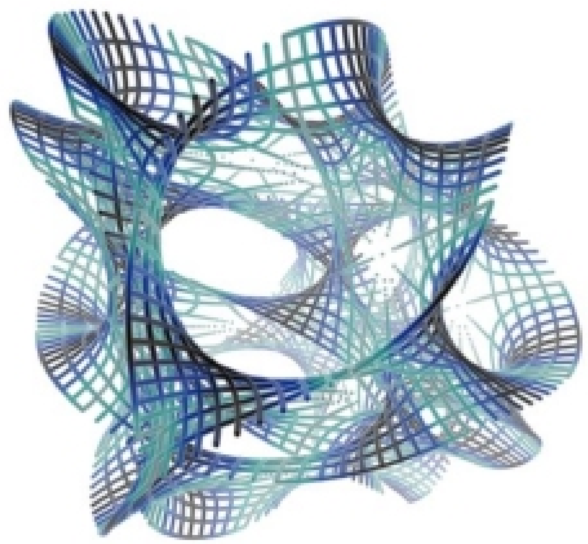
Double Geometry, Generalised Geometry and String Theory



Special Geometric Structures in Mathematics & Physics

Hamburg, 2014

Double Field Theory

Hull & Zwiebach

- From sector of String Field Theory. Features some stringy physics, including T-duality, in simpler setting
- Strings see a doubled space-time
- Needed for non-geometric backgrounds
- Doubled space fully dynamic
- *Strong constraint* restricts to subsector in which extra coordinates auxiliary: get conventional field theory locally. Duality covariant sugra.

What is DFT geometry?

- Metric & fields on doubled space $g_{ij}(x,\tilde{x}),\ b_{ij}(x,\tilde{x}),\ \phi(x,\tilde{x})$
- Novel symmetry, reduces to diffeos + B-field trans. when no dependence on dual coords
- Recent work: find finite gauge transformations & use to understand doubled geometry
- Hohm & Zwiebach: finite gauge transformations with non-associative composition. Non-associative geometry?

Double trouble?

- CMH: doubled space from string theory is manifold, even for non-geometric backgrounds, giving rather different picture
- Recent proposals: try to relate finite DFT gauge transformations to diffeomorphisms of doubled space.
- Problems arise as these are different groups
- Constant 'metric' η in DFT. Is doubled geometry flat?

New Results:

arXiv:1406.7794

- Simple explicit form of finite gauge transformations
- Associative, works for full symmetry group
- Doubled space is a manifold, not flat
- Gives geometric understanding of ‘generalised tensors’ & relation to generalised geometry
- Transition functions give global picture

Strings on Torus

D=n+d

Target space	$\mathbb{R}^{n-1,1} \times T^d$
Coordinates	$x^i = (x^\mu, x^a)$
Momenta	$p_i = (p_\mu, p_a)$
Winding	$w^i = (w^\mu, w^a)$
Dual coordinates (conjugate to winding)	$\tilde{x}_i = (\tilde{x}_\mu, \tilde{x}_a)$
Constant metric and B-field	$E_{ij} = G_{ij} + B_{ij}$

Compact dimensions

p_a, w^a discrete, in Narain lattice, $x^a, \tilde{x}_a$ periodic

Non-compact dimensions x^μ, p_μ continuous

Usually take $w^\mu = 0$ so $\dfrac{\partial}{\partial \tilde{x}_\mu} = 0$, fields $\psi(x^\mu, x^a, \tilde{x}_a)$

String Theory on a Torus

Infinite set of double fields

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x}), \ldots, C_{ijk\ldots l}(x,\tilde{x}), \ldots$$

- Fields on double space satisfy differential constraint
- T-duality symmetry
- Coordinates doubled, but tensor indices not
- Each field carries "level numbers" $N, \bar{N}$

T-Duality

- Interchanges momentum and winding
- Equivalence of string theories on dual backgrounds with very different geometries
- String field theory symmetry, provided fields depend on both $x, \tilde{x}$ Kugo, Zwiebach
- For fields $\psi(x^\mu)$ not $\psi(x^\mu, x^a, \tilde{x}_a)$ Buscher
- Generalise to fields $\psi(x^\mu, x^a, \tilde{x}_a)$

Generalised T-duality Dabholkar & CMH

Free Field Equations (B=0)

$$L_0 + \bar{L}_0 = 2$$

$$p^2 + w^2 = N + \bar{N} - 2$$

Treat as field equation, kinetic operator in doubled space

$$G^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + G_{ij} \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}$$

$$L_0 - \bar{L}_0 = 0$$

$$p_i w^i = N - \bar{N}$$

Treat as constraint on double fields

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} \qquad (\Delta - \mu)\psi = 0$$

Constrained fields $\psi(x^\mu, x^a, \tilde{x}_a)$

$$(\Delta - \mu)\psi = 0 \qquad \mu = N - \bar{N}$$

Momentum space $\psi(p_\mu, p_a, w^a)$ $\qquad \Delta = p_a w^a$

Momentum space: Dimension n+2d
Cone: $p_a w^a = 0$ or hyperboloid: $p_a w^a = \mu$
dimension n+2d-1

DFT: fields on cone or hyperboloid, with discrete p,w
Problem: naive product of fields on cone do not lie on cone. Vertices need projectors

Restricted fields: Fields that depend on d of 2d torus momenta, e.g. $\psi(p_\mu, p_a)$ or $\psi(p_\mu, w^a)$
Simple subsector, no projectors needed, no cocycles.

DFT gives O(D,D) covariant formulation

O(D,D) Covariant Notation

$$X^M \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix} \qquad \partial_M \equiv \begin{pmatrix} \partial^i \\ \partial_i \end{pmatrix}$$

$$\eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad M = 1, ..., 2D$$

$$\Delta \equiv \frac{\partial^2}{\partial x^i \partial \tilde{x}_i} = \frac{1}{2} \partial^M \partial_M$$

Constraint $\partial^M \partial_M A = 0$

on all fields and parameters

Weak Constraint or weak section condition

Arises from SFT constraint

$$L_0^- \Psi = 0, \qquad L_0^- = L_0 - \bar{L}_0$$

Projectors and Cocycles

Naive product of constrained fields doesn't satisfy constraint

$$L_0^- \Psi_1 = 0, L_0^- \Psi_2 = 0 \quad \text{but} \quad L_0^- (\Psi_1 \Psi_2) \neq 0$$

$$\Delta A = 0, \Delta B = 0 \quad \text{but} \quad \Delta(AB) \neq 0$$

String product has explicit projection

Leads to a symmetry that is not a Lie algebra, but is a homotopy lie algebra.

Double field theory requires projections.

SFT has non-local cocycles in vertices, DFT should too

Cocycles and projectors not needed in cubic action

- Weakly constrained DFT non-local
- ALL doubled geometry dynamical, evolution in all doubled dimensions
- Restrict to simpler theory: STRONG CONSTRAINT
- Fields then depend on only half the doubled coordinates
- Locally, just conventional SUGRA written in duality symmetric form, close to generalised geometry approach

Strong Constraint for DFT Hohm, H &Z

$$\partial^M \partial_M (AB) = 0 \qquad\qquad (\partial^M A)\,(\partial_M B) = 0$$

on all fields and parameters

If impose this, then it implies weak form, but product of constrained fields satisfies constraint.

This gives **Restricted DFT,** a subtheory of DFT

Locally, it implies fields only depend on at most half of the coordinates, fields are restricted to null subspace N.
Looks like conventional field theory on subspace N

- If fields supported only on submanifold N of doubled space M, recover Siegel's duality covariant form of (super)gravity on N

- In general get this only locally. In each 2D-dim patch of doubled space, fields supported on D-dim sub-patch, but sub-patches don't fit together to form a manifold with smooth fields.

- In string theory, T-duality acts on torus or fibres of torus fibration, relates local modes and winding

- Winding modes: doubling of torus or fibres

- Other topologies may not have windings, or have different numbers of momenta and windings. No T-duality. No doubling?

- DFT 'background independent' HHZ. Can write on doubling of any space. What is double if not derived from string theory?

Seek **double field theory** for

$$g_{ij}(x,\tilde{x}), b_{ij}(x,\tilde{x}), \phi(x,\tilde{x})$$

$$e^{-2d} = e^{-2\phi}\sqrt{-g}$$

d invariant under Buscher T-duality rules

g,b are DxD matrices depending on 2D coordinates

Impose strong constraint

Generalised T-duality transformations:

$$X'^M \equiv \begin{pmatrix} \tilde{x}'_i \\ x'^i \end{pmatrix} = hX^M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

$$\mathcal{E}_{ij} = g_{ij} + b_{ij}$$

h in O(d,d;Z) acts on toroidal coordinates only

$$\mathcal{E}'(X') = (a\mathcal{E}(X) + b)(c\mathcal{E}(X) + d)^{-1}$$

$$d'(X') = d(X)$$

Buscher if fields independent of toroidal coordinates
Generalisation to case without isometries

DFT in D DIMENSIONS with strong constraint

DFT on $M = \mathbb{R}^D$

O(D,D) symmetry acting on fields and coordinates.
Theory written in doubled space $\mathbb{R}^{2D}$
Gauge transformations written in O(D,D) covariant way, acting on fields, not coordinates

DFT on $M = \mathbb{R}^{n-1,1} \times T^d$

O(D,D) broken to subgroup containing B-shifts and

$$O(n,n) \times O(d,d;\mathbb{Z})$$

B-shifts and $GL(n,\mathbb{R}) \times GL(d,\mathbb{Z})$
arise from local symmetries.

Generalised Metric Formulation

Hohm, H &Z

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij}-b_{ik}g^{kl}b_{lj} \end{pmatrix}.$$

2 Metrics on double space $\quad \mathcal{H}_{MN},\ \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\quad \mathcal{H}^{MP}\mathcal{H}_{PN} \;=\; \delta^{M}{}_{N}$

Generalised Metric Formulation

Hohm, H &Z

$$\mathcal{H}_{MN} \quad = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix} .$$

2 Metrics on double space $\quad \mathcal{H}_{MN}, \; \eta_{MN}$

$$\mathcal{H}^{MN} \equiv \eta^{MP}\mathcal{H}_{PQ}\eta^{QN}$$

Constrained metric $\quad \mathcal{H}^{MP}\mathcal{H}_{PN} \; = \; \delta^M{}_N$

Covariant O(D,D) Transformation

$$h^P{}_M h^Q{}_N \mathcal{H}'_{PQ}(X') \; = \; \mathcal{H}_{MN}(X)$$

$$X' = hX \qquad\qquad h \in O(D,D)$$

O(D,D) covariant action

$$S = \int dx d\tilde{x}\, e^{-2d}\ L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$- 2\,\partial_M d\,\partial_N\mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

- Lagrangian L CUBIC in fields!
- Indices raised and lowered with η_{MN}
- O(D,D) covariant (in $\mathbb{R}^{2D}$)

2-derivative action

$$S \;=\; S^{(0)}(\partial,\partial)+S^{(1)}(\partial,\tilde{\partial})+S^{(2)}(\tilde{\partial},\tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

Gives usual action (+ surface term)

$$\int dx\sqrt{-g}e^{-2\phi}\left[R+4(\partial\phi)^2-\frac{1}{12}H^2\right]$$

$$S^{(0)}=S(\mathcal{E},d,\partial)$$

2-derivative action

$$S \;=\; S^{(0)}(\partial,\partial) + S^{(1)}(\partial,\tilde{\partial}) + S^{(2)}(\tilde{\partial},\tilde{\partial})$$

Write $S^{(0)}$ in terms of usual fields

Gives usual action (+ surface term)

$$\int dx\sqrt{-g}e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]$$

$$S^{(0)} = S(\mathcal{E}, d, \partial)$$

$$S^{(2)} = S(\mathcal{E}^{-1}, d, \tilde{\partial})$$ T-dual!

$S^{(1)}$ strange mixed terms

- Restricted DFT:
  fields independent of half the coordinates
- If independent of $\tilde{x}$, equivalent to usual action
- Duality covariant: duality changes which half of coordinates theory is independent of
- Equivalent to Siegel's formulation Hohm & Kwak
- Good for non-geometric backgrounds

O(D,D) covariant action

$$S \;=\; \int dx d\tilde{x}\, e^{-2d}\; L$$

$$L \;=\frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\,\partial_N\mathcal{H}_{KL}-\frac{1}{2}\mathcal{H}^{MN}\partial_N\mathcal{H}^{KL}\,\partial_L\mathcal{H}_{MK}$$
$$-\,2\,\partial_M d\,\partial_N\mathcal{H}^{MN}+4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Gauge Transformation

$$\delta_\xi\mathcal{H}^{MN} \;=\; \xi^P\partial_P\mathcal{H}^{MN}$$
$$+\,(\partial^M\xi_P-\partial_P\xi^M)\,\mathcal{H}^{PN}+(\partial^N\xi_P-\partial_P\xi^N)\,\mathcal{H}^{MP}$$

Rewrite as “Generalised Lie Derivative”

$$\delta_\xi\mathcal{H}^{MN}=\widehat{\mathcal{L}}_\xi\mathcal{H}^{MN}$$

Generalised Lie Derivative

$$A_{N_1\ldots}^{M_1\ldots}$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N \equiv \xi^P \partial_P A_M{}^N$$
$$+(\partial_M \xi^P - \partial^P \xi_M) A_P{}^N + (\partial^N \xi_P - \partial_P \xi^N) A_M{}^P$$

$$\widehat{\mathcal{L}}_\xi A_M{}^N = \mathcal{L}_\xi A_M{}^N - \eta^{PQ} \eta_{MR}\, \partial_Q \xi^R\, A_P{}^N$$
$$+ \eta_{PQ} \eta^{NR}\, \partial_R \xi^Q\, A_M{}^P$$

Gauge Algebra

Parameters Σ^M

Gauge Algebra $\quad [\delta_{\Sigma_1}, \delta_{\Sigma_2}] = \delta_{[\Sigma_1,\Sigma_2]_C}$

$$\left[\,\widehat{\mathcal{L}}_{\xi_1}\,,\widehat{\mathcal{L}}_{\xi_2}\,\right] = -\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_{\rm C}}$$

C-Bracket:

$$[\Sigma_1, \Sigma_2]_C \equiv [\Sigma_1, \Sigma_2] - \frac{1}{2}\,\eta^{MN}\eta_{PQ}\,\Sigma^P_{[1}\,\partial_N\,\Sigma^Q_{2]}$$

Lie bracket + metric term

Parameters $\Sigma^M(X)$ restricted to N
Decompose into vector + 1-form on N
C-bracket reduces to Courant bracket on N

Same covariant form of gauge algebra found in similar context by Siegel

Jacobi Identities not satisfied!

$$J(\Sigma_1, \Sigma_2, \Sigma_3) \equiv [\,[\Sigma_1, \Sigma_2]\,, \Sigma_3\,] + \text{cyclic} \neq 0$$

for both C-bracket and Courant-bracket

How can bracket be realised as a symmetry algebra?

$$[\,[\delta_{\Sigma_1}, \delta_{\Sigma_2}]\,, \delta_{\Sigma_3}\,] + \text{cyclic} = \delta_{J(\Sigma_1, \Sigma_2, \Sigma_3)}$$

Symmetry is Reducible

Parameters of the form $\Sigma^M = \eta^{MN}\partial_N \chi$
do not act

Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
do not act

Symmetry is Reducible

Parameters of the form $\Sigma^M = \eta^{MN}\partial_N \chi$
do not act

Gauge algebra determined up to such transformations

cf 2-form gauge field $\delta B = d\alpha$
Parameters of the form $\alpha = d\beta$
do not act

Resolution:

$$J(\Sigma_1, \Sigma_2, \Sigma_3)^M = \eta^{MN}\partial_N \chi$$

$\delta_{J(\Sigma_1,\Sigma_2,\Sigma_3)}$ does not act on fields

What is the Geometry of Generalised Tensors?

Doubled space coordinates $X^M = \begin{pmatrix} x^m \\ \tilde{x}_m \end{pmatrix}$

O(D,D) covariant vectors and tensors

$$V^M = \begin{pmatrix} v^m \\ \tilde{v}_m \end{pmatrix} \qquad \mathcal{H}_{MN}$$

Suggestive of tensors on doubled space, but transformations not those of diffeomorphisms on doubled space, as generated by generalised Lie derivative, not usual Lie derivative.

If not tensors on doubled space, what are they?

Finite transformations

Not diffeomorphisms of doubled space, as algebra given by C-bracket, not Lie bracket.

What do you get by exponentiating infinitesimal transformations?

Hohm, Zwiebach

cf exponentiating usual Lie derivative

$$A'_m(x) \;=\; e^{\mathcal{L}_\xi}\, A_m(x)$$

gives transformations induced by coordinate transformation

$$x'^m \;=\; e^{-\xi^k \partial_k}\, x^m$$

HZ write finite transformations for DFT in active form

$$X \to X' = f(X)$$

with generalised vectors transforming as

$$A'_M(X') \; = \; \mathcal{F}_M{}^N A_N(X)$$

$$\mathcal{F}_M{}^N \; \equiv \; \frac{1}{2}\Big(\frac{\partial X^P}{\partial X'^M}\,\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X_P}\,\frac{\partial X^N}{\partial X'^P}\Big)$$

For conventional diffeos, would have

$$\mathcal{F}_M{}^N = \frac{\partial X^N}{\partial X'^M}$$

Important property: η_{MN} invariant

Looks a bit like a conventional geometry.
But there's a catch....
Exponentiating gen. Lie derivative

$$A'_M(X) \; = \; e^{\widehat{\mathcal{L}}_\xi} \, A_M(X) \, ,$$

gives transformations of fields that form a group
(violation of Jacobi's doesn't act on fields)

These induce transformations of coordinates

$$X'^M \; = \; e^{-\Theta^K(\xi)\partial_K} X^M \qquad\qquad \Theta^K(\xi) \; \equiv \; \xi^K + \mathcal{O}(\xi^3) \, ,$$

Not a group. Strange composition law.
Non-associative geometry? Hohm, Lust, Zwiebach

- Attempts ‘realising’ DFT gauge transformations as diffeomorphisms of doubled space’

- DFT gauge group and diffeomorphism group not homomorphic, so leads to trouble

- Attempts realising b-field transforms $\delta b = d\lambda$ as coordinate shifts: $\delta\tilde{x}_i = \lambda_i$

- This would only work for trivial gerbes (with b-field transitions that are trivial cocycles in triple overlaps)

Finite Transformations and Geometry

Constraint $\partial^M \partial_M A = 0$

Strong Constraint for restricted DFT

$$\partial^M \partial_M (AB) = 0 \qquad\qquad (\partial^M A)\,(\partial_M B) = 0$$

Generic solution in patch $\hat{U}$: fields and parameters independent of half the coordinates:

$$\tilde{\partial}^i = 0$$

$$X^M = \begin{pmatrix} x^m \\ \tilde{x}_m \end{pmatrix} \qquad \partial_M = \begin{pmatrix} \partial_m \\ \tilde{\partial}^m \end{pmatrix} \qquad \eta_{MN} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Fields live on null patch U, coordinates x: $\phi(x^m)$

U 'physical' spacetime

Vectors

$$V^M = \begin{pmatrix} v^m \\ \tilde{v}_m \end{pmatrix}$$

Generalised Lie derivative

$$\widehat{\mathcal{L}}_V W^M \;=\; V^P \partial_P W^M + W^P \left(\partial^M V_P - \partial_P V^M \right)$$

Vectors

$$V^M = \begin{pmatrix} v^m \\ \tilde{v}_m \end{pmatrix}$$

Generalised Lie derivative

$$\widehat{\mathcal{L}}_V W^M \; = \; V^P \partial_P W^M + W^P \left(\partial^M V_P - \partial_P V^M\right)$$

has the components

$$(\widehat{\mathcal{L}}_V W)^m = \mathcal{L}_v w^m$$

$$(\widehat{\mathcal{L}}_V W)_m = \mathcal{L}_v \tilde{w}_m + w^p (\partial_m \tilde{v}_p - \partial_p \tilde{v}_m)$$

$\mathcal{L}_v$ is usual Lie derivative

$$\mathcal{L}_v w^m = v^p \partial_p w^m - w^p \partial_p v^m$$
$$\mathcal{L}_v \tilde{w}_m = v^p \partial_p \tilde{w}_m + \tilde{w}_p \partial_m v^p$$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

$$\delta w^m = \mathcal{L}_v w^m$$

$$\delta \tilde{w}_m = \mathcal{L}_v \tilde{w}_m + w^p(\partial_m \tilde{v}_p - \partial_p \tilde{v}_m)$$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

$$\delta w^m = \mathcal{L}_v w^m$$
$$\delta \tilde{w}_m = \mathcal{L}_v \tilde{w}_m + w^p(\partial_m \tilde{v}_p - \partial_p \tilde{v}_m)$$

Introduce a gerbe connection b with transformations

$$\delta_v b_{mn} = \mathcal{L}_v b_{mn} + \partial_m \tilde{v}_n - \partial_n \tilde{v}_m$$

Define $\hat{w}_m = \tilde{w}_m - b_{mn} w^n$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

$$\delta w^m = \mathcal{L}_v w^m$$
$$\delta \tilde{w}_m = \mathcal{L}_v \tilde{w}_m + w^p (\partial_m \tilde{v}_p - \partial_p \tilde{v}_m)$$

Introduce a gerbe connection b with transformations

$$\delta_v b_{mn} = \mathcal{L}_v b_{mn} + \partial_m \tilde{v}_n - \partial_n \tilde{v}_m$$

Define $\hat{w}_m = \tilde{w}_m - b_{mn} w^n$

Then $\delta \hat{w}_m = \mathcal{L}_v \hat{w}_m$

Under infinitesimal transformation $\delta W^M = \widehat{\mathcal{L}}_V W^M$

$$\delta w^m = \mathcal{L}_v w^m$$
$$\delta \tilde{w}_m = \mathcal{L}_v \tilde{w}_m + w^p(\partial_m \tilde{v}_p - \partial_p \tilde{v}_m)$$

Introduce a gerbe connection b with transformations

$$\delta_v b_{mn} = \mathcal{L}_v b_{mn} + \partial_m \tilde{v}_n - \partial_n \tilde{v}_m$$

Define $\hat{w}_m = \tilde{w}_m - b_{mn} w^n$

Then $\delta \hat{w}_m = \mathcal{L}_v \hat{w}_m$

$\hat{w}$ transforms as 1-form under v-transformations and is invariant under $\tilde{v}$ transformations!

COVARIANT TRANSFORMATIONS

Then given $$W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}$$

can define $$\hat{W}^M = \begin{pmatrix} w^m \\ \hat{w}_m \end{pmatrix} = \begin{pmatrix} w^m \\ \tilde{w}_m - b_{mn} w^n \end{pmatrix}$$

$$\delta \hat{W}^M = \mathcal{L}_v \hat{W}^M$$

It is invariant under $\tilde{v}$ transformations

COVARIANT TRANSFORMATIONS

Then given $$W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}$$

can define $$\hat{W}^M = \begin{pmatrix} w^m \\ \hat{w}_m \end{pmatrix} = \begin{pmatrix} w^m \\ \tilde{w}_m - b_{mn} w^n \end{pmatrix}$$

$$\delta \hat{W}^M = \mathcal{L}_v \hat{W}^M$$

It is invariant under $\tilde{v}$ transformations

<u>Gives finite transformations!</u>

$$x \to x'(x) = e^{-v^m \partial_m} x$$

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n} \qquad \hat{w}'_m(x') = \hat{w}_n(x) \frac{\partial x^n}{\partial x'^m}$$

Can also find the transformation of $\tilde{w}$

Standard finite transformations of gerbe connection:

$$b'_{mn}(x') = [b_{pq}(x) + (\partial_p \tilde{v}_q - \partial_q \tilde{v}_p)(x)] \frac{\partial x^p}{\partial x'^m} \frac{\partial x^q}{\partial x'^n}$$

gives

$$\tilde{w}'_m(x') = \Big[\tilde{w}_n(x) + (\partial_n \tilde{v}_q - \partial_q \tilde{v}_n) w^q(x)\Big] \frac{\partial x^n}{\partial x'^m}$$

$$w'^m(x') = w^n(x) \frac{\partial x'^m}{\partial x^n}$$

DFT and GENERALISED GEOMETRY

Consider case fields restricted to submanifold N of M

w transforms as a tangent vector on N and $\hat{w}$ transforms as a cotangent vector under diff(N).
Both invariant under $\tilde{v}$ transformations.

$w \oplus \hat{w}$ is a section of $(T \oplus T^*)N$

This is the generalised tangent bundle on N

$$w \oplus \tilde{w}$$

is section of E, which is $T \oplus T^*$ twisted by a gerbe

$$0 \to T^* \to E \to T \to 0$$

Then ‘generalized vectors’ $W^M = \begin{pmatrix} w^m \\ \tilde{w}_m \end{pmatrix}$

are not really vectors on doubled space, but are sections of generalised tangent bundle over ‘physical space’ N, twisted by a gerbe

$v^m(x)$ symmetries are diffeomorphisms of N

$\tilde{v}_m(x)$ symmetries are b-field gauge transformations on N

Gauge symmetry of DFT

$$\mathrm{Diff}(N) \ltimes \Lambda^2_{closed}(N)$$

Global O(D,D)

2D dimensional doubled space M, D dim. subspace N

3 kinds of vectors $V^M(X)$

Vector fields on M:

Sections of TM,

transform under diff(M)

Hatted generalised vector fields $\hat{W}$ on M:

Sections of $(T \oplus T^*)N$

transform under diff(N)

Generalised vector fields W on M

Sections of E(N)

transform under $\mathrm{Diff}(N) \ltimes \Lambda^2_{closed}(N)$

Extends to tensors, generalised tensors and untwisted generalised tensors

Generalised Metric

$$\mathcal{H}_{MN} = \begin{pmatrix} g_{mn} - b_{mk}g^{kl}b_{ln} & b_{mk}g^{kn} \\ -g^{mk}b_{kn} & g^{mn} \end{pmatrix}$$

Finite transformations give usual ones for g,b

Untwisted form of generalised metric

$$\hat{\mathcal{H}}_{MN} = \begin{pmatrix} g_{mn} & 0 \\ 0 & g^{mn} \end{pmatrix}$$

Natural metric on $T \oplus T^*$

Constant O(D,D) Metric

Matrix with constant components:

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If this is tensor on M, then it is flat metric and this would greatly restrict possible M. Not invariant under Diff(M)

Constant O(D,D) Metric

Matrix with constant components:

$$\eta_{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

If this is tensor on M, then it is flat metric and this would greatly restrict possible M. Not invariant under Diff(M)

If it is generalised tensor, section of $E^* \otimes E^*(N)$

$$\hat{\eta}_{MN} = \eta_{MN}$$

Invariant under DFT gauge transformations, natural object in DFT. Metric for E(N), not T(M)
No restriction on geometry

Global Structure

- Each patch U of M has D-dim sub-patch V
- Local symmetries: diff(V) and b-field gauge trans on V
- Using these gives D-dim manifold & b-field
- If U has T^{2d} fibration, O(d,d;Z) symmetry
- Using this also in transition functions gives T-fold

Conclusions

- Doubled space M is manifold, need not be flat
- If fields live on submanifold N, DFT gives conventional field theory on N
- Generalised tensors in $E \otimes E \cdots \otimes E(N)$ not $T \otimes T \cdots \otimes T(M)$
- E(N) is $(T \oplus T^*)N$ twisted by gerbe
- DFT gauge transformations just diffeos and b-field gauge transformations on N

- DFT then gives sugra in duality symmetric formulation, using generalised geometry on N

- DFT gives covariant formulation of generalised geometry, independent of choice of duality frame

- More generally, all this is only true locally in patches. Globally, patching using symmetries of DFT. If duality used, non-geometric.

- Then DFT gives extension of field theory to non-geometric spaces, e.g. T-folds