The Higgs bundle moduli space

Nigel Hitchin
Mathematical Institute
University of Oxford

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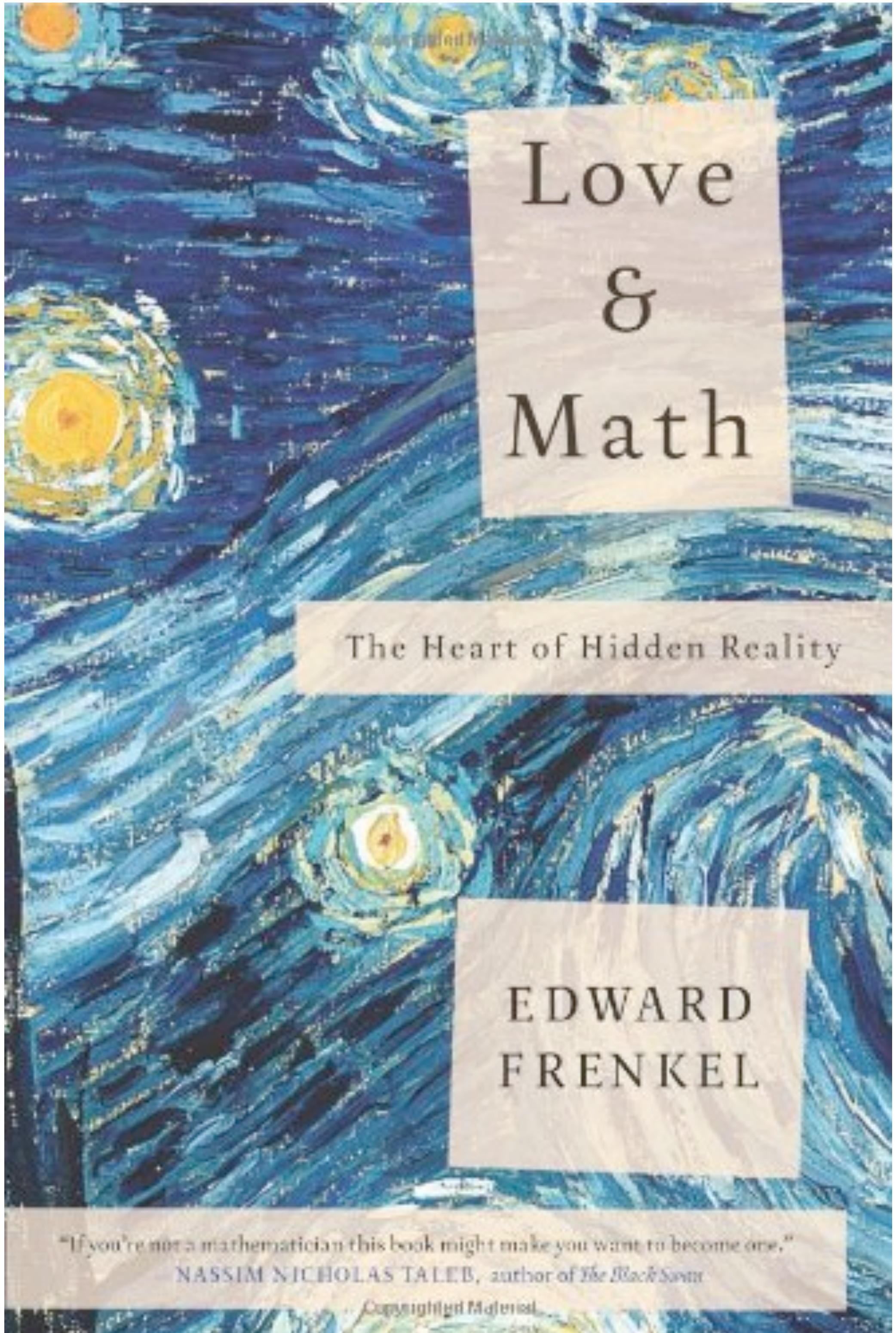
UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics


Love
&
Math
The Heart of Hidden Reality
EDWARD
FRENKEL
"If you're not a mathematician this book might make you want to become one."
NASSIM NICHOLAS TALEB, author of *The Black Swan*
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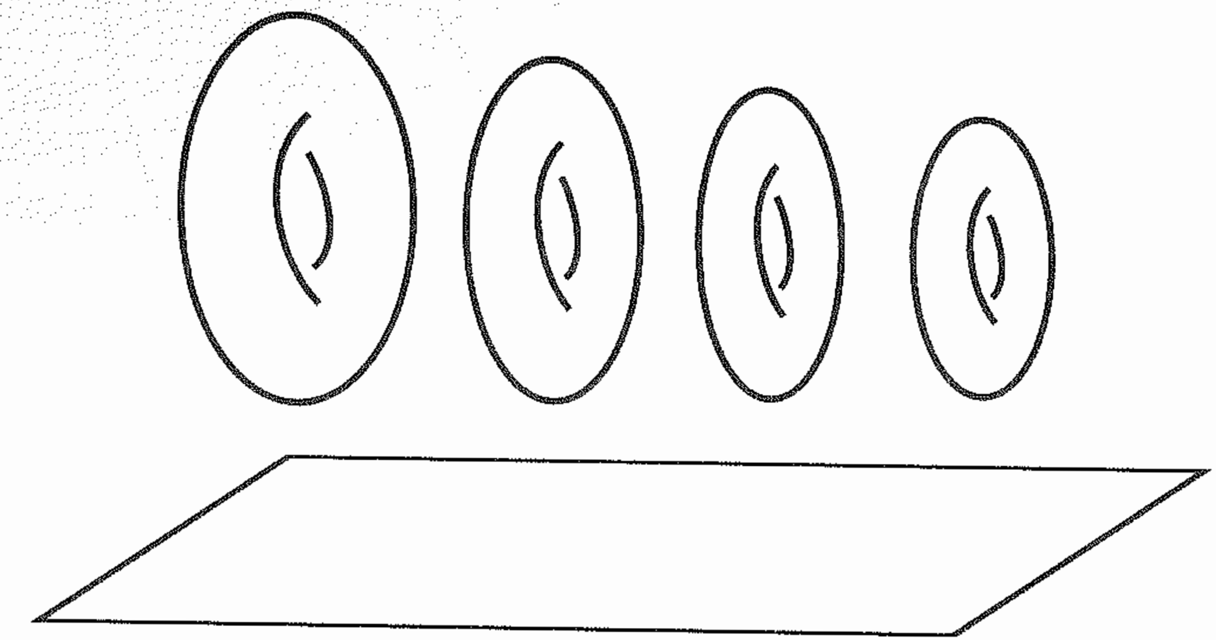

Think of the Hitchin fibration as a box of donuts, except that there are donuts attached not only to a grid of points in the base of the carton box, but to *all* points in the base. So we have infinitely many donuts – Homer Simpson would sure love that!

1977.... INSTANTONS

- connection on $\mathbf{R}^4$, $A_i(x)$ $n \times n$ skew Hermitian

- covariant derivative

$$\nabla_i = \frac{\partial}{\partial x_i} + A_i$$

- curvature $F_{ij} = [\nabla_i, \nabla_j]$

- equations

$$F_{12} + F_{34} = 0 = F_{13} + F_{42} = 0 = F_{14} + F_{23}$$

- anti-self-dual Yang-Mills $F + *F = 0$
- boundary conditions $\int_{\mathbf{R}^4} \|F\|^2 < \infty$
- gauge equivalence $g : \mathbf{R}^4 \to U(n) \qquad \nabla_i \mapsto g^{-1}\nabla_i g$

M.F.Atiyah,N.J.Hitchin,V.G.Drinfeld & Yu.I.Manin: *Construction of instantons*, Phys. Lett. A 65 (1978) 185-187.

1981 .... MONOPOLES

- connection on $\mathbf{R}^3$

- $A_i(x)$ + Higgs field ϕ, $n \times n$ skew Hermitian

- curvature $F_{ij} = [\nabla_i, \nabla_j]$

- equations

  $$F_{12} + [\nabla_3, \phi] = 0 = F_{23} + [\nabla_1, \phi] = 0 = F_{31} + [\nabla_2, \phi]$$

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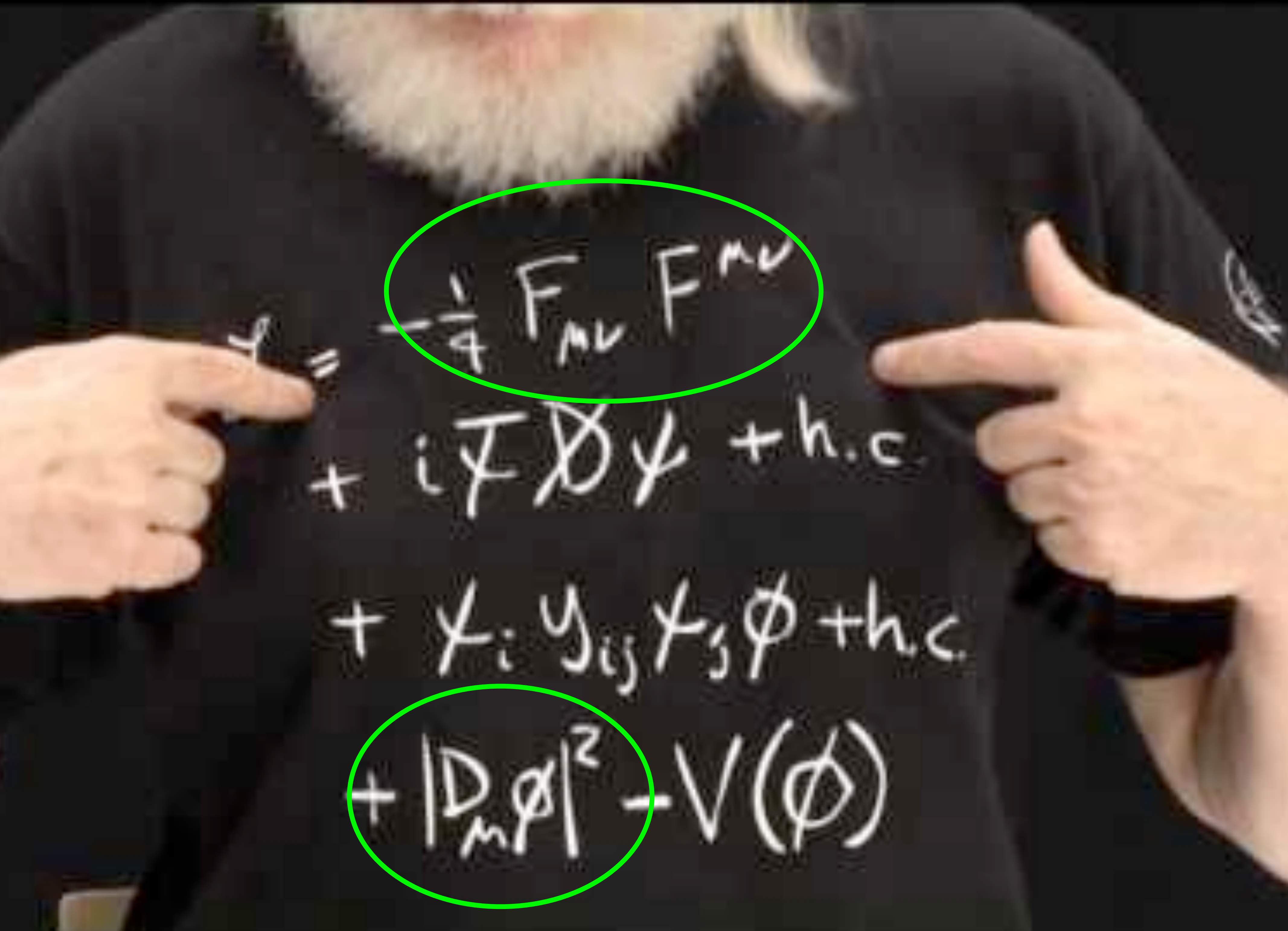
$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
$+ i \bar{\psi} \not{D} \psi + h.c.$
$+ \psi_i y_{ij} \psi_j \phi + h.c.$
$+ |D_\mu \phi|^2 - V(\phi)$

- anti-self-dual Yang-Mills, invariant under x_4-translation

- $\phi \sim A_4$

- boundary conditions $\displaystyle\int_{\mathbf{R}^3} \|F\|^2 + \|\nabla\phi\|^2 < \infty$

N.J.Hitchin *On the construction of monopoles*, Comm.Math.Phys. **83** (1982) 579–602.

TWO DIMENSIONS....

- connection on $\mathbf{R}^2$
- $A_i(x)$ + Higgs fields ϕ_1, ϕ_2, $n \times n$ skew Hermitian
- curvature $F_{ij} = [\nabla_i, \nabla_j]$
- equations

  $$F_{12} + [\phi_1, \phi_2] = 0 = [\nabla_1, \phi_2] + [\nabla_2, \phi_1] = [\nabla_1, \phi_1] + [\nabla_2, \phi_2]$$

TWO- AND THREE-DIMENSIONAL INSTANTONS

M.A. LOHE

Theoretical Physics, Blackett Laboratory, Imperial College, London SW7 2BZ, U.K.

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The four-dimensional Yang-Mills Lagrangian implies corresponding structures in lower dimensions. Instantons, characterized by a zero energy-momentum tensor as well as finite action, emerge as the solutions of coupled first order equations. For the Abelian case all such solutions are determined by the non-linear Poisson-Boltzmann equation.

term have acquired special values. One can obtain first order equations which imply the field equations, and these are deduced from eqs. (6):

$$F_{ij}^{a} = \pm e\, \epsilon_{ij} \epsilon^{abc} \psi^{b} \phi^{c} \ ,$$

$$D_i \psi^a \pm \epsilon_{ij} D_j \phi^a = 0 \ . \tag{11}$$

Unfortunately, in this case the model is not interesting because the action is always zero.

- $[\nabla_1 + i\nabla_2, \phi_1 + i\phi_2] = 0$ Cauchy-Riemann

- $z = x_1 + ix_2 \in \mathbf{C}$

- $\Phi = (\phi_1 + i\phi_2)d(x_1 + ix_2)$

- $F + [\Phi, \Phi^*] = 0$ conformally invariant

  $\Rightarrow$ define on any Riemann surface

- C^∞ Hermitian vector bundle E

- connection $A+$ Higgs field $\Phi \in \Omega^{1,0}(\mathrm{End}\, E)$

- Equations: $\bar{\partial}_A \Phi = 0, \quad F_A + [\Phi, \Phi^*] = 0$

- moduli space $\mathcal{M}$ = all solutions modulo C^∞ unitary automorphisms of E

THE CASE OF $U(1)$

- $E =$ line bundle L, $\operatorname{End} L$ trivial bundle

- Φ = holomorphic 1-form

- $A =$ flat connection on L

- moduli space $= \operatorname{Jac}(\Sigma) \times H^0(\Sigma, K) = T^* \operatorname{Jac}(\Sigma)$

- $\mathcal{M}$ finite-dimensional
- non-compact
- hyperkähler metric

SOLUTIONS

HOLOMORPHIC VECTOR BUNDLES

- Riemann surface Σ

- + holomorphic vector bundle E

- + holomorphic section Φ of $\mathrm{End}\, E \otimes K$

HOLOMORPHIC VECTOR BUNDLES

- Riemann surface Σ
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Hermitian metric on $E \Rightarrow$ compatible connection ∇

.... find a metric such that $F_A + [\Phi, \Phi^*] = 0$

- (E, Φ) stable $\Rightarrow$ there exists a unique solution

- NJH, *The self-duality equations on a Riemann surface,* Proc. London Math. Soc. (3) **55** (1987), 59–126.

  (modelled on Donaldson's proof of Narasimhan-Seshadri)

- C.Simpson, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 595.

  (modelled on Uhlenbeck-Yau's theorem on Hermitian Yang-Mills)

- $V \subset E$ subbundle

- Φ-invariance = $\Phi(V) \subset V \otimes K \subset E \otimes K$

- **stable** = for each Φ-invariant subbundle V

$$\frac{\deg(V)}{\operatorname{rk} V} < \frac{\deg(E)}{\operatorname{rk} E}$$

FLAT VECTOR BUNDLES

- Riemann surface Σ

- flat connection ∇ on vector bundle E

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Hermitian metric on $E \Rightarrow \pi_1$-equivariant map $f : \tilde{\Sigma} \to GL(n, \mathbf{C})/U(n)$

..... find an equivariant harmonic map f.

- ∇ irreducible (irreducible representation $\pi_1(\Sigma) \to GL(n, \mathbf{C})$)
$\Rightarrow$ there exists a unique solution

- S.K.Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. **55** (1987) 12713.

- K.Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), 361382.

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Equations $\nabla^{0,1}\Phi = 0, \quad F_A + [\Phi, \Phi^*] = 0$

$$\Rightarrow \nabla_A + \Phi + \Phi^* \text{ is a flat connection}$$

ANOTHER VIEWPOINT

- charge one $SU(2)$-instanton on $\mathbf{R}^4$

- $SO(4)$-invariant solution

- quaternion $\mathbf{x} = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$

- connection

$$A = \mathrm{Im}\left(\frac{\mathbf{x}d\bar{\mathbf{x}}}{1 + |\mathbf{x}|^2}\right)$$

- curvature

$$F_A = \left(\frac{d\mathbf{x} \wedge d\bar{\mathbf{x}}}{(1 + |\mathbf{x}|^2)^2}\right)$$

- translate from centre 0 to $\mathbf{a_i}$

- $A_1 + \ldots + A_k$ approximate charge k solution if $\mathbf{a_i}$ far apart

- Taubes grafting $\Rightarrow$ exact solution

- charge one $SU(2)$-monopole on $\mathbf{R}^3$

- $SO(3)$-invariant solution

- $\mathbf{x} = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$

- connection $$A = \left(\frac{1}{\sinh r} - \frac{1}{r}\right)\frac{1}{r}\mathbf{x} \times d\mathbf{x}$$

  Higgs field $$\phi = \left(\frac{1}{\tanh r} - \frac{1}{r}\right)\frac{1}{r}\mathbf{x}$$

- translate from centre 0 to $\mathbf{a_i}$
- $A_1 + \ldots + A_k$ approximate charge k solution if $\mathbf{a_i}$ far apart
- Taubes grafting $\Rightarrow$ exact solution
- A.Jaffe & C.Taubes, *Vortices and monopoles,* Birkhäuser, Boston (1980)

- $SU(2)$ Higgs bundle equations on $\mathbf{R}^2$
- $SO(2)$-invariant?
- $=$ anti-self-dual Yang-Mills on $\mathbf{R}^4$ invariant by $\mathbf{R}^2 \times SO(2)$

L.Mason & N.Woodhouse, *Self-duality and the Painlevé transcendents,* Nonlinearity **6** (1993), 569–581.

P_{III}: Two translations and a rotation:

$$X = \partial_{\tilde{\tau}} \qquad Y = \xi\partial_{\xi} - \tilde{\xi}\partial_{\tilde{\xi}} \qquad Z = \partial_{\tau}.$$

The reduction by Y and $X + Z$ gives the Ernst equation, which is known to have a further reduction to P_{III} (as well as to P_{V}): see the papers cited in [1, p 344].

- Ernst equation: stationary axisymmetric spacetimes

- Painlevé III: $\quad \psi_{xx} = \dfrac{1}{2}e^{2x}\sinh 2\psi$

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- D.Gaiotto, G.Moore, & A.Neitzke *Wall crossing, Hitchin systems and the WKB approximation* Ad in Math. **234** (2013) 239–403.

A SINGULAR SOLUTION

- connection

$$A = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right)$$

- $U(1)$-connection $F_A = dA = 0$

- Higgs field Φ, need $[\Phi, \Phi^*] = 0$

$$\Phi = \begin{pmatrix} 0 & r^{1/2} \\ zr^{-1/2} & 0 \end{pmatrix} dz$$

normal using the constant Hermitian metric

A REGULAR SOLUTION

- connection $$A_t = f_t(r) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}} \right)$$

- Higgs field $$\Phi = \begin{pmatrix} 0 & r^{1/2} e^{h_t(r)} \\ e^{i\theta} r^{1/2} e^{-h_t(r)} & 0 \end{pmatrix} dz$$

- $$h_t(r) = \psi(\log(8tr^{3/2}/3)) \qquad f_t(r) = \frac{1}{8} + \frac{1}{4} r \frac{dh_t}{dr}$$

- ψ special solution to Painlevé III

Thm (MSWW)

- (A_t, Φ_t) is smooth at the origin and converges exponentially in t, uniformly in $r > r_0$ to the singular solution.

- (A_t, Φ_t) satisfy the equations $\bar{\partial}_A \Phi_t = 0, F_{A_t} + t^2[\Phi_t, \Phi_t^*] = 0$

- $\Rightarrow (A_t, t\Phi_t)$ satisfies the Higgs bundle equations.

COMPACT RIEMANN SURFACE $g > 1$

- compact Riemann surface Σ
- can't widely separate approximate solutions
- but...if (V, Φ) is stable, so is $(V, t\Phi)$ $t \neq 0$

  (stability condition on Φ-invariant subbundles)
- $\Rightarrow$ if (A, Φ) is a solution, then there is also a solution $(A_t, t\Phi)$

- study solutions $(A_t, t\Phi)$ as $t \to \infty$

- **Note 1:** $(A, e^{i\theta}\Phi)$ is a solution if (A, Φ) is

$$([e^{i\theta}\Phi, (e^{i\theta}\Phi)^*] = [\Phi, \Phi^*])$$

- **Note 2:** Consider $V = L \oplus L^*$

$$\Phi = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \qquad t\Phi = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t^{-1/2} & 0 \\ 0 & t^{1/2} \end{pmatrix}$$

defines same point in moduli space

- Higgs field $\Phi = \begin{pmatrix} 0 & r^{1/2}e^{h_t(r)} \\ e^{i\theta}r^{1/2}e^{-h_t(r)} & 0 \end{pmatrix} dz$

- $\det\Phi = -z dz^2$

- in general, for $SU(2)$, $\det\Phi$ is a holomorphic quadratic differential on the Riemann surface Σ

- invariant by gauge transformations $\Phi \mapsto g^{-1}\Phi g$

- $\Rightarrow$ defined by equivalence class in the moduli space.

- quadratic differential = holomorphic section of K^2
- $\deg K^2 = 4g-4$, $g =$ genus
- generically $\det \Phi$ has $4g-4$ simple zeros.

- **Definition**: A limiting configuration for a Higgs bundle (A, Φ) where $\det \Phi$ has simple zeros is (A_∞, Φ_∞) satisfying

$$F_{A_\infty} = 0, \quad [\Phi_\infty, \Phi_\infty^*] = 0$$

and behaving like the model singular solution near each zero of $\det \Phi$.

$$\Phi = \begin{pmatrix} 0 & r^{1/2} \\ zr^{-1/2} & 0 \end{pmatrix} dz$$

- put $z = w^2$, $\det \Phi = -z dz^2 = -(2w^2 dw)^2$
- eigenvalues of Φ $\pm 2w^2 dw$
- holomorphic 1-form $\alpha = 2w^2 dw$
- $\sigma(w) = -w$, $\sigma^* \alpha = -\alpha$

GLOBALLY...

- $\det \Phi = -q$, q holomorphic section of K^2
- K defines a double covering S branched over the zeros of q
- S compact Riemann surface genus $g_S = 4g - 3$

GLOBALLY...

- $\det \Phi = -q$, q holomorphic section of K^2
- K defines a double covering S branched over the zeros of q
- S compact Riemann surface genus $g_S = 4g - 3$
- $x = \sqrt{q}$ defines a holomorphic 1-form α with double zeros on the ramification points
- involution $\sigma : S \to S$ exchanges sheets, $\sigma^* \alpha = -\alpha$

- on S let L be a flat line bundle with $\sigma^* L \cong L^*$
- rank 2 vector bundle $E = L \oplus L^*$
- Higgs field $\Phi = (\alpha, -\alpha)$ – satisfies equations with $[\Phi, \Phi^*] = 0$
- (E, Φ) is σ-invariant

- σ-invariant Higgs bundle on $S =$

- orbifold Higgs bundle on Σ

- parabolic Higgs bundle, singular connection ....

Thm (MSWW)

- Take a limiting configuration (A_∞, Φ_∞).

- Then there exists a family $(A_t, t\Phi_t)$ of solutions to the Higgs bundle equations for large t where $(A_t, \Phi_t) \to (A_\infty, \Phi_\infty)$

- If (A_∞, Φ_∞) is associated to a solution (A_0, Φ_0) then (A_t, Φ_t) is gauge-equivalent to (A_0, Φ_0).

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- describes $[A_0, t\Phi_0]$ in moduli space as $t \to \infty$

GEOMETRY OF THE MODULI SPACE

- $\mathcal{M}$ has a natural complete hyperkähler metric
- complex structures I, J, K
- Kähler forms $\omega_1, \omega_2, \omega_3$
- circle action fixing ω_1, rotating ω_2, ω_3

HIGGS BUNDLES

- Σ compact Riemann surface
- V smooth vector bundle with Hermitian metric
- $\mathcal{A} =$ infinite-dimensional affine space of $\bar{\partial}$-operators on V

$$\bar{\partial}_A : \Omega^0(V) \to \Omega^{0,1}(V) \qquad \bar{\partial}_A(fs) = f\bar{\partial}_A(s) + \bar{\partial}fs$$

- $\bar{\partial}_A - \bar{\partial}_B \in \Omega^{0,1}(\mathrm{End}(V))$

- $M = \mathcal{A} \times \Omega^{1,0}(\mathrm{End}(V))$
- $T_A M = \Omega^{0,1}(\mathrm{End}(V)) \oplus \Omega^{1,0}(\mathrm{End}(V))$
- Hermitian form $\omega_1 \sim \int_\Sigma (\mathrm{tr}\, aa^* + \mathrm{tr}\, \phi\phi^*)$

$$\omega_2 + i\omega_3 \sim \int_\Sigma \mathrm{tr}\, a\phi$$

- flat hyperkähler manifold

- G = group of $U(n)$ gauge transformations

- $\mathfrak{g} = \{\psi \in \Omega^0(\mathrm{End}(V)), \psi^* = -\psi\}$

- $\mathfrak{g}^* = \{\omega \in \Omega^2(\mathrm{End}(V)), \omega^* = -\omega\}$

- G = group of $U(n)$ gauge transformations
- $\mathfrak{g} = \{\psi \in \Omega^0(\mathrm{End}(V)), \psi^* = -\psi\}$
- $\mathfrak{g}^* = \{\omega \in \Omega^2(\mathrm{End}(V)), \omega^* = -\omega\}$
- moment map $\mu(\bar{\partial}_A, \Phi) = (F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$
- F_A= curvature of Hermitian connection with $\nabla^{0,1} = \bar{\partial}_A$

- $\mu(\bar{\partial}_A, \Phi) = (F_A + [\Phi, \Phi^*], \bar{\partial}_A \Phi)$

- hyperkähler quotient $= \mu^{-1}(0)/G =$ moduli space

- $\bar{\partial}_A \Phi = 0 =$ holomorphic Higgs field $\Phi \in \mathrm{End}(V) \otimes K$

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- $\nabla + \Phi + \Phi^*$ connection

$$F = [\nabla^{1,0} + \Phi, \nabla^{0,1} + \Phi^*] = 0$$

flat $GL(n, \mathbf{C})$ connection

- $\mu(\bar\partial_A, \Phi) = (F_A + [\Phi, \Phi^*], \bar\partial_A \Phi)$

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complex structure I

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flat $GL(n, \mathbf{C})$ connection

complex structure J

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(well separated monopoles $\sim$ simple zeros of $\det \Phi$.)

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(higher rank groups)

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(higher rank groups)

R.Bielawski, *Monopoles and clusters.,* Comm. Math. Phys. **284** (2008), 675–712.

(clusters $\sim$ multiple zeros of $\det \Phi$.)