Lecture 2

Anton Kapustin

California Institute of Technology

July 18, 2017

Goals

- More systematic classification based on TQFT
- Incorporate symmetry breaking
- Evaluate partition functions
- Determine the group structure on $Inv_1(G)$ and $Inv_1^f(G,\rho)$.

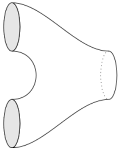
Axiomatic TQFT

Oriented case:

- A f.d. complex vector space V_M for every closed oriented 1-manifold M.
- $V_{\bar{M}} = V_M^*$, where $\bar{M}$ is orientation reversal of M.
- A linear map $\Phi(\Sigma, M, M') : V_M \to V_{M'}$ for every oriented bordism Σ from M to M'.
- Empty M maps to $\mathbb{C}$.
- Disjoint union maps to tensor product

Special cases

- If $M = S^1$, get a "TQFT space of states", which I denote $\mathcal{A}$.
- If $M = S^1 \sqcup S^1$ and $M' = S^1$, and Σ is "pair of pants",

  get a linear map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which makes $\mathcal{A}$ into a commutative algebra.
- If M is empty, get a vector in $V_{M'}$ which depends on Σ.
- If Σ is closed, get $Z(\Sigma) \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ (the partition function for Σ).

Unitary TQFT (Turaev)

- Each V_M has a Hilbert space structure, so we are given anti-linear isomorphisms $V_M \simeq V_M^*$.
- Consider $\Phi(\Sigma, M, M') : V_M \to V_{M'}$ and $\Phi(\bar{\Sigma}, M', M) : V_{M'} \to V_M$. These maps must be adjoint.

One can show that in a unitary TQFT $\mathcal{A}$ is a C^*-algebra and thus is semi-simple.

Since $\mathcal{A}$ is also commutative, it must be a sum of N copies of $\mathbb{C}$. Thus all unitary 1+1d TQFTs are sums of N copies of the trivial TQFT.

An $N > 1$ leads to ground-state degeneracy, but this degeneracy is accidental (not protected by any symmetry).

State-sum construction of a unitary 1+1d TQFT

Let A be a f.d. semi-simple algebra, with basis e_j, $j = 1, \ldots, N$.

$$e_j \cdot e_k = C_{jk}^{\ell} e_{\ell}.$$

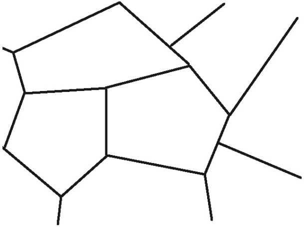
Let

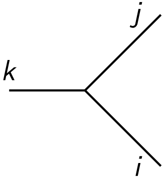
$$\eta_{jk} = \mathrm{Tr}_A e_j e_k = C_{jm}^{\ell} C_{k\ell}^{m}.$$

This is a non-degenerate metric on A. Let $C_{jkm} = \eta_{m\ell} C_{jk}^{\ell}$. C_{jkm} is cyclically-symmetric.

Let Σ be a closed oriented 2d manifold. Choose a trivalent graph Σ whose complement is a bunch of disks and compute the partition function using "Feynman rules". Show later that the partition function does not depend on the triangulation.

"Feynman rules" for the partition function





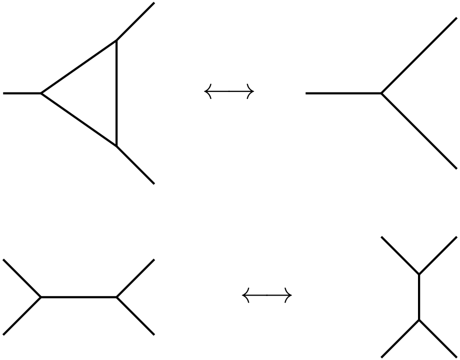
$$= C_{ijk}$$

$$= \eta^{ij}$$

Remarks on "Feynman rules"

- In the end need to sum over labelings of the graph ("states"), hence the name
- The cyclic order on edges issuing from a vertex comes from orientation of Σ
- No need to orient edges, since η^{jk} (the inverse of η_{jk}) is symmetric
- Any two trivalent surface graphs can be connected by a sequence of Pachner moves, so only need to check invariance under Pachner moves

Pachner moves



State-sum construction of the space of states

- Attach to a circle with n points the space $A^{\otimes n}$.
- A cylinder bordism between two such circles is a projector P_n
- Its image is isomorphic to $Z(A)$, for any n.
- To any bordism from M to M' with a graph we can attach a map from $Z(A)^{\otimes k}$ to $Z(A)^{\otimes \ell}$ using the same Feynman rules

Thus $\mathcal{A} = Z(A)$. It is automatically semi-simple (since A is semi-simple).

All 1+1d TQFT are boring

- Any f.d. semi-simple algebra is a direct sum of matrix algebras (Wedderburn theorem)
- The TQFT corresponding to a matrix algebra $A = Mat(N, \mathbb{C})$ is almost the same as for $A = \mathbb{C}$: $\mathcal{A} = \mathbb{C}$, and the partition functions are $N^{\chi(\Sigma)}$, where $\chi(\Sigma) = F - E + V$ is the Euler characteristic of Σ.
- There is a special case of a family of unitary invertible TQFTs with $\mathcal{A} = \mathbb{C}$ and the partition function $\lambda^{\chi(\Sigma)}$, $\lambda \in \mathbb{R}$.
- So the TQFT corresponding to a matrix algebra A can be deformed to the trivial TQFT.
- Up to such deformations, any unitary TQFT is equivalent to a sum of trivial TQFTs

G-equivariant field theory

When a system has symmetry G, can try to couple to a background G-gauge field.

In general, a gauge-invariant coupling may not be possible ('t Hooft anomalies). But if G acts ultralocally (separately on every site), then no 't Hooft anomaly.

From the continuum QFT viewpoint, 't Hooft anomalies are obstructions for discretizing the system so that the symmetry acts ultralocally.

For a finite G which acts by internal symmetries, a G gauge field is the same as a G-bundle, or a map to BG.

Lattice gauge fields

A lattice gauge field is an assignment of $g \in G$ to every oriented edge so that for any face the product of all group elements is 1.

From these data, one can reconstruct a map to BG up to a homotopy.

A gauge transformation at a vertex multiplies group elements for all outgoing edges by an $h \in G$. The corresponding maps to BG are homotopic. The partition function must be invariant under gauge transformations.

N.B. There is a dual construction, where the product of all group elements for every vertex is 1. Gauge transformations then live on faces and multiply all group elements by $h \in G$.

G-equivariant algebras

A G-equivariant algebra A is an algebra with a linear action of G:

$$g : a \mapsto R_g(a), \quad R_g R_h = R_{gh}, \quad \forall g, h \in G,$$

such that

$$R_g(a) \cdot R_g(b) = R_g(a \cdot b).$$

We will now construct a G-equivariant TQFT starting from a G-equivariant algebra A.

N.B. If we use a dual construction for a lattice gauge field, then instead of a G-equivariant algebra we need to use a G-graded algebra (Turaev). For our purposes G-equivariant algebras are more convenient.

G-equivariant vs. G-graded algebras

A G-graded algebra is an algebra A with a decomposition $A = \oplus_{g \in G} A_g$ such that $A_g \cdot A_h \subset A_{gh}$.

Note also that for $G = \mathbb{Z}_2$ a G-equivariant algebra is exactly the same as a G-graded one. But in general they are different.

There is a state-sum construction of G-equivariant 1+1d TQFTs with starts with a G-graded algebra (Turaev). The gauge field then satisfies a constraint on vertices instead of faces.

The existence of two different constructions of the same kind of objects seems puzzling, but has a deep reason for this.

Categorical Morita equivalence

We have two natural tensor categories associated with a finite group G: Vec_G and $Rep(G)$. The tensor product in Vec_G looks as follows:

$$(V \otimes W)_g = \bigotimes_{xy=g} V_x \otimes W_y$$

They are not equivalent, and their properties are rather different for general G. For example, $Rep(G)$ is symmetric, but Vec_G is not symmetric for nonabelian G.

A G-equivariant algebra is an algebra-object in $Rep(G)$, a G-graded algebra is an algebra-object in Vec_G.

While Vec_G and $Rep(G)$ are not equivalent, 2-categories of module categories over them are equivalent. This is a special of categorical Morita equivalence.

The approach using G-equivariant algebras is more convenient, because the stacking operation looks more natural.

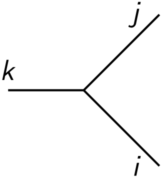
State-sum construction of G-equivariant TQFTs

Let e_j be a basis for A, as before. Define $C_{jk\ell}$ and η_{jk} as before. Note that R_g is orthogonal with respect to η_{jk}:

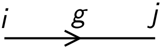
$$\mathrm{Tr}_A R_g(e_j) R_g(e_k) = \mathrm{Tr}_A e_j e_k = \eta_{jk}.$$

Let $\eta_{jk}(g) = \mathrm{Tr}_A \left(e_j R_g(e_k)\right)$. Then $\eta_{jk}(g) = \eta_{kj}(g^{-1})$.

Equivariant Feynman rules:



$= C_{ijk}$



$= \eta^{ij}(g)$

Classification of G-equivariant semi-simple algebras (Ostrik)

Let $H \subset G$ be a subgroup. Let $Q_h : U \to U, h \in H$, be a projective representation of H.

Consider the space of functions on G with values in $End(U)$ satisfying an H-equivariance constraint:

$$f(gh) = Q_h^{-1} f(g) Q_h.$$

They form an algebra A with respect to pointwise multiplication. G acts on A via left translations:

$$R_g(f)(g') = f(g^{-1}g').$$

With this G-action, A becomes a G-equivariant algebra.

This is the most general simple G-equivariant algebra. Any semi-simple one is a sum of such algebras.

Special cases

- $U = \mathbb{C}$: A is the space of complex functions on G/H.
- $H = G$: $A = End(U)$, $R_g(a) = Q_g a Q_g^{-1}$.

In the 1st case, the partition function vanishes unless the G-connection reduces to an H-connection. If this is the case, the partition function is $|G/H|$ regardless of the gauge field. This corresponds to a "Landau" phase with G spontaneously broken down to H

In the 2nd case, the partition function is nonzero for any gauge field, and the TQFT is invertible. The partition function is equal to

$$Z = \exp(2\pi i \int_\Sigma \omega).$$

Here $\omega \in H^2(\Sigma, \mathbb{R}/\mathbb{Z})$ is a pull-back of a 2-cocycle Ω on BG via the map $\Sigma \to BG$ corresponding to the G-connection. The 2-cocycle Ω "measures" the projectiveness of Q_g:

$$Q_g Q_{g'} = \exp(2\pi i \Omega(g, g')) Q_{gg'}.$$

Bosonic phases with symmetry G, no time-reversal

More generally, one can show that the TQFT depends only on H and the class $[\Omega] \in H^2(H, \mathbb{R}/\mathbb{Z})$ which "measures" the projectiveness of Q_h.

H tells us which subgroup of G is unbroken. $[\Omega]$ parameterizes invertible phases with symmetry H.

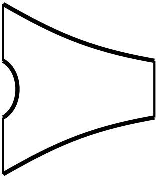
In particular, $Inv_1(G) \simeq H^2(G, U(1))$, provided G does not contain time-reversing symmetries (Chen-Gu-Liu-Wen).

The meaning of A

Consider topological boundary conditions ("branes") for a TQFT. This means extending the axioms, so that M can have boundaries and bordism Σ can have corners.

A is the vector space corresponding to the interval I with a particular boundary condition (the same one on each end).

Multiplication $A \otimes A \to A$ arises from the following bordism between $I \sqcup I$ and I:



TQFTs with time-reversing symmetries

Convenient to compose all anti-unitary (time-reversing) symmetries with CPT, so that they become unitary (and space-reversing).

Such symmetries act on A by anti-automorphisms:

$$R_g(a \cdot b) = R_g(b) \cdot R_g(a), \quad \sigma(g) = -1.$$

Symmetries g with $\sigma(g) = 1$ act on A by automorphisms, as before.

We can use A with such a G-action to construct a G-equivariant unorientable TQFT.

Example: time-reversal symmetry only

Let $G = \mathbb{Z}_2$. If A is an algebra of $m \times m$ matrices, there are two options for R:

- $R(a) = Ja^T J^{-1}$ where $J = J^T$
- $R(a) = Ja^T J^{-1}$ where $J = -J^T$

The first option gives the trivial TQFT.

The second option gives a nontrivial invertible unorientable TQFT with the partition function $(-1)^{\chi(\Sigma)}$.

The Freed-Hopkins theorem for $G = \mathbb{Z}_2^T$

This is a special case of a special case of the Freed-Hopkins theorem:

Deformation classes of unitary invertible $(d+1)$-dimensional TQFTs with an orientation reversing symmetry $\mathbb{Z}_2^T$ are in 1-1 correspondence with the Poincare-dual of the torsion subgroup of unoriented bordisms of degree $d+1$.

Recall that an unoriented bordism group Ω_n^O is the group generated by n-dimensional closed unoriented manifolds, modulo those which are boundaries of compact $(n+1)$-dimensional unoriented manifolds.

$\Omega_2^O = \mathbb{Z}_2$, because all closed 2-manifolds with even χ are null-bordant, while those with odd χ are bordant to $\mathbb{RP}^2$.

In other words, the only nontrivial cobordism is $\int_\Sigma w_2 = \int_\Sigma w_1^2$, where $w_i \in H^i(\Sigma, \mathbb{Z}_2)$ is the i^{th} Stiefel-Whitney class of Σ.

State-sum construction for unorientable TQFTs

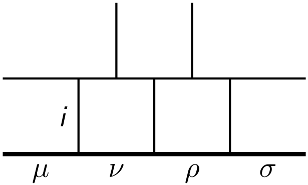
- We do not have a preferred cyclic order for edges coming out of a vertex, so pick an arbitrary orientation/cyclic order
- Each edge is assigned ± 1 depending on whether the orientation near its ends agree or disagree
- This gives a 1-cocycle on the surface graph representing w_1
- Use the same "Feynman rules" as before

Branes from modules

A "closed" TQFT is described by an algebra A, a brane corresponds to a module M over A.

A module is a vector space M together with a map $T : A \to End(M)$ s.t. $T(a)T(b) = T(a \cdot b)$.

Suppose a surface graph has univalent vertices lying on a brane boundary.



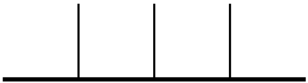
We label boundary edges by basis elements $m_\mu \in M$.

New rule: attach $T(e_i)^\nu_\mu$ to each boundary vertex.

States from branes

To each brane M one can attach a state $\phi_M \in \Phi(S^1)$. The corresponding bordism is an annulus whose interior boundary is a brane boundary.

If the outer boundary is subdivided into N intervals, get an element of $A^{\otimes N}$ of the following form:



This state is automatically in the image of the cylinder projector $A^{\otimes N} \to A^{\otimes N}$.

Matrix Product States

Explicitly, the state $\phi_M \in A^{\otimes N}$ is

$$\sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[T(e_{i_1})T(e_{i_2})\ldots T(e_{i_N})\right]|i_1 i_2 \ldots i_N\rangle.$$

States of this form are known as Matrix Product States. TQFT gives MPS states of a special block-diagonal form:

$$T(e_i) = \mathrm{diag}(T^1(e_i), T^2(e_i), \ldots, T^K(e_i)),$$

where the α^{th} block is spanned by $T^\alpha(e_i)$. Each block corresponds to an "injective" MPS state.

Such MPS are invariant under real-space RG transformations.

Generalized MPS

More general states are obtained by inserting a boundary observable $X \in End_A(M)$ on the interior (brane) boundary:

$$\sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[X T(e_{i_1}) T(e_{i_2}) \ldots T(e_{i_N})\right] |i_1 i_2 \ldots i_N\rangle.$$

Such states span the space of states of TQFT.

If A is a sum of K matrix algebras, we expect K linearly independent state. Can get them by taking any faithful module over A and a suitable projector X.

If M is irreducible, all nonzero X give the same state on a circle (up to a factor).

States on an interval

Let A be a matrix algebra $End(U)$. A module over A has the form $Hom(W, U)$, where W is a vector space. M is irreducible iff W is one-dimensional.

If an interval has been subdivided into $N + 1$ intervals, "Feynman rules" associate to it an element of $Hom(M, A^{\otimes N} \otimes M)$.

A rectangle projector projects to a subspace isomorphic to $Hom_A(M, M) \simeq Hom(W, W)$.

If W is one-dimensional, the space of states on an interval is non-degenerate.

Basis elements of W can be thought of as "Chan-Paton labels" in string theory.

Equivariant MPS

In the G-equivariant case, M should be an equivariant module, i.e. M should be a representation $\tilde{Q}_g \in End(M)$ of G such that

$$T(R_g(a)) = \tilde{Q}_g T(a) \tilde{Q}_g^{-1}.$$

Let us specialize our TQFT to a G-equivariant SPT: $A = End(U)$ where U is a projective representation of G.

Equivariant modules over such A have the form $M = Hom(W, U)$, where W is a projective representation of G with the same cocycle as U.

$A = End(U)$ acts on $M = Hom(W, U)$ by left multiplication.

The space of state on an interval is $Hom_A(M, M) = Hom(W, W)$.

SPT phases and ground-state degeneracy on an interval

Now we can see where the boundary zero modes and ground-state degeneracy come from.

Suppose the smallest irreducible projective representation of G with a 2-cocycle $[\Omega] \in H^2(G, U(1))$ has dimension n.

Then W has dimension at least n, and the space of ground states on an interval has dimension at least n^2.

For example, for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and a nontrivial cocycle, we have degeneracy at least 4.

Larger degeneracy is not required by symmetries and thus is unnatural.