Higgs bundles for diffeomorphism groups

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GAUGE THEORY AND DIFFEOMORPHISMS

- gauge theory – Lie group G, connection, curvature, equations....

- replace G by Diff(X)

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- replace G by Diff(X)

- bundle: manifold $E \to M$, fibre X

- connection: horizontal distribution $H \subset TE$

- flat iff integrable = transverse foliation

NAHM'S EQUATIONS FOR SDiff(M^3)

- 3-manifold M^3 with volume form

- X_1, X_2, X_3 volume-preserving vector fields on M^3

- $$\frac{dX_1}{dt} = [X_2, X_3] \text{ etc.}$$

- $\Rightarrow$ hyperkähler metric on $M^3 \times (a, b)$

A. Ashtekar, T. Jacobson & L. Smolin, *A new characterization of half-flat solutions to Einstein's equation*, Commun. Math. Phys. **115** (1988) 631– 648.

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- $\Rightarrow$ hypercomplex structure on $M^3 \times (a, b)$

NJH, *Hypercomplex manifolds and the space of framings*, in "The geometric universe" 930, OUP 1998

HIGGS BUNDLES

- compact Riemann surface Σ, compact group G

- principal G-bundle P + connection A

- Higgs field $\Phi \in \Omega^{1,0}(\Sigma, \mathfrak{g}^c)$

- equations

$$F_A + [\Phi, \Phi^*] = 0, \quad \bar{\partial}_A \Phi = 0$$

- G^c-connection $\nabla_A + \Phi + \Phi^*$

- equations $\Rightarrow$ flat G^c-connection

- conversely, given a reductive representation $\pi_1(\Sigma) \to G^c$

- a harmonic section of $\tilde{\Sigma} \times_{\pi_1} G^c/G$

- ... defines a solution to the Higgs bundle equations

- G^c connection $\nabla_A + \Phi + \Phi^*$

- equations $\Rightarrow$ flat connection

- real form $G^r \subset G^c$, max compact H

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

- flat G^r connection if A reduces to H and $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$

EXAMPLE: $G = SU(2), G^c = SL(2,\mathbf{C}), G^r = SL(2,\mathbf{R})$

- $V = K^{-1/2} \oplus K^{1/2}$ $\qquad \Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

- $U(1)$-connection A on $K^{1/2}$

- $F_A + [\Phi, \Phi^*] = 0 \Rightarrow K = -1$ (Gaussian curvature

- uniformization: $\pi_1(\Sigma) \to SL(2,\mathbf{R})$

THE GROUP $SU(\infty)$

- SDiff(S^2) = group of symplectic diffeomorphisms of S^2

- $df = i_X\omega$ Hamiltonian vector fields

- Lie algebra = $C^\infty(S^2)/const.$

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- Lie algebra $= C^\infty(S^2)/const.$

- $SU(2) \subset \mathrm{SDiff}(S^2)$

- spherical harmonics $C^\infty/const. = \mathbf{3} + \mathbf{5} + \mathbf{7} + \ldots$

- $SU(2) \to SU(n)$ irreducible representation

- $\mathfrak{su}(n) = \mathbf{3} + \mathbf{5} + \mathbf{7} + \ldots + \mathbf{(2n-1)}$

- $SU(\infty) \overset{\text{def}}{=} \mathrm{SDiff}(S^2)$

- Poisson bracket $\neq$ Lie bracket (except on $SU(2)$)

PROPERTIES

- invariant metric

$$(f, g) = \int_{S^2} f g \omega$$

- invariant polynomials

$$p_n(f) = \int_{S^2} f^n \omega$$

- $\sim$ compact Lie group G

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- ... but no complexification G^c

$SU(\infty)$-CONNECTION

- 2-sphere bundle $p : M^4 \to \Sigma$

- non-vanishing section ω_F of $\Lambda^2 T_F^*$

- horizontal subbundle $H \subset TM$

- such that for each horizontal lift of a vector field X on M...

- .... $\mathcal{L}_X \omega_F = 0$

EXAMPLE

- 2-sphere bundle $p : M^4 \to \Sigma$

- symplectic form ω such that fibres are symplectic

- define horizontal subbundle H

  $=$ symplectic orthogonal to fibres

$SU(\infty)$ HIGGS BUNDLES

$SU(\infty)$-HIGGS FIELD

- Locally $\phi_1 dx_1 + \phi_2 dx_2$

- ϕ_i functions on Σ with values in $C^\infty(S^2)$

- $\sim$ functions on M

- $\Phi = (\phi_1 dx_1 + \phi_2 dx_2)^{1,0}$ section of $p^* K$ on M

- connection: $\quad \dfrac{\partial}{\partial x} + A_1, \quad \dfrac{\partial}{\partial y} + A_2$

  A_1, A_2 Hamiltonian vector fields on S^2 depending on x, y

- Higgs field:

  Φ_1, Φ_2 Hamiltonian vector fields on S^2 depending on x, y

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- $$\left[\frac{\partial}{\partial x} + A_1 - i\Phi_2, \frac{\partial}{\partial y} + A_2 + i\Phi_1\right] = 0$$

  (*cf.* $\nabla_A + \Phi + \Phi^*$ flat)

- complex vector fields

$$X_1 = \frac{\partial}{\partial x} + A_1 - i\Phi_2, \quad X_2 = \frac{\partial}{\partial y} + A_2 + i\Phi_1$$

$$[X_1, X_2] = 0$$

- $\Rightarrow$ integrable complex structure

- as long as $X_1, \bar{X}_1, X_2, \bar{X}_2$ are linearly independent

- iff Hamiltonian vector fields Φ_1, Φ_2 are linearly independent

- Hamiltonian functions ϕ_1, ϕ_2 for vector fields Φ_1, Φ_2

- linear dependence where $\{\phi_1, \phi_2\} = 0$ (Poisson bracket)

- defines a hypersurface N^3

$$\left[\frac{\partial}{\partial x} + A_1 + i\left(\frac{\partial}{\partial y} + A_2\right), \Phi_1 + i\Phi_2\right] = 0$$

$$\left[\frac{\partial}{\partial x} + A_1 + i\Phi_2, \frac{\partial}{\partial y} + A_2 - i\Phi_1\right] = 0$$

- two more complex structures $\Rightarrow$ hypercomplex manifold
- symplectic $\Rightarrow$ hyperkähler

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- ω_1-symplectic-orthogonal to fibres = SDiff(S^2)-connection

THE CANONICAL MODEL

- $V = K^{-1/2} \oplus K^{1/2}$ $\quad\quad$ $\Phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

- $SU(2) \subset SU(\infty)$

- defines the canonical folded hyperkähler metric

- .... the extension of the hyperbolic metric on Σ

- real form $SO(2) \subset SU(2), SL(2, \mathbf{R}) \subset SL(2, \mathbf{C})$

THE CANONICAL MODEL

Theorem (Feix, Kaledin) Given a real analytic Kähler metric on M there is a unique S^1-invariant hyperkähler extension to a neighbourhood of the zero section in T^*M.

- $\omega_2 + i\omega_3 =$ canonical complex symplectic form

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- $M = S^2$ complete metric (Eguchi-Hanson)

- $M = \Sigma$ surface of genus $g > 1$ incomplete (complete $\Rightarrow$ polynomial growth in π_1)

- take $\boldsymbol{\Sigma}$ with hyperbolic metric

- $T^*\boldsymbol{\Sigma} \subset \mathbf{P}(K \oplus 1)$

- the hyperkähler extension is defined on the unit disc bundle in $T^*\boldsymbol{\Sigma}$

- .... and extends to a folded hyperkähler metric on the S^2-bundle $\mathbf{P}(K \oplus 1)$

J.D.Gegenberg & A.Das, *Stationary Riemannian space-times with self-dual curvature,* Gen. Relativity Gravitation **16** (1984) 817–829.

H.Pedersen & B.Nielsen, *On some Euclidean Einstein metrics,* Lett.Math.Phys. **12** (1986) 277–282.

S.K.Donaldson, *Moment maps in differential geometry,* Surv. Differ. Geom., **8** Int. Press, Somerville, MA, 2003 171-189.

- Kähler form ω_1 on fibre $\quad i\dfrac{dwd\bar{w}}{4(1-|w|^2)^{1/2}}$

$$i\frac{dwd\bar{w}}{4(1-|w|^2)^{1/2}} = \frac{dx_1 \wedge dx_2}{2(1-x_1^2-x_2^2)^{1/2}} = \frac{dx_1 \wedge dx_2}{2x_3}$$

- well-defined on S^2

- $\omega_1, \omega_2, \omega_3$ well-defined on $M^4 \stackrel{S^2}{\to} \Sigma$

- take Σ with hyperbolic metric

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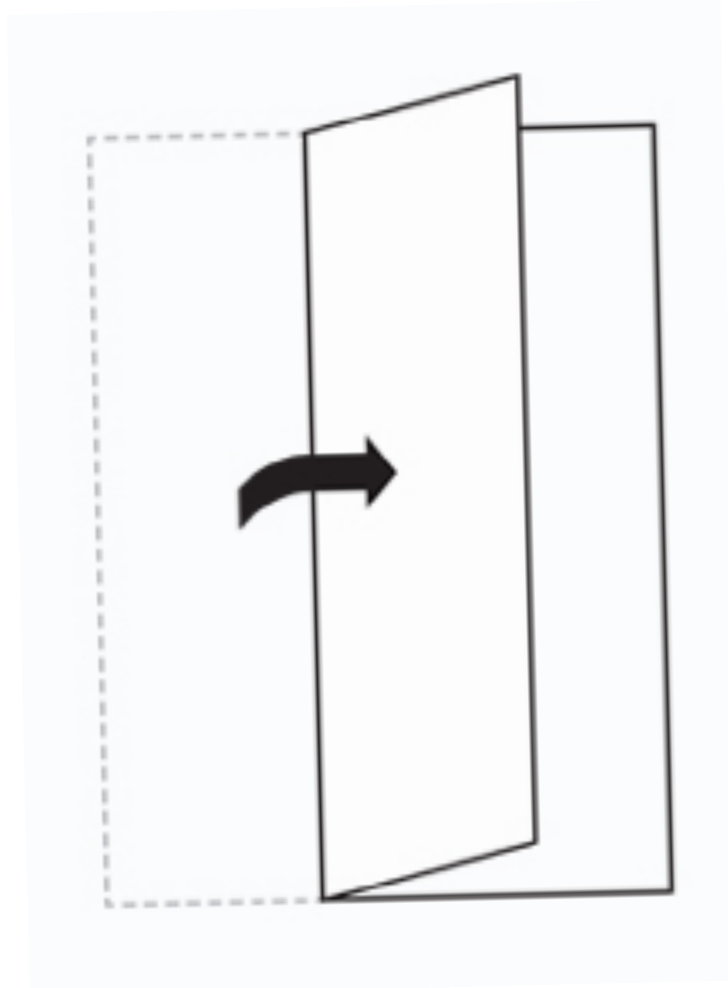
FOLDING



- $f : \mathbf{R}^2 \to \mathbf{R}^2 \qquad f(x, y) = (x^2, y)$

- $f^*(dx \wedge dy) = 2x dx \wedge dy$

- symplectic manifold M^{2m}: closed 2-form ω, $\omega^m \neq 0$

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- ... and $\omega|_N$ has maximal rank

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- ... and $\omega|_N$ has maximal rank

- normal form $\quad x dx \wedge dy + \sum_1^{m-1} du_i \wedge dv_i$

Theorem: Any compact oriented 4-manifold admits a folded Kähler structure.

R I Baykur, *Kähler decompositions of 4-manifolds*, AGT **6** (2006) 1239–1265.

(symplectic geometry of Stein surfaces)

- $M^4 = M^+ \cup N^3 \cup M^-$

- Kähler metric $\pm$ definite on $M^\pm$

HYPERKÄHLER GEOMETRY

4D HYPERKÄHLER MANIFOLD

- metric g, complex structures I, J, K

- Kähler forms $\omega_1, \omega_2, \omega_3$

- $\omega_1^2 = \omega_2^2 = \omega_3^2$

- $\omega_1\omega_2 = \omega_2\omega_3 = \omega_3\omega_1 = 0$

- metric $g = \omega_1\omega_2^{-1}\omega_3$

FOLDED HYPERKÄHLER

- closed 2- forms $\omega_1, \omega_2, \omega_3$
- $\omega_1^2 = \omega_2^2 = \omega_3^2$
- $\omega_1\omega_2 = \omega_2\omega_3 = \omega_3\omega_1 = 0$
- $\omega_1^2 = 0$ defines a smooth hypersurface N^3

- at $x \in N$, suppose $\omega_1, \omega_2, \omega_3$ are linearly independent in $\Lambda^2 T_x^* M$
- 3-dimensional subspace $V_x \subset \Lambda^2 T_x^* M$
- $\omega \in V_x \Rightarrow \omega^2 = 0 \Rightarrow$ decomposable $\omega = \alpha_1 \wedge \alpha_2$
- $\Rightarrow$ projective line in $\mathbf{P}(T_x^*)$

- α-planes and β-planes in the Klein quadric

- $\Leftrightarrow$ lines in a plane in $\mathbf{P}(T_x^*)$ or...

- .. lines through a point.

- all folded: $\omega_i = xdx \wedge \alpha_i + \beta_i \wedge \gamma_i$
- $=$ β-plane
- $=$ lines in $\mathbf{P}(T_x N)$

α-PLANES

- $\omega_1 = dx \wedge \varphi + x d\varphi$

  $\omega_2 = x dx \wedge \alpha_1 + \beta_1 \wedge \varphi$

  $\omega_3 = x dx \wedge \alpha_2 + \beta_2 \wedge \varphi$

- $[\varphi] \in \mathbf{P}(T^*)$

THE GROUP $SO(\infty)$

- $S^2 = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$

- involution $\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3)$

- $SO(\infty) = \{f \in \mathsf{SDiff}(S^2) : f\sigma = \sigma f\}$

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- involution $\sigma(x_1, x_2, x_3) = (x_1, x_2, -x_3)$
- $SO(\infty) = \{f \in \mathsf{SDiff}(S^2) : f\sigma = \sigma f\}$
- Lie algebra = odd functions on S^2
- $\mathfrak{m}$ = even functions

- $SL(\infty)$-Higgs bundle
- ϕ_1, ϕ_2 even $\Rightarrow \{\phi_1, \phi_2\}$ odd
- $\Rightarrow \{\phi_1, \phi_2\}$ vanishes on circle bundle
- $\Rightarrow$ fold $=$ circle bundle

- $SO(\infty)$ preserves the fixed point set $x_3 = 0$

- homomorphism $SO(\infty) \to \mathrm{Diff}(S^1)$

- $SL(\infty)$-Higgs bundle $\Rightarrow$ $\mathrm{Diff}(S^1)$-connection

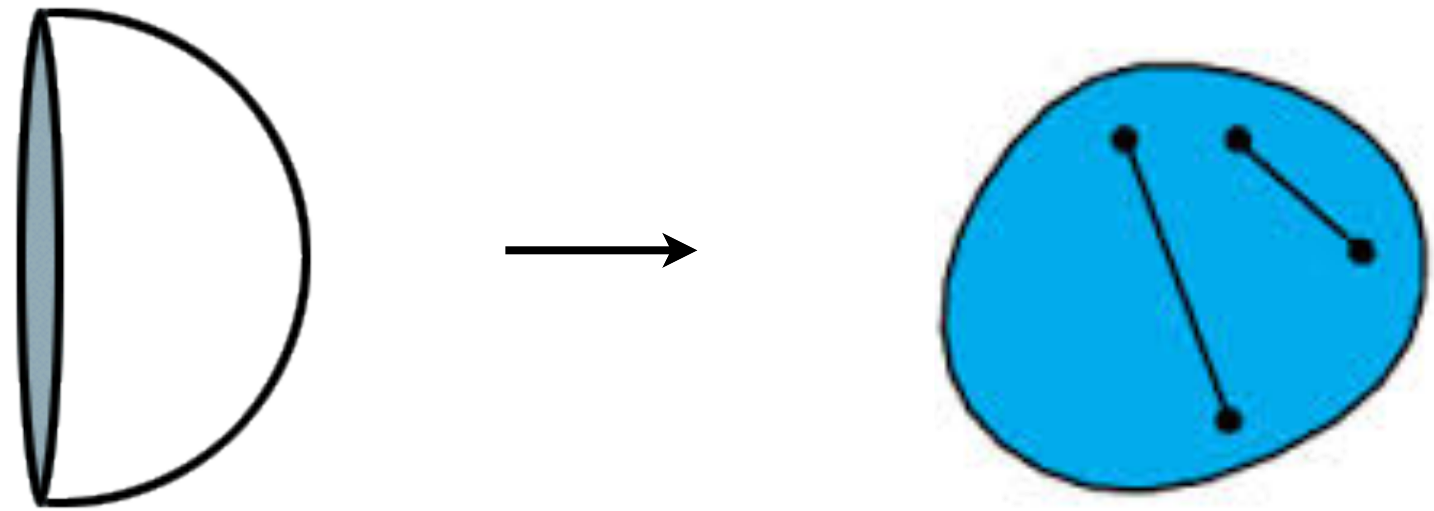
GEOMETRY OF THE FOLD

- Higgs field Φ section of p^*K

- ... assume it defines a diffeomorphism from one disc bundle in $f : M \to T^*\Sigma$

- ..then $\phi_1 dx_1 + \phi_2 dx_2 = f^*\theta$ canonical one-form

- and $\omega_2 + i\omega_3 = f^*(dw \wedge dz)$



- f maps the fold to a (non-quadratic) circle bundle in $T^*\Sigma$
- Finsler geometry $=$ circle bundle in $T\Sigma$
- Legendre transform $\Rightarrow T^*\Sigma$

- on $f(N)$, annihilator of β_1, φ = one-dimensional foliation

- $\sim$ Hamiltonian flow

- β_2 restricts to a parameter on the integral curve

- annihilator of φ = horizontal subspaces = Diff(S^1)-connection

- hyperkähler forms near $\{\phi_1, \phi_2\} = x = 0$

  $\omega_1 = dx \wedge \varphi + x d\varphi$

  $\omega_2 = x dx \wedge \alpha_1 + \beta_1 \wedge \varphi$

  $\omega_3 = x dx \wedge \alpha_2 + \beta_2 \wedge \varphi$

- on the fold N^3:

  closed 2-forms $\beta_1 \wedge \varphi, \beta_2 \wedge \varphi$

- on $f(N)$ restrictions of real and imaginary parts of $dw \wedge dz$

EXAMPLE: CANONICAL MODEL

- N = unit (cotangent) circle bundle for hyperbolic metric

- foliation = geodesic flow

- $\beta_2 = ds$ length

A QUESTION

- for each group $SL(n, \mathbf{R})$ there is a distinguished component of $\mathrm{Hom}(\pi_1(\Sigma), SL(n, \mathbf{R}))/SL(n, \mathbf{R})$

- ... Teichmüller space for $n = 2$

- .... $\cong \bigoplus_{m=2}^{n} H^0(\Sigma, K^m)$ using Higgs bundles

- Is there an analogue for $SL(\infty, \mathbf{R})$ and does it parametrize generalized geodesic structures?

EVIDENCE 1

- circle action $\Phi \mapsto e^{i\theta}\Phi$

- for higher Teichmüller space unique fixed point

- ... = uniformizing representation

$$\pi_1(\Sigma) \to SL(2,\mathbf{R}) \stackrel{S^m}{\to} SL(m+1,\mathbf{R})$$

Is the canonical model the only S^1-invariant folded hyperkähler manifold of this type?

- S^1-invariance = $SU(\infty)$ Toda equation

- locally $(e^u)_{tt} + u_{xx} + u_{yy} = 0$

- globally:

$$\frac{\partial^2 g}{\partial t^2} - Kg = 0$$

- $g_{tt} = Kg \Rightarrow$ volume is quadratic in t

- rescale g to constant volume metric h

- put $h = fg_H$, g_H=hyperbolic metric

$$t(2-t)f_{tt} + 4(1-t)f_t = \Delta_H \log f.$$

- boundary conditions + maximum $\Rightarrow f = const.$

EVIDENCE 2

- deformations of fixed point of circle action

- for higher Teichmüller space $\sim$ holomorphic sections of K^2, K^3,

- Also for $SL(\infty, \mathbf{R})$?

- θ canonical holomorphic 1-form on $T^*\Sigma$

- π Poisson tensor

- α holomorphic section of K^m

- h Hermitian form of hyperbolic metric

complex vector field $X^c = \pi(\alpha h^{-(k-1)}\bar{\theta}^{k-1})$

- X= real part of X^c
- $\Rightarrow$ the closed 2-forms $\mathcal{L}_X\omega_i$ are anti-self-dual
- first order deformation $\dot{\omega}_1 = 0, \dot{\omega}_2 = \mathcal{L}_X\omega_2, \dot{\omega}_3 = \mathcal{L}_X\omega_3$
- deformation of hyperkähler metric
- deformation of polynomial invariant $\sim \alpha$