Lecture 1

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Microscopic Hamiltonians vs. phases

Goal: study and hopefully classify "phases of quantum matter", i.e. phases at zero temperature.

Microscopic Hilbert space: $V = \otimes_i V_i$.

Microscopic Hamiltonian $H = \sum_i H_i$ is assumed to be short-range. This means that $||[H_i, O_j]||$ decays rapidly with distance (faster than any power of the distance $d(i,j)$). Here O_j is localized at site j.

Two microscopic Hamiltonians belong to the same "phase" (equivalence class) if there is a deformation connecting them which preserves macroscopic properties.

Usually assumed: Macroscopic properties are described by effective QFT.

Not an obvious assumption, there are some fairly simple counter-examples (J. Haah, PhD thesis). These are translationally-invariant Hamiltonians on a 3d cubic lattice whose ground-state degeneracy depends on the size of the lattice.

Gapped vs. gapless

A phase is called gapped if the gap between the energies of the ground state(s) and the excited states is bounded by $\epsilon > 0$ as one takes the spatial volume to infinity. Phases which are not gapped are called gapless.

A QFT for a gapless phases reduces to a nontrivial scale-invariant QFT in the IR limit. Usually a CFT.

A QFT for a gapped phase usually reduces to a TQFT in the IR limit. This TQFT describes the ground states and the response of the ground state to external probes (sources).

Gapped phases described by a QFT are also known as topological phases.

Equivalence classes of gapped Hamiltonians (Wen, Kitaev)

For gapped Hamiltonians, can try to give a more precise definition of the equivalence relation(s).

- Homotopy between gapped local Hamiltonians
- Adding new sites with decoupled degrees of freedom

Existence of the large-volume limit ???

TQFTs vs. quasi-TQFTs

An important subtlety: sometimes the stress-energy tensor $T_{\mu\nu}$ is a nonzero c-number. Such QFTs will be called quasi-topological.

$$T_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}}.$$

In a quasi-TQFT, $\Gamma(g)$ is a nontrivial function of the background metric.

$\Gamma(g)$ is defined up to integrals of local diffeomorphism-invariant expressions.

Nontrivial $\Gamma(g)$ exist only for $d = 4k - 2$, $k \in \mathbb{Z}$ (gravitational Chern-Simons actions S_{CS}^{grav}). In practice, this subtlety is important only for $d = 2$:

$$S_{CS}^{grav} = c \int \mathrm{Tr} \left(\omega d\omega + \frac{2}{3} \omega^3 \right).$$

Invariants of (quasi)topological phases

To tell apart phases, we need invariants. Some examples:

- The coefficient of S_{CS}^{grav} for $d = 2$.
- Dimension of the space of ground states (on a torus T^d or some other compact d-manifold).
- Dimension of the space of sources localized at a point.
- Partition function on a compact $(d + 1)$-manifold.
- "Anomalous" boundary behavior

Symmetries

A more refined classification takes into account symmetries. We will focus on symmetries which act trivially on the spatial lattice.

- Each V_i is a representation of a group G, $g \mapsto R_i(g) \in End(V_i)$.
- $R_i(g)$ is either unitary or anti-unitary. In the latter case g is time-reversing.
- The Hamiltonian H is invariant under G.
- Only allow deformations of Hamiltonians which preserve G.

Note that each H_i need not be separately invariant.

G-equivariant TQFT and SET phases

If a microscopic theory has a symmetry G acting locally, it can be coupled to a background G gauge field.

Hence the corresponding TQFT can be coupled to a background G gauge field. Such TQFTs are called equivariant TQFTs.

The corresponding topological phases are called Symmetry Enhanced Topological (SET) phases.

Symmetry breaking

In an infinite volume, symmetry is often spontaneously broken (the vacuum is not invariant under all elements of G). The unbroken symmetry group $K \subset G$ is another simple invariant of a phase.

In the 50s and 60s, this was the only known invariant which distinguished gapped phases (L.D. Landau's theory of phase transitions). But now we know many more such invariants. Below I will provide a complete set of invariant for $d = 1$ phases.

Long-range vs. short-range entanglement

Symmetry-breaking phases are examples of phases with long-range entanglement: some correlators of local quantities do not decay as long distances. For example, consider the 1d quantum Ising chain:

$$H = -\sum_i Z_i Z_{i+1} + h \sum_i X_i, \quad X_i = \sigma_i^x, Y_i = \sigma_i^y, Z_i = \sigma_i^z.$$

For small enough h, $\langle Z_i Z_{i+N} \rangle$ tends to a nonzero number for $N \to \infty$.

For $d = 1$, all long-range entanglement comes from symmetry breaking, but in higher dimensions this is not true.

Put differently, TQFT correlators describe precisely the IR limit of correlators. In $d = 1$, there are no nontrivial indecomposable TQFTs (if we ignore symmetry). For $d > 1$ this is not true.

SRE phases can be nontrivial

Is a gapped phase with only a short-range entanglement (SRE phase) trivial? Not necessarily!

The trivial phase has a ground state which is a product state: $|0\rangle = \otimes v_i, \quad v_i \in V_i$. The trivial phase is an SRE phase, but the opposite is not true, in general!

Certain invariants can be non-vanishing even for an SRE phase.

- For $d = 2$, the coefficient c of S_{CS}^{grav} can be nonzero.
- For $d = 1$, the edge zero modes might transform in a projective representation of G (but the whole system transforms in a genuine representation).
- Partition function on a $(d + 1)$-dimensional compact manifold
- Partition function on a $(d + 1)$-dimensional compact manifold with a background G gauge field

Invertible phases

It is a bit tricky to define SRE phases. One necessary property is that the ground state should be unique on a compact manifold of any topology.

Kitaev: an SRE phase is the same as an invertible phase.

- Phases can be tensored ("stacking")
- This operation is associative, commutative, and has a neutral element (the trivial phase)
- Invertible elements in this commutative monoid form an abelian group. This is the group of invertible phases Inv_d.

Important problem: compute Inv_d for any spatial dimension d. More generally, compute $Inv_d(G)$ for any spatial dimension d and any symmetry G.

Invertible TQFTs

Analogous notion in TQFT: invertible TQFTs. Usually one assumes unitarity too.

One can show that the partition function of a unitary invertible TQFT is nonzero for any compact $(d+1)$-manifold M.

The group of unitary invertible TQFTs has been computed (Freed, Hopkins, 2016), for any G and d, and turns out closely related to cobordisms of BG, the classifying space of G. More on this later.

SPT phases vs SRE phases

A symmetry-protected topological phase (SPT phase) is a nontrivial element of $Inv_d(G)$ which becomes trivial when mapped to Inv_d.

For some d, Inv_d is trivial, so an SPT phase with a symmetry G is the same as an invertible (or SRE) phase with symmetry G.

But it is best not to conflate them. We will see an example of an SRE phase which is not an SPT phase momentarily.

Why SRE phases are important

Basically, because they are easier to understand.

General topological phases are very complicated already for $d = 2$: any modular tensor category gives rise to a $d = 2$ phase, and these are hopeless to classify.

For $d > 2$ we do not even know a complete set of equations which describe general topological phases (not to speak about SET phases).

Bosons and fermions

One may allow V_i to be $\mathbb{Z}_2$-graded, for all i. The grading operator is denoted $P = (-1)^F$. The group it generates is denoted $\mathbb{Z}_2^F$.

$PO = OP$ means the observable O is even.

$PO = -OP$ means the observable O is odd.

If odd observables on different sites anti-commute rather than commute, the system contains fermions.

Phases with fermions are called fermionic phases. Phases without fermions are called bosonic.

Fermionic phases form a commutative monoid, invertible elements in it form an abelian group Inv_d^f.

Symmetries of fermionic systems

Symmetry of a fermionic system always includes fermion parity P. The total global symmetry group $\hat{G}$ contains $\mathbb{Z}_2^F$ as a central subgroup (I disregard supersymmetries).

Since all physical quantities are bosonic, they are in a faithful representation of $G = \hat{G}/\mathbb{Z}_2^F$.

It is often assumed that $\hat{G} \simeq G \times \mathbb{Z}_2^F$, we will call this a split case. In general, we have a central extension

$$0 \to \mathbb{Z}_2^F \to \hat{G} \to G \to 0.$$

Instead of this extension, one can specify an element $\rho \in H^2(G, \mathbb{Z}_2)$. We will denote by $Inv_d^f(G, \rho)$ the corresponding group of invertible phases.

Bosonic SRE phases in $d = 1$, no time reversing symmetries

Since nothing interesting happens in the bulk, let's focus on the boundary behavior.

On the left boundary, there may be zero modes which act in a boundary Hilbert space U_L. On the right boundary, we have U_R.

The group G acts on $U_L \otimes U_R$. It acts independently on the left and right zero modes, so both U_L and U_R must carry a representation of G. But these representations might be projective, with opposite 2-cocycles:

$$Q_L(g)Q_L(g') = \exp(2\pi i \Omega(g, g'))Q_L(gg'),$$
$$Q_R(g)Q_R(g') = \exp(-2\pi i \Omega(g, g'))Q_R(gg').$$

It is clear that adding G-invariant impurities can change U_L and U_R, but not Ω. This suggests that $Inv_1(G) \simeq H^2(G, U(1))$.

Group cohomology

Let $C^n(G, A)$ the set of functions $f(g_1, \ldots, g_n)$ with values in an abelian group A. Consider the maps

$$\delta^n : C^n(G, A) \to C^{n+1}(G, A),$$

$$(\delta^n f)(g_1, \ldots, g_{n+1}) = f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(\ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

They satisfy $\delta^n \delta^{n-1} = 0$. Group cohomology of G with coefficients in A is

$$H^n(G, A) = \mathrm{ker}\delta^n / \mathrm{im}\delta^{n-1}.$$

Elements of $\mathrm{ker}\delta^n$ are called n-cocycles, elements of $\mathrm{im}\delta^{n-1}$ are called coboundaries.

$(\delta^0 f) = 0$, hence $H^0(G, A) = A$.

$(\delta^1 f)(g_1, g_2) = f(g_2) - f(g_1 g_2) + f(g_2)$, hence $H^1(G, A) = Hom(G, A)$.

2-cocycles on G with values in A describe possible group laws for a central extension of G by A:

$$(g_1, a_1) \circ (g_2, a_2) = (g_1 g_2, a_1 + a_2 + f(g_1, g_2)).$$

If f is a 2-coboundary, this central extension is isomorphic to $G \times A$. Hence $H^2(G, A)$ classifies isomorphism classes of central extensions of G by A.

$H^3(G, A)$ classifies crossed modules based on (G, A).

Classifying space of G

Group cohomology of G with values in A has a topological interpretation.

For any finite group G there is a certain cell complex BG such that $\pi_1(BG) = G$ and $\pi_i(BG) = 0$ for $i \neq 1$.

In other words, it is an Eilenberg-MacLane space of type $K(G, 1)$.

One can show that $H^n(G, A) = H^n(BG, A)$.

The space BG looks as follows: it has an n-simplex for any $n + 1$-tuple $(g_1, \ldots, g_{n+1})$ satisfying $g_1 \ldots g_{n+1} = 1$.

Thus it has one 0-cell, a 1-cell for every $g \in G$, a 2-cell for every pair (g_1, g_2), etc. It is an infinite-dimensional simplical complex.

One can show that for any space X homotopy classes of maps $X \to BG$ classify isomorphism classes of G-bundles over X.

Twisted group cohomology

Let G act on A by automorphisms, $a \mapsto g \cdot a$. We can define twisted differentials:

$$\delta^n : C^n(G, A) \to C^{n+1}(G, A),$$
$$(\delta^n f)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) +$$
$$\sum_{i=1}^{n} (-1)^i f(\ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots) + (-1)^{n+1} f(g_1, \ldots, g_n).$$

The corresponding cohomology groups are still denoted $H^n(G, A)$, where A is now regarded as a G-module.

Topological interpretation: cohomology of a certain flat bundle over BG with fiber A. The monodromy of the bundle along a 1-cell g is $a \mapsto g \cdot a$.

Twisted group cohomology: example

We will only need the case $A = U(1)$, and an action of the form

$$g : e^{2\pi i\phi} \mapsto e^{i\sigma(g)2\pi i\phi},$$

where $\sigma : G \to \{+1, -1\}$ is a homomorphism. We will use the notation $H^n_\sigma(G, U(1))$.

For example, a twisted 1-cocycle $\phi(g) \in \mathbb{R}/\mathbb{Z}$ satisfies

$$\sigma(g_1)\phi(g_2) - \phi(g_1 g_2) + \phi(g_1) = 0.$$

A twisted coboundary has the form $\sigma(g)\phi - \phi$ for some $\phi \in \mathbb{R}/\mathbb{Z}$.

Suppose we have a system with a unique ground state $|0\rangle$. How can G act on the ground state?

A non-time-reversing g: $|0\rangle \mapsto e^{2\pi i\phi(g)}|0\rangle$.

A time-reversing g: $|0\rangle \mapsto e^{2\pi i\phi(g)}K|0\rangle$, where K is complex conjugation.

Imposing group law tells us that $\phi(g)$ is a twisted 1-cocycle.

Redefining $|0\rangle$ by a phase $\exp(2\pi i\alpha)$ modifies this twisted 1-cocycle by a twisted 1-coboundary.

One can say that $H^1_\sigma(G, U(1))$ classifies $d = 0$ SRE phases.

Bosonic SRE phases in $d = 1$, general symmetry

Consider again zero modes on the left boundary. An element $g \in G$ conjugates them by

$$M_L(g)K^{\epsilon(g)},$$

where $M_L(g)$ is a unitary matrix, K is complex conjugation, and $\epsilon(g) = 1$ if $\sigma(g) = -1$ and $\epsilon(g) = 0$ otherwise.

Now require the group law to hold up to a phase. (This phase is then canceled by a phase from the right boundary) Then associativity of matrix multiplication implies that that the phase is a twisted 2-cocycle.

Redefining $M_L(g)$ by a phase $\exp(2\pi i\alpha(g))$ does not affect the action on zero modes, but it modifies the twisted 2-cocycle by a twisted coboundary.

Hence bosonic SRE phases in $d = 1$ are classified by $H^2_\sigma(G, U(1))$.

Some examples

For $G = \mathbb{Z}_2$, we have $H^2(G, U(1)) = 0$. Hence no nontrivial SRE phases.

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ we have $H^2(G, U(1)) = \mathbb{Z}_2$. Hence there is a unique nontrivial SRE phase (Haldane spin chain). The basic projective representation of G has dimension 2, hence on an interval we have quadruple degeneracy (two from each end).

For $G = \mathbb{Z}_2^T$ (time-reversal symmetry) we have $H^2_\sigma(G, U(1)) = \mathbb{Z}_2$. Hence a unique nontrivial SRE phase. Each endpoint is doubly degenerate ("Kramers doublet"), on an interval we have quadruple degeneracy.

Kramers doublets and Wigner's theorem

Every unitary transformartion can be diagonalized, and eigenvalues have the form $e^{i\theta}$.

E. Wigner showed that every anti-unitary can be block-diagonalized, with 1 x 1 blocks of the form K (complex conjugation), and 2 x 2 blocks of the form

$$W_\theta = \begin{pmatrix} 0 & e^{i\theta/2} \\ e^{-i\theta/2} & 0 \end{pmatrix} K, \quad 0 < \theta \leq \pi.$$

If we want W_θ^2 be proportional to identity, then $\theta = \pi$, and $W_\pi = \sigma_2 K$.

If $T^2 = -1$, it acts as W_π, and each state has a partner (its time-reversal). This pair of states is called a Kramers doublet.

Fermionic SRE phases in $d = 1$

Now let us discuss fermionic SRE phases in $d = 1$, but only for $G = 0$ (and $\hat{G} = \mathbb{Z}_2^F$).

The algebra of zero modes on the left boundary can be either supermatrices, i.e. all operators on a $\mathbb{Z}_2$-graded vector space U_L, or a tensor product of such algebra with $\mathrm{Cl}(1)$.

I remind that $\mathrm{Cl}(n)$ is an algebra generated by n Dirac matrices Γ_i. For even n it is isomorphic to an algebra of supermatrices, for odd n it is isomorphic to a tenspr product of the algebra of supermatrices and $\mathrm{Cl}(1)$.

Adding impurities does affect the presence or absence of a $\mathrm{Cl}(1)$ factor. Note also that $\mathrm{Cl}(1) \otimes \mathrm{Cl}(1) \simeq \mathrm{Cl}(2)$.

Hence fermionic SRE phases in $d = 1$ are classified by $\mathbb{Z}_2$.

Mayorana chain (Kitaev)

Let us examine the unique nontrivial fermionic SRE phase.

On each booundary one has an odd zero model γ_L or γ_R. Together they form $\mathrm{Cl}(2)$.

The basic graded representation of $\mathrm{Cl}(2)$ is two-dimensional, so we have double degeneracy on an interval.

One of these states is bosonic, the other one is fermionic. The degeneracy cannot be lifted by any local modification of the Hamiltonian.

γ_L acts by σ_2, γ_R acts by σ_3.

Hamiltonian for the Majorana chain

Place fermionic $c_j, c_j^\dagger$ on the j^{th} site, $j = 1, \ldots, N$. Let

$$\gamma_j = c_j + c_j^\dagger, \quad \gamma_j' = i(c_j - c_j^\dagger).$$

These are generators of $\mathrm{Cl}(2N)$ ("Majorana fermions"). Let

$$H = -it \sum_{j=1}^{N-1} \gamma_j \gamma_{j+1}'.$$

The unpaired Majorana fermions γ_1' and γ_N can be identified with γ_L and γ_R.

Majorana chain as a topological superconductor

In terms of $c_j, c_j^\dagger$, the Majorana chain Hamiltonian is:

$$H = t \sum_{j=1}^{N-1} \left(c_j^\dagger c_{j+1} + c_j c_{j+1} + h.c. \right).$$

The 1st term is a hopping term, the 2nd term is a superconducting pairing term breaking $U(1)$ (particle number symmetry). Thus the Majorana chain is a 1d topological superconductor.

Stacking two chains gives a system equivalent to the trivial SRE phase. Indeed, two zero modes $\gamma_L, \tilde{\gamma}_L$ can be lifted by a local boundary term

$$H_L = -i t_L \gamma_L \tilde{\gamma}_L,$$

and similarly for the right boundary.

Thus fermionic 1+1d SRE phases without any bosonic symmetries are classified by $\mathbb{Z}_2$.