Rigidity of higher-rank lattices actions

Vincent Pecastaing

Université Côte d'Azur, Nice

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- Higher-rank lattices and Margulis' super-rigidity
- Zimmer's generalization: cocycle super-rigidity
- What is it good for? Actions on G-structures
- Unimodular results (1990's)
- Recent progress on Zimmer's conjectures, non-unimodular results

Zimmer Program: two samples

Theorem (Ghys '99)*

Any $\mathcal{C}^1$ action $\rho : SL_3(\mathbb{Z}) \to Diff^1(\mathbb{S}^1)$ has finite image, i.e. factorizes $SL_3(\mathbb{Z}) \to F \to Diff^1(\mathbb{S}^1)$, with F finite group.

(Related works: Witte 94, Burger-Monod 99, Deroin-Hurtado 20.)
Rk: $SL_3(\mathbb{R})$ acts faithfully on $\mathbb{S}^2 = \{\text{half-lines of } \mathbb{R}^3\}$.

Théorème (Zimmer '86)

If (M^n, g) is a compact Lorentzian manifold, then any isometric action $\rho : SL_3(\mathbb{Z}) \to Isom(M, g)$ has finite image.

The Lorentz signature is "too small" for $SL_3(\mathbb{Z})$.

Rk : $SL_2(\mathbb{R})$ acts on $T^1\Sigma_g$, $g \geqslant 2$, preserving a Lorentzian metric with $K = -1$ (AdS_3).
Rk : If $min(p, q) \geqslant 3$, there are flat metrics on $\mathbb{T}^{p+q}$ of sign. (p, q) which are $SL_3(\mathbb{Z})$-invariant.

(*) Disclaimer: Ghys' theorem is more general. Same for the rest of the talk.

Motivation: Super-rigidity of lattices in real-rank $\geqslant 2$

Setting: Let G be a real, simple, non-compact Lie group. Ex:

$G = SL_n(\mathbb{R})$, $SO(p,q)$, $SU(p,q)$ etc...

Definition

A lattice Γ of G is a discrete subgroup such that $\mathrm{vol}(G/\Gamma) < \infty$.
It is cocompact if G/Γ is compact.

For ex: $\Gamma = SL_n(\mathbb{Z})$ is a (non cocompact) lattice of $G = SL_n(\mathbb{R})$.

$\Gamma = SL_n(\mathbb{Z}[i])$ is a (non cocompact) lattice of $G = SL_n(\mathbb{C})$.

$\underbrace{\qquad\qquad}_{\text{Arithmetic groups } \Gamma = G_{\mathbb{Z}} \text{ when } G \text{ algebraic defined over } \mathbb{Q}}$

$\Gamma = \pi_1(M)$, with M^n = closed hyperbolic manifold.

$\mathrm{hol}(\Gamma) \subset G = SO(1,n)$ is a torsion-free cocompact lattice.

Définition

The real-rank of G is the dimension of its maximal $\mathbb{R}$-split tori.

Ex: $\mathrm{Rk}_{\mathbb{R}}\, SL_n(\mathbb{R}) = n-1$, $\mathrm{Rk}_{\mathbb{R}}\, SO(p,q) = \min(p,q)$

Margulis ('74) :

$$\boxed{\text{If } \mathrm{Rg}_{\mathbb{R}} G \geqslant 2, \text{ then all its lattices } \Gamma < G \text{ are arithmetic.}}$$

$$\Uparrow$$

Theorem (Margulis' super-rigidity, '74)

Let G be a simple real Lie group, with real-rank $\geqslant 2$, and $\Gamma < G$ a lattice. Let H be a simple real algebraic group. Let $\rho : \Gamma \to H$ be a group homomorphism. If $\rho(\Gamma)$ is Zariski-dense, then there exists a Lie group homomorphism $\overline{\rho} : G \to H$ such that $\rho = \overline{\rho}|_{\Gamma}$.

$\Rightarrow$ Finite dimensional representations of Γ are essentially determined by those of G. It is classic Lie theory.

Rk: A p-adic version is needed for arithmeticity theorem.
Rk: It does not work for $G = \mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. But it does for $\mathrm{Sp}(n,1)$ and $F_{4(-20)}$ (Corlette 92, Gromov-Schoen 92)

Geometric interpretation

For the rest of the talk, G = simple Lie group with $\mathrm{Rk}_{\mathbb{R}}\, G \geqslant 2$ and $\Gamma < G$ a lattice , typically $G = \mathrm{SL}_n(\mathbb{R})$, $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ with $n \geqslant 3$.

Let H be as above, say $\mathrm{GL}_k(\mathbb{R})$. To any $\rho : \Gamma \to H$ corresponds a principal H-bundle $P^\rho \to G/\Gamma$, on which G acts by automorphisms.
Definition: $P^\rho = (G \times H)/\Gamma$, where $(g, h) \sim (g\gamma, \rho(\gamma^{-1})h)$.

Conversely, if $P \to G/\Gamma$ acted upon by G, there is $\rho : \Gamma \to H$ such that $P \simeq P^\rho$, G-equivariantly.

How does the conclusion of Margulis' theorem read? Extending ρ to $\overline{\rho} : G \to H$ yields a global trivialization $P^\rho = G/\Gamma \times H$ diagonalizing the G-action: $g.(g_0\Gamma, h) = (gg_0\Gamma, \overline{\rho}(g)h)$.

Margulis' super-rigidity can be seen as a structure result for G-actions on H-principal bundles over G/Γ.

Zimmer's generalization

Zimmer ($\simeq$ '80) : Extends this to H-principal bundles $P \to X$, with *non-homogeneous base* X. Instead, (X, μ) = ergodic G-space.

Let G, Γ, H be as above, *e.g.* $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{Z})$, $\mathrm{SL}_m(\mathbb{R})$, $n \geqslant 3$.

Theorem (Zimmer's cocycle super-rigidity)

Let $P \to M$ be an H-principal bundle, with M = smooth manifold. Assume that G acts on P by automorphisms, by preserving a finite Borel measure μ on M, with algebraic hull H.
Then there exist $P \simeq M \times H$ a *measurable* trivialization and $\rho : G \to H$ such that $g.(x, h) = (g.x, \rho(g).h)$ for μ-a.e. x.

Rephrasing: any measurable cocycle $c : G \times M \to H$ is measurably cohomologous to a "ρ-simple" cocycle.
Rk: Same theorem with $G \leftarrow \Gamma$.
Rk: The asumption μ is restrictive, *e.g.* $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{R}P^{n-1}$ without *any* finite invariant measure.

What is it good for?

If M^n is a smooth manifold, its linear frame bundle $\mathcal{F}(M) \to M$ is a $\mathrm{GL}_n(\mathbb{R})$ principal bundle, all $f \in \mathrm{Diff}(M)$ acts naturally on $\mathcal{F}(M)$.

Corollary

Let G be a simple Lie group of rank $\geqslant 2$, $\Gamma < G$ a lattice. Let $\Gamma \curvearrowright (M^n, \mu)$ be a differentiable action, preserving a finite Borel measure μ.
Then there exists $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ a Lie group homomorphism s.t. $\forall \gamma \in \Gamma$, the Lyapunov exponents of γ are the $\log(|\lambda|)$, λ eigenvalue of $\rho(\gamma)$.

Lyapunov exponents of $f \in \mathrm{Diff}(M)$ = exponential growth rate of the eigenvalues of Df^k as $k \to \infty$
$\Rightarrow$ If G is "too large", then $\rho \equiv \mathrm{id} \Rightarrow$ all exponents of all elements of Γ vanish. For instance when $\Gamma = \mathrm{SL}_m(\mathbb{Z})$, $m > n$.

And this, for all Γ-invariant measure μ. Does it indicate that the action is trivial?

Unimodular G-structures

Another natural context: A pseudo-Riemannian metric g of signature (p, q) on M^n is a smooth assignement of quadratic forms of signature (p, q) on tangent spaces of M. This is the same as an $O(p, q)$-reduction $\mathcal{O}(M) \subset \mathcal{F}(M)$, the orthonormal frame bundle.

Isometries of (M, g) are diffeomorphisms preserving $\mathcal{O}(M)$, and preserve the volume element $\mathrm{d}vol^g$.

If $m \geqslant 3$ and $\rho : \mathrm{SL}_m(\mathbb{Z}) \to \mathrm{Iso}(M, g)$ is an isometric action, and if $\mathrm{SL}_m(\mathbb{R})$ does not embed into $O(p, q)$, then we obtain the same dynamical conclusion.

Raising the same question: can the action be non-trivial in such a situation?

Zimmer's conjectures ($\mathrm{SL}_m(\mathbb{Z})$ case)

Conjecture 1 (Zimmer $\sim$ '85)

Let (M^n, ω) be a compact manifold, with a volume form ω.
Let $\rho : \mathrm{SL}_m(\mathbb{Z}) \to \mathrm{Diff}(M, \omega)$ be a volume-preserving action.
If $n < m$, then ρ has finite image.

This is optimal as $\mathrm{SL}_n(\mathbb{Z})$ acts on $(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n, \mathrm{vol}_{\mathrm{euc}})$.

Conjecture 2 (Farb-Shalem '99)

Let M^n be a compact manifold. Let $\rho : \mathrm{SL}_m(\mathbb{Z}) \to \mathrm{Diff}(M)$.
If $n < m - 1$, then ρ has finite image.

Idem, as $\mathrm{SL}_n(\mathbb{R})$ acts projectively on $\mathbb{S}^{n-1}$ and $\mathbb{R}P^{n-1}$.

Theorem (Brown-Fisher-Hurtado '17)

Both conjectures are true (in regularity $\mathcal{C}^2$).

A global rigidity result for unimodular actions

Action(s) at the critical dimension? Uniqueness of $\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathbb{T}^n$?
This action preserves a finite volume **and** a flat *connection*.

If ∇ = linear connection on M, $f \in \mathrm{Diff}(M)$ is affine if $f^*\nabla = \nabla$
$\Rightarrow$ for all geodesic γ, $f(\gamma(t))$ is also geodesic.
$\rightsquigarrow$ affine dynamics are locally linear.

Theorem (Zeghib '97)

Let $\rho : \mathrm{SL}_n(\mathbb{Z}) \to \mathrm{Diff}(M^n, \nabla, \omega)$, with ∇ affine connection, ω volume form.
If ρ has infinite image, then (M, ∇) is covered by $\mathbb{T}^n$.

(In the continuity of Zimmer (86), Feres (92), Goetze (94).)

Rk: Wrong without ∇ (Katok-Lewis 96).

Projective actions of cocompact lattices

Characterization of actions at the critical dimension for non-volume preserving actions?

Définition

Two linear connections ∇, ∇' are projectively equivalent if they define the same geodesics up to parametrization $\rightsquigarrow$ proj. class $[\nabla]$.

$\rightsquigarrow$ $f \in \mathrm{Diff}(M)$ is projective w.r.t. ∇ if it sends geodesic locus to geodesic locus. We note $\mathrm{Proj}(M, [\nabla])$ the group of projective diffeomorphisms.

Théorème (P. '19)

Let $(M^n, [\nabla])$ compact projective manifold; and $\Gamma < \mathrm{SL}_{n+1}(\mathbb{R})$ cocompact. Let $\rho : \Gamma \to \mathrm{Proj}(M, [\nabla])$ be a projective action

If $\rho(\Gamma)$ is infinite, then $(M, [\nabla]) \simeq \mathbb{R}P^n$ or $\mathbb{S}^n$.

Conjecture: this shall be true without assuming ρ projective.

Ideas of proof

Difficulty: for such structures, no *natural* finite invariant measure (contrarily to isometric actions for instance).

An individual $f \in \mathrm{Diff}(M)$ always preserve $\mu < \infty$ when acting continuously on a compact space X. This is the *amenability* of $\mathbb{Z}$. Other amenable groups: compact, solvable, their extensions.

But $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{Z})$ (non-compact semi-simple Lie groups G, and their lattices) are not amenable. How to use super-rigidity then?

By contradiction when no such μ shall exist.

Lifting invariant measures

If $f \in \mathrm{Proj}(M, [\nabla])$, then f is not linearizable. But it is 2-rigid: it is determined by its 2-jet at any point.

$\Rightarrow \mathrm{Proj}(M^n, [\nabla])$ acts freely on a sub-bundle $B \subset \mathcal{F}^2(M)$ of the bundle of 2-frames, the associated *Cartan bundle* $\pi : B \to M$. It has structure group $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$.

Assume a subgroup $\Gamma = \mathrm{SL}_k(\mathbb{Z}) \subset \mathrm{Proj}(M, [\nabla])$ preserves μ. If $k > n$, then $\mathrm{SL}_k(\mathbb{R})$ does not embed into $\mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$. Super-rigidity $\Rightarrow \Gamma$ preserves a measurable subbundle $M \times K \subset B$, with $K \subset \mathrm{GL}_n(\mathbb{R}) \ltimes \mathbb{R}^n$ compact. So Γ preserves $\mu \otimes \mathrm{Haar}_K$ on B. But the action is free on B: contradiction.

Conclusion: for $k > \dim M$, $\Gamma = \mathrm{SL}_k(\mathbb{Z})$ cannot preserve any finite measure on $(M, [\nabla])$.

Building Γ-invariant measures

Let $\rho : \Gamma \to \mathrm{Diff}(M)$, with $\Gamma < G = \mathrm{SL}_k(\mathbb{R})$ **cocompact** lattice.
Problem: Difficult to work with Γ-orbits. We prefer G-orbits.

Suspension of ρ: $M^\rho = (G \times M)/\Gamma$, $\gamma.(g,x) = (g\gamma, \gamma^{-1}.x)$.
Fibered action of G on M^ρ: $g'.[(g,x)] = [(g'g,x)]$.
Γ cocompact $\Rightarrow M^\rho$ compact.

Lemma

$\exists \nu$ Γ-invariant on $M \iff \exists\ \mu$ G-invariant on M^ρ.

Let $A < G$ be a maximal $\mathbb{R}$-split torus, *e.g.*

$$A = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \right\} \subset \mathrm{SL}_k(\mathbb{R}).$$

Since $A \simeq \mathbb{R}^{k-1}$ is amenable, there are A-invariant measures.
Brown, Rodriguez-Hertz, Wang gave sufficient criteria for these measures to be G-invariant.

Higher-rank Oseledets' theorem

Let M be a compact manifold, and $\mathbb{R}^k =: A \curvearrowright M$ a differentiable action.

Theorem (Oseledets)

Let μ be a finite A-invariant, A-ergodic measure. Then there exist $\Lambda \subset M$ with $\mu(\Lambda) = 1$, and a measurable splitting $TM|_\Lambda$

$$T_x M = E_1(x) \oplus \cdots \oplus E_r(x), \ \ A - \text{invariant}$$

and linear forms $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$ s.t. $\forall x \in \Lambda$, $1 \leqslant i \leqslant r$,

$$\forall X \in \mathfrak{a}, \ \ v \in E_i(x) \setminus \{0\}, \ \ \frac{1}{t} \log \|D_x \varphi_X^t v\| \xrightarrow[t \to \pm\infty]{} \chi_i(X).$$

Definition

$\chi_1, \ldots, \chi_r$ are the Lyapunov forms (of μ).

Maximal entropy argument

Let μ be a finite A-invariant, A-ergodic measure on M^ρ, projecting to the Haar measure of G/Γ.
Let $\chi_1, \ldots, \chi_r \in \mathfrak{a}^*$ be its vertical Lyapunov forms.

Proposition (Brown, Rodriguez-Hertz, Wang 2016)

If there exists a non-zero $X \in \mathfrak{a}$ s.t. $\chi_1(X) = \ldots = \chi_r(X) = 0$, then μ is G-invariant.

For $G = \mathrm{SL}_k(\mathbb{R})$, with $k > n = \dim M$, no such measure exists. As $r \leqslant n = \dim M$ and $\dim A = k - 1$, we must have $k - 1 \leqslant n$ and if equality, $r = n$ and $(\chi_1, \ldots, \chi_n)$ linearly independent.

It is what happens when $\Gamma < \mathrm{SL}_{n+1}(\mathbb{R}) = G$ acts on $(M^n, [\nabla])$, for all A-invariant measure on M^ρ proj. to the Haar measure of G/Γ.

Contracting dynamics

From $\mathrm{Vect}(\chi_1, \dots, \chi_n)^\perp = 0$, we get $\exists ! X \in \mathfrak{a}$ with $\chi_1(X) = \cdots = \chi_n(X) = -1$. Let φ_X^t the corresponding flow on M^ρ.

Stable manifolds of φ_X^t, $\rightsquigarrow (\gamma_k) \in \Gamma$, times $(T_k) \to +\infty$, $x \in M$, a neighborhood $U \ni x$ s.t.

- $\{\gamma_k U\} \to \{x\}$.
- $\forall v \in T_x M$ non-zero, $\frac{1}{T_k} \log \|D_x \gamma_k v\| \to -1$.

Rigidity of $[\nabla] \Rightarrow$ valid for all $y \in U$.

Morally, (γ_k) is "asymptotically homothetic" on U.

Standard arguments $\Rightarrow$ vanishing of Cartan curvature on U. Valid in the neighborhood of every compact Γ-invariant. Globalisation.

Thank you for your attention!