it must be a subset either of U or of V since it is connected: in particular, no such f can be in b. $\square$

5.3 Graph-like spaces inducing matroids

In this section we will explain what it means for a graph-like space to induce a matroid and prove some fundamental facts about graph-like spaces inducing matroids which we will need in Section 5.4 and Section 5.6.

If for a graph-like space G there is a matroid M on $E(G)$ whose circuits are precisely the topological circuits of G and whose cocircuits are precisely the topological bonds of G, then we say that G *induces* M, and we may denote M by $M(G)$. Note that there can only be one such matroid since a matroid is uniquely defined by its set of circuits.

Example 5.3.1. For any finitely separable graph G the space $|G|$ induces the topological cycle matroid $M_C(G)$. The one-point compactification of a locally finite graph G induces the algebraic cycle matroid $M_A(G)$; if G is not locally finite and does not include a subdivision of the Bean graph, a similar construction can be used to construct a noncompact graph-like space that induces $M_A(G)$. Finally, the geometric realisation of G induces the finite cycle matroid $M_{FC}(G)$.

Lemma 5.3.2. *Let G be a graph-like space, and suppose G induces a matroid M. Then for any $C, D \subseteq E(M)$, the graph-like space $G/C\backslash D$ induces $M/C\backslash D$.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}^*$ be respectively the collection of topological circuits and the collection of topological cuts of $G/C\backslash D$. We will show that every circuit of $M/C\backslash D$ is in $\mathcal{C}$, and that every cocircuit of $M/C\backslash D$ is in $\mathcal{C}^*$. Lemma 5.2.18 states that for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$, so it will follow by Lemma 1.3.7 that the topological circuits of $G/C\backslash D$ are the circuits of $M/C\backslash D$ and that the minimal topological cuts (i.e. the topological bonds) of $G/C\backslash D$ are the cocircuits of $M/C\backslash D$, completing the proof.

Let o be a circuit of $M/C\backslash D$. By Lemma 1.2.7 there is a circuit o' of M such that $o \subseteq o' \subseteq o \cup C$. Since o' is a circuit of M, there is a pseudo-circle O in G with edge-set o'. Let $f_C : G \to G/C$ be as in the definition of the contraction G/C. Then $f_C\restriction_O$ is a map of graph-like spaces from O to a subspace of $G/C\backslash D$ that has edge-set o. If it describes a contraction of $O \cap C$, then Lemma 5.2.13 implies that o is a circuit of $G/C\backslash D$ as required. Otherwise, some vertex of

$G/C\backslash D$ must contain two vertices p and q of O such that their deletion from the pseudo-circle O leaves two elements e and f of o in different components of $O - p - q$. Then by Lemma 1.3.5 there is a cocircuit b of $M/C\backslash D$ with $o \cap b = \{e, f\}$. Using the dual of Lemma 1.2.7, there is a cocircuit b' of M with $b \subseteq b' \subseteq b \cup D$, so that $o' \cap b' = \{e, f\}$. b' is a topological bond of G not meeting C and with p and q on opposite sides, contradicting the assumption that they are identified when we contract C.

Let b be a cocircuit of $M/C\backslash D$. It follows by the dual of Lemma 1.2.7 that there is a cocircuit b' of M (hence also a topological cut of G) such that $b \subseteq b' \subseteq b \cup D$. Let U, V be the disjoint open sets in G that partition $V(G)$ so that the set of edges with an end in each of U and V is b'. Let $f_C : G \mapsto G/C$ be the map of graph-like spaces describing the contraction of C from G. Since b' is disjoint from C, f_C does not identify any element of U with any element of V. Thus $f_C(U), f_C(V)$ are open sets in $G/C\backslash D$, and b is the set of edges with an end in each, showing that b is a topological cut of $G/C\backslash D$, as required.

□

Definition 5.3.3. A *switching sequence* for a base s in a matroid with ground set E is a finite sequence $(e_i | 1 \leq i \leq n)$ whose terms are alternately in s and not in s and where for $i < n$ if $e_i \in s$ then $e_{i+1} \in b_{e_i}$ and if $e_i \notin s$ then $e_{i+1} \in o_{e_i}$.

Lemma 5.3.4. *Let M be a connected matroid with a base s, and e and f be edges of M. Then there is a switching sequence with first term e and last term f.*

Proof. This is immediate from Lemma 1.5.7 □

Proposition 5.3.5. *Let G be a graph-like space inducing a connected matroid M with a base s. Then for any edges e and f of M, and any endvertices v of e and w of f, there is a unique pseudo-arc from v to w that uses only edges in s.*

Proof. By Lemma 5.3.4, we can find a switching sequence $(e_i | 1 \leq i \leq n)$ for s with first term e and last term f. Pick a sequence $(v_i | 1 \leq i \leq n)$, with first term v and last term w, where for each i the vertex v_i is an endvertex of e_i. Then for any $i < n$ we can find a pseudo-arc from v_i to v_{i+1} using only edges of s: if $e_i \in s$ then we take an interval of the pseudo-arc $\overline{o_{e_{i+1}}} \setminus e_{i+1}$, and if $e_i \notin s$ then we take an interval of the pseudo-arc $\overline{o_{e_i}} \setminus e_i$. Repeatedly applying Lemma 5.2.12 we find the desired pseudo-arc from v to w.

To show uniqueness, we suppose for a contradiction that there are 2 distinct such pseudo-arcs R_1 and R_2. Then without loss of generality there is an edge e_0 in $R_1 \setminus R_2$.

Let $a \in R_1 \cap R_2$ be the $\leqq_{R_1}$-smallest point that is still $\leqq_{R_1}$-bigger than any point on e_0; such a point exists as the intersection of the two pseudo-arcs is closed. Similarly, let $b \in R_1 \cap R_2$ be the $\leqq_{R_1}$-biggest point that is still $\leqq_{R_1}$-smaller than any point on e_0. Then aR_1b and bR_2a are internally disjoint. Therefore aR_1bR_2a is a pseudo-circle all of whose edges are in s, a contradiction. □

Remark 5.3.6. *The proof of uniqueness above does not make use of the assumption that v and w are endvertices of edges.*

Let us call the pseudo-arc whose uniqueness is noted above vsw by analogy to the special case where s is a pseudo-arc. Next, we give a precise description of vsw.

Proposition 5.3.7. *The pseudo-arc vsw contains precisely those edges of s whose fundamental cocircuit with respect to s separates v from w. Its linear order is given by $e \leq f$ if and only if e lies on the same side as v of the fundamental cocircuit b_f of f.*

Proof. Let R be the pseudo-arc from v to w using edges in s only. Since R is connected, it must contain all edges whose fundamental cocircuit with respect to s separates v from w.

On the other hand let e be an edge on R. Let z_1 and z_2 be the endvertices of e, with $z_1 \leqq_R z_2$. Then by the above we can join v to z_1 by the pseudo-arc vRz_1 and w to z_2 by the pseudo-arc wRz_2. In G with the fundamental cocircuit of e removed, z_1 and z_2 lie on different sides, which we will call A_1 and A_2. Since $vRz_1 \subseteq A_1$ and $wRz_2 \subseteq A_2$, the fundamental cocircuit of e separates v from w, which completes the proof of the first part.

The second part is immediate from the definitions. □

5.4 Existence

Let G be a graph-like space inducing a matroid M. Then every finite minor of M is induced by a finite minor of G (finite in the sense that it only has finitely many edges) by Lemma 5.3.2. But this finite minor must consist simply of a graph, together with a (possibly infinite) collection of spurious vertices, by Lemma 5.1.2 applied to the closure of the set of edges. In particular, every finite minor of M is graphic. We also know that M has to be tame, by Lemma 5.2.19. The aim of this section is to prove that these conditions are also sufficient to show that M is induced by some graph-like space. More precisely, we wish to show:

Theorem 5.4.1. *Let M be a matroid. The following are equivalent.*

1. *There is a graph-like space G inducing M.*

2. *M is tame and every finite minor of M is the cycle matroid of some graph.*

The forward implication was proved above. The rest of this section will be devoted to proving the reverse implication. The strategy is as follows: we consider an extra structure that can be placed on certain matroids, with the following properties:

- There is such a structure on any matroid induced by a graph-like space (in particular, there is such a structure on any finite graphic matroid).

- Given such a structure on a matroid M, we can obtain a graph-like space inducing M.
- The structure is finitary.

Then we proceed as follows: given a tame matroid all of whose finite minors are graphic, we obtain a graph framework on each finite minor. Then the finitariness of the structure, together with the tameness of the matroid, allows us to show by a compactness argument that there is a graph framework on the whole matroid. From this graph framework, we build the graph-like space we need.

5.4.1 Graph frameworks

A *signing* for a tame matroid M is a choice of functions $c_o \colon o \to \{-1, 1\}$ for each circuit o of M and $d_b \colon b \to \{-1, 1\}$ for each cocircuit b of M such that for any circuit o and cocircuit b we have

$$\sum_{e \in o \cap b} c_o(e) d_b(e) = 0 \,,$$

where the sums are evaluated over $\mathbb{Z}$. The sums are all finite since M is tame. A tame matroid is *signable* if it has a signing.

Signings for finite matroids were introduced in [70], where it was shown that a finite matroid is signable if and only if it is regular, i.e. representable over any field. This result was extended to tame infinite matroids, for a suitable infinitary notion of representability, in [13]. In [12] it is shown that the standard matroids associated to graphs are all signable. The construction for a graph G is as follows: we begin by choosing some orientation for each edge of G (equivalently, we choose some digraph whose underlying graph is G). We also choose a cyclic orientation of each circuit of the matroid and an orientation of each bond used as a cocircuit of the matroid. Then $c_o(e)$ is 1 if the orientation of e agrees with the orientation of o and -1 otherwise. Similarly, $d_b(e)$ is 1 if the orientation of e agrees with that of b and -1 otherwise. Then the terms $c_o(e)d_b(e)$ are independent of the orientation of e: such a term is 1 if o traverses b at e in a forward direction, and -1 if o traverses b at e in the reverse direction. Since o must traverse b the same number of times in each direction, all the sums in the definition evaluate to 0.

We therefore think of a signing, in a graphic context, as providing information about the cyclic orderings of the circuits and about the direction in which each edge in a given bond points relative to that bond. In order to reach the notion of a graph framework, we need to modify the notion of a signing in two ways. Firstly, we need to add some extra information specifying on which side of a bond b each edge not in b lies. Secondly, we need to add some conditions saying that these data induce well-behaved cyclic orderings on the circuits.

Recall that if s has a cyclic order R, then we say that $p, q \in s$ are *clockwise adjacent* in R if $[p, q, g]_R$ is in the cyclic order for all $g \in s - p - q$.

Definition 5.4.2. A *graph framework* on a matroid M consists of a signing of M and a map $\sigma_b : E \setminus b \to \{-1, 1\}$ for every cocircuit b, which we think of as telling us which side of the bond b each edge lies on, satisfying certain conditions. First, we require that these data induce a cyclic order R_o for each circuit o of M: For distinct elements e, f and g of M, we take $[e, f, g]_{R_o}$ if and only if both $e, f, g \in o$ and there exists a cocircuit b of M such that $b \cap o = \{e, f\}$ and $\sigma_b(g) = c_o(f)d_b(f)$. That is, we require that each such relation R_o satisfies the axioms for a cyclic order given in Definition 5.2.15. In particular, by asymmetry and totality, we require that this condition is independent from the choice of b: if o is a circuit with distinct elements e, f and g, and b and b' are cocircuits such that $o \cap b = o \cap b' = \{e, f\}$, then $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $\sigma_{b'}(g) = c_o(f)d_{b'}(f)$. Let o be a circuit, b be a cocircuit and s be a finite set with $b \cap o \subseteq s \subseteq o$. Then $s \subseteq o$ inherits a cyclic order $R_o\restriction_s$from o. Our final conditions are as follows: for any two $p, q \in s$ clockwise adjacent in $R_o\restriction_s$ we require:

1. If $p, q \in b$, then $c_o(p)d_b(p) = -c_o(q)d_b(q)$.
2. If $p, q \notin b$, then $\sigma_b(p) = \sigma_b(q)$.
3. If $p \in b$ and $q \notin b$, then $c_o(p)d_b(p) = \sigma_b(q)$.
4. If $p \notin b$ and $q \in b$, then $c_o(q)d_b(q) = -\sigma_b(p)$.

Graph frameworks behave well with respect to the taking of minors. Let M be a matroid with a graph framework, and let $N = M/C\backslash D$ be a minor of M. For any circuit o of N we may choose by Lemma 1.2.7 a circuit o' of M with $o \subseteq o' \subseteq o \cup C$. This induces a function $c_{o'}\restriction_o \colon o \to \{-1, 1\}$. Similarly for any cocircuit b of N we may choose a cocircuit b' of N with $b \subseteq b' \subseteq b \cup D$, and this induces functions $d_{b'}\restriction_b \colon b \to \{-1, 1\}$ and $\sigma_{b'}\restriction_{E(N)\setminus b} \colon E(N) \setminus b \to \{-1, 1\}$. Then these choices comprise a graph framework on N, with R_o given by the restriction of $R_{o'}$ to o.

Next we show that every matroid induced by a graph-like space has a graph framework. Let M be a matroid induced by a graph-like space G. Fix for each topological bond of G a pair (U_b, V_b) of disjoint open sets in G inducing b, and fix an orientation $R'_{\overline{o}}$ of the pseudo-circle $\overline{o}$ inducing each topological circle o (recall from Section 5.2 that an orientation of a pseudo-circle is a choice of one of the two canonical cyclic orders of the set of points). For each topological circuit o, let the function $c_o \colon o \to \{-1, 1\}$ send e to 1 if $[\iota_e(0), \iota_e(0.5), \iota_e(1)]_{R'_{\overline{o}}}$, and to -1 otherwise. For each topological bond d_b, let the function $d_b \colon b \to \{-1, 1\}$ send e to 1 if $\iota_e(0) \in U_e$ and to -1 if $\iota_e(0) \in V_e$. Finally, for each topological bond d_b, let the function $\sigma_b \colon E \setminus b \to \{-1, 1\}$ send e to -1 if the end-vertices of e are both in U_b and to 1 if they are both in V_b.

Lemma 5.4.3. *The c_o, d_b and σ_b defined above give a graph framework on M.*

Proof. The key point will be that the cyclic ordering R_o we obtain on each circuit o will be that induced by the chosen orientation $R'_{\overline{o}}$. So let o be a topological circuit of G. First we show that for any distinct edges e, f and g in o and any topological bond b with $o \cap b = \{e, f\}$ we have $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R_{\overline{o}}}$. For any edge $e \in b$ we define $\iota_e^b\colon [0,1] \to G$ to be like ι_e but with the orientation changed to match b. That is, we set $\iota_e^b(r) = \iota_e(r)$ if $\iota_e(0) \in U_b$ and $\iota_e^b(r) = \iota_e(1-r)$ if $\iota_e(0) \in V_b$.

Since the pseudo-circle $\overline{o}$ with edge set o is compact, there can only be finitely many edges in o with both endpoints in U_b but some interior point not in U_b, so by adding the interiors of those edges to U_b if necessary we may assume without loss of generality that there are no such edges, and similarly we may assume that if an edge of o has both endpoints in V_b then all its interior points are also in V_b. Thus the two pseudo-arcs obtained by removing the interior points of e and f from $\overline{o}$ are both entirely contained in $U_b \cup V_b$. Since each of these two pseudo-arcs is connected and precisely one endvertex of e is in U_b, we must have that one of these pseudo-arcs, which we will call R^U is included in U_b. And the other, which we will call R^V, is included in V_b. The end-vertices of R^U must be $\iota_e^b(0)$ and $\iota_f^b(0)$, and those of R^V must be $\iota_e^b(1)$ and $\iota_f^b(1)$.

Suppose first of all that $\sigma_b(g) = 1$. Let R be the pseudo-arc $\iota_f^b(0) f \iota_f^b(1) R^V \iota_e^b(1)$. Then $c_o(f)d_b(f) = 1$ if and only if the ordering along R agrees with the orientation of $\overline{o}$, which happens if and only if $[\iota_f(0.5), \iota_g(0.5), \iota_e(0.5)]_{R'_{\overline{o}}}$, which is equivalent to $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R'_{\overline{o}}}$. The case that $\sigma_b(g) = -1$ is similar. This completes the proof that for any distinct edges e, f and g in o and any topological bond b with $o \cap b = \{e, f\}$ we have $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R'_{\overline{o}}}$.

In particular, the construction of Definition 5.4.2 really does induce cyclic orders on all the circuits. We now show that these cyclic orders satisfy (1)-(4). Let o, b, s, p and q be as in Definition 5.4.2. Without loss of generality $\overline{o}$ is the whole of G. We may also assume without loss of generality that all edges e are oriented so that $c_o(e) = 1$. Since $\overline{o}$ is compact we may as before assume that all interior points of edges not in s are in either U_b or V_b. Thus each of the pseudo-arcs obtained by removing the interior points of the edges in s, as in Corollary 5.2.17, is entirely included in U_b or V_b. Since they both lie on one of these pseudo-arcs, $\iota_p(1)$ and $\iota_q(0)$ are either both in U_b or both in V_b. We shall deal with the case that both are in V_b: the other is similar. In case (1), we get $d_b(p) = 1$ and $d_b(q) = -1$. In case (2), we get $\sigma_b(p) = \sigma_b(q) = 1$. In case (3), we get $d_b(p) = 1$ and $\sigma_b(q) = 1$. Finally in case (4) we get $\sigma_b(p) = 1$ and $d_b(q) = -1$. Since we are assuming that $c_o(p) = c_o(q) = 1$, in each case the desired equation is satisfied. This completes the proof. $\square$

Since a graph framework is a finitary structure, we can lift it from finite minors to the whole matroid.

Lemma 5.4.4. *Let M be a tame matroid such that every finite minor is a cycle matroid of a finite graph. Then M has a graph framework.*

Proof. By Lemma 5.4.3 we get a graph framework on each finite minor of M. We will construct a graph framework for M from these graph frameworks by a compactness argument. Let $\mathcal{C}$ and $\mathcal{C}^*$ be the sets of circuits and of cocircuits of

M. Let $H = \bigcup_{o \in \mathcal{C}} o \times \{o\} \sqcup \bigcup_{b \in \mathcal{C}^*} b \times \{b\} \sqcup \bigcup_{\tilde{b} \in \mathcal{C}^*} (E \setminus \tilde{b}) \times \{\tilde{b}\} \sqcup \bigcup_{o \in \mathcal{C}} o \times o^3$. Endow $X = \{-1, 1\}^H$ with the product topology. Any element in X encodes a choice of functions $c_o \colon e \mapsto x(o, e)$ for every circuit o, functions $d_b \colon e \mapsto x(b, e)$ and $\sigma_b \colon e \mapsto x(\tilde{b}, e)$ for every cocircuit $\tilde{b}$, and ternary relations $R_o = \{(e, f, g) \in o^3 | x(e, f, g) = 1\}$ for each circuit o.

To comprise a graph framework, these function have to satisfy several properties. These will be encoded by the following six types of closed sets.

For any circuit o and cocircuit b, let $C_{o,b} = \{x \in X | \sum_{e \in o \cap b} x(o, e)x(b, e) = 0\}$. Note that the functions c_o and d_b corresponding to any x in the intersection of all these closed sets will form a signing.

Secondly, for every circuit o, distinct edges $e, f, g \in o$ and cocircuit b such that $o \cap b = \{e, f\}$, let $C_{o,b,g} = \{x \in X | x(o, e, f, g) = x(\tilde{b}, g)x(o, f)x(b, f)\}$. So x is in the intersection of these closed sets if and only if the cyclic orders encoded by x are given as in Definition 5.4.2.

Thirdly any circuit o and distinct elements e, f, g of o we set $C_{o,e,f,g,\mathrm{Cyc}} = \{x \in X | x(o, e, f, g) = x(o, f, g, e)\}$. Note that for any x and o in the intersection of all these closed sets the relation R_o derived from x will satisfy the Cyclicity axiom. Similarly we get sets $C_{o,e,f,g,\mathrm{AT}}$ encoding the Asymmetry and Totality axioms and $C_{o,e,f,g,h,\mathrm{Trn}}$ encoding the Transitivity axiom.

Finally, for every circuit o, cocircuit b, finite set s with $o \cap b \subseteq s$, and $p, q \in s$ distinct, let $C_{b,o,s,p,q}$ denote the set of those x such that, if p and q are clockwise adjacent with respect to $R_o\restriction_s$, then the appropriate condition of (1)-(4) from Definition 5.4.2 is satisfied.

By construction, any x in the intersection of all those closed sets gives rise to a graph framework. As X has the finite intersection property, it remains to show that any finite intersection of those closed sets is nonempty. Given a finite family of those closed sets, let B and O be the set of all those cocircuits and circuits, respectively, that appear in the index of these sets. Let F be the set of those edges that either appear in the index of one of those sets or are contained in some set s or appear as the intersection of a circuit in O and a cocircuit in B. As the family is finite and M is tame, the sets B,O and F are finite.

By Lemma 4.6 from [13] we find a finite minor M' of M satisfying the following.

> For every M-circuit $o \in O$ and every M-cocircuit $b \in B$, there are M'-circuits o' and M'-cocircuits b' with $o' \cap F = o \cap F$ and $b' \cap F = b \cap F$ and $o' \cap b' = o \cap b$.

By Lemma 5.4.3 M' has a graph framework $((c'_o | o \in \mathcal{C}(M')), (d'_b | b \in \mathcal{C}^*(M')), (\sigma'_b | b \in \mathcal{C}^*(M')))$, giving cyclic orders $R'_{o'}$ on the circuits o'. Now by definition any x with $c_o\restriction_F = c'_o\restriction_F$ and $d_b\restriction_F = d'_b\restriction_F$ and $\sigma_b\restriction_F = \sigma'_b\restriction_F$ and $R_o\restriction_{o'} = R_{o'}$ for $o \in O$ and $b \in B$ will lie in the intersection of all the closed sets in the finite family, as required. This completes the proof. $\square$

5.4.2 From graph frameworks to graph-like spaces

In this subsection, we prove the following lemma, which, together with Lemma 5.4.4, gives the reverse implication of Theorem 5.4.1.

Lemma 5.4.5. *Let M be a tame matroid with a graph framework $\mathcal{F}$. Then there exists a graph-like space $G = G(M, \mathcal{F})$ inducing M.*

We take our notation for the graph framework as in Definition 5.4.2.

We begin by defining G. The vertex set will be $V = \{-1, 1\}^{\mathcal{C}^*(M)}$, and of course the edge set will be $E(M)$. As in Definition 5.1.1, the underlying set of the topological space G will be $V \sqcup ((0,1) \times E)$.

Next we give a subbasis for the topology of G. First of all, for any open subset U of $(0,1)$ and any edge $e \in E(M)$ we take the set $U \times \{e\}$ to be open. The other sets in the subbasis will be denoted $U_b^i(\epsilon_b)$ where $i \in \{-1, 1\}$, $b \in \mathcal{C}^*(M)$ and $\epsilon_b : b \to (0,1)$. Roughly, $U_b^1(\epsilon_b)$ should contain everything that is above b and $U_b^{-1}(\epsilon_b)$ should contain everything that is below b, so that removing the edges of b from G disconnects G. In other words, $G \setminus (\bigcup_{e \in b} (0,1) \times \{e\})$ should be disconnected because the open sets $U_b^1(\epsilon_b)$ and $U_b^{-1}(\epsilon_b)$ should partition it (for every ϵ_b). Formally, we define $U_b^i(\epsilon_b)$ as follows.

$$\begin{aligned} U_b^i(\epsilon_b) \quad &= \quad \{v \in V | v(b) = i\} \cup \bigcup_{e \in E \setminus b, \sigma_b(e) = i} (0,1) \times \{e\} \\ &\cup \bigcup_{e \in b, d_b(e) = i} (1 - \epsilon_b(e), 1) \times \{e\} \cup \bigcup_{e \in b, d_b(e) = -i} (0, \epsilon_b(e)) \times \{e\} \end{aligned}$$

To complete the definition of G, it remains to define the maps ι_e for every $e \in E(M)$. For each $r \in (0,1)$, we must set $\iota_e(r) = (r, e)$. For $r \in \{0, 1\}$, we let:

$$\iota_e(0)(b) = \begin{cases} \sigma_b(e) & \text{if } e \notin b \\ -d_b(e) & \text{if } e \in b \end{cases} ; \iota_e(1)(b) = \begin{cases} \sigma_b(e) & \text{if } e \notin b \\ d_b(e) & \text{if } e \in b \end{cases} ;$$

Note that ι_e is continuous and $\iota_e \restriction_{(0,1)}$ is open. This completes the definition of G. Next, we check the following.

Lemma 5.4.6. *G is a graph-like space.*

Proof. The only nontrivial thing to check is that for any distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$. Indeed, if $v \neq v'$, there is some $b \in \mathcal{C}^*$ such that $v(b) \neq v'(b)$, and then for any ϵ_b with $\epsilon_b(e) \leq 1/2$ for each $e \in E(M)$, the sets $U_b^1(\epsilon_b)$ and $U_b^{-1}(\epsilon_b)$ have all the necessary properties.

□

Having proved that G is a graph-like space, it remains to show that G induces M. This will be shown in the next few lemmas.

Lemma 5.4.7. *Any circuit o of M is a topological circuit of G.*

The proof, though long, is simply a matter of unwinding the above definitions, and may be skipped.

Proof. By the symmetry of the construction of G, we may assume without loss of generality that $c_o(e) = 1$ for all $e \in o$. The graph framework of M induces a cyclic order R_o on o. From this cyclic order we get a corresponding pseudo-circle C with edge set o by Remark 5.2.16. We begin by defining a map f of graph-like spaces from C to G as follows. First we define $f(v)$ for a vertex v by specifying $f(v)(b)$ for each cocircuit b of M.

If $b \cap o = \emptyset$, then $(f(v))(b) = \sigma_b(e)$ for some $e \in o$. This is independent of the choice of e by condition (2) in the definition of graph frameworks. This ensures that $f^{-1}(U_b^i(\epsilon_b)) = C$ if $i = \sigma_b(e)$, and $f^{-1}(U_b^i(\epsilon_b)) = \emptyset$ if $i = -\sigma_b(e)$.

If $b \cap o =: s$ is nonempty, then s is finite as M is tame. The cyclic order of o induces a cyclic order on $s \cup \{v\}$: choose $p_{v,b}$ so that $p_{v,b}$ and v are clockwise adjacent in this cyclic order. We take $(f(v))(b) = d_b(p_{v,b})$.

Finally, we define the action of f on interior points of edges by $f(\iota_e^C(r)) = \iota_e^G(r)$ for $r \in (0,1)$. We may check from the definitions above that this formula also holds at $r = 0$ and $r = 1$. First we deal with the case that $r = 0$. We check the formula pointwise at each cocircuit b of M. In the case that $b \cap o = \emptyset$, we have $f(\iota_e^C(0))(b) = \sigma_b(e) = \iota_e^G(0)(b)$. Next we consider those b with $e \in b$. Let $s = o \cap b$, so that $p_{\iota_e^C(0),b}$ and e are clockwise adjacent in s. Thus $f(\iota_e^C(0))(b) = d_b(p_{\iota_e^C(0),b}) = -d_b(e) = \iota_e^G(0)(b)$ by condition (1) in the definition of graph frameworks and our assumption that $c_o(f) = 1$ for any $f \in o$. The other possibility is that $b \cap o$ is nonempty but $e \notin b$. In this case, let $s = b \cap o + e$, so that $p_{\iota_e^C(0),b}$ and e are clockwise adjacent in s. Thus $f(\iota_e^C(0))(b) = d_b(p_{\iota_e^C(0)}) = \sigma_b(e) = \iota_e^G(0)$ by condition (3) in the definition of graph frameworks and our assumption on c_o. The equality $f(\iota_e^C(1)) = \iota_e^G(1)$ may also be checked pointwise. The cases with $e \notin b$ are dealt with as before, but the case $e \in b$ needs a slightly different treatment: we note that in this case $p_{\iota_e^C(1),b} = e$, so that $f(\iota_e^C(1))(b) = d_b(e) = \iota_e^G(1)$.

It is clear by definition that f is injective on interior points of edges. To see that f is injective on vertices, let v and w be vertices of C such that $f(v) = f(w)$ and suppose for a contradiction that $v \neq w$. Since C is a pseudo-circle, there are two edges e and f in C such that v and w lie in different components of $C \backslash \{e, f\}$. By Lemma 1.3.5, there is a cocircuit b of M with $o \cap b = \{e, f\}$. Without loss of generality we have $e = p_{v,b}$. It follows that $f = p_{w,b}$. Since e and f are clockwise adjacent in the induced cyclic order on $\{e, f\}$, we have $f(v)(b) = d_b(e) = -d_b(f) = -f(w)(b)$ by condition (1) in the definition of graph frameworks and our assumption that $c_o(f) = 1$ for any $f \in o$. This is the desired contradiction. So f is injective.

To see that f is continuous, we consider the inverse images of subbasic open sets of G. It is clear that for any edge e and any open subset U of $(0,1)$, $f^{-1}(\{e\} \times U) = \{e\} \times U$ is open in C, so it remains to check that each set of the form $f^{-1}(U_b^i(\epsilon_b))$ is open in C. If $b \cap o = \emptyset$ then this set is either empty or the whole of C. So suppose that $b \cap o \neq \emptyset$, and let $x \in f^{-1}(U_b^i(\epsilon_b))$. If x is an interior point of an edge e then it is clear that some open neighborhood of x of the form $\{e\} \times U$ is included in $f^{-1}(U^i_b(\epsilon_b))$.

We are left with the case that x is a vertex and $s = b \cap o \neq \emptyset$. By Corollary 5.2.17, the component of $C \backslash s$ containing x is the pseudo-arc A consisting of all points y on C with $[a, y, b]_{R_C}$, together with a and b, for some vertices $a = \iota^C_p(1)$ and $b = \iota^C_q(0)$, where for any vertex v of A we have $p_{v,b} = p$ and where p and q are clockwise adjacent in the restriction of R_o to s. Since $f(x) \in U^i_b(\epsilon_b)$, we have $i = f(x)(b) = d_b(p)$ and so for any other vertex v of A we also have $f(v)(b) = d_b(p) = i$, so that $f(v) \in U^i_b(\epsilon_b)$. For any edge e of A, applying condition (3) in the definition of graph frameworks to p and e in the set $s + e$ gives $\sigma_b(e) = d_b(p) = i$, so that $f''(0,1) \times e = (0,1) \times e \subseteq U^i_b(\epsilon_b)$. By definition, we have $(1 - \epsilon_b(p), 1) \times \{p\} \subseteq U^i_b(\epsilon_b)$, and using condition (1) in the definition of graph frameworks we get $d_b(q) = -d_b(p) = -i$, so that $(0, \epsilon_b(q)) \times \{q\} \subseteq U^i_b(\epsilon_b)$. We have now shown that every point y of C with $[\iota^C_p(1 - \epsilon_b(p)), y, \iota^C_q(\epsilon_b(q))]_{R_C}$ is in $f^{-1}(U^i_b(\epsilon_b))$. But the set of such points is open in C, which completes the proof of the continuity of f.

We have shown that the map f is a map of graph-like spaces from the pseudo-circle C to G and that the edges in its image are exactly those in o, so that o is a topological circuit of G as required.

□

It is clear that any cocircuit of M is a topological cut of G, as witnessed by the sets $U^{-1}_b(\frac{1}{2})$ and $U^1_b(\frac{1}{2})$. Combining this with Lemmas 5.4.7 and 5.2.18, we are in a position to apply Lemma 1.3.7 with $\mathcal{C}$ the set of topological circuits and $\mathcal{D}$ the set of topological cuts in G. The conclusion is Lemma 5.4.5, which together with Lemma 5.4.4 gives us Theorem 5.4.1.

5.5 A forbidden substructure

The next lemma gives a useful forbidden substructure for graph-like spaces inducing matroids.

Lemma 5.5.1. *Let G be a graph-like space, and let v be a vertex in it. Let $\{Q_n | n \in \mathbb{N}\}$ be a set of pseudo-arcs starting at v, and vertex-disjoint apart from that. Suppose also that the union of the edge sets of the Q_n is independent. Let y be a point in the closure of the set of their endvertices. Assume there is a nontrivial v-y-pseudo-arc P that is vertex-disjoint from all the $Q_n - v$.*

Then G does not induce a matroid.

Proof. First, we shall show that $\left(\bigcup_{n \in \mathbb{N}} Q_n\right) \cup P$ does not include a pseudo-circle. Suppose for a contradiction that it includes a pseudo-circle K. Then K must include some edge e from P and some edge f from Q_m for some $m \in \mathbb{N}$. Going along K starting from f until we first hit the closed set P, we get two disjoint pseudo-arcs L_1 and L_2, one for each cyclic order of K. Formally, we consider the pseudo-arc $K - f$ endowed with the linear order $\leqq_{K-f}$. Let s be its start vertex and t be its endvertex. Let l_1 be the first point of $K - f$ in P, and let l_2 be the last point of $K - f$ in P. Then $L_1 = s(K - f)l_1$ and $L_2 = l_2(K - f)t$.

We shall show that each of these pseudo-arcs contains v. Since f and $P - v$ are in different components of $(P \cup Q_m) - v$, each L_i contains either v or some edge f' in some Q_l with $l \in \mathbb{N} - m$. Note that fL_if' is included in $\bigcup_{n\in\mathbb{N}} Q_n$ and is an f-f'-pseudo-arc. By the independence of $\bigcup_{n\in\mathbb{N}} Q_n$ and Remark 5.3.6, it must be that $fL_if' = fQ_mvQ_lf'$. In particular, $v \in L_i$, as desired. This contradicts that L_1 and L_2 are disjoint. Thus $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ does not include a pseudo-circle.

Now suppose for a contradiction that G induces a matroid M. We pick $e \in P$ arbitrarily. Since $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ is M-independent as shown above, there must be a cocircuit meeting $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ precisely in e (for example, the fundamental cocircuit with respect to any base extending this set).

This cocircuit defines a topological cut of G with the two endvertices of e on different sides. This contradicts that $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup (P - e)$ is connected. $\square$

Figure 5.2:

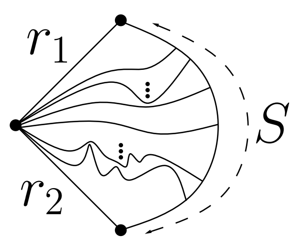


Figure 5.2: The situation of Lemma 5.5.2.

Lemma 5.5.2. *Let G be a graph-like space in which there is a pseudo-circle C with a vertex v of C that is indicent with two edges r_1 and r_2 of C. Let S be the pseudo-arc with edge set $E(C) - r_1 - r_2$. Assume there are infinitely many pseudo-arcs Q_n starting at v to points in S that are vertex-disjoint aside from v.*

If $\bigcup_{n\in\mathbb{N}} Q_n$ does not include a pseudo-circle, then G does not induce a matroid.

Proof. Without loss of generality, we may assume that the pseudo-arcs Q_n only meet S in their end-vertices. By Ramsey's theorem there is an infinite subset N of $\mathbb{N}$ such that the endpoints in S of the Q_n for $n \in N$ form a sequence that is either increasing or decreasing with respect to the linear order $\leqq_S$ of the pseudo-arc S. Let y be their limit point. Let P be the v-y-pseudo-arc included in C that avoids all the endpoints of those Q_n with $n \in N$. Note that P is nontrivial since it has to include either r_1 or r_2. Applying Lemma 5.5.1 now gives the desired result. $\square$

Corollary 5.5.3. *Let G be a graph-like space, C a pseudo-circle of G, and r_1 and r_2 distinct edges of C. Let S_1 and S_2 be the two components of $C \setminus \{r_1, r_2\}$. If there is an infinite set W of edges of G each with one end-vertex in S_1 and the other in S_2 and with all of their end-vertices in S_2 distinct, then G does not induce a matroid.*

Proof. Let G' be the graph-like space obtained from G by contracting all edges of S_1. Then in G', there is a vertex v that is endvertex of all edges in W. On the other hand, the other endvertices are distinct for any two edges in W. Indeed, let b be the cocircuit meeting C in precisely r_1 and r_2. Then $W \subseteq b$ and no two endvertices in S_2 are identified.

The set $\overline{W}$ cannot include a pseudo-circle with at least 3 edges since then v would be an endvertex of at least 3 edges of that pseudo-circle, which is impossible. So by Lemma 5.5.2 with each of the Q_n given by a single edge of W, we obtain that G' does not induce a matroid. By Lemma 5.3.2, nor does G. $\square$

5.6 Countability of circuits in the 3-connected case

Our aim in this section is to prove the following:

Theorem 5.6.1. *Any topological circuit in a graph-like space inducing a 3-connected matroid is countable.*

For the remainder of the section we fix such a graph-like space G, inducing a 3-connected matroid M, and we also fix a pseudo-circle C of G, whose edge set gives a circuit o of M.

We begin by taking a base s of M/o, and letting $G' = G/s$. Thus by Lemma 5.3.2 G' induces the matroid $M' = M/s$ in which o is a spanning circuit. For any $e \in o$, $o - e$ is a base of o and so $s \cup o - e$ is a base of M, which we shall denote s^e. We shall call the edges of $E(M') \setminus o$ which are not loops *bridges*. We denote the set of bridges by Br. The endpoints of each bridge lie on the pseudo-circle C' corresponding to o in G'. The edges of C' are the same as those of C, but the vertices are different: recall that the vertices of the contraction $G' = G/s$ were defined to be equivalence classes of vertices of G. Each of these can contain at most one vertex of C, since o is a circuit of M'. Thus each vertex of C' contains a unique vertex of C.

Lemma 5.6.2. *Let $g \in o$ and let f be a bridge with endpoints v' and w' in G'. Let v be the vertex of C contained in v', and w the vertex of C contained in w'. Let x be the endvertex of f in G contained in v', and y the endvertex of f in G in contained in w'. Then the fundamental circuit o_f of f with respect to the base s^g of M is given by concatenating 4 pseudo-arcs: the first, from x to y, consists of only f. The second, from y to w, contains only edges of s. The third, from w to v contains only edges of o - it is the interval of $C - g$ from w to v. The fourth, from v to x, contains only edges of s.*

Proof. $o_f \cap o$ must consist of the fundamental circuit of f with respect to the base $o - g$ of M' - that is, of the interval of $C' - g$ from w' to v'. So the pseudo-arc $v(C-g)w$, which is the closure of this set of edges, lies on the pseudo-circle $\bar{o}_f$. So $(\bar{o}_f - f) \setminus v(C-g)w$ consists of two pseudo-arcs joining v and w to x and y. These two pseudo-arcs use edges from s only. Since v and y lie in different connected components of $G\restriction_s$, we must have that the first goes from v to x, and the second goes from w to y. This completes the proof. □

Lemma 5.6.3. *For any distinct edges e and f of C, there is a bridge whose endvertices separate e from f in C.*

Proof. Since M is 3-connected, $\{e, f\}$ is not a bond of M, so we can pick some $g \notin \{e, f\}$ in the fundamental bond of f with respect to the base s^e. Then f lies in the fundamental circuit o_g of g, which is therefore not a subset of $s + g$. Thus g is a bridge, and since the fundamental circuit of g with respect to the base $o - e$ of M' contains f but not e the endpoints of g separate e from f. □

Given that we are aiming to prove Theorem 5.6.1, we may as well assume that o has at least 2 elements, and by Lemma 5.6.3 we obtain that there is at least one bridge. We now fix a particular bridge e_0, and make use of the 3-connectedness of M to build a tree structure capturing the way the endpoints of the bridges divide up C'. We will call this tree the *partition tree*, and define it in terms of certain auxiliary sequences $(I_n \subseteq \mathrm{Br})$, $(J_n \subseteq V(C'))$ and (K_n) indexed by natural numbers, given recursively as follows:

We always construct J_n from I_n as the set of endvertices of elements of I_n, and K_n as the set of components of $C' \setminus J_n$. We take I_0 to be $\{e_0\}$, and I_{n+1} to be the set of bridges that have endvertices in different elements of K_n or at least one endvertex in J_n.

Then the nodes of the tree at depth n will be the elements of K_n, with p a child of q if and only if it is a subset of q.

Lemma 5.6.4. *Every bridge is in some I_n.*

Proof. Suppose not, for a contradiction, and let e be any bridge which is in no I_n. In particular, the endpoints of e both lie in the same component of $C - J_0$, so there is a pseudo-arc joining them in C that meets neither endvertex of e_0. Let f be any edge of this pseudo-arc. Let v_0' be any endvertex of e_0, and let v_0 be the unique vertex of C contained in v_0'.

For each n, let B_n be the element of K_n of which f is an edge, and let $B = \bigcap_{n \in \mathbb{N}} B_n$ and $A = C \setminus B$. Note that any 2 vertices in B are joined by a unique pseudo-arc in B, and that A has the same property. Since the two endvertices of e_0 (in G') avoid B_1, they are both in A. Since e is in no I_n, its two endvertices lie in B.

Let A_V be the set of endvertices v of edges of G such that the first point of $vs^f v_0$ on C is contained in a vertex in A. Let A_E be the set of edges of G that have both endvertices in A_V, and let $B_E = E(M) \setminus A_E$. Note that for any vertex $v \in A_V$, all edges of the unique v-C-path included in s_f lie in A_E. And for any $v \notin A_V$, all edges of the unique v-C-path included in s_f lie in B_E.

We shall show that (A_E, B_E) is a 2-separation of M, which will give the desired contradiction since we are assuming that M is 3-connected.

First, we show that $s^f \cap A_E$ is a base of A_E. It is clearly independent. Let g be any edge in $A_E \setminus s^f$. Suppose first of all that g is a bridge. We decompose the fundamental circuit of g as in Lemma 5.6.2, taking the notation from that lemma. Then since each of the endpoints x and y of g is in A_V, every edge of this fundamental circuit is in A_E, as required.

So suppose instead that g isn't a bridge, that is, g is a loop in M'. Let R_1 and R_2 be the pseudo-arcs from the endpoints x and y of g to v_0 which use only edges from s^f. Let z be the first point of R_1 to lie on R_2. Then zR_1v_0 and zR_2v_0 must be identical, as both are pseudo-arcs from z to v_0 using only edges of s^f. Let k be the first point on this pseudo-arc that is in C. By assumption, $k \in A$. Also, xR_1zR_2y is a pseudo-arc from x to y using only edges from s^f, so must form (with g) the fundamental circuit of g with respect to s^f, so can meet C at most in a single vertex (since g is a loop in M'). Thus all edges in this fundamental circuit lie on either xR_1k or yR_2k, and so are in A_E, as required.

Next, we show that $(s_f \cap B_E) + f$ is a base of B_E. It is independent since A includes some edge as e_0 is a bridge. Let g be any edge in $B_E \setminus s^f - f$. If g isn't a bridge we can proceed as before, so we suppose it is a bridge. We decompose the fundamental circuit of g as in Lemma 5.6.2, taking the notation from that Lemma. At least one of v' and w' lies in B: without loss of generality it is v'. Suppose for a contradiction that w' is in A. Then either w' is in some J_n or it is an element of some K_n not containing f. In either case, $g \in I_{n+1}$ and so $v' \in J_{n+1}$, giving the desired contradiction since we are assuming $v' \in B$. Thus w' is also in B. Let R be the pseudo-arc from v to w in B. Then g is spanned by the pseudo-arc $xs^f vRws^f y$, which uses only edges of $s_f \cap B_E + f$. To see this we apply Lemma 5.6.2 with some edge not in B_1 in place of f of that lemma.

Since each of A_E and B_E has at least 2 elements, and the union of the bases for them given above only contains one more element than the base s^f of M, this gives a 2-separation of M, completing the proof. $\square$

Lemma 5.6.5. *Every node of the Partition-tree has at most countably many children.*

Proof. Let $x \in K_n$ be a node of the Partition-tree. Then the closure $\bar{x}$ of the set of interior points of edges of x is a pseudo-arc. Let $\hat{x}$ be the set obtained from this pseudo-arc by removing its end-vertices. An x-bridge is a bridge with one endvertex in $\hat{x}$ and one in its complement. Thus every element of $J_{n+1} \cap x$ must be an endvertex of an x-bridge or of $\bar{x}$.

Let v_1 and v_2 be vertices of $\hat{x}$ with $v_1 \leqq_{\bar{x}} v_2$. Suppose for a contradiction that there are infinitely many elements of J_{n+1} between v_1 and v_2. Pick a corresponding set W of infinitely many x-bridges with different attachment points between v_1 and v_2. Since neither of v_1 and v_2 is an endpoint of $\bar{x}$, there are edges e_1 and e_2 in x such that all points of e_1 are $\leqq_{\bar{x}}$-smaller than v_1, and similarly all points of e_2 are $\leqq_{\bar{x}}$-bigger than v_2. Then by Corollary 5.5.3 with $r_1 = e_1$ and $r_2 = e_2$, G' does not induce a matroid, which gives the desired contradiction.

We have established that between any two elements of $J_{n+1} \cap \hat{x}$ there are only finitely many others. Hence $J_{n+1} \cap \hat{x}$ is finite or has the order type of $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$. In all these cases there are only countably many children of x, since these children are the connected components of $x \setminus (J_{n+1} \cap x)$. $\square$

We now consider rays in the partition tree: a *ray* consists of a sequence $(k_n \in K_n | n \in \mathbb{N})$ such that for each n the node k_{n+1} is a child of k_n. Given such a ray, we call the set $\bigcap_{n \in \mathbb{N}} k_n$ its *partition class*.

Lemma 5.6.6. *The partition class of any ray includes at most one edge.*

Proof. Suppose for a contradiction that there is some ray (k_n) whose partition class includes 2 different edges e and f. Then by Lemma 5.6.3 there is a bridge g whose endvertices separate e from f in C. By Lemma 5.6.4, g lies in some I_n. But then e and f lie in different elements of K_n, so can't both lie in k_n, which is the desired contradiction. $\square$

For any element k of K_n with $n \geq 1$, the *parent* $p(k)$ is the unique element of K_{n-1} including k.

An element k of K_n with $n \geq 2$ is *good* if no bridge in I_n has endvertices in two different components of $\overline{p(p(k)) \setminus k}$. Note that $\overline{p(p(k)) \setminus k}$ has at most two components. Note that if k is not good, there have to be two vertices in different components of not only $\overline{p(p(k)) \setminus k}$ but also $p(p(k)) \setminus k$.

Lemma 5.6.7. *Every node of the Partition-tree has at most one good child.*

Proof. Suppose for a contradiction that some $x \in K_n$ with $n \geq 1$ has two good children y_1 and y_2. Since they are different, there is an element i of J_{n+1} separating them, and a bridge e in I_{n+1} of which i is an endvertex. Since $i \notin J_n$, $e \notin I_n$ and so the other endvertex j of e must lie in $p(x) = p(p(y_1)) = p(p(y_2))$. Now the two endvertices of e have to be in different components of $\overline{p(p(y_1)) \setminus y_1}$ or $\overline{p(p(y_2)) \setminus y_2}$. Hence y_1 and y_2 cannot both be good at the same time, a contradiction. $\square$

Lemma 5.6.8. *Let (k_n) be a ray whose partition class includes an edge. Then all but finitely many nodes on it are good.*

Proof. Let e be the edge in the partition class of this ray. Let f be any edge of $C \setminus k_0$.

Suppose for a contradiction that there is an infinite set N of natural numbers such that k_n is not good for any $n \in N$. Let N' be an infinite subset of N that does not contain 0, 1 or any pair of consecutive natural numbers. For each $n \in N'$, pick a bridge e_n in I_n with endvertices in both components of $\overline{p(p(k_n)) \setminus k_n}$, which is possible since k_n is not good. The endvertices of e_n are in J_n but not J_{n-2} and so we cannot find $m \neq n \in N'$ such that e_m and e_n share an endvertex. Applying Corollary 5.5.3 with $r_1 = e$, $r_2 = f$ and $W = \{e_n | n \in \mathbb{N}\}$ yields that G' does not induce a matroid, a contradiction. This completes the proof. $\square$

Proof of Theorem 5.6.1. For each edge of C there is a unique ray whose partition class contains that edge. By Lemma 5.6.8, we can find a first node on that ray such that it and all successive nodes are good. This gives a map from the edges of C to the nodes of the partition tree. By Lemma 5.6.7 and Lemma 5.6.6, this map is injective. By Lemma 5.6.5 the partition tree has only countably many nodes. □

5.7 Planar graph-like spaces

A nice consequence of Theorem 5.6.1 is the following.

Corollary 5.7.1. *Let M be a tame 3-connected matroid such that all finite minors are planar. Then $E(M)$ is at most countable.*

Proof. Let e be some edge. By Lemma 5.3.4, there is a switching sequence from e to any other edge. Hence it suffices to show that there are only countably many different switching sequences starting at e. We show by induction that there are only countably many switching sequences of length n for each n. The case $n = 1$ is obvious. The first $n - 1$ elements of a switching sequence of length n form a switching sequence of length $n - 1$. On the other hand, there are only countably many ways to extend a given switching sequence of length $n - 1$ to one of length n since all circuits and cocircuits of M are countable by Theorem 5.6.1. Hence there are only countably many switching sequences of length n. This completes the proof. □

This raises the question how to embed the graph-like space constructed from a tame matroid all of whose finite minors are planar in the plane. However, we shall construct such a matroid that does not seem to be embeddable in this sense the plane. Let N be the matroid whose circuits are the edge sets of topological circles in the topological space depicted in Figure 5.3. We omit the proof that this gives a matroid - it can be found in [29]. However, much of the complication of this matroid was introduced to make it 3-connected, and if we do not require 3-connectedness then it is easy to construct other simpler examples sharing the essential property of this matroid: it is tame and all finite minors are planar, but the topology of the graph-like space it induces has no countable basis of neighbourhoods for the vertex at the apex, so it cannot be embedded into the plane.

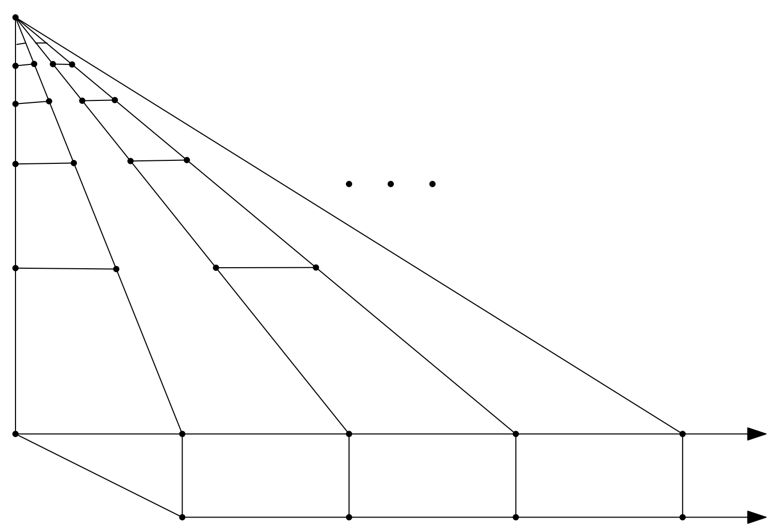


Figure 5.3: The matroid N.

Chapter 6

The Packing/Covering Conjecture

The well-known finite matroid intersection theorem of Edmonds states that for any two finite matroids M and N the size of a biggest common independent set is equal to the minimum of the rank sum $r_M(E_M) + r_N(E_N)$, where the minimum is taken over all partitions $E = E_M \dot\cup E_N$. The same statement for infinite matroids is true, but for a silly reason [30], which suggests that more care is needed in extending this statement to the infinite case.

Nash-Williams [3] proposed the following for finitary matroids.

Conjecture 6.0.2 (The Matroid Intersection Conjecture)**.** *Any two matroids M and N on a common ground set E have a common independent set I admitting a partition $I = J_M \cup J_N$ such that $\mathrm{Cl}_M(J_M) \cup \mathrm{Cl}_N(J_N) = E$.*

For finite matroids this is easily seen to be equivalent to the intersection theorem, which is why we refer to Conjecture 6.0.2 as the Matroid Intersection Conjecture. If for a pair of matroids M and N on a common ground set there are sets I, J_M and J_N as in Conjecture 6.0.2, we say that M and N have the *Intersection property*, and that I, J_M and J_N *witness* this.

In [5], it was shown that this conjecture implies the celebrated Aharoni-Berger-Theorem [1], also known as the Erdős-Menger-Conjecture. The conjecture is true in the cases where M is finitary and N is co-finitary [5].[1] Aharoni and Ziv [3] proved the conjecture for one matroid finitary and the other a countable direct sum of finite rank matroids.

In this chapter we will demonstrate that the Matroid Intersection Conjecture is a natural formulation by showing that it is equivalent to several other new conjectures in unexpectedly different parts of infinite matroid theory.

Suppose we have a family of matroids $(M_k | k \in K)$ on the same ground set E. A *packing* for this family consists of a spanning set S_k for each M_k such that the S_k are all disjoint. Note that not all families of matroids have a packing.

[1] In fact in [5] the conjecture was proved for a slightly larger class.

More precisely, the well-known finite base packing theorem states that if E is finite then the family has a packing if and only if for every subset $Y \subseteq E$ the following holds.

$$\sum_{k \in K} r_{M_k.Y}(Y) \leq |Y|$$

The Aharoni-Thomassen graphs [2, 34] show that this theorem does not extend verbatim to finitary matroids. However, the base packing theorem extends to finite families of co-finitary matroids [4]. This implies the topological tree packing theorems of Diestel and Tutte. Independently from our main result, we close the gap in between by showing that the base packing theorem extends to arbitrary families of co-finitary matroids (for example, topological cycle matroids).

Similar to packings are coverings: a *covering* for the family $(M_k|k \in K)$ consists of an independent set I_k for each M_k such that the I_k cover E. And analogously to the base packing theorem, there is a base covering theorem characterising the finite families of finite matroids admitting a covering.

We are now in a position to state our main conjecture, which we will show is equivalent to the intersection conjecture. Roughly, the finite base packing theorem says that a family has a packing if it is very dense. Similarly, the finite base covering theorem says roughly that a family has a covering if it is very sparse. Although not every family of matroids has a packing and not every family has a covering, we could ask if it is always possible to divide the ground set into a "dense" part, which has a packing, and a "sparse" part, which has a covering?

Definition 6.0.3. We say that a family of matroids $(M_k|k \in K)$ on a common ground set E, has the *Packing/Covering* property if E admits a partition $E = P\dot{\cup}C$ such that $(M_k\restriction_P|k \in K)$ has a packing and $(M_k.C|k \in K)$ has a covering.

Conjecture 6.0.4. *Any family of matroids on a common ground set has the Packing/Covering property.*

Here $M_k\restriction_P$ is the restriction of M_k to P and $M_k.C$ is the contraction of M_k onto C. Note that if $(M_k\restriction_P|k \in K)$ has a packing, then $(M_k.P|k \in K)$ has a packing, so we get a stronger statement by taking the restriction here. Similarly, we get a stronger statement by contracting to get the family which should have a covering than we would get by restricting.

For finite matroids, we show that this new conjecture is true and implies the base packing and base covering theorems. So the finite version of Conjecture 6.0.4 unifies the base packing and the base covering theorem into one theorem.

For infinite matroids, we show that Conjecture 6.0.4 and the intersection conjecture are equivalent, and that both are equivalent to Conjecture 6.0.4 for pairs of matroids. In fact, for pairs of matroids, we show that (M, N) has the Packing/Covering property if and only if M and N^* have the Intersection property. As the Packing/Covering property is preserved under duality for pairs of matroids, this shows the less obvious fact that the Intersection property is also preserved under duality:

Corollary 6.0.5. *If M and N are matroids on the same ground set then M and N have the intersection property if and only if M^* and N^* do.*

Conjecture 6.0.4 also suggests a base packing conjecture and a base covering conjecture which we show are equivalent to the intersection conjecture but not to the above mentioned rank formula formulation of base packing for infinite matroids.

The various results about when intersection is true transfer via these equivalences to give results showing that these new conjectures also hold in the corresponding special cases. For example, while the rank-formulation of the covering theorem is not true for all families of co-finitary matroids, the new covering conjecture is true in that case. This yields a base covering theorem for the algebraic cycle matroid of any locally finite graph and the topological cycle matroid of any graph. Similarly, we immediately obtain in this way that the new packing and covering conjectures are true for finite families of finitary matroids. Thus we get packing and covering theorems for the finite cycle matroid of any graph.

For finite matroids, the proofs of the equivalences of these conjectures simplify the proofs of the corresponding finite theorems.

We show that Conjecture 6.0.4 might be seen as the infinite analogue of the rank formula of the matroid union theorem. It should be noted that there are two matroids whose union is not a matroid [4], so there is no infinite analogue of the finite matroid union theorem as a whole.

This new point of view also allows us to give a simplified account of the special cases of the intersection conjecture and even to extend the results a little bit. Our result includes the following:

Theorem 6.0.6. *Any family of matroids $(M_k | k \in K)$ on the same ground set E for which there are only countably many sets appearing as circuits of matroids in the family has the Packing/Covering property.*

We also prove the following special case:

Theorem 6.0.7. *Let (T, N) and (T, N') be trees of matroids of overlap 1 such that $N(t)$ and $N'(t)$ have the same ground set for any node t of T. Then for any Borel sets Ψ and Ψ' of ends of T the two matroids $M_\Psi(T, N)$ and $M_{\Psi'}(T, N')$ have the Packing/Covering property*

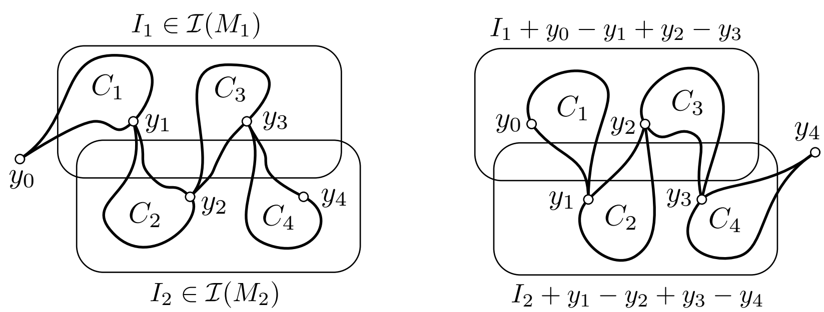
Corollary 6.0.8. *Let G be a locally finite graph with a tree-decomposition into finite parts of adhesion at most 2, and let Ψ_1 and Ψ_2 be Borel sets of ends of G. Then the pair $(M_{\Psi_1}(G), M_{\Psi_2}(G))$ satisfies the Packing/Covering Conjecture.*

Combining this with the basic structural theory of tame matroids developed above, according to which any matroid has a canonical decomposition over its 2-separations into such a tree structure, we have the first beginnings of a structural attack on the Packing/Covering conjecture. First, assuming the axiom of determinacy, we can show that if M is a connected matroid all of whose 3-connected minors are finite then (M, M) satisfies Packing/Covering. Without the axiom of determinacy we do not have this result in general, but we still have it for well-enough behaved matroids.

This chapter is closely based on two joint papers with Johannes Carmesin [15, 17]

6.1 Exchange chains

Below, we will need a modification of the concept of exchange chains introduced in [4]. The only modification is that we need not only exchange chains for families with two members but more generally exchange chains for arbitrary families, which we define as follows: Let $(M_k|k \in K)$ be a family of matroids and let $B_k \in \mathcal{I}(M_k)$. A $(B_k|k \in K)$-*exchange chain (from y_0 to y_n)* is a tuple $(y_0, k_0; y_1, k_1; \ldots; y_n)$ where $B_{k_l} + y_l$ includes an M_{k_l}-circuit containing y_l and y_{l+1}. A $(B_k|k \in K)$-exchange chain from y_0 to y_n is called *shortest* if there is no $(B_k|k \in K)$-exchange chain $(y'_0, k'_0; y'_1, k'_1; \ldots; y'_m)$ with $y'_0 = y_0$, $y'_m = y_n$ and $m < n$. A typical exchange chain is shown in Figure 6.1.



(a) Before the exchange (b) After the exchange

Figure 6.1: An (I_1, I_2)-exchange chain of length 4.

Lemma 6.1.1. *Let $(M_k|k \in K)$ be a family of matroids and let $B_k \in \mathcal{I}(M_k)$. If $(y_0, k_0; y_1, k_1; \ldots; y_n)$ is a shortest $(B_k|k \in K)$-exchange chain from y_0 to y_n, then $B'_k \in \mathcal{I}(M_k)$ for every k, where*

$$B'_k := B_k \cup \{y_l|k_l = k\} \setminus \{y_{l+1}|k_l = k\}$$

Moreover, $\mathrm{Cl}_{M_k} B_k = \mathrm{Cl}_{M_k} B'_k$.

Proof (Sketch). The proof that the B'_k are independent is done by induction on n and is that of Lemma 4.2 in [4]. To see the second assertion, first note that $\{y_l|k_l = k\} \subseteq \mathrm{Cl}_{M_k} B_k$ and thus $B'_k \subseteq \mathrm{Cl}_{M_k} B_k$. Thus it suffices to show that $B_k \subseteq \mathrm{Cl}_{M_k} B'_k$. For this, note that the reverse tuple $(y_n, k_{n-1}; y_{n-1}, k_{n-2}; \ldots; y_0)$ is a B'_k-exchange chain giving back the original B_k, so we can apply the preceding argument again. □

Lemma 6.1.2. *Let M be a matroid and $I, B \in \mathcal{I}(M)$ with B maximal and $B \setminus I$ finite. Then $|I \setminus B| \leq |B \setminus I|$.*

Lemma 6.1.3. *Let $(M_k|k \in K)$ be a family of matroids, let $B_k \in \mathcal{I}(M_k)$ and let C be a circuit for some M_{k_0} such that $C \setminus B_{k_0}$ only contains one element, e. If there is a $(B_k|k \in K)$-exchange chain from x_0 to e, then for every $c \in C$, there is a $(B_k|k \in K)$-exchange chain from x_0 to c.*

Proof. Let $(y_0 = x_0, k_0; y_1, k_1; \ldots; y_n = e)$ be an exchange chain from x_0 to e. Then $(y_0 = x_0, k_0; y_1, k_1; \ldots; y_n = e, k_0; c)$ is the desired exchange chain. □

6.2 The Packing/Covering conjecture

The matroid union theorem is a basic result in the theory of finite matroids. It gives a way to produce a new matroid $M = \bigvee_{k\in K} M_k$ from a finite family $(M_k|k \in K)$ of finite matroids on the same ground set E. We take a subset I of E to be M-independent iff it is a union $\bigcup_{k\in K} I_k$ with each I_k independent in the corresponding matroid M_k. The fact that this gives a matroid is interesting, but a great deal of the power of the theorem comes from the fact that it gives an explicit formula for the ranks of sets in this matroid:

$$r_M(X) = \min_{X=P\dot\cup C} \sum_{k\in K} r_{M_k}(P) + |C| \tag{6.1}$$

Here the minimisation is over those pairs (P, C) of subsets of X which partition X.

For infinite matroids, or infinite families of matroids, this theorem is no longer true [4], in that M is no longer a matroid. However, it turns out, as we shall now show, that we may conjecture a natural extension of the rank formula to infinite families of infinite matroids.

First, we state the formula in a way which does not rely on the assumption that M is a matroid:

$$\max_{I_k\in\mathcal{I}(M_k)} \left|\bigcup_{k\in K} I_k\right| = \min_{E=P\dot\cup C} \sum_{k\in K} r_{M_k}(P) + |C| \tag{6.2}$$

Note that this is really only the special case of (6.1) with $X = E$. However, it is easy to deduce the more general version by applying (6.2) to the family $(M_k\restriction_X|k \in K)$.

Note also that no value $|\bigcup_{k\in K} I_k|$ appearing on the left is bigger than any value $\sum_{k\in K} r_{M_k}(P) + |C|$ appearing on the right. To see this, note that $|\bigcup_{k\in K}(I_k \cap P)| \leq \sum_{k\in K} r_{M_k}(P)$ and $\bigcup_{k\in K}(I_k \cap C) \subseteq C$. So the formula is equivalent to the statement that we can find $(I_k|k \in K)$ and P and C with $P\dot\cup C = E$ so that

$$\left|\bigcup_{k\in K} I_k\right| = \sum_{k\in K} r_{M_k}(P) + |C|\,. \tag{6.3}$$

For this, what we need is to have equality in the two inequalities above, so we get

$$\left|\bigcup_{k\in K} (I_k \cap P)\right| = \sum_{k\in K} r_{M_k}(P) \text{ and } \bigcup_{k\in K} (I_k \cap C) = C\,. \tag{6.4}$$

The equation on the left can be broken down a bit further: it states that each $I_k \cap P$ is spanning (and so a base) in the appropriate matroid $M_k\restriction_P$, and that all these sets are disjoint. This is the familiar notion of a packing:

Definition 6.2.1. Let $(M_k|k \in K)$ be a family of matroids on the same ground set E. A *packing* for this family consists of a spanning set S_k for each M_k such that the S_k are all disjoint.

So the $I_k \cap P$ form a packing for the family $(M_k\restriction_P|k \in K)$. In fact, in this case, each $I_k \cap P$ is a base in the corresponding matroid. In Definition 6.2.1, we do not require the S_k to be bases, but of course if we have a packing we can take a base for each S_k and so obtain a packing employing only bases.

Dually, the right hand equation in (6.4) corresponds to the presence of a covering of C:

Definition 6.2.2. Let $(M_k|k \in K)$ be a family of matroids on the same ground set E. A *covering* for this family consists of an independent set I_k for each M_k such that the I_k cover E.

It is immediate that the sets $I_k \cap C$ form a covering for the family $(M_k\restriction_C|k \in K)$. In fact we get the stronger statement that they form a covering for the family $(M_k.C|k \in K)$ where we contract instead of restricting, since for each k we have that $I_k \cap P$ is an M_k-base for P, and we also have that I_k, which is the union of $I_k \cap C$ with $I_k \cap P$, is M_k-independent.

Putting all of this together, we get the following self-dual notion:

Definition 6.2.3. Let $(M_k|k \in K)$ be a family of matroids on the same ground set E. We say this family has the *Packing/Covering property* iff there is a partition of E into two parts P (called the *packing side*) and C (called the *covering side*) such that $(M_k\restriction_P|k \in K)$ has a packing, and $(M_k.C|k \in K)$ has a covering.

We have established above that this property follows from the rank formula for union, but the argument can easily be reversed to show that in fact Packing/Covering is equivalent to the rank formula, where that formula makes sense. However, Packing/Covering also makes sense for infinite matroids, where the rank formula is no longer useful. We are therefore led to the following conjecture:

Conjecture 6.0.4. *Every family of matroids on the same ground set has the Packing/Covering property.*

Because of this link to the rank formula, we immediately get a special case of this conjecture:

Theorem 6.2.4. *Every finite family of finite matroids on the same ground set has the Packing/Covering property.*

Packing/Covering for pairs of matroids is closely related to another property which is conjectured to hold for all pairs of matroids.

Definition 6.2.5. A pair (M, N) of matroids on the same ground set E has the *Intersection property* iff there is a subset J of E, independent in both matroids, and a partition of J into two parts J^M and J^N such that

$$\mathrm{Cl}_M(J^M) \cup \mathrm{Cl}_N(J^N) = E\,.$$

Conjecture 6.0.2. *Every pair of matroids on the same ground set has the Intersection property.*

We begin by demonstrating a link between Packing/Covering for pairs of matroids and Intersection.

Proposition 6.2.6. *Let M and N be matroids on the same ground set E. Then M and N have the Intersection property iff (M, N^*) has the Packing/Covering property.*

Proof. Suppose first of all that (M, N^*) has the Packing/Covering property, with packing side P decomposed as $S^M \dot\cup S^{N^*}$ and covering side C decomposed as $I^M \dot\cup I^{N^*}$. Let J^M be an M-base of S^M, and J^N an N-base of $C \setminus I^{N^*}$. $J = J^M \cup J^N$ is independent in M since $J^N \subseteq I^M$ is independent in $M.C$ and J^M is independent in $M\restriction_P$. Similarly J is independent in N since $J^M \subseteq P \setminus S^{N^*}$ is independent in $N.P$ and J^N is independent in $N\restriction_C$. But also

$$\mathrm{Cl}_M(J^M) \cup \mathrm{Cl}_N(J^N) = \mathrm{Cl}_M(S^M) \cup \mathrm{Cl}_N(C \setminus I^{N^*}) \supseteq P \cup C = E\,.$$

Now suppose instead that M and N have the Intersection property, as witnessed by $J = J^M \dot\cup J^N$. Let $J^M \subseteq P \subseteq \mathrm{Cl}_M(J^M)$ and $J^N \subseteq C \subseteq \mathrm{Cl}_N(J^N)$ be a partition of E (this is possible since $\mathrm{Cl}_M(J^M) \cup \mathrm{Cl}_N(J^N) = E$). We shall show first of all that $M\restriction_P$ and $N^*\restriction_P$ have a packing, with the spanning sets given by $S^M = J^M$ and $S^{N^*} = P \setminus J^M$. J^M is spanning in $M\restriction_P$ since $P \subseteq \mathrm{Cl}_M(J^M)$, so it is enough to check that $P \setminus J^M$ is spanning in $N^*\restriction_P$, or equivalently that J^M is independent in $N.P$. But this is true since J^N is an N-base of C and $J^M \cup J^N$ is N-independent.

Similarly, J^N is independent in $M.C$, and since $C \subseteq \mathrm{Cl}_N(J^N)$ J^N is spanning in $N\restriction_C$ and so $C \setminus J^N$ is independent in $N^*.C$. Thus the sets $I^M = J^N$ and $I^{N^*} = C \setminus J^N$ form a covering for $(M.C, N^*.C)$. $\square$

Corollary 6.2.7. *If M and N are matroids on the same ground set then (M, N) has the Packing/Covering property iff (M^*, N^*) does.* $\square$

This corollary is not too hard to see directly. However, the following similar corollary is less trivial.

Corollary 6.0.5. *If M and N are matroids on the same ground set then M and N have the Intersection property iff M^* and N^* do.* $\square$

Proposition 6.2.6 shows that Conjecture 6.0.2 follows from Conjecture 6.0.4, but so far we would only be able to use it to deduce that any pair of matroids has the Packing/Covering property from Conjecture 6.0.2. However, this turns out to be enough to give the whole of Conjecture 6.0.4.

Proposition 6.2.8. *Let $(M_k|k \in K)$ be a family of matroids on the same ground set E, and let $M = \bigoplus_{k\in K} M_k$, on the ground set $E \times K$. Let N be the matroid on the same ground set given by $\bigoplus_{e\in E} U^*_{1,K}$. Then the M_k have the Packing/Covering property iff M and N do.*

Proof. First of all, suppose that the M_k have the Packing/Covering property and let P, C, S_k and I_k be as in Definition 6.2.3. We can partition $E \times K$ into $P' = P \times K$ and $C' = C \times K$. Let $S^M = \bigcup_{k\in K} S_k \times \{k\}$, and let $S^N = P' \setminus S^M$. S^M is spanning in $M\restriction_{P'}$ by definition, and since the sets S_k are disjoint, there is for each $e \in P$ at most one $k \in K$ with $(e, k) \notin S^N$. Thus S^N is spanning in $N\restriction_{P'}$. Similarly, let $I^M = \bigcup_{k\in K} I_k \times \{k\}$ and let $I^N = C' \setminus I^M$. I^M is independent in $M.C'$ by definition, and since the sets I_k cover C there is for each $e \in E$ at least one $k \in K$ with $(e, k) \notin I^N$. Thus I^N is independent in $N.C'$.

Now suppose instead that M and N have the Packing/Covering property, with packing side P decomposed as $S^M \dot\cup S^N$ and covering side C decomposed as $I^M \dot\cup I^N$. First we modify these sets a little so that the packing and covering sides are given by $\overline{P} \times K$ and $\overline{C} \times K$ for some sets $\overline{P}$ and $\overline{C}$. To this end, we let $\overline{P} = \{e \in E|(\forall k \in K)(e, k) \in P\}$, and $\overline{C} = \{e \in E|(\exists k \in K)(e, k) \in C\}$, so that $\overline{P}$ and $\overline{C}$ form a partition of E. Let $\overline{S}^N = S^N \cap (\overline{P} \times K)$ and $\overline{I}^N = I^N \cup ((\overline{C} \times K) \setminus C)$. We shall show that $(S^M, \overline{S}^N)$ is a packing for $(M\restriction_{\overline{P}\times K}, N\restriction_{\overline{P}\times K})$ and $(I^M, \overline{I}^N)$ is a covering for $(M.(\overline{C} \times K), N.(\overline{C} \times K))$.

For any $e \in \overline{C}$, the restriction of the corresponding copy of $U^*_{1,K}$ to $P \cap (\{e\} \times K)$ is free, and so since the intersection of S^N with this set is spanning there, it must contain the whole of $P \cap (\{e\} \times K)$. So since $S^M \subseteq P$ is disjoint from S^N, it can't contain any (e, k) with $e \in \overline{C}$. That is, $S^M \subseteq \overline{P} \times K$. It also spans $\overline{P} \times K$ in M, since it spans the larger set P. For each $e \in \overline{P}$, $\overline{S}^N \cap (\{e\} \times K) = S^N \cap (\{e\} \times K)$ N-spans $\{e\} \times K$. Thus $\overline{S}^N$ N-spans $\overline{P} \times K$, so $(S^M, \overline{S}^N)$ is a packing for $(M\restriction_{\overline{P}\times K}, N\restriction_{\overline{P}\times K})$.

To show that $(I^M, \overline{I}^N)$ is a covering for $(M.(\overline{C} \times K), N.(\overline{C} \times K))$, it suffices to show that $\overline{I}^N$ is $N.(\overline{C} \times K)$-independent. For each $e \in \overline{C}$, the set $C \cap (\{e\} \times K)$ is nonempty, so the contraction of the corresponding copy of $U^*_{1,K}$ to this set consists of a single circuit, so there is some point in this set but not in I^N. Then that same point is also not in $\overline{I}^N$, and so $\overline{I}^N \cap (\{e\} \times K)$ is independent in the corresponding copy of $U^*_{1,K}$, so $\overline{I}^N$ is indeed $N.(\overline{C} \times P)$-independent.

Now that we have shown that $\overline{P} \times K$, $\overline{C} \times K$, $(S^M, \overline{S}^N)$ and $(I^M, \overline{I}^N)$ also witness that M and N have the Packing/Covering property, we show how we can construct a packing and a covering for $(M_k\restriction_{\overline{P}}|k \in K)$ and $(M_k.\overline{C}|k \in K)$ respectively.

For each $k \in K$ let $I_k = \{e \in E|(e, k) \in I^M\}$. Since, as we saw above, I^M meets each of the sets $\{e\} \times K$ with $e \in \overline{C}$, the union of the I_k is $\overline{C}$. Since also each I_k is independent in $M_k.\overline{C}$, they form a covering for $(M_k.\overline{C}|k \in K)$. Similarly, let $S_k = \{e \in E|(e, k) \in S^M\}$. Since the intersection of $\overline{S}^N$ with

$\{e\} \times K$ is spanning in the corresponding copy of $U^*_{1,k}$ for any $e \in \overline{P}$, it follows that for such e it misses at most one point of this set, so that there can be at most one point in $S^M \cap (\{e\} \times K)$, so the S_k are disjoint. Thus they form a packing of $(M_k \upharpoonright_{\overline{P}} | k \in K)$. $\square$

Corollary 6.2.9. *The following are equivalent:*

(a) Any two matroids have the Intersection property (Conjecture 6.0.2).

(b) Any two matroids in which the second is a direct sum of copies of $U_{1,2}$ have the Intersection property.

(c) Any pair of matroids has the Packing/Covering property.

(d) Any pair of matroids in which the second is a direct sum of copies of $U_{1,2}$ has the Packing/Covering property.

(e) Any family of matroids has the Packing/Covering property (Conjecture 6.0.4).

Proof. We shall prove the following equivalences.

$$\begin{tikzcd}
(b) \arrow[r, leftrightarrow] & (d) \arrow[d, leftrightarrow] & \\
(a) \arrow[r, leftrightarrow] & (c) \arrow[r, leftrightarrow] & (e)
\end{tikzcd}$$

The equivalences of (a) with (c) and (b) with (d) both follow from Proposition 6.2.6. (c) evidently implies (d), but we can also get (c) from (d) by applying Proposition 6.2.8. Similarly, (e) evidently implies (c) and we can get (e) from (c) by applying Proposition 6.2.8. $\square$

6.3 A special case of the Packing/Covering conjecture

In [3], Aharoni and Ziv prove a special case of the intersection conjecture. Here we employ a simplified form of their argument to prove a special case of the Packing/Covering conjecture. Our simplification also yields a slight strengthening of their theorem.

Key to the argument is the notion of a wave.

Definition 6.3.1. Let $(M_k | k \in K)$ be a family of matroids all on the ground set E. A *wave* for this family is a subset P of E together with a packing $(S_k | k \in K)$ of $(M_k \upharpoonright_P | k \in K)$. In a slight abuse of notation, we shall sometimes refer to the wave just as P or say that elements of P are in the wave. A wave is a *hindrance* if the S_k don't completely cover P. The family is *unhindered* if there is no hindrance, and *loose* if the only wave is the empty wave.

Remark 6.3.2. *Those familiar with Aharoni and Ziv's notion of wave should observe that if $(P, (S_1, S_2))$ is a wave as above and we let F be an M_2-base of S_2 then F is not only M_2-independent but also $M_1^*.P$-independent, since $S_1 \subseteq P \setminus F$ is $M_1\restriction_P$-spanning. Now since $P \subseteq \mathrm{Cl}_{M_2}(F)$, we get that F is also $M_1^*.\mathrm{Cl}_{M_2}(F)$-independent. Thus F is a wave in the sense of Aharoni and Ziv for the matroids M_1^* and M_2. There is a similar correspondence of the other notions defined above.*

Similarly, they say that the pair (M_1, M_2) is matchable *iff there is a set which is M_1-spanning and M_2-independent. Those interested in translating between the two contexts should note that there is a covering for (M_1, M_2) iff (M_1^*, M_2) is matchable.*

We define a partial order on waves by $(P, (S_k|k \in K)) \leq (P', (S'_k|k \in K))$ iff $P \subseteq P'$ and for each $k \in K$ we have $S_k \subseteq S'_k$. We say a wave is *maximal* iff it is maximal with respect to this partial order.

Lemma 6.3.3. *Let $(M_k|k \in K)$ be a family of matroids on the same ground set E, and let $((P^\beta, (S_k^\beta|k \in K))|\beta < \alpha)$ a family of waves indexed by some ordinal α. Then there is a wave $(P, (S_k, |k \in K))$ with $P = \bigcup_{\beta<\alpha} P^\beta$ and $P \geq P_0$.*

Proof. For each $\beta < \alpha$, let $Y^\beta = P^\beta \setminus \bigcup_{\gamma<\beta} P^\gamma$. For $k \in K$, let $S_k = \bigcup_{\beta<\alpha}(Y^\beta \cap S_k^\beta)$. These are clearly disjoint subsets of P: we aim to show that they form a packing. We shall show by induction on $\beta < \alpha$ that for each $k \in K$ we have $P^\beta \subseteq \mathrm{Cl}_{M_k}(S_k)$. By the induction hypothesis, we have that $S_k^\beta \setminus Y^\beta \subseteq \bigcup_{\gamma<\beta} P^\gamma \subseteq \mathrm{Cl}_{M_k}(S_k)$, so $P^\beta \subseteq \mathrm{Cl}_{M_k}(S_k^\beta) \subseteq \mathrm{Cl}_{M_k}(\mathrm{Cl}_{M_k}(S_k)) = \mathrm{Cl}_{M_k}(S_k)$.

It follows that $P \subseteq \mathrm{Cl}_{M_k}(S_k)$, so the S_k form a packing for $(M_k\restriction_P)$ as desired. □

Corollary 6.3.4. *For any wave P there is a maximal wave $P_{\max} \geq P$.*

Proof. We apply Lemma 6.3.3 to a family of waves with P as the first element and which includes all waves to obtain a new wave $P' \geq P$ such that any element of any wave is an element of P'. We can extend P to a maximal wave by assigning the elements of $P' \setminus \bigcup_{k\in K} S'_k$ in any way to the sets S'_k. □

Corollary 6.3.5. *If $P_{\max}$ is a maximal wave then anything in any wave P is in $P_{\max}$.*

Proof. We apply Lemma 6.3.3 to the pair $(P_{\max}, P)$. □

Lemma 6.3.6. *For any $e \in E$ and $k \in K$, any maximal wave P satisfies $e \in \mathrm{Cl}_{M_k} P$ whenever there is any wave P' with $e \in \mathrm{Cl}_{M_k} P'$.*

In particular, if e is not contained in any wave, there are at least two k such that, for every wave P', $e \notin \mathrm{Cl}_{M_k} P'$.

Proof. Let $(P, (S_k|k \in K))$ be a maximal wave. By Corollary 6.3.5 for any wave $(P', (S'_k|k \in K))$ we have $S'_k \subseteq \mathrm{Cl}_{M_k} S_k$. Thus $e \in \mathrm{Cl}_{M_k} P' = \mathrm{Cl}_{M_k} S'_k$ implies $e \in \mathrm{Cl}_{M_k} P$, as desired.

For the second assertion, assume toward contradiction that there is at most one k_0 such that, for every wave P', $e \notin \mathrm{Cl}_{M_{k_0}} P'$. Then $e \in \mathrm{Cl}_{M_k} P$ for all $k \neq k_0$. But then the following is a wave and contains e:
$X := (P + e, (\overline{S}_k | k \in K))$ where $\overline{S}_{k_0} = S_{k_0} + e$ and $\overline{S}_k = S_k$ for other values of k. This is a contradiction. $\square$

Lemma 6.3.7. *Let $(P, (S_k | k \in K))$ be a wave for a family $(M_k | k \in K)$ of matroids. Let $(P', (S'_k | k \in K))$ be a wave for the family $(M_k/P | k \in K)$. Then $(P \cup P', (S_k \cup S'_k | k \in K))$ is a wave for the family $(M_k | k \in K)$. If either P or P' is a hindrance then so is $P \cup P'$.*

Remark 6.3.8. *In fact, though we will not need this, a similar statement can be shown for an ordinal indexed family of waves P^β, with P^β a wave for the family $(M_k / \bigcup_{\gamma < \beta} P^\gamma | k \in K)$.*

Proof. For each k, the set S'_k is spanning in $M_k \restriction_{P \cup P'} / P$ and S_k is spanning in $M_k \restriction_{P \cup P'} \restriction P$, so each set $S_k \cup S'_k$ spans $P \cup P'$, and they are clearly disjoint. If the S_k don't cover some point of P then the $S_k \cup S'_k$ also don't cover that point, and the argument in the case where P' is a hindrance is similar. $\square$

Corollary 6.3.9. *For any maximal wave $P_{\max}$, the family $(M_k / P_{\max} | k \in K)$ is loose.*

We are now in a position to present another Conjecture equivalent to the Packing/Covering Conjecture. It is for this new form that we shall present our partial proof.

Conjecture 6.3.10. *Any unhindered family of matroids has a covering.*

Proposition 6.3.11. *Conjecture 6.3.10 and Conjecture 6.0.4 are equivalent.*

Proof. First of all, suppose that Conjecture 6.0.4 holds, and that we have an unhindered family $(M_k | k \in K)$ of matroids. Using Conjecture 6.0.4, we get P, C, S_k and I_k as in Definition 6.2.3. Then $(P, (S_k | k \in K))$ is a wave, and since it can't be a hindrance the sets S_k cover P. They must also all be independent, since otherwise we could remove a point from one of them to obtain a hindrance. So the sets $S_k \cup I_k$ give a covering for $(M_k | k \in K)$.

Now suppose instead that Conjecture 6.3.10 holds, and let $(M_k | k \in K)$ be any family of matroids on the ground set E. Then let $(P, (S_k | k \in K))$ be a maximal wave. By Corollary 6.3.9, $(M_k / P | k \in K)$ is loose, and so in particular this family is unhindered. So it has a covering $(I_k | k \in K)$. Taking covering side $C = E \setminus P$, this means that the M_k have the Packing/Covering property. $\square$

Lemma 6.3.12. *Suppose that we have an unhindered family $(M_k | k \in K)$ of matroids on a ground set E. Let $e \in E$ and $k_0 \in K$ such that for every wave P we have $e \notin \mathrm{Cl}_{M_{k_0}} P$. Then the family $(M'_k | k \in K)$ on the ground set $E - e$ is also unhindered, where $M'_{k_0} = M_{k_0}/e$ but $M'_k = M_k \backslash e$ for other values of k.*

Proof. Suppose not, for a contradiction, and let $(P, (S_k|k \in K))$ be a hindrance for $(M'_k|k \in K)$. Without loss of generality, we assume that the S_k are bases of P. Let $\overline{S}_k$ be given by $\overline{S}_{k_0} = S_{k_0} + e$ and $\overline{S}_k = S_k$ for other values of k. Note that $\overline{S}_{k_0}$ is independent because otherwise, by the M_{k_0}/e-independence of S_{k_0}, we must have $e \in \mathrm{Cl}_{M_{k_0}}(S_{k_0})$ (in fact, $\{e\}$ must be an M_{k_0}-circuit), so that $P \subseteq \mathrm{Cl}_{M_{k_0}}(S_{k_0})$, and thus $(P, (S_k|k \in K))$ is a wave for the M_k with $e \in \mathrm{Cl}_{M_{k_0}} P$. Let P' be the set of $x \in P$ such that there is no $(\overline{S}_k|k \in K)$-exchange chain from x to e.

Let $x_0 \in P \setminus \bigcup_{k \in K} S_k$. If $x_0 \in P'$, then we will show that $(P', (P' \cap \overline{S}_k|k \in K))$ is a wave containing x_0. This contradicts the assumption that $(M_k|k \in K)$ is unhindered. We must show for every k that every $x \in P' \setminus P' \cap \overline{S}_k$ is M_k-spanned by $P' \cap \overline{S}_k$. Since $e \notin P'$ we cannot have $x = e$. Let C be the unique circuit contained in $x + \overline{S}_k$. If $x \in P'$, then $C \subseteq P'$ by Lemma 6.1.3, so $x \in \mathrm{Cl}_{M_k}(P' \cap \overline{S}_k)$, as desired.

If $x_0 \notin P'$, there is a shortest $(\overline{S}_k|k \in K)$-exchange chain

$$(y_0 = x_0, k_0; y_1, k_1; \ldots ; y_n = e)$$

from x_0 to e. Let $\overline{S}'_k := \overline{S}_k \cup \{y_l|k_l = k\} \setminus \{y_{l+1}|k_l = k\}$. By Lemma 6.1.1, $\overline{S}'_k$ is M_k-independent and $\mathrm{Cl}_{M_k} \overline{S_k} = \mathrm{Cl}_{M_k} \overline{S_k}'$ for all $k \in K$. Thus each $\overline{S}'_k$ M_k-spans P but avoids e, in other words: $(P, (\overline{S}'_k|k \in K))$ is an $(M_k|k \in K)$-wave. But also $e \in \mathrm{Cl}_{M_{k_0}} P$ since $e \in \overline{S}_{k_0}$, a contradiction. □

We will now discuss those partial versions of Conjecture 6.3.10 which we can prove. We would like to produce a covering of the ground set by independent sets - and that means that we don't want any of the sets in the covering to include any circuits for the corresponding matroid. First of all, we show that we can at least avoid *some* circuits. In fact, we'll prove a slightly stronger theorem here, showing that we can specify a countable family of sets, which are to be avoided whenever they are dependent. In all our applications, the dependent sets we care about will be circuits.

Theorem 6.3.13. *Let $(M_k|k \in K)$ be an unhindered family of matroids on the same ground set E. Suppose that we have a sequence of subsets o_n of E. Then there is a family $(I_k|k \in K)$ whose union is E and such that for no $k \in K$ and $n \in \mathbb{N}$ do we have both $o_n \subseteq I_k$ and o_n dependent in M_k.*

Proof. If some wave includes the whole ground set, then as the family is unhindered, this wave would yield the desired covering. Unfortunately, we may not assume this. Instead, we recursively build a family $(J_k|k \in K)$ of disjoint sets such that some wave $(P, (S_k|k \in K))$ for the $M_k/J_k \setminus \bigcup_{l \neq k} J_l$ includes enough of $E \setminus \bigcup_k J_k$ that any family $(I_k|k \in K)$ whose union is E and with $I_k \cap (P \cup \bigcup_{k \in K} J_k) = S_k \cup J_k$ will work.

We construct J_k as the nested union of some $(J_k^n|n \in \mathbb{N} \cup \{0\})$ with the following properties. Abbreviate $M_k^n := M_k/J_k^n \setminus \bigcup_{l \neq k} J_l^n$.

(a) J_k^n is independent in M_k.

(b) For different k, the sets J_k^n are disjoint.

(c) $(M_k^n | k \in K)$ is unhindered.

(d) Either the set $o_n \setminus \bigcup_{k\in K} J_k^n$ is included in some $(M_k^n|k \in K)$-wave or there are distinct l, l' such that there is some $e \in o_n \cap J_l^n$ and some $e' \in o_n \cap J_{l'}^n$.

Put $J_k^0 := \emptyset$ for all k. These satisfy (a)-(c), and (d) is vacuous since there is no term o_0 (we are following the convention that 0 is not a natural number). Assume that we have already constructed J_k^n satisfying (a)-(d).

If (d) with o_{n+1} in place of o_n is already satisfied by the $(J_k^n|k \in K)$ we can simply take $J_k^{n+1} := J_k^n$ for all k.

Otherwise, if we let P_{max} be a maximal wave, there is some $e \in o_{n+1} \setminus \bigcup_{k\in K} J_k^n$ not in P_{max} and so not in any $(M_k^n|k \in K)$-wave. By Lemma 6.3.6, there are at least two $k \in K$ such that $e \notin \mathrm{Cl}_{M_k} P'$ for every wave P'. In particular, e is not a loop ($\{e\}$ is independent) in M_k for those two k. Let l be one of these two values of k. Now let $\overline{J_l^{n+1}} := J_l^n + e$ and $\overline{J_k^{n+1}} := J_k^n$ for $k \neq l$. Then the $\overline{J_k^{n+1}}$ satisfy (a) and (b). By Lemma 6.3.12 and the choice of e, we also have (c).

If the $\overline{J_k^{n+1}}$ already satisfy (d), then we are done. Else, to obtain (d), repeat the induction step so far and find $e' \in o_{n+1} \setminus \bigcup_{k\in K} \overline{J_k^{n+1}}$ not in any $(\overline{M_k^n}|k \in K)$-wave. Here $\overline{M_k^n}$ is M_k^n/e if $k = l$ and $M_k^n \backslash e$ otherwise. Further we find, $l' \neq l$ such that $\{e'\}$ is independent in $\overline{M_{l'}^n}$ and $e' \notin \mathrm{Cl}_{M_l} P'$ for every wave P'. Now let $J_{l'}^{n+1} := \overline{J_{l'}^{n+1}} + e'$ and $J_k^{n+1} := \overline{J_k^{n+1}}$ for $k \neq l'$. Then the J_k^{n+1} satisfy (a) and (b) and now also (d). By Lemma 6.3.12 and the choice of e', we also have (c).

We now define a new family of matroids by $M'_k := M_k/J_k \backslash \bigcup_{l\neq k} J_l$, and we construct an $(M'_k|k \in K)$-wave $(P, (S_k|k \in K))$. We once more do this by taking the union of a recursively constructed nested family. Explicitly, we take $S_k = \bigcup_{n\in\mathbb{N}} S_k^n$ and $P = \bigcup_{n\in\mathbb{N}} P^n$, where for each n the wave $W^n = (P^n, (S_k^n|k \in K))$ is a maximal wave for $(M_k^n|k \in K)$ and the S_k^n are nested. We can find such waves using Corollary 6.3.4: for each n we have that W^n is also a wave for $(M_k^{n+1}|k \in K)$ since in our construction we never contract or delete anything which is in a wave.

Now let $(I_k|k \in K)$ be chosen so that $\bigcup I_k = E$ and for each $k_0 \in K$ we have $I_{k_0} \cap (P \cup \bigcup_{k\in K} J_k) = S_{k_0} \cup J_{k_0}$. Suppose for a contradiction that for some pair (k_0, n) we have $o_n \subseteq I_{k_0}$ and o_n is dependent in M_{k_0}. Then by (d), either the set $o_n \setminus \bigcup_{k\in K} J_k^n$ is included in some $(M_k^n|k \in K)$-wave or there are distinct l, l' such that there is some $e \in o_n \cap J_l^n$ and some $e' \in o_n \cap J_{l'}^n$. In the second case, clearly $o_n \nsubseteq I_{k_0}$.

In the first case, we will find a hindrance for $(M_k^n|k \in K)$, which contradicts (c). It suffices to show that $S_{k_0}^n$ is dependent in $M_{k_0}^n$, since then we can obtain a hindrance by removing a point from $S_{k_0}^n$ in W^n. Let $o = o_n \backslash \bigcup_{k\in K} J_k^n = o_n \backslash J_{k_0}^n$. Note that o is dependent in $M_{k_0}^n$, since o_n is dependent in $M_{k_0}^n$ but $J_{k_0}^n$ is not by (a). By assumption, $o \subseteq P^n$, and so since also $o \subseteq o_n \subseteq I_{k_0}$ we have $o \subseteq I_{k_0} \cap P^n = S_{k_0}^n$, so that $S_{k_0}^n$ is $M_{k_0}^n$-dependent as required.

□

Note that, in particular, if we have a countable family of matroids each with only countably many circuits then Theorem 6.3.13 applies in order to prove Conjecture 6.0.4 in that special case. Requiring only countably many circuits might seem quite restrictive, but there are many cases where it holds:

Proposition 6.3.14. *A matroid of any of the following types on a countable ground set has only countably many circuits:*

(a) A finitary matroid.

(b) A matroid whose dual has finite rank.

(c) A direct sum of matroids each with only countably many circuits.

Proof. (a) follows from the fact that the countable ground set has only countably many finite subsets. For (b), since every base B has finite complement, there are only countably many bases. As every circuit is a fundamental circuit for some base, there can only be countably many circuits, as desired. For (c), there can only be countably many nontrivial summands in the direct sum since the ground set is countable, and the result follows. □

In particular, Theorem 6.3.13 applies to any countable family of matroids each of which is a direct sum of matroids that are finitary or whose duals have finite rank. This includes the main result of Aharoni and Ziv in [3], if the ground set E is countable, by Proposition 6.2.6.

If we have a family of sets $(I_k|k \in K)$ which does not form a covering, because some elements aren't independent, how might we tweak it to make them more independent? Suppose that the reason why I_k is dependent is that it contains a circuit o of M_k, but that o also includes a cocircuit for another matroid $M_{k'}$ from our family. Then we could move some point from I_k into $I_{k'}$ to remove this dependence without making $I_{k'}$ any more dependent.[2] We are therefore not so worried about circuits including cocircuits in this way as we are about other sorts of circuits. Therefore we now consider cases where most circuits do include such cocircuits:

Definition 6.3.15. Let $(M_k|k \in K)$ be a family of matroids on the same ground set E. For each $k \in K$ we let W_k be the set of all M_k-circuits that do not contain an $M_{k'}$-cocircuit with $k' \neq k$. Call the family $(M_k|k \in K)$ of matroids *at most countably weird* if $\bigcup W_k$ is at most countable.

Note that if E is countable then $(M_k|k \in K)$ is at most countably weird if and only if $\bigcup W_k^\infty$ is countable where W_k^∞ is the subset of W_k consisting only of the infinite circuits in W_k.

Theorem 6.3.16. *Any unhindered and at most countably weird family $(M_k|k \in K)$ of matroids has a covering.*

[2] We may assume that the I_k are disjoint. Then any new circuits in $I_{k'}$ would have to meet the cocircuit in just one point, which is impossible.

Proof. Apply Theorem 6.3.13 to $(M_k|k \in K)$ where the o_n enumerate $\bigcup W_k$ where the W_k are defined as in Definition 6.3.15.

So far $(I_k|k \in K)$ is not necessarily a covering since each I_k might still contain circuits. But by the choice of the family of circuits each circuit contained in I_k contains an $M_{k'}$-cocircuit with $k' \neq k$.

In the following, we tweak $(I_k|k \in K)$ to obtain a covering $(L_k|k \in K)$. First extend I_k into a minimal M_k-spanning set B_k by $(IM)^*$. We obtain L_k from B_k by removing all elements in $I_k \cap \bigcup_{l \neq k} B_l$. We can suppose without loss of generality $(I_k|k \in K)$ was a partition of E, and so the family $(L_k|k \in K)$ covers E. It remains to show that L_k is independent. For this, assume for a contradiction that L_k contains an M_k-circuit C. By the choice of B_k, the circuit C is contained in I_k. In particular, C contains an M_l-cocircuit X for some $l \neq k$. By construction B_l meets X and thus C. As $C \subseteq I_k$, the circuit C is not contained in L_k, a contradiction. So $(L_k|k \in K)$ is the desired covering. □

Since by Lemma 1.2.7 for any set P the family $(M_k/P|k \in K)$ is at most countably weird if $(M_k|k \in K)$ is, we can now apply the argument of Proposition 6.3.11 to obtain the following:

Corollary 6.3.17. *Any at most countably weird family $(M_k|k \in K)$ of matroids has the Packing/Covering property.* □

However, there are still some important open questions here.

Definition 6.3.18 ([5])**.** The *finitarisation of a matroid M* is the matroid M^{fin} whose circuits are precisely the finite circuits of M.[3] A matroid is called *nearly finitary* if every base misses at most finitely elements of some base of the finitarization.

From Proposition 6.2.6 and the corresponding case of Matroid Intersection [5] we obtain the following:

Corollary 6.3.19. *Any pair of nearly finitary matroids has the Packing/Covering property.*

By Proposition 6.2.8 Corollary 6.3.19 implies the that any finite family of nearly finitary matroids has the Packing/Covering property. We do not know the answer to the following question.

Open question 6.3.20. *Must every (countably) infinite family of nearly finitary matroids have the Packing/Covering property?*

In a similar way, we have the following question.

Open question 6.3.21. *Must every family of finitary matroids have the Packing/Covering property?*

[3]It is easy to check that M^{fin} is indeed a matroid [5].

6.4 Base covering

The well-known base covering theorem reads as follows.

Theorem 6.4.1. *Any family of finite matroids $(M_k|k \in K)$ on a finite common ground set E has a covering if and only if for every finite set $X \subseteq E$ the following holds.*

$$\sum_{k \in K} r_{M_k}(X) \geq |X|$$

Taking the family to contain only one matroid, consisting of one infinite circuit, we see that this theorem does not extend verbatim to infinite matroids. However, Theorem 6.4.1 extends verbatim to finite families of finitary matroids by compactness [4].[4] The requirement that the family is finite is necessary as $(U_k = U_{1,\mathbb{R}}|k \in \mathbb{N})$ satisfies the rank formula but does not have a covering.

In the following, we conjecture an extension of the finite base covering theorem to arbitrary infinite matroids. Our approach is to replace the rank formula by a condition that for finite sets X is implied by the rank formula but is still meaningful for infinite sets. A first attempt might be the following:

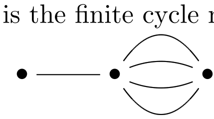
Any packing for the family $(M_k \restriction_X |k \in K)$ is already a covering. (6.5)

Indeed, for finite X, if $(M_k \restriction_X |k \in K)$ has a packing and there is an element of X not covered by the spanning sets of this packing, then this violates the rank formula. However, there are infinite matroids that violate (6.5) and still have a covering, see Figure 6.2.

We propose to use instead the following weakening of (6.5).

If $(M_k \restriction_X |k \in K)$ has a packing, then it also has a covering. (6.6)

To see that (6.6) does not imply the rank formula for some finite X, consider the family (M, M), where M is the finite cycle matroid of the graph



This graph has an edge not contained in any cycle (so that (M, M) does not have a packing) but enough parallel edges to make the rank formula false.

Using (6.6), we obtain the following:

Conjecture 6.4.2 (Covering Conjecture)**.** *A family of matroids $(M_k|k \in K)$ on the same ground set E has a covering if and only if (6.6) is true for every $X \subseteq E$.*

Proposition 6.4.3. *Conjecture 6.0.4 and Conjecture 6.4.2 are equivalent.*

[4]The argument in [4] is only made in the case where all M_k are the same but it easily extends to finite families of arbitrary finitary matroids.

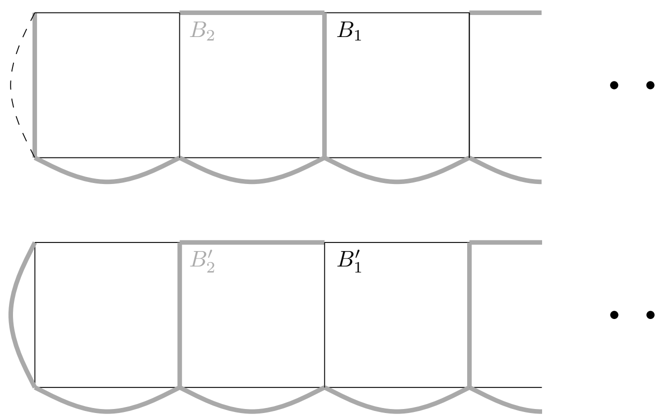


Figure 6.2: Above is a base packing which isn't a base covering. Below that is a base covering for the same matroids, namely the finite cycle matroid for the graph, taken twice.

Proof. For the "only if" direction, note that Conjecture 6.4.2 implies Conjecture 6.3.10, which by Proposition 6.3.11 implies Conjecture 6.0.4.

For the "if" direction, note that by assumption we have a partition $E = P\dot\cup C$ such that there exist disjoint $M_k\restriction_P$-spanning sets S_k and $M_k.C$-independent sets I_k whose union is C. By (6.6), $(M_k\restriction_P | k \in K)$ has a covering with sets B_k, where $B_k \in \mathcal{I}(M_k\restriction_P)$. As $I_k \cup B_k \in \mathcal{I}(M_k)$, the sets $I_k \cup B_k$ form the desired covering. $\square$

As Packing/Covering is true for finite matroids, Proposition 6.4.3 implies the non-trivial direction of Theorem 6.4.1. By Corollary 6.3.17 we obtain the following applications.

Corollary 6.4.4. *Any at most countably weird family of matroids* $(M_k|k \in K)$ *has a covering if and only if (6.6) is true for every* $X \subseteq E$.

Let us now specialise to graphs. In [27], many new matroids associated to infinite graphs are introduced, each with its own notion of tree. This development is so recent that the obvious questions about tree packing and covering for these notions have not yet been addressed. However, the theory developed here allows us to give some basic results. We rely on the fact that the algebraic cycle matroid of any locally finite graph and the topological cycle matroid of any graph are co-finitary.

Definition 6.4.5. The bases of the topological cycle matroid are called *topological trees* and the bases of the algebraic cycle matroid are called *algebraic trees.*

Using this we define *topological tree-packing, topological tree-covering, algebraic tree-packing, algebraic tree-covering.*

Corollary 6.4.6 (Base covering for the topological cycle matroids)**.** *A family of countable graphs $(G_k|k \in K)$ together with an identification of each set $E(G_k)$ with some common ground set E has a topological tree-covering if and only if the following is true for every $X \subseteq E$.*

$$\text{If } (G_k[X]|k \in K) \text{ has a topological tree-packing, then it also has a topological tree-covering.} \tag{6.7}$$

Corollary 6.4.7 (Base covering for the algebraic cycle matroids of locally finite graphs)**.** *A family of locally finite countable graphs $(G_k|k \in K)$ together with an identification of each set $E(G_k)$ with some common ground set E has an algebraic tree-covering if and only if the following is true for every $X \subseteq E$.*

$$\text{If } (G_k[X]|k \in K) \text{ has an algebraic tree-packing, then it also has an algebraic tree-covering.} \tag{6.8}$$

6.5 Base packing

The well-known base packing theorem reads as follows.

Theorem 6.5.1. *Any family of finite matroids $(M_k|k \in K)$ on a finite common ground set E has a packing if and only if for every finite set $Y \subseteq E$ the following holds.*

$$\sum_{k \in K} r_{M_k.Y}(Y) \leq |Y|$$

Aigner-Horey, Carmesin and Fröhlich [4] extended this theorem to families consisting of finitely many copies of the same co-finitary matroid. We extend this to arbitrary co-finitary families.

Theorem 6.5.2. *Any family of co-finitary matroids $(M_k|k \in K)$ on a common ground set E has a packing if and only if for every finite set $Y \subseteq E$ the following holds.*

$$\sum_{k \in K} r_{M_k.Y}(Y) \leq |Y|$$

Proof by a compactness argument. We will think of partitions of the ground set E as functions from E to K - such a function f corresponds to a partition $(S_k^f|k \in K)$, given by $S_k^f = \{e \in E|f(e) = k\}$. Endow K with the co-finite topology where a set is closed iff it is finite or the whole of K. Then endow K^E with the product topology, which is compact since the topology on K is compact.

A set S is spanning for a matroid M if and only if its complement includes no cocircuit, that is, if and only if it meets every cocircuit. So we would like a function f contained in each of the sets $C_{k,B} = \{f|S_k^f \cap B \neq \emptyset\}$, where B is a cocircuit for the matroid M_k. We will prove the existence of such a function by a compactness argument: we need to show that each $C_{k,B}$ is closed in the topology given above and that any finite intersection of them is nonempty.

To show that $C_{k,B}$ is closed, we rewrite it as $\bigcup_{e\in B}\{f|f(e)=k\}$. Each of the sets $\{f|f(e)=k\}$ is closed since their complements are basic open sets, and the union is finite since M_k is co-finitary.

Now let $(k_i|1\le i\le n)$ and $(B_i|1\le i\le n)$ be finite families with each B_i a cocircuit in M_{k_i}. We need to show that $\bigcap_{1\le i\le n} C_{k_i,B_i}$ is nonempty. Let $X=\bigcup_{1\le i\le n} B_i$. Since the rank formula holds for each subset of X, we have by the finite version of the base packing Theorem a packing $(S_k|k\in K)$ of $(M_k.X|k\in K)$. Now any f such that $f(e)=k$ for $e\in S_k$ will be in $\bigcap_{1\le i\le n} C_{k_i,B_i}$, since each B_i is an $M_{k_i}.X$-cocircuit. This completes the proof.

□

Theorem 6.5.1 does not extend verbatim to arbitrary infinite matroids. Indeed, for every integer k there exists a finitary matroid M on a ground set E with no three disjoint bases yet satisfying $|Y|\ge kr_{M.Y}(Y)$ for every finite $Y\subseteq E$ [2, 34].

In the following we conjecture an extension of the finite base packing theorem to arbitrary infinite matroids. This extension uses the following condition, which for finite sets Y is implied by the rank formula of the base packing theorem but is still meaningful for infinite sets:

$$\text{If } (M_k.Y|k\in K) \text{ has a covering, then it also has a packing.} \tag{6.9}$$

Indeed, if $(M_k.Y|k\in K)$ has a covering and there is an element of Y contained in several of the corresponding independent sets, then this violates the rank formula.

Using our new condition, we obtain the following:

Conjecture 6.5.3 (Packing Conjecture)**.** *A family of matroids $(M_k|k\in K)$ on the same ground set E has a packing if and only if (6.9) is true for every $Y\subseteq E$.*

Proposition 6.5.4. *Conjecture 6.0.4 and Conjecture 6.5.3 are equivalent.*

Proof. Since condition (6.9) for a pair of matroids is equivalent to (6.6) for the duals of those matroids and a pair of matroids have a packing if and only if their duals have a covering, Conjecture 6.5.3 implies that any pair of matroids satisfying (6.6) has a covering, and in particular that any unhindered pair of matroids has a covering. As in the proof of (6.3.11), this implies that any pair of matroids has the Packing/Covering property, which implies Conjecture 6.0.4 by Corollary 6.2.9.

The converse is proved as in the proof of Proposition 6.4.3. □

As Packing/Covering is true for finite matroids, Proposition 6.5.4 implies the non-trivial direction of Theorem 6.5.1. By Corollary 6.3.17 we obtain the following applications.

Corollary 6.5.5. *Any at most countably weird family of matroids on ground set E has a packing if and only if (6.9) is true for every $Y \subseteq E$.*

In particular, we obtain the following:

Corollary 6.5.6 (Base packing theorem for the finite cycle matroid)**.** *Any family of countable graphs $(G_k | k \in K)$ with a common edge set E has a tree-packing if and only if (6.10) is true for every $Y \subseteq E$.*

$$\text{If } (M_k.Y | k \in K) \text{ has a tree-covering, then it also has a tree-packing.} \tag{6.10}$$

By Corollary 6.3.19, we also obtain the following.

Corollary 6.5.7 (Base packing theorem for the finite cycle matroid)**.** *Any finite family of graphs $(G_k | k \in K)$ with edge set E has a tree-packing if and only if (6.10) is true for every $Y \subseteq E$.*

A similar result was obtained by Aharoni and Ziv [3]. However, their argument is different and they have the additional assumption that the ground set is countable.

Note that the covering conjecture for arbitrary finitary families is still open and equivalent to Open Question 6.3.21.

6.6 Taking stock

We have shown that a great many natural conjectures are equivalent, which we will review in this section. We are indebted to a reviewer for pointing out the importance of the fact that many of the equivalence we have proved specialise to smaller classes than the class of all matroids. We therefore consider the following conjectures, each of which could be made relative to a class $\mathcal{M}$ of matroids.

The Intersection conjecture: Any two matroids in $\mathcal{M}$ on the same ground set have the Intersection property

The pairwise Packing/Covering conjecture: Any pair of matroids from $\mathcal{M}$ on the same ground set has the Packing/Covering property

The Packing/Covering conjecture: Any family of matroids from $\mathcal{M}$ on the same ground set has the Packing/Covering property

The Packing conjecture: A family of matroids $(M_k \in \mathcal{M} | k \in K)$ on the same ground set E has a packing if and only if the following condition is true for every $Y \subseteq E$:

If $(M_k.Y | k \in K)$ has a covering, then it also has a packing.

The Covering conjecture: A family of matroids $(M_k \in \mathcal{M} | k \in K)$ on the same ground set E has a covering if and only if the following condition is true for every $Y \subseteq E$:

If $(M_k \restriction_Y | k \in K)$ has a packing, then it also has a covering.

Most crudely, if $\mathcal{M}$ is a class of matroids containing $U_{1,2}$ and closed under duality, minors and direct sums then all of the above conjectures are equivalent to each other, with proofs exactly as above. However, particular equivalences only depend on weaker conditions on the class $\mathcal{M}$. For the equivalence of the Intersection conjecture with the pairwise Packing/Covering conjecture, both relative to $\mathcal{M}$, we just need that $\mathcal{M}$ is closed under duality. For the equivalence of the pairwise Packing/Covering conjecture with the Packing/Covering conjecture, we just need that $\mathcal{M}$ contains $U_{1,2}$ and is closed under direct sums. This equivalence also holds for classes of matroids of bounded size:

Lemma 6.6.1. *Let $\mathcal{M}_{<\kappa}$ be the family of all matroids on ground sets of cardinality less than κ for some regular*[5] *cardinal κ. Then the pairwise Packing/Covering conjecture for $\mathcal{M}_\kappa$ is equivalent to the Packing/Covering conjecture for $\mathcal{M}_\kappa$.*

Proof (assuming the axiom of choice). It is clear that the pairwise Packing/Covering conjecture follows from the Packing/Covering conjecture. For the converse, suppose the pairwise Packing/Covering conjecture holds, and let $(M_k | k \in K)$ be a set of matroids on the same ground set E of cardinality less than κ. For each $e \in E$, let K_e be the set of $k \in K$ for which $\{e\}$ is independent in M_k. Let $E' = \{e \in E | \#(K_e) < \kappa\}$, and let $K' = \bigcup_{e \in E'} K_e$. Then K' has cardinality less than κ, so by Proposition 6.2.8 the family $(M_k \restriction_{E'} | k \in K')$ has the Packing/Covering property: call the packing side P and the covering side C, and let the packing and the covering be $(I_k | k \in K')$ and $(S_k | k \in K')$.

Let $C' = E \setminus P$, and for any $k \in K \setminus K'$ let $S_k = \emptyset$, which is spanning in $M_k \restriction E'$ by the definition of K'. Using some well-ordering of $E \setminus E'$, we can choose recursively for each $e \in E \setminus E'$ an element $k(e)$ of K_e such that all of the $k(e)$ are distinct. For each $k \in K \setminus K'$, we now set $I_k = \{e \in E \setminus E' | k(e) = k\}$, which is either empty or has size 1 and is independent in M_k. Then the S_k form a packing of P and the I_k form a covering of C', so $(M_k | k \in K)$ has the Packing/Covering property. $\square$

For the equivalence of the Packing/Covering conjecture with the Covering conjecture, both relative to $\mathcal{M}$, we just need that $\mathcal{M}$ is closed under contraction. For the equivalence of the Packing/Covering conjecture with the Packing conjecture, both relative to $\mathcal{M}$, we just need that $\mathcal{M}$ is closed under deletion. To see this, it is not enough to use the argument in the proof of Proposition 6.5.4, for that argument goes via the pairwise Packing/Covering conjecture. Instead, an argument dual to that for the Covering conjecture must be used, relying on

[5] Recall that an infinite cardinal κ is *regular* if and only if no set of cardinality κ can be expressed as a union of fewer than κ sets, all of cardinality less than κ.

the existence of maximal cowaves, where a cowave is a pair $(C, (I_k | k \in K))$ with the I_k forming a covering of $(M_k.C | k \in K)$. The existence of maximal cowaves can be demonstrated by an argument dual to that for Corollary 6.3.4.

6.7 Sketch of the proof of Theorem 6.0.7

The rest of this chapter will consist of a proof of Theorem 6.0.7. Since the proof is long and technical, let's first of all step back and look at a sketch of how the proof will go.

As we have seen, it is enough to prove for pairs (M, N) of matroids as in Theorem 6.0.7 that for every edge e of the ground set there is either a set P containing e such that $(M\restriction_P, N\restriction_P)$ has a packing (we call such a P a wave) or a set Q containing e such that $(M.Q, N.Q)$ has a covering (we call such a Q a co-wave).

We therefore look at an example of how such a wave P can interact with a common 2-separation of M and N: Assume $M = M_1 \oplus_2 M_2$ and $N = N_1 \oplus_2 N_2$ and $E(M_1) = E(N_1)$ and $E(M_2) = E(N_2)$.[6] We assume that $e \in E(M_1)$ and call the gluing edge f.

Now suppose that in $(M_1 \setminus f, N_1/f)$ there is a wave P_1 containing e with spanning sets S^M and S^N, and in (M_2, N_2) there is a wave P_2 avoiding f with spanning sets T^M and T^N such that f is in the N_2-span of T^N. We can stick together these two waves to give a wave $P = P_1 \cup P_2$ in (M, N) with spanning sets $S^M \cup T^M$ and $S^N \cup T^N$. We imagine the wave P_1 as relying on a promise from P_2 that it will N-span the edge f. This is one of the 6 ways, classified in Section 6.9, in which a wave in (M, N) can be built from waves in the two smaller pairs.

Our result relies on the determinacy of certain games. The first is called the *Packing game*, and is played between two players, called Packer and Coverina: we think of Packer as trying to build a wave and Coverina as trying to stop him. At any point in the game, Packer has a partially built wave, together with a collection of promises on which this 'partial wave' relies. Coverina is allowed to challenge one of these promises, at which point Packer must show that it can be fulfilled by giving a partial wave fulfilling it, which relies on further promises, which Coverina may in turn challenge, etc.

The game is designed to have the property that there is a wave containing e if and only if Packer has a winning strategy in this game. Similarly to the Packing game, we define a Covering game, where Coverina is trying to build a suitable co-wave and Packer is trying to prevent her from doing this. These two games will be determined because Ψ_1 and Ψ_2 are Borel. Thus, it suffices to show that we cannot have both a winning strategy for Packer in the Covering game and a winning strategy for Coverina in the Packing game.

We show that if there were such strategies then it would be possible to in some sense play them off against each other, recursively producing infinite plays in each strategy one of which must be losing. Since the strategies were supposed

[6] Here $\oplus_2$ denotes the 2-sum.

to be winning, this gives our desired contradiction. In the recursive construction we can work locally within particular pairs of finite matroids. However, as often happens, the result about finite matroids which we need to apply is not quite the specialisation of our result to finite matroids. Instead we need a strengthening of the Packing/Covering theorem for finite matroids, explained in Lemma 6.12.1 and Lemma 6.12.5.

In order to produce the impossible plays mentioned in the argument above, we will need the following standard consequence of the axiom of choice.

Lemma 6.7.1 (König's Infinity Lemma [34])**.** *Let $V_0, V_1, \ldots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex in V_n with $n \geq 1$ has a neighbour in V_{n-1}. Then G includes a ray $v_0 v_1 \ldots$ with $v_n \in V_n$ for all n.*

We will use the following notation:

For any vertex t of a rooted tree T other than the root, t^- is the unique neighbour of t which is closer to the root. Whenever, we have a rooted tree T, we will consider the edges to be directed towards the root. The terminal vertex of an edge e is denoted by $t(e)$, and the initial vertex by $s(e)$. For a set X of edges of such a tree, let $T_V(X)$ denote the set of terminal vertices, and $S_V(X)$ the set of starting vertices of edges in X. For a set F of edges, let $V(F)$ be the set of vertices incident with edges in F. For a vertex set Z, we denote by $E(Z)$ the set consisting of those edges with both endvertices in Z.

By π_1 and π_2 we denote the two coordinate-projections for ordered pairs.

A *strategy for the first player* in a game $\mathcal{G}$ is a set σ of finite odd-length plays P such that the following is true for all $P \in \sigma$: Let m be a move of the second player such that Pm is a legal play. Then there is a unique move m' of the first player such that $Pmm' \in \sigma$. Furthermore, we require that σ is closed under *2-truncation*, that is, for every $P \in \sigma$ there are some $P' \in \sigma$ and moves m and m' of the second player and the first player, respectively, such that $P'mm' = P$.

An infinite play *belongs* to a strategy σ for the first player if all its odd length finite initial plays are in σ. A strategy for the first player is *winning* if the first player wins in all infinite plays belonging to σ. Similarly, one defines *strategies* and *winning strategies* for the second player.

6.8 Tooling up

We will need the following lemmas for the main argument.

6.8.1 Waves and cowaves

We know that if (X, S^M, S^N) and (Y, T^M, T^N) are waves for (M, N) then $(X \cup Y, S^M \cup (T^M \setminus X), S^N \cup (T^N \setminus X))$ is a wave. We will denote this wave $X \circ Y$.

If X is a hindrance focusing on e then so is $X \circ Y$. If e is M-spanned by X and not contained in Y then e is M-spanned by $X \circ Y$.

Corollary 6.8.1. *Let (M, N) be a pair of matroids on the same ground set E. If for any set X and any edge $e \in E \setminus X$ there is either a wave in $(M/X, N/X)$ containing e or a cohindrance in $(M/X, N/X)$ focusing on e then (M, N) satisfies the Packing/Covering Conjecture.*

Proof. Let X be a maximal wave. Then by Lemma 6.3.7 there is no nontrivial wave in $(M/X, N/X)$, so by assumption every edge in $E \setminus X$ is at the focus of some cohindrance. So by the dual of Corollary 6.3.4 there is a cowave for this pair whose underlying set is $E \setminus X$. This cowave, together with X, witnesses that (M, N) satisfies the Packing/Covering Conjecture. $\square$

Lemma 6.8.2. *Let (M, N) be a pair of matroids on a common ground set E, and let $f \in E$. If there is a hindrance (X, S^M, S^N) in $(M/f, N \setminus f)$, then in (M, N) either X is a wave or there is a hindrance $X' \subseteq X$.*

Proof. We may assume that f is not a loop in M, and that S^M and S^N are bases of $(M/f)\restriction_X$ and $(N \setminus f)\restriction_X$, respectively. Thus $S^M + f$ is M-independent. Let z be in the focus of the hindrance. Let $X' \subseteq X$ be the set of edges y for which there is some $(S^M + f, S^N)$-chain from z to y. First we consider the case that $f \notin X'$. Then $(X', S^M \cap X', S^N \cap X')$ is a hindrance focusing on z, which is the second outcome of the lemma.

Thus we may assume that there is an $(S^M + f, S^N)$-chain from z to f. Applying Lemma 6.1.1, we get sets $J_M \in \mathcal{I}(M)$ and $J_N \in \mathcal{I}(N)$ such that $S^M \cup S^N + z = J_M \cup J_N$. Moreover, J_M and J_N span X in M and N, respectively. Hence we get the first outcome: (X, J_M, J_N) is a wave. $\square$

Lemma 6.8.3. *Let (M, N) be a pair of matroids on the common ground set E, and let $e \in E$. If there is a hindrance (X, S^M, S^N) for (M, N), then in (M, N) either there is a hindrance focusing on e or there is a hindrance that does not contain e.*

Proof. If e is not in $S^M \cup S^N$, then we are done. So we assume without loss of generality that $e \in S^M$. Then $X - e$ is a hindrance for $(M/e, N \setminus e)$. By Lemma 6.8.2, in (M, N) either $X - e$ is a wave or there is a hindrance $X' \subseteq X - e$. As we are done in the later case, it suffices to show that e is M-spanned and N-spanned by $X - e$. As $e \in S^M$, it is N-spanned by $S^N \subseteq X - e$. If e is not M-spanned by $X - e$, then $(X - e, S^M - e, S^N)$ is a hindrance avoiding e, in which case we are also done. $\square$

Lemma 6.8.4. *Let (M, N) be a pair of finite matroids on a common ground set E, and let $e \in E$. Then either there is a cohindrance focusing on e or e is contained in a wave.*

Proof. We assume that e is not contained in any wave. Let X be a maximal wave. Let Y be a maximal cowave for $(M/(X + e), N/(X + e))$. Since Packing/Covering holds for finite matroids [15], we can apply it to the pair $(M/X \setminus Y, N/X \setminus Y)$. By Lemma 6.3.7, $E \setminus (X \cup Y)$ does not include a wave, so is a cowave. It cannot be a wave, so it is a cohindrance. By the dual of

Lemma 6.3.7, any cohindrance for $(M/X \setminus Y, N/X \setminus Y)$ contains e. So by the dual of Lemma 6.8.3, we get a cohindrance focusing on e for $(M/X\setminus Y, N/X\setminus Y)$, which gives rise to a cohindrance focusing on e for (M,N) by the dual of Lemma 6.3.7. □

Lemma 6.8.5. *Let M and N be two matroids on a common finite ground set E. Let $e, f \in E$ distinct. Assume that every nonempty wave for (M,N) contains e. Then in $(M/f, N \setminus f)$ either $E - f$ is a cowave or there is a hindrance focusing on e.*

Proof. We assume that $E - f$ is not a cowave for $(M/f, N \setminus f)$. Then by Corollary 6.3.4, there is an edge g not in any cowave for $(M/f, N \setminus f)$. By the dual of Lemma 6.8.4, we get that there is a hindrance (X, S^M, S^N) for $(M/f, N \setminus f)$ focusing on g. Now we apply Lemma 6.8.3 to (X, S^M, S^N) and the edge e. To show that there is a hindrance focusing on e, it suffices to show that there cannot be a hindrance $(X', {S'}^M, {S'}^N)$ with $e \notin X'$. If there were, then by Lemma 6.8.2 we would get that X' is a wave for (M,N) or that there is a hindrance $X'' \subseteq X'$ for (M,N). Both of these contradict the assumption that every wave contains e. Thus there is a hindrance focusing on e, which completes the proof. □

6.8.2 Trees of matroids

Lemma 6.8.6. *Let T be a rooted tree with root t_0, let $\mathcal{T} = (T, \overline{M})$ be a tree of matroids of overlap 1 and let Ψ be a Borel set of ends of T. Let X be any subset of $E(M_\Psi(\mathcal{T}))$, and let U be the set of nodes t of T such that $e(t^-t)$ is spanned by $X \cap E(\mathcal{T}_{t^- \to t})$ in $M_\Psi(\mathcal{T}_{t^- \to t})$. Then there is a choice of a Ψ-precircuit $(S_t, \overline{o}_t)$ in $\mathcal{T}_{t^- \to t}$ for each $t \in U$ witnessing this in the sense that $e(t^-t) \in \underline{(S_t, \overline{o}_t)} \subseteq X + e(t^-t)$ and such that for any nodes u, v and w with $w \in S_u \cap S_v$ we have $\overline{o}_u(w) = \overline{o}_v(w)$.*

Proof. We denote the tree-order in T by $\leq_T$.

We construct the pre-circuits $(S_t, \overline{o}_t)$ recursively in the height of t in T, so for each n we choose all $(S_t, \overline{o}_t)$ with t at height n before choosing those with greater heights. When choosing $(S_u, \overline{o}_u)$, we first of all check whether there is some $t <_T u$ with $u \in S_t$. If so, we pick t minimal with this property and let $S_u = S_t \cap T_{u^- \to u}$ and $\overline{o}_u(v) = \overline{o}_t(v)$ for each $v \in S_u$. Otherwise, we pick any $(S_u, \overline{o}_u)$ such that $e(u^-u) \in \underline{(S_u, \overline{o}_u)} \subseteq X + e(u^-u)$: there is some such pre-circuit since $u \in U$.

The only thing to check is that for any nodes u, v and w with $w \in S_u \cap S_v$ we have $\overline{o}_u(w) = \overline{o}_v(w)$. So suppose we have such u, v and w. By construction, $u \leq_T w$ and $v \leq_T w$, so without loss of generality $u \leq_T v$. Since $u \leq_T v \leq_T w$ and both u and w are in S_u, we must also have $v \in S_u$. Let $t \leq_T u$ be minimal with $u \in S_t$. Then by construction, since $v \in S_u$ we also have $v \in S_t$. Further, t is minimal with $v \in S_t$ since if there were $t' <_T t$ with $v \in S_{t'}$ we would also have $u \in S_{t'}$ (since $t' \leq_T u \leq_T v$), contradicting our choice of t. Thus $\overline{o}_u(w) = \overline{o}_t(w) = \overline{o}_v(w)$, as required. □

Our main result will be the following:

Theorem 6.8.7. *Let $(T,\overline{M})$ and $(T,\overline{N})$ be trees of matroids of overlap 1 such that for any $t \in V(T)$, the matroids $\overline{M}(t)$ and $\overline{N}(t)$ have the same finite ground set $E(t)$. Let Ψ_M and Ψ_N be Borel sets of ends of T. Then the pair $(M_{\Psi_M}(T,\overline{M}), M_{\Psi_N}(T,\overline{N}))$ of matroids satisfies the Packing/Covering Conjecture.*

We will go via the following special case of this theorem:

Proposition 6.8.8. *Theorem 6.8.7 holds in the case that Ψ_M and Ψ_N partition the set of ends of T.*

In Section 6.10, we will prove that Theorem 6.8.7 follows from Proposition 6.8.8. However, the heart of our proof is the proof of Proposition 6.8.8, which is the content of Section 6.11 and Section 6.12.

6.9 The Packing game and the Covering game

The purpose of this section is to define the Packing game and the Covering game as discussed in Section 6.7 and prove some basic facts about these games. Throughout this section, we fix a tree T, together with two functions $\overline{M}$ and $\overline{N}$ such that $\mathcal{T}^M = (T,\overline{M})$ and $\mathcal{T}^N = (T,\overline{N})$ are trees of matroids of overlap 1 such that for each $t \in V(T)$ the two matroids $\overline{M}(t)$ and $\overline{N}(t)$ have the same finite ground set $E(t)$. We denote the common underlying set of $(T,\overline{M})$ and $(T,\overline{N})$ by $E(T)$. We also fix some $e \in E(T)$, and $\Psi_M, \Psi_N \subseteq \Omega(T)$. Let t_0 be the unique node of T with $e \in E(t_0)$.

An *arena* consists of matroids M and N on a common finite ground set E, a subset F of E and an element $e \in E \setminus F$. The set F is called the set of *upper edges* and e is called the *lower edge* of the arena.

For $t \in V(T) - t_0$, we shall later on consider the arena

$$A(t) = (M(t), N(t), E(t), F_t, e(tt^-)),$$

where $F_t = e``(X_t)$ and X_t is the set of edges incident with t and not equal to tt^-. For $t = t_0$, we take the same definition of $A(t)$ and X_{t_0} except that we take the lower edge to be e.

The *promise set* $\mathcal{P}$ is $\{\perp, M_-, M_+, N_-, N_+, \top\}$. Members of $\mathcal{P}$ are called *promises*. Given an arena $(M, N, E, \emptyset, e)$, a wave (W, S^M, S^N) in (M, N) *fulfils* a promise P if one of the following is true:

1. $P = \perp$;
2. $P = M_+$ and e is M-spanned by W;
3. $P = M_-$ and $e \in S^N$;
4. $P = N_+$ and e is N-spanned by W;

5. $P = N_-$ and $e \in S^M$;

6. $P = \top$ and e is both M-spanned and N-spanned by W;

Note that 6 just means that $P = \top$ and $W + e$ is a hindrance focusing on e.

For $P, Q \in \mathcal{P}$, we say that $P \geq_{\mathcal{P}} Q$ if and only if the following is true: Whenever there is a wave that fulfils P, there is also a wave that fulfils Q.

For example, if there is a wave (W, S^M, S^N) without e that M-spans e, then $(W + e, S^M, S^N + e)$ is a wave with e on the N-side. So $M_+ \geq_{\mathcal{P}} M_-$. Clearly, $\geq_{\mathcal{P}}$ defines a partial order on $\mathcal{P}$.

$$\begin{tikzcd}[column sep=small, row sep=small]
& \top \arrow[dl, phantom, "\geqslant"] \arrow[dr, phantom, "\geqslant"] & \\
M_+ \arrow[d, phantom, "\geqslant"] & & N_+ \arrow[d, phantom, "\geqslant"] \\
M_- \arrow[dr, phantom, "\geqslant"] & & N_- \arrow[dl, phantom, "\geqslant"] \\
& \bot &
\end{tikzcd}$$

Figure 6.3: The partial order $\leq_{\mathcal{P}}$.

Lemma 6.9.1. *The partial order $\geq_{\mathcal{P}}$ is the one generated from the relations $\top \geq_{\mathcal{P}} M_+, \top \geq_{\mathcal{P}} N_+, M_+ \geq_{\mathcal{P}} M_-, N_+ \geq_{\mathcal{P}} N_-, M_- \geq_{\mathcal{P}} \bot, N_- \geq_{\mathcal{P}} \bot$, see Figure 6.3.*

Proof. It is clear that all these relations hold in $\geq_{\mathcal{P}}$. The arenas $(U_{0,1}, U_{1,1}, \{e\}, \emptyset, e)$ and $(U_{1,1}, U_{0,1}, \{e\}, \emptyset, e)$ show that any $P \in \{M_-, M_+\}$ is incomparable with any $Q \in \{N_-, N_+\}$. [7] These arenas also show that $\top$ is strictly larger than M_+ and M_-, and that $\bot$ is strictly smaller than M_- and N_-.

The arena $(U_{1,2}, U_{1,2}, \{e, f\}, \emptyset, e)$ shows that M_+ is strictly larger than M_- and also that N_+ is strictly larger than N_-. This shows that $\geq_{\mathcal{P}}$ is generated from the relations in the lemma. $\square$

We let $\mathcal{P}^* = \{\bot^*, M_-^*, M_+^*, N_-^*, N_+^*, \top^*\}$ be the set of dual promises. A cowave (W, S^M, S^N) fulfils P^* if one of 1-6 above is true with the word 'spans' replaced by 'cospans'. We let $P^* \leq_{\mathcal{P}^*} Q^*$ if and only if $P \leq_{\mathcal{P}} Q$.

Definition 6.9.2. Let (M, N, E, F, e) be an arena and $\varphi \colon F \to \mathcal{P}$ a function. Let $M' = (M/(\varphi^{-1}\{\top, M_+\}) \setminus (\varphi^{-1}\{\bot, N_+\})$ and $N' = N/(\varphi^{-1}\{\top, N_+\}) \setminus (\varphi^{-1}\{\bot, M_+\})$. Then a *wave relying on* φ is a wave (W, S^M, S^N) for the pair of matroids (M', N') such that $S^M \cap \varphi^{-1}(N_+) = \emptyset$ and $S^N \cap \varphi^{-1}(M_+) = \emptyset$.

Moreover, a wave relying on φ *fulfils* a promise P in the arena (M, N, E, F, e) if it fulfils P in the arena $(M', N', E \setminus \varphi^{-1}\{\bot, N_+, M_+, \top\}, \emptyset, e)$.

We will now explain a construction by means of which a wave for the pair $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$ can be broken down into local waves in the arenas $A(t)$ at vertices t of T, each relying on promises fulfilled by waves higher in the tree.

[7] As usual, we denote by $U_{m,n}$ the uniform matroid of rank m on a set of size n.

Construction 6.9.3. Let $W = (X, S^M, S^N)$ be a wave for $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$ fulfilling some promise P at e. For each $t \in V(T)$, we shall construct a promise $P(t)$. This will induce for each t a function $\varphi_t \colon F_t \to \mathcal{P}$ sending $e(st)$ to $P(s)$. We will also construct for each t a wave $W(t)$ relying on φ_t and fulfilling $P(t)$ in the arena $A(t)$.

First we define $P(t)$. We let $P(t_0) = P$. If $t \neq t_0$, the construction is as follows. We abbreviate $E_t = E(\mathcal{T}_{t^- \to t})$. Very roughly, we take for $P(t)$ the strongest promise fulfilled by the wave $(X \cap E_t, S^M \cap E_t, S^N \cap E_t)$, possibly modified by adding $e(tt^-)$ to one of the sides of the wave. More precisely, If $Z(t) = (X \cap E_t + e(tt^-), S^M \cap E_t, S^N \cap E_t)$ is a hindrance focusing on $e(tt^-)$, we let $P(t) = \top$. Otherwise if $Z(t) = (X \cap E_t, S^M \cap E_t, S^N \cap E_t)$ is a wave such that $S^M \cap E_t$ spans $e(tt^-)$ in $M(\mathcal{T}^M_{t^- \to t})$, we let $P(t) = M_+$. Otherwise if $Z(t) = (X \cap E_t + e(tt^-), S^M \cap E_t, S^N \cap E_t + e(tt^-))$ is a wave, we let $P(t) = M_-$. The cases in which we take $P(t) = N_+$ or $P(t) = N_-$ are like the cases where we take $P(t) = M_+$ or $P(t) = M_-$ but with the roles of the matroids M and N reversed. In all other cases we take $P(t) = \bot$ and $Z(t) = \emptyset$.

Finally, we define $W(t)$. Let $Z(t) = (Y(t), S^M(t), S^N(t))$ be as defined above. Let $F_t(M_-)$ be the set of those $e(st) \in \varphi_t^{-1}(M_-)$ such that $e(st)$ is N-spanned by $S^N \cap E(\mathcal{T}_{s \to t})$ in $M_{\Psi_N}(\mathcal{T}^N_{s \to t})$, and let $F_t(N_-)$ be given in the same way but with the roles of M and N interchanged. We let $W(t) = (Y(t)', S^M(t)', S^N(t)')$ where $Y(t)' = Y(t) \cap E(t) \cup F_t(M_-) \cup F_t(N_-)$ and $S^M(t)' = S^M(t) \cap Y(t)'$ and $S^N(t)' = S^N(t) \cap Y(t)'$.

It is now straightforward to show that $W(t)$ is a wave relying on φ_t in the arena $A(t)$ fulfilling $P(t)$.

Definition 6.9.4. Let $A = (M, N, E, F, e)$ be an arena and let $P \in \mathcal{P}$. Then a *tactic K attaining P at e* consists of a function $\varphi_K \colon F \to \mathcal{P}$ and a wave (W_K, S^M_K, S^N_K) relying on φ_K and fulfilling P, together with sets C^M_K and C^N_K. If $P \in \{\top, M_+, M_-\}$, then we require that $C^M_K \in \mathcal{C}(M)$ and that $e \in C^M_K \subseteq S^M_K \cup \varphi^{-1}\{\top, M_+, M_-\}$. Similarly, if $P \in \{\top, N_+, N_-\}$, then we require that $C^N_K \in \mathcal{C}(N)$ and that $e \in C^N_K \subseteq S^N_K \cup \varphi^{-1}\{\top, N_+, N_-\}$.

By $\mathcal{K} = \mathcal{K}(A, P)$ we denote the set of all tactics K attaining P at e.

Note that $\varphi^{-1}(M_-) \subseteq S^M_K$, so that we could have left out M_- in the term $S^M_K \cup \varphi^{-1}\{\top, M_+, M_-\}$ above. The same remark is true for N_-. A *cotactic* is defined in the same way as a tactic but with a star in the appropriate places. To simplify notation, we will sometimes call cotactics just tactics.

Note that if W is a wave relying on φ and fulfilling P then we can choose some sets C^M and C^N such that (φ, W, C^M, C^N) is a tactic attaining P.

We now return to Construction 6.9.3, which started from a wave for the pair $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$ and gave us a wave in each of the arenas $A(t)$. We now show how these waves can be augmented to tactics, in a way which encodes more precisely how certain edges $e(st)$ were spanned in $M_{\Psi_M}(\mathcal{T}^M_{t \to s})$ and $M_{\Psi_N}(\mathcal{T}^N_{t \to s})$.

Construction 6.9.5. Let (W, S^M, S^N) be a wave for $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$, fulfilling some promise P at e. For each vertex t of T we will construct a tactic

$K(t) = (\varphi_t, W(t), C^M_{K(t)}, C^N_{K(t)})$ attaining $P(t)$, with φ_t and $W(t)$ constructed as in Construction 6.9.3.

As in Lemma 6.8.6, we can pick Ψ_M-precircuits $(S^M_t, \overline{o}^M_t)$ for each $t \in V(T)$ with $P(t) \in \{\top, M_+, M_-\}$, such that if $w \in S^M_u \cap S^M_v$, then $\overline{o}^M_u(w) = \overline{o}^M_v(w)$. Similarly, we find Ψ_N-precircuits $(S^N_t, \overline{o}^N_t)$ such that if $w \in S^N_u \cap S^N_v$, then $\overline{o}^N_u(w) = \overline{o}^N_v(w)$. If $P(t) \in \{\top, M_+, M_-\}$, we take $C^M_{K(t)} = \overline{o}^M_t(t)$. If $P(t) \in \{\top, N_+, N_-\}$, we take $C^N_{K(t)} = \overline{o}^N_t(t)$. We complete the definition of $K(t)$ by assigning $C^M_{K(t)}$ an arbitrary value if $P(t) \notin \{\top, M_+, M_-\}$, similarly for $C^N_{K(t)}$.

We have seen how to break up any wave for $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$ into tactics at each node. In Construction 6.9.7 we will show how to do the reverse: how to build a wave from a collection of local tactics, as long as they fit well together. By 'fit well together' here, we mean that collectively they form a winning strategy for a particular game, which we call the Packing game.

Definition 6.9.6. Let $P_0 \in \mathcal{P}$. The *Packing game* $\mathcal{G}(P_0) = \mathcal{G}(T, \overline{M}, \Psi_M, \overline{N}, \Psi_N, P_0, e)$ is played between two players, called Packer and Coverina, as follows:

Play alternates between the players, with Packer making the first move. At any point in the game there is a *current node* $t_c \in V(t)$, and a *current edge* $e_c \in E(t_c)$, and a *current promise* $P_c \in \mathcal{P}$. Initially we set $e_c = e$ and $t_c = t_0$ to be the node of T with $e_c \in E(t_c)$ and $P_c = P_0$.

For any n the $(2n-1)^{\text{st}}$ move is made by Packer: he must play a tactic $K_n = (\varphi_{K_n}, W_{K_n}, C^M_{K_n}, C^N_{K_n})$ that attains the promise P_c in the arena $A_n = A(t_c)$.

Then the $2n^{\text{th}}$ move is made by Coverina: she must play an edge $f_n \in \varphi^{-1}_{t_c}(\mathcal{P} - \bot)$. After she does this, the current edge is updated to f_n, the current node to the unique node t_n such that $f_n = e(t_{n-1}t_n)$, and the current challenge is updated to $\varphi_{K_n}(f_n)$.

The current challenge f_n is *M-strong* if $\varphi_{K_n}(f_n)$ is in $\{\top, M_+, M_-\}$ and $f_n \in C^M_{K_n}$. Otherwise f_n is *M-weak*. Similarly, one defines *N-strong* and *N-weak*.

If play continues forever, the winner is computed from the end ω of T containing $(t_n | n \in \mathbb{N})$ and the sequences $(\varphi_{K_n}(f_n) | n \in \mathbb{N})$ and $(f_n | n \in \mathbb{N})$. An end ω *is used by M* if all but finitely many f_n are M-strong. Similarly, ω *is used by N* if all but finitely many $\varphi_{K_n}(f_n)$ are in $\{\top, N_+, N_-\}$ and N-strong.

Packer wins if and only if one of the following is true:

1. $\omega \in \Psi_M \cap \Psi_N$;

2. $\omega \in \Psi_M$ and ω is not used by N;

3. $\omega \in \Psi_N$ and ω is not used by M;

4. ω is used by neither M nor N;

The *Covering game* $\mathcal{G}^*(P_0) = \mathcal{G}^*(T, \overline{M}, \Psi_M, \overline{N}, \Psi_N, P_0, e)$ is the game like the dual Packing game $\mathcal{G}(T, \overline{M}^*, \Psi_M{}^{\complement}, \overline{N}^*, \Psi_N{}^{\complement}, P_0^*, e)$, but with the roles of Packer and Coverina reversed. We will also use a different notation for the

Covering game, putting stars on the notation for the Packing game. Thus for example the current edge is denoted e_c^*, and Coverina's $(2n-1)^{\text{st}}$ move is a tactic K_n^*, and in the $2n^{\text{th}}$ move Packer plays some $f_n^* \in \{f^* \in e``F^* \mid \varphi_{K_n^*}(f^*) \neq \bot^*\}$, and the current challenge f_n^* is M^*-*strong* if $\varphi_{K_n^*}(f_n^*)$ is in $\{\top^*, M_+^*, M_-^*\}$ and $f_n^* \in C_{K_n^*}^{M^*}$.

Given a winning strategy σ for the Packing game, we can recover from it a subtree Z of T together with a tactic at each node of Z. We let Z be the set of nodes that appear as current nodes in some play according to σ. For each node t in Z, there is a unique play $s_t \in \sigma$ in which t is the current node - this play arises when Packer plays according to σ and Coverina challenges on edges on the path from t_0 to t. Let $K(t)$ be the last move of Packer in s_t, and $P(t)$ the promise attained by $K(t)$.

We now show how to build a wave attaining P at e from this collection of tactics.

Construction 6.9.7. Let τ be a winning strategy for Packer in the Packing game. By modifying the tactics K played by Packer according to τ, we can build a winning strategy σ with the property that if $\varphi_K(f) \in \{M_-, N_-\}$ then $e(f) \in W_K$. Let Z and the $K(t)$ and $P(t)$ be derived from σ as above. Let $W = (\bigcup_{t\in Z} W_{K(t)}) \cap E$, and $S^M = (\bigcup_{t\in Z} S^M_{K(t)}) \cap E$, and $S^N = (\bigcup_{t\in Z} S^N_{K(t)}) \cap E$.

First we show that (W, S^M, S^N) is a wave. Let $x \in W \setminus S^M$ be arbitrary. Our aim is to find some M_{Ψ_M}-circuit o such that $x \in o \subseteq S^M + x$. For this we need some definitions, which are illustrated in Figure 6.4.

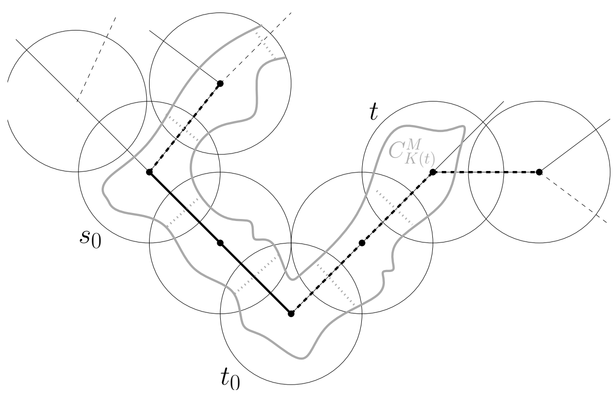


Figure 6.4: The construction of o. Here the highlighted path Q from s_0 to t_0 has length 2 and all its edges are in U_1. The edges in U_2 are drawn dashed. The precircuit $(L, \overline{o})$ is drawn in grey.

Let $s_0 \in Z$ be such that $x \in E(M(s_0))$. Let Q be the unique path from t_0 to s_0. Note that $Q \subseteq E(Z)$.

Let U_1 be the set of those edges tu on Q such that the promise fulfilled by $K(t)$ is N_-. Let $U_2 \subseteq E(Z)$ be the set of those edges tu not on Q such that the promise fulfilled by $K(t)$ is in $\{\top, M_+, M_-\}$.

In order to build o, it suffices to build a Ψ_M-precircuit $(L, \overline{o})$ such that $x \in \underline{(L, \overline{o})} \subseteq S^M + x$. We shall ensure that $L \subseteq T[U_1 \cup U_2]$.

For this we first define for each $t \in T_V(U_1) \cup S_V(U_2) + s_0$ an $M(t)$-circuit $o_t \subseteq S^M_{K(t)} \cup e''(U_1 \cup U_2) + x$. If $t = s_0$, there is such an o_t with the additional property that $x \in o_t$. Next we consider the case that $t \in T_V(U_1)$ so that there is some node t' with $t't \in U_1$. Since $\varphi_{K(t)}(e(t't)) \in \{M_-, N_-\}$, the dummy edge $e(t't)$ is in $W_{K(t)}$, and thus there is such a circuit o_t containing $e(t't)$.

Finally, we consider the case that $t \in S_V(U_2)$ so that there is some node u with $tu \in U_2$. Here we can just take $o_t = C^M_{K(t)}$, which has the additional property that it contains $e(tu)$.

Next we define L. For this we define a sequence $(L_n | n \in \mathbb{N})$ of sets $L_n \subseteq V(T[U_1 \cup U_2])$ with distance n from s_0. We start with $L_0 = \{s_0\}$. Assume that L_n is already constructed. Let L_{n+1} be the set of those nodes w that have distance $n+1$ from s_0 such that there is some $t \in L_n$ with $e(tw) \in o_t$, where we consider tw as an undirected edge. Having defined the L_n, we take L to be the subtree of $T[U_1 \cup U_2]$ with vertex set $\bigcup_{n \in \mathbb{N}} L_n$. Then $(L, t \mapsto o_t)$ is a precircuit.

To see that all ends of L are in Ψ_M, let ω be an end of L and $R \subseteq L$ a ray converging to ω. Let R' be the ray from t_0 which shares a tail with R. Let p be the infinite play according to σ obtained when Packer plays according to σ and Coverina always challenges on edges of R'. Then the challenges are eventually all on edges of U_2, and so are M-strong. Thus we get an infinite play belonging to σ which M-uses ω. As σ is winning, it must be that $\omega \in \Psi_M$. Thus $(L, t \mapsto o_t)$ is a Ψ_M-precircuit, giving rise to a circuit o, which witnesses that S^M M_{Ψ_M}-spans x. Thus S^M M_{Ψ_M}-spans W. Similarly one proves that S^N N_{Ψ_N}-spans W. So (W, S^M, S^N) is a wave.

It remains to show that (W, S^M, S^N) fulfils P at e. If $P \in \{\top, M_-, N_-, \bot\}$, this follows from the fact that $(W_{K(t_0)}, S^M_{K(t_0)}, S^N_{K(t_0)})$ fulfils P at e. If $P = M_+$, then we construct an M_{Ψ_M}-circuit o_e with $e \in o_e \subseteq S^M + e$ in a similar way to that described above. The case $P = N_+$ is similar. Thus (W, S^M, S^N) fulfils P at e, which completes the construction.

Lemma 6.9.8. *Packer has a winning strategy σ in the Packing game $\mathcal{G}(P)$ if and only if there is a wave for $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$ fulfilling P at e.*

Proof. First assume that there is a wave (W, S^M, S^N) for $(M_{\Psi_M}(T, \overline{M}), M_{\Psi_N}(T, \overline{N}))$ fulfilling P at e. Then Packer has the following winning strategy: at the node v he plays the tactic $K(v)$ defined in Construction 6.9.5. If Coverina challenges at some dummy edge f, then the new challenge is $\varphi_{K(v)}(f) = P_f$. It is straightforward to check that this is a winning strategy.

Conversely, if Packer has a winning strategy σ then Construction 6.9.7 gives us a wave fulfilling P at e. □

By duality, we get the following:

Lemma 6.9.9. *Coverina has a winning strategy σ^* in the Covering game $\mathcal{G}^*(P^*)$ if and only if there is a cowave for $(M_{\Psi_M}(T,\overline{M}), M_{\Psi_N}(T,\overline{N})$ fulfilling P^* at e.* $\square$

Lemma 6.9.10. *If Ψ_M and Ψ_N are Borel, then the Packing game is determined.*

Proof. Let $\mathcal{X}$ be the set of infinite plays in the Packing game. We endow $\Omega(T)$ with the topology inherited from the Freudenthal compactification of T. For each infinite play P, the moves of the second player form a ray of T. Let ω_P be the end this ray belongs to. Then the function f mapping P to ω_P is continuous. Thus both $f^{-1}(\Psi_M)$ and $f^{-1}(\Psi_N)$ are Borel sets.

By $S_{\{\top,M_+,M_-\}}$ we denote the set of those infinite plays whose challenges are eventually in $\{\top, M_+, M_-\}$ and M-strong. The set $S_{\{\top,M_+,M_-\}}$ is a countable union of closed sets and thus Borel. Similarly, by $S_{\{\top,N_+,N_-\}}$ we denote the Borel set of those infinite plays whose challenges are eventually in $\{\top, N_+, N_-\}$ and N-strong.

Now we are in a position to write the set W of infinite plays in which Packer wins as a Borel set:

$$W = [f^{-1}(\Psi_M) \cap f^{-1}(\Psi_N)] \cup [f^{-1}(\Psi_M) \setminus S_{\{\top,N_+,N_-\}}] \cup [f^{-1}(\Psi_N) \setminus S_{\{\top,M_+,M_-\}}] \cup$$
$$[S_{\{\top,M_+,M_-\}} \cup S_{\{\top,N_+,N_-\}}]^{\complement}$$

$\square$

6.10 Blocking sets

The purpose of this section is to prove that Proposition 6.8.8 implies Theorem 6.8.7. First, we turn our attention to play in an arena without upper edges: we analyse which collections of promises (or co-promises) can be fulfilled by waves (or cowaves) in such an arena. For any arena $A = (M, N, E, \emptyset, e)$, we let $\mathcal{A}(A)$ be the set of promises or co-promises fulfillable in A.

Lemma 6.10.1. *There are precisely 5 possible values for $\mathcal{A}(A)$, as follows:*

1. $\mathcal{P} + \perp^*$
2. $\mathcal{P}^* + \perp$
3. $\{\perp, M_-, M_+, \perp^*, N^*_-, N^*_+\}$
4. $\{\perp, N_-, N_+, \perp^*, M^*_-, M^*_+\}$
5. $\{\perp, M_-, N_-, \perp^*, M^*_-, N^*_-\}$.

Proof. First we show that all 5 values are possible. We can get all but the last value from arenas with $E = \{e\}$: for the first value we take both M and N to be $U_{0,1}$, for the second we take both to be $U_{1,1}$, and for the third and fourth we take one to be $U_{0,1}$ and the other $U_{1,1}$. For the final value, we may take M and N to both be $U_{1,2}$.

Next we show that no other value is possible. We begin by showing that $\mathcal{A}(A)$ cannot contain both M_+ and M^*_-. Suppose for a contradiction that it did, and let (W, S^M, S^N) be a wave fulfilling M_+ and (X, T^M, T^N) a co-wave fulfilling M^*_-. By removing edges outside $W + e$ and/or contracting edges outside X if necessary, we may assume without loss of generality that $W + e = X = E$. Thus $|E| \geq |S^M| + |S^N| + 1 \geq r(M) + r(N \backslash e) + 1$. But also since T^N is co-spanning in N, $T^N - e$ is co-spanning in $N \backslash e$, and so $|E| \geq |T^M| + |T^N - e| + 1 \geq r(M^*) + r((N \backslash e)^*) + 1$, so $2|E| \geq r(M) + r(M^*) + r(N \backslash e) + r((N \backslash e)^*) + 2 = 2|E| + 1$, which is the desired contradiction.

We continue by showing that at least one of M_+ and M^*_- must be contained in $\mathcal{A}(A)$. We begin by taking a maximal wave (W, S^M, S^N) for $(M \setminus e, N \setminus e)$. If $e \in \mathrm{Cl}_M(W)$ then W fulfills M_+. Otherwise, by contracting W if necessary, we may assume that every nonempty wave contains e. Now we apply the Packing/Covering Theorem for finite matroids to $(M/e, N \backslash e)$, obtaining a partition $E - e = P \dot\cup Q$ with a packing of P and a covering of Q. Then the packing of P isn't a hindrance since if it were then by Lemma 6.8.2 there would be a nontrivial wave for the pair (M, N) not containing e. So it is also a covering, so that there is a cowave $(E - e, T^M, T^N)$. Now if e is an M-loop then the empty wave fulfills M_+ and if not then e is in the M^*-span of T^M and so $(E, T^M, T^N + e)$ witnesses M^*_-.

So far we have shown that $\mathcal{A}(A)$ must contain precisely one of M_+ and M^*_-. Similarly, it must contain precisely one element of each of the sets $\{M_-, M^*_+\}$, $\{N_+, N^*_-\}$ or $\{N_-, N^*_+\}$. So if it is not given by the fifth option above, it must contain one of M_+, N_+, M^*_+ and N^*_+: without loss of generality let us say it contains M_+. If it also contains N^*_+ then, since it must be down-closed by the definition of $\leq_{\mathcal{P}}$, it can only be the third option above. But if not then it must contain N_-. Now let W be a wave fulfilling M_+ and let X be a wave fulfilling N_-. Then $W \circ X$ is a wave fulfilling $\top$ and so, by down-closure again and the fact that $\perp^*$ is witnessed by the empty cowave, $\mathcal{A}(A)$ must be the first option above. $\square$

Remark 6.10.2. *The only place where the finiteness of E was used in this argument was in the application of the Packing/Covering Theorem to minors of (M, N). So we get the same result without assuming finiteness of E, on the assumption that all minors of (M, N) satisfy the Packing/Covering conjecture.*

Definition 6.10.3. A *challenger* to a promise P in an arena $A = (M, N, E, F, e)$ is a function γ assigning an element of F to each tactic K in $\mathcal{K}(A, P)$. For any tactic K we denote by $\overline{\gamma}(K)$ the promise $\varphi_K(\gamma(K))$. For any $f \in F$, we denote by $\overline{\gamma}[f]$ the up-closure of the set $\{\overline{\gamma}(K) | K \in \mathcal{K}(A, P)$ and $\gamma(K) = f\}$.

Challengers are important in the analysis of winning strategies in the Packing and Covering games. Let σ be a winning strategy for Coverina in the Packing game $\mathcal{G}(P)$, and let $s \in \sigma$ be a finite play of length $2n$. Let $\overline{s}$ be P if s has length 0 and $\varphi_{s_{2n-1}}(s_{2n})$ otherwise (so after the play s we have $P_c = \overline{s}$). Since

σ is winning, we may define a challenger γ_s^σ to $\overline{s}$ in $A_{n+1} = A(t_c)$ by sending each tactic K attaining $\overline{s}$ in A_{n+1} to the unique $f \in e``F$ such that $s \cdot K \cdot f \in \sigma$. We omit the superscript σ when it is clear from the context which strategy we are working with.

Definition 6.10.4. A subset of $\mathcal{P} \cup \mathcal{P}^*$ is *blocking* if it meets all the possible values of $\mathcal{A}(A)$ listed in Lemma 6.10.1.

That is, a set of promises and co-promises is blocking if for any arena with $F = \emptyset$ there exists a promise in the blocking set attainable in this arena.

Lemma 6.10.5. *Let $A = (M, N, E, F, e)$ be an arena and ρ a function assigning to each $f \in F$ a subset of $\mathcal{P} \cup \mathcal{P}^*$. Let F' be the set of $f \in F$ at which $\rho(f)$ is blocking. Then there is an arena $A' = (M', N', E', F', e)$ such that for each $P \in \mathcal{P} \cup \mathcal{P}^*$ and tactic K' attaining P at e in A' there is a tactic K attaining P at e in A for which the function φ_K extends $\varphi_{K'}$ and $C_K^M \cap F' = C_{K'}^M \cap F'$ and $C_K^N \cap F' = C_{K'}^N \cap F'$ and for each $f \in F \setminus F'$ we have $\varphi_K(f) \notin \rho(f)$.*

Remark 6.10.6. *As a consequence, for any promise P and any challenger γ to P in A such that $\gamma[f] \subseteq \rho(f)$ for each $f \in F$, there is a challenger γ' to P in A' such that for each tactic K attaining P at e in A' there is a tactic K attaining P at e in A for which $\gamma_P(K) = \gamma'_P(K')$, the function φ_K extends $\varphi_{K'}$, $C_K^M \cap \backslash F' = C_{K'}^M \cap F'$ and $C_K^N \cap F' = C_{K'}^N \cap F'$.*

Proof of Lemma 6.10.5. For each $f \in F \setminus F'$, choose one of the 5 sets from Lemma 6.10.1 which $\rho(f)$ fails to meet, and let F_i be the set of those $f \in F \setminus F'$ for which the i^{th} element of the list was chosen. Let $E' = E \backslash (F_1 \cup F_2 \cup F_3 \cup F_4)$, $M' = M/(F_1 \cup F_3) \backslash (F_2 \cup F_4)$ and $N' = N/(F_1 \cup F_4) \backslash (F_2 \cup F_3)$. As in the statement, let $A' = (M', N', E', F', e)$.

Let K' be a tactic attaining some promise P at e in A'. We define the corresponding tactic K in A as follows: let $W_K = W'_{K'}$, and let φ_{K_n} be obtained as the extension of φ'_{K_n} to F taking the value $\top$ on F_1, $\bot$ on F_2, M_+ on F_3, N_+ on F_4, M_- on $F_5 \cap S^M$ and N_- on $F_5 \setminus S^M$. Let C_K^M be an extension of $C_{K'}^M$ whose new elements all come from $F_1 \cup F_3$ and C_K^N be an extension of $C_{K'}^N$ whose new elements all come from $F_1 \cup F_4$. $\square$

Lemma 6.10.7. *Let B be a blocking set and $A = (M, N, E, F, e)$ an arena. For each $P \in B$, let γ_P be a challenger to P in A. Then there is some $f \in F$ such that $\bigcup_{P \in B} \overline{\gamma}_P[f]$ is blocking.*

Proof. Suppose for a contradiction that there is no such f. Then we apply Lemma 6.10.5 with $\rho \colon f \mapsto \bigcup_{P \in B} \overline{\gamma}_P[f]$ and get an arena with no upper edges in which none of the promises in B can be attained by any tactic, contradicting the fact that B is blocking. $\square$

This useful property makes it worth looking at blocking sets in detail, and we will now pause to analyse their structure more carefully. However, we shall only consider up-closed blocking sets: note that a set is blocking if and only if its up-closure is and by the definitions of challenger and of $\leq_{\mathcal{P}}$, if we have a challenger to every element of some blocking set then we also get a challenger to every element of its up-closure.

Lemma 6.10.8. *An up-closed set is blocking if and only if it includes one of the following sets as a subset:* $\{\bot\}$, $\{\bot^*\}$, $\{M_+, M_-^*\}$, $\{M_-, M_+^*\}$, $\{N_+, N_-^*\}$, $\{N_-, N_+^*\}$, $\{M_+, N_-, \top^*\}$, $\{M_-, N_+, \top^*\}$, $\{M_+^*, N_-^*, \top\}$ *or* $\{M_-^*, N_+^*, \top\}$.

Proof. Each of the listed sets is clearly blocking. Conversely, let B be an up-closed blocking set. Since it meets $\{\bot, M_-, N_-, \bot^*, M_-^*, N_-^*\}$ and is up-closed it must contain one of M_-, N_-, M_-^* or N_-^*: by symmetry we may assume without loss of generality that it contains M_-. Since it meets $\mathcal{P}^* \cup \bot$ it must contain one of $\top^*$ and $\bot$ and since it meets $\{\bot, N_-, N_+, \bot^*, M_-^*, M_+^*\}$ it must contain one of N_+ and M_+^*. Now if it contains $\bot$ then it includes $\{\bot\}$, if it contains M_+^* then it includes $\{M_-, M_+^*\}$, and if it contains neither then it contains both of $\top^*$ and N_+ and so includes $\{M_-, N_+, \top^*\}$. $\square$

We are now ready to prove the main result of this section:

Proof that Proposition 6.8.8 implies Theorem 6.8.7. Suppose that we have two trees of matroids as in the statement of Theorem 6.8.7. By Corollary 6.8.1, it suffices to prove that every edge either lies in some wave or else is the focus of some cohindrance. So let e be some edge. We now consider the Packing and Covering games $\mathcal{G}(M_-)$, $\mathcal{G}(N_-)$ and $\mathcal{G}^*(\top^*)$, taking our notation as in Definition 6.9.6. If Packer has a winning strategy in either of $\mathcal{G}(M_-)$ or $\mathcal{G}(N_-)$ or Coverina has a winning strategy in $\mathcal{G}^*(\top^*)$ then we are done by Lemma 6.9.8 or Lemma 6.9.9. So we suppose for a contradiction that there are no such strategies.

By the determinacy of these games (Lemma 6.9.10) we get winning strategies σ_{M_-} and σ_{N_-} for Coverina in $\mathcal{G}(M_-)$ and $\mathcal{G}(N_-)$ and a winning strategy $\sigma_{\top^*}$ for Packer in $\mathcal{G}^*(\top^*)$. Let σ be the union of these three strategies. For any finite play s, let $l(s)$ be the last move of s. Note that if $s \in \sigma$ then $l(s)$ is always of the form $e(f)$ for some $f \in E(T)$. For any edge tt' of T, let $\sigma[tt']$ be $\{\overline{s} | s \in \sigma \text{ and } l(s) = e(tt')\}$. Let U be the set of edges tt' of T at which $\sigma[tt']$ is blocking, and let T' be the subtree of T on the vertices which can be joined to t_0 via a path all of whose edges are in U.

We now define two trees of matroids on T', to which we will apply Proposition 6.8.8 to obtain the desired contradiction. For each $u \in T'$, we apply Lemma 6.10.5 to the arena $A(u)$ and the function $\rho\colon e(tu) \mapsto \sigma[tu]$ to get a new arena $A'(u) = (\overline{M}'(u), \overline{N}'(u), E'(u), F'_u, e(uu^-)$, where we choose the underlying sets $E'(u)$in such a way that all the sets $E'(u) \setminus (F_u + e(uu^-))$ are disjoint and contain no dummy edges. Then $(T', \overline{M}')$ and $(T', \overline{N}')$ are trees of matroids.

Now we consider the Packing game $\mathcal{G}'(M_-) = \mathcal{G}(T', \overline{M}', \Psi_M \cap \Omega(T'), \overline{N}', \Psi_N \cap \Omega(T'), M_-, e)$. We will use a slightly different notation for this game than for $\mathcal{G}(M_-)$, putting dashes on the notation used in $\mathcal{G}(M_-)$. Thus, for example, the current promise at any point is denoted by P'_c. We can convert σ_{M_-} into a winning strategy for Coverina in $\mathcal{G}'(M_-)$ as follows: Coverina should imagine an auxiliary play in the game $\mathcal{G}(M_-)$, in which she plays according to σ_{M_-}, and for which she should ensure that at any point the current node, edge and promise agree with those in $\mathcal{G}'(M_-)$. When Packer plays a tactic K'_n, Coverina should choose a tactic K_n attaining $P_c = P'_c$ in A_c as in Lemma 6.10.5 and she should let her response f'_n be the move f_n prescribed by σ_{M_-} in response to K_n.

By Lemma 6.9.8, the existence of this winning strategy entails that there is no wave fulfilling M_- in $(M_{\Psi_M \cap \Omega(T')}(T', \overline{M}'), (M_{\Psi_N \cap \Omega(T')}(T', \overline{N}'))$. Similarly, there is no wave fulfilling N_- and no cowave fulfilling $\top^*$ for this pair. By Remark 6.10.2, since the set $\{M_-, N_-, \top^*\}$ is blocking, there is some minor of this pair for which Packing/Covering fails to hold. Thus in order to obtain the desired contradiction by applying Proposition 6.8.8 to this minor, we just need to show that every end of T' is in $\Psi_M \triangle \Psi_N$.

Let $\omega = (t_n | n \in \mathbb{N})$ be an end of T'. For each n the set $\sigma[t_n t_{n+1}]$ is blocking and does not contain $\bot^*$, so must meet $\mathcal{P}$. Thus there must be some play $s \in \sigma_{M_-} \cup \sigma_{N_-}$ with $l(s) = e(t_n t_{n+1})$. Since there are only finitely many such plays for each n, we obtain by Lemma 6.7.1 that there is some infinite play $\hat{s}$ according to one of σ_{M_-} or σ_{N_-} with $\hat{s}_{2n} = e(t_n t_{n+1})$ for each n. But then since these strategies are winning for Coverina, it follows that ω must be in at most one of Ψ_M and Ψ_N. A similar argument shows that it is also in at least one of Ψ_M and Ψ_N, so that it is in $\Psi_M \triangle \Psi_N$ as required.

□

6.11 Main result

As we have just shown, in order to prove our main result it remains to prove the special case given in Proposition 6.8.8.

Throughout this section we fix two trees of matroids as in the statement of Proposition 6.8.8. Our aim is to show that the pair $(M_{\Psi_M}(T, M), M_{\Psi_N}(T, N))$ satisfies matroid intersection. We shall suppose that it does not, and in the remainder of this section we will derive a contradiction from that supposition. However, it will become clear during the course of the proof that we must rely on two technical lemmas, whose proofs we defer to the next section.

By Corollary 6.8.1, we may assume that there is some edge e of $E(T)$ which is not in any wave or cowave for our pair of matroids. We now consider the Packing and Covering games $\mathcal{G}(M_-)$ and $\mathcal{G}^*(M_+^*)$, taking our notation as in Definition 6.9.6. By Lemma 6.9.8 Packer does not have a winning strategy in $\mathcal{G}(M_-)$, and by Lemma 6.9.9 Coverina does not have a winning strategy in $\mathcal{G}^*(M_+^*)$. So by the determinacy of these games, there are winning strategies σ_{M_-} for Coverina in $\mathcal{G}(M_-)$ and $\sigma_{M_+^*}$ for Packer in $\mathcal{G}^*(M_+^*)$. Let σ be the union of these strategies.

Let t_0 be the unique vertex of T with $e \in E(t_0)$. In order to get a contradiction, we shall recursively construct two infinite plays s_{M_-} and $s_{M_+^*}$ in $\mathcal{G}(M_-)$ and $\mathcal{G}^*(M_+^*)$ respectively. We shall construct s_{M_-} and $s_{M_+^*}$ such that they are both played along the same ray $(t_i | i \in \mathbb{N})$ from t_0 and such that either Packer wins s_{M_-} or Coverina wins $s_{M_+^*}$.

More explicitly, let us say that a finite or infinite play s is (M, i)-weak if there is some $j \geq i$ such that the challenge s_{2j} is defined and M-weak. We define (N, i)-weak, (M^*, i)-weak and (N^*, i)-weak similarly. We shall recursively build a ray $(t_i | i \in \mathbb{N})$ from t_0 in T and sequences $(B_i | i \in \mathbb{N})$ of blocking sets and $(\lambda_i \colon B_i \to \sigma | i \in \mathbb{N})$ of functions, with the following properties:

1. $B_0 = \{M_-, M^*_+\}$ and λ_0 sends both elements to trivial plays.
2. Each of the sets B_i is one of the blocking sets listed in Lemma 6.10.8.
3. For any $P \in B_i$ with $i > 0$ the play $\lambda_i(P)$ is a play in σ with last move $e(t_{i-1}t_i)$ and $\overline{\lambda_i(P)} \leq P$.
4. For any $P, Q \in B_i$ with $\overline{\lambda_i(P)} = \overline{\lambda_i(Q)}$ we have $\lambda_i(P) = \lambda_i(Q)$.
5. For any $i > 0$ and any $P \in B_i$ there is some $P' \in B_{i-1}$ such that $\lambda_i(P)$ is an extension of $\lambda_{i-1}(P')$.
6. For any i there is some $j \geq i$ such that one of the following is true:
   - For each $P \in B_j \cap \mathcal{P}$ the play $\lambda_j(P)$ is (N, i)-weak.
   - For each $P \in B_j \cap \mathcal{P}^*$ the play $\lambda_j(P)$ is (M^*, i)-weak.
7. For any i there is some $j \geq i$ such that one of the following is true:
   - For each $P \in B_j \cap \mathcal{P}$ the play $\lambda_j(P)$ is (M, i)-weak.
   - For each $P \in B_j \cap \mathcal{P}^*$ the play $\lambda_j(P)$ is (N^*, i)-weak.

It is possible to recursively build a sequence satisfying 1-5 by Lemma 6.10.7. To get the additional conditions, we will need to make use of the results of Section 6.12. But before we do this, we will explain why the existence of such a sequence would result in a contradiction. As each end of T is in precisely one of Ψ_M and Ψ_N, we may without loss of generality suppose that the end $(t_i | i \in \mathbb{N})$ of T is in $\Psi_N \setminus \Psi_M$. Since each B_i is finite, by Lemma 6.7.1, we can find an infinite play s_{M_-} such that for each $i \in \mathbb{N}$ the restriction $s_{M_-} \restriction_{2i}$ is in both σ_{M_-} and the image of λ_i. Thus s_{M_-} is an infinite play according to σ_{M_-}. Since this is a winning strategy for Coverina, there must be some i_{M_-} such that $(s_{M_-})_{2j}$ is never an N-weak challenge for $j \geq i_{M_-}$. Similarly, we can build an infinite play $s_{M^*_+}$ such that for each $i \in \mathbb{N}$ the restriction $s_{M^*_+} \restriction_{2i}$ is in both $\sigma_{M^*_+}$ and the image of λ_i, and there is some $i_{M^*_+}$ such that $(s_{M^*_+})_{2i}$ is never an M^*-weak challenge for $j \geq i_{M^*_+}$. Now let i be whichever of i_{M_-} and $i_{M^*_+}$ is larger, and apply condition 6 above. If the first option holds, then s_{M_-} is (N, i)-weak, contrary to the construction of i. But if the second option holds then $s_{M^*_+}$ is (M^*, i)-weak, which is again a contradiction.

So to complete our proof it remains to show how we can ensure that the sequence we recursively construct satisfies the 6th and 7th conditions above. In order to do this, it is enough to show how, given choices of t_k, B_k and λ_k for $k \leq i$ satisfying 1-5 we can extend these finite sequences to longer finite sequences $(t_k | k \leq j)$, $(B_k | k \leq j)$ and $(\lambda_k | k \leq j)$ for some $j \geq i$ such that one of the following is true:

- For each $P \in B_j \cap \mathcal{P}$ the play $\lambda_j(P)$ is (N, i)-weak.
- For each $P \in B_j \cap \mathcal{P}^*$ the play $\lambda_j(P)$ is (M^*, i)-weak.

For if we can do this, then we can use a symmetrical construction to further extend our sequences to $(B_k | k \leq j')$ and $(\lambda_k | k \leq j')$ for some $j' \geq j$ such that one of the following is true:

- For each $P \in B_j \cap \mathcal{P}$ the play $\lambda_j(P)$ is (M, i)-weak.
- For each $P \in B_j \cap \mathcal{P}^*$ the play $\lambda_j(P)$ is (N^*, i)-weak.

Repeatedly carrying out this pair of constructions and, if they don't make the sequences longer, extending them using Lemma 6.10.7, we will obtain infinite sequences satisfying all the conditions above.

So suppose that we are given choices of t_k, B_k and λ_k for $k \leq i$ and that we wish to extend these sequences to satisfy condition 6 at i. The way we do this depends on the value of B_i. We cannot, by the construction of the Packing and Covering games, have $B_i = \{\bot\}$ or $B_i = \{\bot^*\}$. If $B_i = \{M_+, M_-^*\}$, then we are done, since the play $\lambda_i(M_+)$ is necessarily N-weak. The cases where B_i is one of $\{M_-, M_+^*\}$, $\{N_+, N_-^*\}$ or $\{N_-, N_+^*\}$ are dealt with similarly.

The next case, $B = \{M_-, N_+, \top^*\}$, is a little trickier. Here we may be forced to extend the sequence. The object we need in order to do this is encoded in the following definition:

Definition 6.11.1. Let $A = (M, N, E, F, e)$, and γ_{M_-}, γ_{N_+} and $\gamma_{\top^*}$ be challengers to the respective promises. A *tactician*$^+$ for a blocking set B at an edge $f \in F$ in A is a function μ sending each P in B to a pair (Q, K), where $Q \in \{M_-, N_+, \top^*\}$ and K is a tactic attaining Q in A and $\varphi_K(f) \leq P$ and $\gamma_Q(K) = f$.

Note that in the context of Definition 6.11.1, F cannot be empty since $\{M_-, N_+, \top^*\}$ is blocking.

We are interested in the case where $\gamma_{M_-} = \gamma^{\sigma_{M_-}}_{\lambda_i(M_-)}$ (the challenger determined by the strategy σ_{M_-} after the finite play $\lambda_i(M_-)$), $\gamma_{N_+} = \gamma^{\sigma_{M_-}}_{\lambda_i(N_+)}$ and $\gamma_{\top^*} = \gamma^{\sigma_{M_+^*}}_{\lambda_i(\top^*)}$ In this context, given such f, B and μ, we can extend our sequences as follows: we choose t_{i+1} with $f = e(t_i t_{i+1})$, we choose B_{i+1} to be B, and for each $P \in B_{i+1}$ we take $\lambda_{i+1}(P)$ to be the play consisting of $\lambda_i(\pi_1(\mu(P)))$ followed by the tactic $\pi_2(\mu(P))$ and then the edge f. We must be able to find some extension like this by the following lemma:

Lemma 6.11.2. *For each $f \in F$ and blocking set B included in $\overline{\gamma}_{M_-}[f] \cup \overline{\gamma}_{N_+}[f] \cup \overline{\gamma}_{\top^*}[f]$, there is a tactician$^+$ μ_B for B at f.*

Proof. For each $P \in B$, the promise P is in $\overline{\gamma}_Q[f]$ for some $Q \in \{M_-, N_+, \top^*\}$. Then there is a tactic K fulfilling Q at e such that $\varphi_K(f) = P$ and $\gamma_{M_-}(K) = f$. We let $\mu_B(P) = (Q, K)$. $\square$

In order to ensure that our extension is helpful, we use the following lemma, to be proved in the next section:

Lemma 6.11.3. *Let $A = (M, N, E, F, e)$ be an arena, γ_{M_-}, γ_{N_+} and let $\gamma_{\top^*}$ be challengers as in Definition 6.11.1. Then there are a blocking set B, an edge $f \in F$ and a tactician$^+$ μ for B at f in A such that one of the following holds:*

(i) Double Extension case: $B = \{M_-, N_+, \top^*\}$ and $\pi_1(\mu(P)) = P$ for each $P \in B$;

(ii) Weak Challenge case in the Packing game: For any tactic K with (N_+, K) in the image of μ, the edge f is an N-weak challenge to K;

(iii) Weak Challenge case in the Covering game: For any tactic K with $(\top^*, K)$ in the image of μ, the edge f is an M^*-weak challenge to K.

The Weak Challenge cases are self-explanatory: for example, if we have the Weak Challenge case in the Packing game then this ensures that for each $P \in B_{i+1} \cap \mathcal{P}$ the play $\lambda_{i+1}(P)$ is (N, i)-weak. The Double Extension case is more subtle. In this case, we find ourselves in the same situation we were in before, with $B_{i+1} = \{M_-, N_+, \top^*\}$, but we don't seem to have made any progress. However, we can apply the Lemma again repeatedly to get a contradiction as follows:

Suppose for a contradiction that there is no finite $j \geq i$ for which there are extensions $(B_k | k \leq j)$ and $(\lambda_k | k \leq j)$ of our sequences which satisfy condition 6 at i and j. Then we recursively build sequences $(B_j | j > i)$, $(\lambda_j | j > i)$ and $(t_j | j > i)$, where for each $j > i$ we choose t_j, B_j and λ_j as above using Lemma 6.11.3. Since as we have noted by our supposition we never have either challenge case, we get that $B_j = \{M_-, N_+, \top^*\}$, $\lambda_j(M_-)$ extends $\lambda_{j-1}(M_-)$ and $\lambda_j(N_+)$ extends $\lambda_{j-1}(N_+)$ for each $j > i$. So there are two infinite plays u_{M_-} and u_{N_+} according to σ_{M_-} such that, for each $j \geq i$, each u_P extends $\lambda_j(P)$. Let ω be the end $(t_k | k \in \mathbb{N})$. As σ_{M_-} is winning and in u_{M_-} all challenges are eventually N-weak, we must have $\omega \notin \Psi_M$. Similarly $\omega \notin \Psi_N$, which is the desired contradiction.

Thus there is some finite $j \geq i$ for which there are extensions $(B_k | k \leq j)$ and $(\lambda_k | k \leq j)$ of our sequences which satisfy condition 6 at i and j, as required. This completes our treatment of the case $B = \{M_-, N_+, \top^*\}$. The case $B = \{M_+^*, N_-^*, \top\}$ is similar, using the dual of Lemma 6.11.3.

The case $B = \{M_+, N_-, \top^*\}$, is very similar, but there is an additional complexity. Once more we may be forced to extend the sequences (t_k), (B_k) and (λ_k). This time the object we need in order to do this is encoded in the following definition:

Definition 6.11.4. Let $A = (M, N, E, F, e)$, and γ_{M_+}, γ_{N_-} and $\gamma_{\top^*}$ be challengers to the respective promises. A *tactician*$^-$ for a blocking set B at an edge $f \in F$ in A is a function μ sending each P in B to a pair (Q, K), where $Q \in \{M_+, N_-, \top^*\}$ and K is a tactic attaining Q in A and $\varphi_K(f) \leq P$ and $\gamma_Q(K) = f$.

Note that in the context of Definition 6.11.4, F cannot be empty since $\{M_+, N_-, \top^*\}$ is blocking.

We are interested in the case where $\gamma_{M_+} = \gamma^{\sigma_{M_-}}_{\lambda_i(M_+)}$, $\gamma_{N_-} = \gamma^{\sigma_{M_-}}_{\lambda_i(N_-)}$ and $\gamma_{\top^*} = \gamma^{\sigma_{M^*_+}}_{\lambda_i(\top^*)}$. In this context, given such f, B and μ, we can extend our sequences as follows: we choose t_{i+1} with $f = e(t_i t_{i+1})$, we choose B_{i+1} to be B, and for each $P \in B_{i+1}$ we take $\lambda_{i+1}(P)$ to be the play consisting of $\lambda_i(\pi_1(\mu(P)))$ followed by the tactic $\pi_2(\mu(P))$ and then the edge f. We must be able to find some extension like this by the following lemma, which can be proved similarly to Lemma 6.11.2:

Lemma 6.11.5. *For each $f \in F$ and blocking set B included in $\overline{\gamma}_{M_+}[f] \cup \overline{\gamma}_{N_-}[f] \cup \overline{\gamma}_{\top^*}[f]$, there is a tactician$^-$ μ_B for B at f.* □

In order to ensure that our extension is helpful, we will once more rely on a technical lemma, to be proved in the next section:

Lemma 6.11.6. *Let $A = (M, N, E, F, e)$ be an arena, γ_{M_+}, γ_{N_-} and let $\gamma_{\top^*}$ be challengers as in Definition 6.11.4. Then there are a blocking set B, an edge $f \in F$ and a tactician$^-$ μ for B at f in A such that one of the following holds:*

(i) Double Extension case: $B = \{M_+, N_-, \top^*\}$ and $\pi_1(\mu(P)) = P$ for each $P \in B$;

(ii) Weak Challenge case in the Packing game: For any tactic K with (N_-, K) in the image of μ, the edge f is an N-weak challenge to K;

(iii) Weak Challenge case in the Covering game: For any tactic K with $(\top^*, K)$ in the image of μ, the edge f is an M^*-weak challenge to K;

(iv) Improvement case 1: $B = \{M_-, N_+, \top^*\}$;

(v) Improvement case 2: $B = \{N^*_-, M^*_+, \top\}$.

The Weak Challenge cases are once more self-explanatory, and the Double Extension case can be dealt with as before. But Improvement cases 1 and 2 reduce the situation to one in which the current blocking set is $\{M_-, N_+, \top^*\}$ or $\{M^*_+, N^*_-, \top\}$, and both of these situations have been dealt with above.

This completes our treatment of the case $B = \{M_+, N_-, \top^*\}$. The case $B = \{M^*_-, N^*_+, \top\}$ is similar, using the dual of Lemma 6.11.3. We have now dealt with all cases which can arise, and this completes the proof of Proposition 6.8.8 and hence of Theorem 6.8.7.

6.12 Proof of Lemmas 6.11.3 and 6.11.6

6.12.1 Proof of Lemma 6.11.3

The aim of this subsection is to prove Lemma 6.11.3. First we need some intermediate lemmas. We start with a lemma on waves in finite pairs of matroids.

Lemma 6.12.1. *Let M and N be two matroids on the same finite ground set E. Let $G, H, J \subseteq E$ disjoint and $e \in E \setminus (G \cup H \cup J)$. Then one of the following is true.*

1. *There is a wave with e on the N-side in $(M/(H \cup J), N/(H \cup J))$.*
2. *There is a wave N-spanning e in $(M \setminus (G \cup J), N \setminus G/J)$.*
3. *There is some $G' \subseteq G$ and a cohindrance (Y, T^M, T^N) focusing on e in $(M \setminus (G' \cup J), N \setminus G'/J)$ such that there is some M-cocircuit b with $e \in b \subseteq (T^M + e) \setminus H$.*

If $G = H = J = \emptyset$, then this lemma just says that $\{M_-, N_+, \top^*\}$ is blocking. So this lemma can be seen as an extension of this fact.

Proof. We assume that we do not have outcome 1 or 2 and aim to show that then we get outcome 3. Thus as $\{M_-, N_+, \top^*\}$ is blocking by Lemma 6.10.8, in the pair $(M', N') = (M \setminus (G \cup J), N \setminus G/J)$ it must be that the promise $\top^*$ is attainable: There is a cohindrance (X, S^M, S^N) focusing on e.

Now we tweak this cohindrance a little to get outcome 3. As $\{M_-, M_+^*\}$ is blocking by Lemma 6.10.8 and we do not have outcome 1, in the pair $(M/(H \cup J), N/(H \cup J))$ there is a cowave (Y, T^M, T^N) that M-cospans e. In particular, there is an M-cocircuit b with $e \in b \subseteq (T^M + e)$. So b avoids $H \cup J \cup G'$, where $G' = G \setminus Y$. Then $(X \setminus Y, S^M \setminus Y, S^N \setminus Y)$ is a cohindrance focusing on e in the pair $(M' \setminus Y, N' \setminus Y)$. By the dual of Lemma 6.3.3 $(X \cup Y, (S^M \setminus Y) \cup T^M, (S^N \setminus Y) \cup T^N)$ is a cohindrance in $(M \setminus (G' \cup J), N \setminus G'/J)$, and together with b it witnesses that we have outcome 3. □

Lemma 6.12.1 is the main principle we use in the proof of Lemma 6.11.3. The work of bridging from Lemma 6.12.1 to Lemma 6.11.3 is done in the following lemma.

Lemma 6.12.2. *Let $\overline{\gamma}_{M_-}, \overline{\gamma}_{N_+} \subseteq \mathcal{P} - \perp$ and $\overline{\gamma}_{\top^*} \subseteq \mathcal{P}^* - \perp^*$ be up-closed such that $\overline{\gamma} = \overline{\gamma}_{M_-} \cup \overline{\gamma}_{N_+} \cup \overline{\gamma}_{\top^*}$ is blocking. Then one of the following is true.*

1. *One of the 4 sets $\{M_+, M_-^*\}$, $\{M_-, M_+^*\}$, $\{N_+, N_-^*\}$ or $\{N_-, N_+^*\}$ is a subset of $\overline{\gamma}$;*
2. *$M_- \in \overline{\gamma}_{M_-}$ and $\{N_+, \top^*\} \subseteq \overline{\gamma}$;*
3. *$N_- \in \overline{\gamma}_{M_-}$ and $\{M_+, \top^*\} \subseteq \overline{\gamma}$;*
4. *$\top \in \overline{\gamma}_{M_-}$ and one of $\{M_-^*, N_+^*\} \subseteq \overline{\gamma}_{\top^*}$ or $\{M_+^*, N_-^*\} \subseteq \overline{\gamma}_{\top^*}$;*
5. *$\overline{\gamma}_{M_-} \subseteq \{M_+, N_+, \top\}$ and $\overline{\gamma}_{\top^*} \subseteq \{M_+^*, N_+^*, \top^*\}$ and one of $\{M_-, N_+\} \subseteq \overline{\gamma}$ or $\{M_+, N_-\} \subseteq \overline{\gamma}$;*
6. *$\overline{\gamma}_{M_-} = \emptyset$ and $N_-^* \notin \overline{\gamma}_{\top^*}$ and $\{M_-^*, N_+^*, \top\} \subseteq \overline{\gamma}$ and $\overline{\gamma}_{N_+} \subseteq \{N_+, \top\}$;*
7. *$\overline{\gamma}_{M_-} = \emptyset$ and $\{M_+^*, N_-^*, \top\} \subseteq \overline{\gamma}$ and $\overline{\gamma}_{N_+} \subseteq \{M_+, \top\}$;*

Proof. Since $\overline{\gamma} \cap (\mathcal{P} + \bot^*)$ is nonempty and $\bot^* \notin \overline{\gamma}$, we get that $\overline{\gamma} \cap \mathcal{P}$ is nonempty, thus $\top \in \overline{\gamma}$. Similarly, $\top^* \in \overline{\gamma}$. Now suppose for a contradiction that we do not have one of the outcomes 1-7. By Lemma 6.10.8, one of the 4 sets $\{M_+, N_-, \top^*\}$, $\{M_+^*, N_-^*, \top\}$, $\{M_-, N_+, \top^*\}$ or $\{M_-^*, N_+^*, \top\}$ is a subset of $\overline{\gamma}$.

Case 1: $\{M_+, N_-, \top^*\} \subseteq \overline{\gamma}$ or $\{M_-, N_+, \top^*\} \subseteq \overline{\gamma}$. Then M_+ and N_+ are in $\overline{\gamma}$, so M_-^* and N_-^* are not as we do not have outcome 1. Also, $M_- \notin \overline{\gamma}_{M_-}$ as we do not have outcome 2, and $N_- \notin \overline{\gamma}_{M_-}$ as we do not have outcome 3. Thus we have outcome 5, which is the desired contradiction.

Case 2: $\{M_+^*, N_-^*, \top\} \subseteq \overline{\gamma}$. Then M_- and N_+ cannot be in $\overline{\gamma}_{N_+}$ as we do not have outcome 1. Also, $\overline{\gamma}_{M_-}$ must be empty as we do not have outcome 4. Thus we have outcome 7, which is the desired contradiction.

Case 3: $\{M_-^*, N_+^*, \top\} \subseteq \overline{\gamma}$ but $\{M_+^*, N_-^*, \top\} \not\subseteq \overline{\gamma}$. Then M_+ and N_- cannot be in $\overline{\gamma}_{N_+}$ as we do not have outcome 1. By assumption, $N_-^* \notin \overline{\gamma}_{\top^*}$. Also, $\overline{\gamma}_{M_-}$ must be empty as we do not have outcome 4. Thus we have outcome 6, which is the desired contradiction. □

Now we are in a position to prove Lemma 6.11.3.

Proof of Lemma 6.11.3. Suppose for a contradiction that there are an arena $A = (M, N, E, F, e)$ and challengers γ_{M_-}, γ_{N_+} and $\gamma_{\top^*}$ for which Lemma 6.11.3 is false. We pick these such that the set F of upper edges is of minimal size. Although we will not need it, it is worth noting that F is nonempty since $\{M_-, N_+, \top^*\}$ is blocking. We abbreviate $\overline{\gamma}[f] = \overline{\gamma}_{M_-}[f] \cup \overline{\gamma}_{N_+}[f] \cup \overline{\gamma}_{\top^*}[f]$.

Sublemma 6.12.3. *For each $f \in F$ the set $\overline{\gamma}[f]$ is blocking.*

Proof. This is immediate by the minimality of $|F|$ and Lemma 6.10.5 and Remark 6.10.6.

□

Sublemma 6.12.4. *For each $f \in F$ one of the following three conditions from Lemma 6.12.2 is true.*

5. $\overline{\gamma}_{M_-}[f] \subseteq \{M_+, N_+, \top\}$ *and* $\overline{\gamma}_{\top^*}[f] \subseteq \{M_+^*, N_+^*, \top^*\}$ *and one of* $\{M_-, N_+\} \subseteq \overline{\gamma}[f]$ *or* $\{M_+, N_-\} \subseteq \overline{\gamma}[f]$*;*

6. $\overline{\gamma}_{M_-}[f] = \emptyset$ *and* $N_-^* \notin \overline{\gamma}_{\top^*}[f]$ *and* $\{M_-^*, N_+^*, \top\} \subseteq \overline{\gamma}[f]$ *and* $\overline{\gamma}_{N_+}[f] \subseteq \{N_+, \top\}$*;*

7. $\overline{\gamma}_{M_-}[f] = \emptyset$ *and* $\{M_+^*, N_-^*, \top\} \subseteq \overline{\gamma}[f]$ *and* $\overline{\gamma}_{N_+}[f] \subseteq \{M_+, \top\}$*;*

Proof. By Sublemma 6.12.3 and Lemma 6.12.2 the sets $\overline{\gamma}_{M_-}[f]$, $\overline{\gamma}_{N_+}[f]$ and $\overline{\gamma}_{\top^*}[f]$ fulfil one of the outcomes of Lemma 6.12.2. If they satisfy 5,6 or 7 we are done. Otherwise they satisfy one of the conditions 1-4 of Lemma 6.12.2.

Case 1: $\overline{\gamma}_{M_-}[f]$, $\overline{\gamma}_{N_+}[f]$ and $\overline{\gamma}_{\top^*}[f]$ satisfy 1: Let B be one of $\{M_+, M_-^*\}$, $\{M_-, M_+^*\}$, $\{N_+, N_-^*\}$ or $\{N_-, N_+^*\}$ such that $B \subseteq \overline{\gamma}[f]$. Then we pick a tactician$^+$ μ_B as in Lemma 6.11.2. If $B = \{M_+, M_-^*\}$ or $B = \{M_-, M_+^*\}$, we get the Weak Challenge case in the Packing game. If $B = \{N_-, N_+^*\}$ or $B = \{N_+, N_-^*\}$, we get the Weak Challenge case in the Covering game. Thus we get a contradiction in this case.

Case 2: $\overline{\gamma}_{M_-}[f]$, $\overline{\gamma}_{N_+}[f]$ and $\overline{\gamma}_{\top^*}[f]$ satisfy 2: Then $M_- \in \overline{\gamma}_{M_-}[f]$ and $\{N_+, \top^*\} \subseteq \overline{\gamma}[f]$. We let $B = \{M_-, N_+, \top^*\}$. If $N_+ \in \overline{\gamma}_{N_+}[f]$, then we can define some μ as in the Double Extension case. Otherwise, $N_+ \in \overline{\gamma}_{M_-}[f]$, so that without loss of generality, the μ_B from Lemma 6.11.2 is such that $\pi_1\mu_B(N_+) = M_-$. Thus $\mu_B^{-1}(N_+, K) = M_-$ for every tactic K where this is defined. So we get the Weak Challenge case in the Packing game. Thus we get a contradiction in this case.

Case 3: $\overline{\gamma}_{M_-}[f]$, $\overline{\gamma}_{N_+}[f]$ and $\overline{\gamma}_{\top^*}[f]$ satisfy 3: Then $B = \{M_+, N_-, \top^*\} \subseteq \overline{\gamma}[f]$. Furthermore, there is some tactic K_1 fulfilling M_- at e such that $\varphi_{K_1}(f) = N_-$, and some tactic K_2 fulfilling M_- or N_+ at e with $\varphi_{K_2}(f) = M_+$. So that without loss of generality the μ_B from Lemma 6.11.2 is such that $\mu_B(N_-) = (M_-, K_1)$ and one of $\mu_B(M_+) = (M_-, K_2)$ or $\mu_B(M_+) = (N_+, K_2)$. In particular, for every tactic K the set $\mu_B^{-1}(N_+, K)$ is either empty or is the singleton $\{M_+\}$. So we get the Weak Challenge case in the Packing game. Thus we get a contradiction in this case.

Case 4: $\overline{\gamma}_{M_-}[f]$, $\overline{\gamma}_{N_+}[f]$ and $\overline{\gamma}_{\top^*}[f]$ satisfy 4: Then there is some tactic K fulfilling M_- at e with $\varphi_K(f) = \top$, and either $\{M_-^*, N_+^*\} \subseteq \overline{\gamma}_{\top^*}[f]$ or $\{M_+^*, N_-^*\} \subseteq \overline{\gamma}_{\top^*}[f]$. If $\{M_-^*, N_+^*\} \subseteq \overline{\gamma}_{\top^*}[f]$, then we let $B = \{M_-^*, N_+^*, \top\}$. So that without loss of generality the μ_B from Lemma 6.11.2 is such that $\mu_B(\top) = (M_-, K)$. So we get the Weak Challenge case in the Packing game. The case $\{M_+^*, N_-^*\} \subseteq \overline{\gamma}_{\top^*}[f]$ is similar. Thus we get a contradiction in this case. □

Sublemma 6.12.4 motivates the following definition: Let $G \subseteq F$ be the set of those $f \in F$ that satisfy 5 and let $H \subseteq F$ be the set of those $f \in F \setminus G$ that satisfy 6. Finally let $J = F \setminus G \setminus J$. Note that any $f \in J$ satisfies 7 by Sublemma 6.12.4. Now we apply Lemma 6.12.1 to G, H and J. According to which outcome we get, we now split into cases.

Case 1: We get outcome 1 of Lemma 6.12.1: There is a wave with e on the N-side in $(M/(H \cup J), N/(H \cup J))$. This wave gives rise to a tactic K fulfilling M_- at e such that:

$$\varphi_K(f) = \begin{cases} \top \text{ if } f \in H \cup J \\ M_- \text{ or } N_- \text{ or } \bot \text{ if } f \in G \end{cases}$$

As γ_{M_-} is a challenger, there is some $f \in F$ such that $\gamma_{M_-}(K) = f$. As $\overline{\gamma}_{M_-}[x] = \emptyset$ for each $x \in H \cup J$, f cannot be in $H \cup J$ and it cannot be in

G either since $\overline{\gamma}_{M_-}[x] \subseteq \{M_+, N_+, \top\}$ for each $x \in G$. This is the desired contradiction.

Case 2: We get outcome 2 of Lemma 6.12.1: There is a wave N-spanning e in $(M \setminus (G \cup J), N \setminus G/J)$. This wave gives rise to a tactic K fulfilling N_+ at e such that:

$$\varphi_K(f) = \begin{cases} \perp \text{ if } f \in G \\ M_- \text{ or } N_- \text{ or } \perp \text{ if } f \in H \\ N_+ \text{ if } f \in J \end{cases}$$

As γ_{N_+} is a challenger, there is some $f \in F$ such that $\gamma_{N_+}(K) = f$. Note that $f \notin G$. As $\overline{\gamma}_{N_+}[x] \subseteq \{N_+, \top\}$ for each $x \in H$, f cannot be in H and it cannot be in J either since $\overline{\gamma}_{N_+}[x] \subseteq \{M_+, \top\}$ for each $x \in J$. This is the desired contradiction.

Case 3: We get outcome 3 of Lemma 6.12.1: There is some $G' \subseteq G$ and a cohindrance (Y, T^M, T^N) focusing on e in $(M \setminus (G' \cup J), N \setminus G'/J)$ such that there is some M-cocircuit b with $e \in b \subseteq (T^M + e) \setminus H$. This cohindrance gives rise to a tactic K fulfilling $\top^*$ at e with $C_K^{M^*} = b$ such that:

$$\varphi_K(f) = \begin{cases} \top^* \text{ if } f \in G' \\ M_-^* \text{ or } N_-^* \text{ or } \perp^* \text{ if } f \in H \cup (G \setminus G') \\ M_+^* \text{ if } f \in J \end{cases}$$

Let $f = \gamma_{\top^*}(K)$. If $f \in G$, then it is in G' because $\overline{\gamma}_{\top^*}[x] \subseteq \{M_+^*, N_+^*, \top^*\}$ for each $x \in G$. Then we let $B = \{M_-, N_+, \top^*\}$ (if $\{M_-, N_+\} \subseteq \overline{\gamma}[f]$) or $B = \{M_+, N_-, \top^*\}$ (if $\{M_+, N_-\} \subseteq \overline{\gamma}[f]$). We pick μ_B such that $\mu_B(\top^*) = (\top^*, K)$. Thus we get the Weak Challenge case in the Covering game as $C_K^{M^*}$ does not meet G'.

If $f \in H$, then $\varphi_K(f) = M_-^*$ as $N_-^* \notin \overline{\gamma}_{\top^*}[f]$ and $\gamma_{\top^*}$ is a challenger. Thus we let $B = \{M_-^*, N_+^*, \top\}$ and we pick μ_B such that $\mu_B(M_-^*) = (\top^*, K)$. Thus we get the Weak Challenge case in the Covering game.

If $f \in J$, we let $B = \{M_+^*, N_-^*, \top\}$ and we pick μ_B such that $\mu_B(M_+^*) = (\top^*, K)$. Thus we get the Weak Challenge case in the Covering game. This completes the proof. $\square$

6.12.2 Proof of Lemma 6.11.6

The aim of this subsection is to prove Lemma 6.11.6. The structure of the proof will be as in the last subsection.

Lemma 6.12.5. *Let M and N be two matroids on the same finite ground set E. Let $H, J \subseteq E$ disjoint and $e \in E \setminus H \setminus J$. Then one of the following is true.*

1. *There is some $H' \subseteq H$ such that there is a wave M-spanning e in $(M/H'/J, N \setminus H'/J)$.*

2. *There is some $J' \subseteq J$ and a wave (X, S^M, S^N) with e on the M-side in $(M/J', N/J')$ such that there is an N-circuit o with $e \in o \subseteq (S^N + e) \setminus H$.*
3. *There is some $H' \subseteq H$ and a cohindrance (Y, T^M, T^N) focusing on e in $(M/H', N \setminus H')$ such that there is some M-cocircuit b with $e \in b \subseteq (T^M + e) \setminus J$.*

If $H = J = \emptyset$, then this lemma just says that $\{M_+, N_-, \top^*\}$ is blocking. So this lemma can be seen as an extension of this fact.

Proof. We prove this lemma by induction on the size of E.

Case 1: $H = \emptyset$.

Subcase 1.1: There is a nonempty wave (Z, U^M, U^N) in (M, N) avoiding e. Now we apply the induction hypothesis to $(M/Z, N/Z)$ and $\emptyset$ and $J \setminus Z$. If we have outcome 3, we immediately get outcome 3 in (M, N). If we get a wave as in outcome 1 or 2, we stick (Z, U^M, U^N) onto that wave by Lemma 6.3.3 and get outcome 1 or 2, respectively, in (M, N).

To complete Case 1, it remains to consider the following subcase.

Subcase 1.2: Every nonempty wave in (M, N) contains e. If $J = \emptyset$, we can just use the fact that $\{M_+, N_-, \top^*\}$ is blocking by Lemma 6.10.8. So let $j \in J$. Now we apply the induction hypothesis to $(M/j, N \setminus j)$. If we get outcome 1 or 2, we get outcome 1 or 2, respectively, in (M, N). In particular, there is no hindrance in $(M/j, N \setminus j)$ focusing on e. So by Lemma 6.8.5, $E - j$ is a cowave in $(M/j, N \setminus j)$. And we may assume that we get a cohindrance (Y, T^M, T^N) and an M-cocircuit as in outcome 3. By sticking $E - j$ onto (Y, T^M, T^N) if necessary, we may assume that $Y = E - j$. The edge j is not an M-loop since otherwise $\{j\}$ would be a nonempty wave not containing e, contrary to our assumption. Thus $(E, S^M, S^N + j)$ is a cohindrance in (M, N), which together with b witnesses outcome 3.

Case 2: $H \neq \emptyset$.

Subcase 2.1: There is a nonempty cowave (Z, U^M, U^N) in $M/J, N/J$ avoiding e. Now we apply the induction hypothesis to $(M \setminus Z, N \setminus Z)$ and $H \setminus Z$ and J. Just as in Subcase 1.1, one checks that if one gets outcome 1, 2 or 3 in the minor, then one gets outcome 1, 2 or 3 in (M, N), respectively.

Thus it remains to consider the following subcase:

Subcase 2.2: Every cowave in $(M/J, N/J)$ contains e. As we are in Case 2, there is some $h \in H$, and we apply the induction hypothesis to $(M/h, N \setminus h)$ and $H - h$ and J. If we get outcome 1 or 3, we get outcome 1 or 3, respectively, in (M, N). So we may assume that we have outcome 2: There is some $J' \subseteq J$ and a wave (X, S^M, S^N) with e on the M-side in $(M/(J' + h), N/J' \setminus h)$ such that there is an N-circuit o with $e \in o \subseteq (S^N + e) \setminus (H - h) = (S^N + e) \setminus H$. By adding the edges in $J \setminus (X \cup J')$ to J' if necessary, we may assume that $J' = J \setminus X$.

The wave (X, S^M, S^N) is almost the wave we are looking for, except that it lives in the wrong pair of matroids. Next, we shall extend (X, S^M, S^N) to a wave living in the right pair of matroids. By the dual of Lemma 6.8.5, either there is a cohindrance focusing on e in $(M/(J + h), N/J \setminus h)$ or $E \setminus (J + h)$ is a wave in $(M/(J + h), N/J \setminus h)$ with spanning sets T^M and T^N. We may assume that the second occurs since the first gives us outcome 3. Then $(E \setminus (J \cup X + h), T^M \setminus X, T^N \setminus X)$ is a wave in $(M/(J \cup X + h), N/(J \cup X) \setminus h)$. By sticking X onto that wave, we get that $(E \setminus (J' + h), S^M \cup T^M \setminus X, S^N \cup T^N \setminus X)$ is a wave in $(M/(J' + h), N/J' \setminus h)$.

The edge h is not an N-coloop since otherwise $\{h\}$ would be a cowave, contrary to our assumption in this subcase. So h is not an N/J'-coloop either. Thus $(E \setminus J', (S^M + h) \cup T^M \setminus X, S^N \cup T^N \setminus X)$ is a wave in $(M/J', N/J')$ with e on the M-side. Moreover the circuit o witnesses that we have outcome 2. □

Lemma 6.12.5 is the main principle we use in the proof of Lemma 6.11.6. The work of bridging from Lemma 6.12.5 to Lemma 6.11.6 is done in the following lemma.

Lemma 6.12.6. *Let $\overline{\gamma}_{M_+}, \overline{\gamma}_{N_-} \subseteq \mathcal{P} - \bot$ and $\overline{\gamma}_{\top^*} \subseteq \mathcal{P}^* - \bot^*$ be up-closed such that $\overline{\gamma} = \overline{\gamma}_{M_+} \cup \overline{\gamma}_{N_-} \cup \overline{\gamma}_{\top^*}$ is blocking. Then one of the following is true.*

1. *One of the 6 sets $\{M_+, M_-^*\}$, $\{M_-, M_+^*\}$, $\{N_+, N_-^*\}$, $\{N_-, N_+^*\}$, $\{M_-, N_+, \top^*\}$ or $\{M_+^*, N_-^*, \top\}$ is a subset of $\overline{\gamma}$;*
2. *$\{M_+, N_-, \top^*\} \subseteq \overline{\gamma}$ and $\overline{\gamma}_{M_+}$ meets $\{M_+, N_-\}$;*
3. *$\top \in \overline{\gamma}_{M_+}$ and $\{M_-^*, N_+^*\} \subseteq \overline{\gamma}_{\top^*}$;*
4. *$\overline{\gamma}_{M_+} \subseteq \{\top, N_+\}$ and $\overline{\gamma}_{N_-} = \{\top, M_+, N_+, N_-\}$ and $\top^* \in \overline{\gamma}_{\top^*} \subseteq \{\top^*, M_+^*\}$;*
5. *$\overline{\gamma}_{M_+} = \emptyset$ and $\top \in \overline{\gamma}_{N_-} \subseteq \{\top, N_+\}$ and $\overline{\gamma}_{\top^*} = \{\top^*, M_+^*, M_-^*, N_+^*\}$;*

Proof. Since $\overline{\gamma} \cap (\mathcal{P} + \bot^*)$ is nonempty and $\bot^* \notin \overline{\gamma}$, we get $\overline{\gamma} \cap \mathcal{P}$ is nonempty, thus $\top \in \overline{\gamma}$. Similarly, $\top^* \in \overline{\gamma}$. Now suppose for a contradiction that we do not have one of the outcomes 1-5. By Lemma 6.10.8, either $\{M_+, N_-, \top^*\} \subseteq \overline{\gamma}$ or $\{M_-^*, N_+^*, \top\} \subseteq \overline{\gamma}$ as we do not have outcome 1.

If $\{M_+, N_-, \top^*\} \subseteq \overline{\gamma}$, then $N_+ \in \overline{\gamma}$. So $\overline{\gamma}$ contains neither M_-^* nor N_+^* nor M_- as we do not have outcome 1. Since we do not have outcome 2, it must be that $\overline{\gamma}_{M_+}$ avoids $\{M_+, N_-\}$. Thus $\overline{\gamma}_{M_+} \subseteq \{\top, N_+\}$ and so $\overline{\gamma}_{N_-} = \{\top, M_+, N_+, N_-\}$ and $\top^* \in \overline{\gamma}_{\top^*} \subseteq \{\top^*, M_+^*\}$, as they are both up-closed. Thus we have outcome 4, which is a contradiction in this case.

Hence it suffices to consider the case that $\{M_-^*, N_+^*, \top\} \subseteq \overline{\gamma}$. Then $M_+^* \in \overline{\gamma}$. So $\overline{\gamma}$ contains neither M_+ nor N_- nor N_-^* as we do not have outcome 1. Since we do not have outcome 3, $\overline{\gamma}_{M_+} = \emptyset$. Thus $\top \in \overline{\gamma}_{N_-} \subseteq \{\top, N_+\}$ and $\overline{\gamma}_{\top^*} = \{\top^*, M_+^*, M_-^*, N_+^*\}$ as they are both up-closed. Thus we have outcome 5, which is the desired contradiction. □

Now we are in a position to prove Lemma 6.11.6.

Proof of Lemma 6.11.6. Suppose for a contradiction that there are an arena $A = (M, N, E, F, e)$ and challengers γ_{M_+}, γ_{N_-} and $\gamma_{\top^*}$ for which Lemma 6.11.6 is false. We pick these such that the set F of upper edges is of minimal size. Although we will not need it, it is worth noting that F is nonempty since $\{M_+, N_-, \top^*\}$ is blocking. We abbreviate $\overline{\gamma}[f] = \overline{\gamma}_{M_+}[f] \cup \overline{\gamma}_{N_-}[f] \cup \overline{\gamma}_{\top^*}[f]$. The following sublemma may be proved in a similar way to Sublemma 6.12.3.

Sublemma 6.12.7. *For each $f \in F$ the set $\overline{\gamma}[f]$ is blocking.*

Sublemma 6.12.8. *For each $f \in F$ one of the following two conditions from Lemma 6.12.6 is true.*

4. $\overline{\gamma}_{M_+}[f] \subseteq \{\top, N_+\}$ *and* $\overline{\gamma}_{N_-}[f] = \{\top, M_+, N_+, N_-\}$ *and* $\top^* \in \overline{\gamma}_{\top^*}[f] \subseteq \{\top^*, M^*_+\}$*;*
5. $\overline{\gamma}_{M_+}[f] = \emptyset$ *and* $\top \in \overline{\gamma}_{N_-}[f] \subseteq \{\top, N_+\}$ *and* $\overline{\gamma}_{\top^*}[f] = \{\top^*, M^*_+, M^*_-, N^*_+\}$*;*

Proof. By Sublemma 6.12.7 and Lemma 6.12.6 the sets $\overline{\gamma}_{M_+}[f]$, $\overline{\gamma}_{N_-}[f]$ and $\overline{\gamma}_{\top^*}[f]$ fulfil one of the outcomes of Lemma 6.12.6. If they satisfy 4 or 5, we are done. Otherwise they satisfy one of the conditions 1-3 of Lemma 6.12.6.

First suppose for a contradiction that they satisfy 1: Let B be one of $\{M_+, M^*_-\}$, $\{M_-, M^*_+\}$, $\{N_+, N^*_-\}$, $\{N_-, N^*_+\}$, $\{M_-, N_+, \top^*\}$ or $\{\top, N^*_-, M^*_+\}$ such that $B \subseteq \overline{\gamma}[f]$. Then we pick a tactician$^-$ μ_B as in Lemma 6.11.5. If $B = \{M_-, N_+, \top^*\}$, we get Improvement case 1. If $B = \{\top, N^*_-, M^*_+\}$, we get Improvement case 2. If $B = \{M_+, M^*_-\}$ or $B = \{M_-, M^*_+\}$, we get the Weak Challenge case in the Packing game. If $B = \{N_-, N^*_+\}$ or $B = \{N_+, N^*_-\}$, we get the Weak Challenge case in the Covering game. Thus we get a contradiction in this case.

Next, we consider the case that $\overline{\gamma}_{M_+}[f]$, $\overline{\gamma}_{N_-}[f]$ and $\overline{\gamma}_{\top^*}[f]$ satisfy 2: $\{M_+, N_-, \top^*\} \subseteq \overline{\gamma}[f]$ and $\overline{\gamma}_{M_+}[f]$ meets $\{M_+, N_-\}$. We let $B = \{M_+, N_-, \top^*\}$. If $\overline{\gamma}_{M_+}[f] \cap \{M_+, N_-\} = \{M_+\}$, then $N_- \in \overline{\gamma}_{N_-}[f]$, and so we can define some μ as in the Double Extension case. Otherwise $N_- \in \overline{\gamma}_{M_+}[f]$, so that without loss of generality μ_B is such that $\pi_1(\mu_B(N_-)) = M_+$. Thus $\mu_B^{-1}(N_-, K) = M_+$ for every tactic K where this is defined. So we get the Weak Challenge case in the Packing game.

Thus, it remains to consider the case that $\overline{\gamma}_{M_+}[f]$, $\overline{\gamma}_{N_-}[f]$ and $\overline{\gamma}_{\top^*}[f]$ satisfy 3: $\top \in \overline{\gamma}_{M_+}[f]$ and $\{M^*_-, N^*_+\} \subseteq \overline{\gamma}_{\top^*}[f]$. We let $B = \{\top, M^*_-, N^*_+\}$ and define the tactician$^-$ μ_B such that $\pi_1(\mu_B(\top)) = M_+$. So we have the Weak Challenge case in the Packing game. This completes the proof. □

Sublemma 6.12.8 motivates the following definition: Let $H \subseteq F$ be the set of those $f \in F$ that satisfy 4 and let $J = F \setminus H$. Note that any $j \in J$ satisfies 5 by Sublemma 6.12.8. Now we apply Lemma 6.12.5 to J and H. According to which outcome we get, we now split into cases.

Case 1: We get outcome 1 of Lemma 6.12.5: There is some $H' \subseteq H$ such that there is a wave M-spanning e in $(M/H'/J, N \setminus H'/J)$. This wave gives rise to a tactic K fulfilling M_+ at e such that:

$$\varphi_K(f) = \begin{cases} M_+ \text{ if } f \in H' \\ M_- \text{ or } N_- \text{ or } \bot \text{ if } f \in H \setminus H' \\ \top \text{ if } f \in J \end{cases}$$

As γ_{M_+} is a challenger, there is some $f \in F$ such that $\gamma_{M_+}(K) = f$. As $\overline{\gamma}_{M_+}[j] = \emptyset$ for each $j \in J$, f cannot be in J and it cannot be in H either since $\overline{\gamma}_{M_+}[h] \subseteq \{\top, N_+\}$ for each $h \in H$. This is the desired contradiction.

Case 2: We get outcome 2 of Lemma 6.12.5: There is some $J' \subseteq J$ and a wave (X, S^M, S^N) with e on the M-side in $(M/J', N/J')$ such that there is an N-circuit o with $e \in o \subseteq (S^N + e) \setminus H$. This wave gives rise to a tactic K fulfilling N_- at e with $C_K^N = o$ such that:

$$\varphi_K(f) = \begin{cases} \top \text{ if } f \in J' \\ M_- \text{ or } N_- \text{ or } \bot \text{ if } f \in H \cup (J \setminus J') \end{cases}$$

Let $f = \gamma_{N_-}(K)$. If $f \in H$, we let $B = \{M_+, N_-, \top^*\}$ and we pick μ_B such that $\mu_B(N_-) = (N_-, K)$. Thus we get the Weak Challenge case in the Packing game. If $f \in J$, then it must be in J' because $\overline{\gamma}_{N_-}[j] \subseteq \{\top, N_+\}$ for each $j \in J$. Then we let $B = \{M_-^*, N_+^*, \top\}$ and we pick μ_B such that $\mu_B(\top) = (N_-, K)$. Thus we get the Weak Challenge case in the Packing game.

Case 3: We get outcome 3 of Lemma 6.12.5: There is some $H' \subseteq H$ and a cohindrance (Y, T^M, T^N) focusing on e in $(M/H', N \setminus H')$ such that there is some M-cocircuit b with $e \in b \subseteq (T^M + e) \setminus J$. This cohindrance gives rise to a tactic K fulfilling $\top^*$ at e with $C_K^{M^*} = b$ such that:

$$\varphi_K(f) = \begin{cases} N_+^* \text{ if } f \in H' \\ M_-^* \text{ or } N_-^* \text{ or } \bot^* \text{ if } f \in J \cup (H \setminus H') \end{cases}$$

Let $f = \gamma_{\top^*}(K)$. If $f \in H$, then it is in H' because $\overline{\gamma}_{\top^*}[h] \subseteq \{\top^*, M_+^*\}$ for each $h \in H$. Then we let $B = \{M_+, N_-, \top^*\}$ and we pick μ_B such that $\mu_B(\top^*) = (\top^*, K)$. Thus we get the Weak Challenge case in the Covering game. If $f \in J$, we let $B = \{M_-^*, N_+^*, \top\}$ and we pick μ_B such that $\mu_B(M_-^*) = (\top^*, K)$. Thus we get the Weak Challenge case in the Covering game.

□

6.13 Concluding remarks

There does not seem to be any reason, in principle, why the methods of this chapter should not extend to trees of matroids with larger overlap. However, we have found that the most naive attempt to do this results in an explosion of the number of cases which must be dealt with. This puts the necessary computations far beyond the bounds of what we could reasonably attempt.

It is clear that the success of our argument provides only weak evidence for the truth of the Packing/Covering conjecture in general. However, we think further evidence for this conjecture is given by the fact that our argument only just succeeds: an argument which resolved this case more straightforwardly would have suggested that the conclusion was an artifact of the tree structure, rather than relying on the matroidal structure.

There are some subtle issues of descriptive set theory surrounding the question of how much we have shown. If we have matroids $M_{\Psi_M}(T, M)$ and $M_{\Psi_N}(T, N)$ which are matroids *because* the sets Ψ_M and Ψ_N are Borel then our results allow us to deduce that $(M_{\Psi_M}(T, M), M_{\Psi_N}(T, N))$ satisfies Packing/Covering. The same comment applies if Ψ_M and Ψ_N are taken from some other class of determined sets closed under basic operations such as inverse images under continuous functions, for example the class of analytic sets if there is a measurable cardinal. But if $M_{\Psi_M}(T, M)$ and $M_{\Psi_N}(T, N)$ just happen to be matroids then it is not clear that they must satisfy Packing/Covering, because the games on whose determinacy our argument relies are quite different from the games whose determinacy witnesses that $M_{\Psi_M}(T, M)$ and $M_{\Psi_N}(T, N)$ are matroids.

If it could be shown that the determinacy of the latter collection of games implies that of the former, then this would significantly strengthen our result. On the other hand, a counterexample to this implication would give a counterexample to the Packing/Covering conjecture. Although for these reasons we are eager to resolve this issue, only the bare beginnings of an investigation into questions like this (regarding the relationship of determinacy of particular games outside well-behaved classes) has been made, in papers such as [49] and [47].

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