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Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

Numerical Computation of Optimal Trajectories for Coplanar, Aeroassisted Orbital Transfer

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Abstract. This paper is concerned with the problem of optimal, planar aero-assisted orbital transfer from a high Earth orbit to a low Earth orbit. It is assumed that initial and final orbits are circular and coplanar, and that the gravitational field is central and governed by the inverse square law. The whole trajectory is assumed to consist of two impulsive changes of the velocity at the begin and at the end of the trajectory and of one interior atmospheric subarc, where the vehicle is controlled via the lift coefficient.

The problem is reduced to the atmospheric part of the trajectory, thus arriving at an optimal control problem with free final time and lift coefficient as the only (bounded) control variable. For this problem, the necessary conditions of optimal control theory are derived. Applying multiple shooting techniques two trajectories with different control structures are computed. The first one is characterized by a lift coefficient which is chosen on its minimum value during the whole atmospheric pass. For the second trajectory an optimal control history with a boundary subarc followed by a free subarc is chosen. It turns out, that this second trajectory satisfies the minimum principle whereas the first one fails to satisfy this necessary condition.

In the second part of the paper the assumption of impulsive velocity changes is dropped. Instead, a more realistic modeling with two finite thrust subarcs in the nonatmospheric part of the trajectory is considered. The resulting optimal control problem now describes the whole maneuver including the nonatmospheric parts. It contains the thrust, the thrust angle, and the lift coefficient as control variables. Further, the mass of the vehicle occurs as an additional state variable. For this optimal control problem numerical solutions are presented. They are compared with the solutions of the first model.

Key Words. Aeroassisted Orbital Transfer, Optimal Control Theory, Minimum Principle, Multiple Shooting Techniques.

AMS (MOS) subject classifications. 34B10, 34H05, 49-99, 65L10.

1. Introduction

In space-travel problems one of the important tasks is to minimize the fuel consumption or often equivalent, to maximize the pay-load. Since the pioneering paper of London (Ref. 1), it is well known that the use of aerodynamic forces may reduce the energy consumption for a certain maneuver considerably against a pure propulsive mode. This type of maneuver which uses a combination of propulsive forces in space and aerodynamic forces in atmospheric subarcs is called an aeroassisted orbital transfer (AOT). Standard applications of AOT, which are considered extensively in literature, are orbital plane changes, planetary aerobraking and coplanar and noncoplanar orbital transfer. For these tasks optimal solutions are constructed and guidance schemes are developed. Recent surveys on the topic AOT are given by Walberg (Ref.2), Mease (Ref.3), Naidu (Ref.4) and Miele (Ref.5).

In the first part of this paper, we reconsider the basic problem of a fuel-minimal coplanar orbit transfer from a high Earth orbit (HEO) to a low Earth orbit (LEO) by means of AOT. This problem has been considered often in the literature (Ref. 6–9). Two different control structures for the aerodynamic lift coefficient are investigated. In order to prove the optimality of the control structure, the necessary conditions of optimal control theory are applied. A solution with a new control structure is computed which satisfies the minimum principle.

In general, AOT–problems have been treated under the assumption of impulsive propulsion outside of the atmosphere. In the second part of the paper, we drop this assumption and change the model by including finite, limited thrust subarcs instead of the impulsive burns. The problem here is not reduced to the atmospheric part in order to verify the control structure with respect to the thrust. Numerical results for a rather low maximal thrust value are presented and they are compared with the impulsive case.

2. Problem description

We consider the problem to determine optimal control functions for a fuel–minimal coplanar aeroassisted orbital transfer from a HEO to a LEO. In particular, HEO is chosen as the equatorial geosynchron Earth orbit (GEO) which is assumed to be circular with radius r_{00} and correspondence altitude h_{00}.

The final orbit LEO may approximate the circular orbit of the international space station ISS with prescribed radius r_{11} and corresponding altitude h_{11}.

We assume that these initial and final orbits are coplanar and that the whole transfer takes place in this common plane. I.e. the bank angle is chosen zero. Further, the gravitational field is assumed to be central and governed by the inverse square law. The initial and final velocity vectors of the vehicle are tangential to the orbits; the magnitudes v_{00} and v_{11}, respectively, are given in correspondence to the radii by $v = \sqrt{\mu/r}$, where μ is the Earth's gravitational constant.

For the first part of the paper, we assume that the orbital transfer is controlled by two impulsive changes of the velocity and the lift control in the atmospheric part of the transfer. One impulse is employed at the exit point (index 00) in HEO and one at the entry point (index 11) in LEO. Both impulses are tangential and are given by the characteristic velocities Δv_{00} and Δv_{11}, respectively. I.e., the initial and the final velocity of the vehicle are changed instantaneously in the following way

$$v(t_{00}) \;=\; v_{00} \;-\; \Delta v_{00}\,, \quad v_{11} \;=\; v(t_{11}) \;+\; \Delta v_{11}\,. \tag{1}$$

The fuel consumption is measured by the total characteristic velocity

$$\Delta V \;:=\; \Delta v_{00} \;+\; \Delta v_{11}\,. \tag{2}$$

By dropping the constant additive terms, ΔV may be replaced by the functional to be minimized

$$I \;:=\; -(v(t_{00}) + v(t_{11}))\,. \tag{3}$$

The total maneuver consists of three parts, two nonatmospheric subarcs and one interior atmospheric portion, starting at point t_0 (entry into the atmosphere) and ending at point t_1 (exit from the atmosphere). During this atmospheric subarc, the vehicle is controlled via the lift coefficient C_L. For simplicity, in this paper the influence of Earth's rotation is neglected. Similar results can be obtained, if Earth's rotation is taken into account. These recent results will be published elsewhere, cf. Baumann (Ref. 10).

With respect to an Earth-fixed coordinate system, the equations of motion are given as follows

$$dh/dt \;=\; v\,\sin\gamma\,, \tag{4a}$$

$$dv/dt \;=\; -D/m \;-\; \mu/r^2\,\sin\gamma\,, \tag{4b}$$

$$d\gamma/dt \;=\; L/(mv) \;+\; (v/r - \mu/(r^2 v))\cos\gamma\,. \tag{4c}$$

Here, h denotes the altitude, v is the velocity and γ is the flight–path angle of the vehicle. The distance to the gravitational center of Earth is $r = r_E + h$, where r_E denotes Earth's radius. Further, m is the constant mass of the vehicle.

The aerodynamic forces drag D and lift L are given by

$$D \;=\; 0.5\,S\,\rho\,v^2\,C_D\,,\;\; L \;=\; 0.5\,S\,\rho\,v^2\,C_L\,, \tag{5}$$

where $\rho = \rho(h)$ denotes the air density, S is the reference surface area, and C_D, C_L are the atmospheric drag and lift coefficients. We assume the usual exponential law for the air density

$$\rho(h) \;=\; \rho_0\exp(-\beta h)\,. \tag{6}$$

Further, we assume a quadratic polar

$$C_D \;=\; C_{D0} \;+\; C_{D2}\,C_L^2\,, \tag{7}$$

where the lift coefficient C_L is considered as a control variable. The dependence of C_D and C_L on the Mach and the Reynolds number is disregarded. The data for these approximations are given in the following Table 1. They correspond to the data for the spacecraft S1 in the paper by Miele (Ref. 6).

Let h_a denote an appropriate value of the thickness of the atmosphere. Then, for altitudes $h > h_a$ the air density is set to zero. The choice $h_a = 120\,\mathrm{km}$ leads to a discontinuity jump of the air density of about $3.4 \times 10^{-8}\,\mathrm{kg/m^3}$, which, for practical reasons, can be neglected.

With these approximations the problem of a fuel-optimal aeroassisted orbital transfer can be formulated as the following optimal control problem.

Problem P1. Determine the free transfer time t_{11}, and the control $C_L(t), t_{00} \le t \le t_{11}$, such that the functional

$$I = -(v(t_{00}) + v(t_{11})) \tag{8}$$

is minimized subject to the state equations (4a-c), the boundary conditions

$$h(t_{00}) = h_{00}\,, \qquad h(t_{11}) = h_{11}\,, \tag{9a}$$

$$\gamma(t_{00}) = 0\,, \qquad \gamma(t_{11}) = 0\,, \tag{9b}$$

and the control constraints

$$C_{L\,\min} \le C_L \le C_{L\,\max}\,. \tag{10}$$

In correspondence to Eq.(1) the initial and the final velocity are kept free. Further, the initial time can be fixed, for example $t_{00} = 0$.

The characteristic data for Problem P1 can be found in the following Table 1.

Table 1. Parameters of Problem P1

r_{00}	$42,164\,\mathrm{km}$	r_{11}	$6,708\,\mathrm{km}$
h_{00}	$35,786\,\mathrm{km}$	h_{11}	$330\,\mathrm{km}$
v_{00}	$3.074664580\,\mathrm{km/s}$	v_{11}	$7.708539821\,\mathrm{km/s}$
μ	$3.986 \times 10^5\,\mathrm{km^3/s^2}$	r_E	$6,378\,\mathrm{km}$
ρ_0	$1.225\,\mathrm{kg/m^3}$	β	$1/6.9\,\mathrm{km}$
r_a	$6,498\,\mathrm{km}$	h_a	$120\,\mathrm{km}$
C_{D0}	0.1	C_{D2}	1.11
$C_{L\,\min}$	-0.9	$C_{L\,\max}$	0.9
m/S	$300\,\mathrm{kg/m^2}$		

3. Reduction on the Atmospheric Subarc

The control function C_L affects the state equations only on the atmospheric subarc. The biggest part of the maneuver takes place however on the outside of the atmosphere. For numerical purposes it is customary to reduce the problem onto the atmospheric part.

The reduction can be done by use of the conservation of energy and angular momentum on the nonatmospheric parts.

The conservation of angular momentum $\mathcal{L} = r\, v\, \cos\gamma$ implies for the nonatmospheric parts $[0, t_0]$ and $[t_1, t_{11}]$:

$$v(t_{00}) \;=\; (r_a/r_{00})\, v_0\, \cos\gamma_0\,, \quad v(t_{11}) \;=\; (r_a/r_{11})\, v_1\, \cos\gamma_1\,. \tag{11}$$

Here, (h_0, v_0, γ_0) and (h_1, v_1, γ_1) denote the state variables at the entry point t_0 and at the exit point t_1 of the atmospheric part, respectively.

In an analogous way the conservation of energy $\mathcal{E} = mv^2/2 - \mu m/r$ can be applied on the nonatmospheric subarcs. Using the relation (11) one obtains

$$v_0^2\left[1 \;-\; (r_a/r_{00})^2\, \cos^2\gamma_0\right] \;+\; 2\mu\, (1/r_{00} \;-\; 1/r_a) \;=\; 0\,, \tag{12a}$$

$$v_1^2\left[1 \;-\; (r_a/r_{11})^2\, \cos^2\gamma_1\right] \;+\; 2\mu\, (1/r_{11} \;-\; 1/r_a) \;=\; 0\,. \tag{12b}$$

Herewith, Problem P1 can be reformulated as follows

Problem P1'. Determine the free final time t_1, and the control $C_L(t),\; t_0 \le t \le t_1$, such that the functional

$$I \;=\; -\,\left[(r_a/r_{00})\, v_0\, \cos\gamma_0 \;+\; (r_a/r_{11})\, v_1\, \cos\gamma_1\right] \tag{13}$$

is minimized subject to the state equations (4a–c), the boundary conditions (12a–b),

$$h(t_0) \;=\; h_a\,, \quad h(t_1) \;=\; h_a\,, \tag{14}$$

and the control constraints (10).

Of course, the initial time t_0 may be fixed arbitrarily, for example $t_0 = 0$.

For numerical reasons the physical quantities are scaled or normalized in an appropriate way, cf. Miele (Ref. 7), Naidu (Ref. 4), or Oberle and Taubert (Ref. 11). We define the new state variables

$$\overline{h}(\overline{t}) \;:=\; h(a\overline{t})/h_a\,, \quad \overline{v}(\overline{t}) \;:=\; av(a\overline{t})/r_a\,, \quad \overline{\gamma}(\overline{t}) \;:=\; \gamma(a\overline{t})\,, \tag{15}$$

and the new control function

$$\eta(\overline{t}) \;:=\; C_L(a\overline{t})/K\,, \tag{16}$$

where

$$\overline{t} \;:=\; t/a\,, \quad a \;:=\; \sqrt{r_a^3/\mu}\,, \quad K \;:=\; \sqrt{C_{D0}/C_{D2}}\,. \tag{17}$$

The scaling takes the thickness of the atmosphere h_a as reference altitude. The time scaling is such that one turning around Earth in altitude h_a and with orbital velocity takes 2π time units.

With the abbreviations

$$b := r_a/h_a, \qquad \delta(h) := \exp(-\beta\, h_a\, h)\,, \tag{18a}$$

$$A_1 := 0.5\, C_{D0}\,(S/m)\rho_0\, r_a, \qquad A_2 := 0.5\, K\,(S/m)\rho_0\, r_a\,, \tag{18b}$$

$$\eta_{\min} := C_{L\,\min}/K, \qquad \eta_{\max} := C_{L\,\max}/K\,, \tag{18c}$$

$$a_{00} := r_{00}/r_a, \qquad a_{11} := r_{11}/r_a \tag{18d}$$

one gets the following reformulation of Problem 1'. Note that the performance measure in this formulation corresponds only up to a positive factor with the performance measure of Problem 1'. For simplicity, the bars are omitted and the initial time is fixed to $t_0 = 0$.

Problem P1". Determine the free final time t_1, and the lift control $\eta(t)$, $0 \le t \le t_1$, such that the functional

$$I = -\,[v(0)\cos\gamma(0)/a_{00} + v(t_1)\,\cos\gamma(t_1)/a_{11}] \tag{19}$$

is minimized subject to the state equations

$$dh/dt = b\,v\,\sin\gamma\,, \tag{20a}$$

$$dv/dt = -A_1\,\delta\,(1+\eta^2)\,v^2 - (b/(b-1+h))^2\,\sin\gamma\,, \tag{20b}$$

$$d\gamma/dt = A_2\,\delta\,\eta\,v + \Big[(b/(b-1+h))v - (b/(b-1+h))^2/v\Big]\cos\gamma\,, \tag{20c}$$

the two-point-boundary conditions

$$h(0) = h(t_1) = 1\,, \tag{21a}$$

$$\Big(a_{00}^2 - \cos^2\gamma(0)\Big)\,v(0)^2 + 2\,a_{00}\,(1-a_{00}) = 0\,, \tag{21b}$$

$$\Big(a_{11}^2 - \cos^2\gamma(t_1)\Big)\,v(t_1)^2 + 2\,a_{11}\,(1-a_{11}) = 0\,, \tag{21c}$$

and the control constraint

$$\eta_{\min} \le \eta(t) \le \eta_{\max}\,. \tag{22}$$

4. Necessary Conditions for Problem P1

We derive the necessary conditions of optimal control theory for the formulation Problem P1". The notation used is taken from Bryson, Ho (Ref. 12). Our aim is to derive a boundary value problem (BVP) for state- and adjoint variables. To this end we assume the regularity of an optimal control function η^* of the problem.

Let λ_h, λ_v, and λ_γ denote the adjoint variables with respect to the state variables h, v, and γ, respectively. The Hamiltonian is then given by

$$H = \lambda_h \, dh/dt + \lambda_v \, dv/dt + \lambda_\gamma \, d\gamma/dt \, . \tag{23}$$

From Eq.(23) one obtains the adjoint differential equations

$$\begin{aligned} d\lambda_h/dt &= \lambda_v \left[A_1 \delta'(h)(1+\eta^2)v^2 - 2\,(b^2/c^3)\, \sin\gamma \right] \\ &+ \lambda_\gamma \left[-A_2 \delta'(h) \eta v + (bv/c^2 - 2b^2/(c^3 v))\, \cos\gamma \right] , \end{aligned} \tag{24a}$$

$$\begin{aligned} d\lambda_v/dt &= -\lambda_h b \sin\gamma + 2 \lambda_v A_1 \delta(h)(1+\eta^2) v \\ &- \lambda_\gamma \left[A_2 \delta(h) \eta + (b/c + b^2/(c^2 v^2))\, \cos\gamma \right] , \end{aligned} \tag{24b}$$

$$d\lambda_\gamma/dt = -\lambda_h b v \cos\gamma + \lambda_v (b/c)^2 \cos\gamma + \lambda_\gamma \left[(b/c) v - b^2/(c^2 v) \right] \sin\gamma \, , \tag{24c}$$

where c abbreviates the term $c := b - 1 + h$, and $\delta'(h)$ denotes the derivative of the (scaled) atmospheric density with respect to the (scaled) altitude.

The optimal control is characterized by the minimum principle. First, from $\partial H/\partial \eta = 0$ one obtains the free control ($\lambda_v \neq 0$ is assumed)

$$\eta_{\text{free}} = \lambda_\gamma A_2 / (2\, \lambda_v \, A_1 \, v) \, . \tag{25}$$

Now taking into account the control constraints (22) and the fact that the Hamiltonian is quadratic with respect to η, one obtains a complete description for the optimal control by the minimum principle as follows

$$\text{if} \quad \lambda_v < 0 : \qquad \eta^* = \begin{cases} \eta_{\min} \, , & \text{if} \ \eta_{\text{free}} < \eta_{\min} \\ \eta_{\text{free}} \, , & \text{if} \ \eta_{\min} < \eta_{\text{free}} < \eta_{\max} \\ \eta_{\max} \, , & \text{if} \ \eta_{\text{free}} > \eta_{\max} \end{cases} \tag{26a}$$

$$\text{if} \quad \lambda_v = 0 : \qquad \eta^* = \begin{cases} \eta_{\min} \, , & \text{if} \ \lambda_\gamma > 0 \\ \eta_{\max} \, , & \text{if} \ \lambda_\gamma < 0 \end{cases} \tag{26b}$$

$$\text{if} \quad \lambda_v > 0 : \qquad \eta^* = \begin{cases} \eta_{\min} \, , & \text{if} \ \eta_{\text{free}} > (\eta_{\min} + \eta_{\max})/2 \\ \eta_{\max} \, , & \text{if} \ \eta_{\text{free}} < (\eta_{\min} + \eta_{\max})/2 \end{cases} \tag{26c}$$

For Eq.(26a–c) one needs only the assumption that the adjoint variables λ_v and λ_γ do not vanish identically both on a nontrivial interval. In fact for the numerical solutions presented in the following, this assumption holds. Both adjoint variables have only isolated roots and we found that $\sqrt{\lambda_v^2 + \lambda_\gamma^2} > 4 \times 10^{-6}$ holds for all $t \in [0, t_1]$.

Corresponding to the nonlinear boundary conditions (21b, c) one obtains the following (nonlinear) natural boundary conditions

$$v(0) \sin\gamma(0) \left(a_{00} - \lambda_v(0) \cos\gamma(0)\right) \; + \; \lambda_\gamma(0) \left(a_{00}^2 - \cos^2\gamma(0)\right) \; = \; 0\,, \tag{27a}$$

$$v(t_1) \sin\gamma(t_1) \left(a_{11} + \lambda_v(t_1) \cos\gamma(t_1)\right) \; - \; \lambda_\gamma(t_1) \left(a_{11}^2 - \cos^2\gamma(t_1)\right) \; = \; 0\,. \tag{27b}$$

Finally, the free final time t_1 is fixed by the natural boundary condition

$$H[t_1] \; = \; 0\,. \tag{28}$$

We have established altogether a BVP for the numerical solution of Problem P1. It consists of the six ode's (20a–c), (24a–c) and an additional trivial differential equation for the treatment of the free final time by means of a linear transformation onto a fixed interval. The corresponding boundary conditions are given by the Eqs. (21a–c), (27a, b), and (28).

5. Numerical Results for Problem P1

In this section we present numerical results for Problem P1. These are obtained by solving the BVP established in Section 4 by means of the multiple shooting code BNDSCO; see Bulirsch (Ref. 13), Stoer and Bulirsch (Ref. 14), and Oberle and Grimm (Ref. 15).

For the application of this technique one has to estimate the control structure, that is the number and the relative position of free and boundary subarcs of the control η. The switching points which separate those subarcs become additional variables of the BVP. Corresponding to these variables one has to add certain switching conditions (interior boundary conditions) to the BVP. Mainly these switching conditions are given by the minimum principle or by Eq.(26) respectively. In the case of Eq.(26a) for example, a switching of the control history from $\eta_{\min}$ to η_{free} is characterized by the switching condition

$$\eta_{\text{free}}(\tau) \; = \; \eta_{\min}\,, \tag{29}$$

where τ denotes the (unknown) switching point and η_{free} is evaluated according to Eq.(25).

Starting point for the numerical computation is the control structure

$$\textbf{Structure A:} \qquad \eta(t) \; \equiv \; \eta_{\min}\,, \quad \text{for all } t \in [0, t_1]\,, \tag{30}$$

which corresponds to the control structure of solutions presented in Miele (Ref. 8).

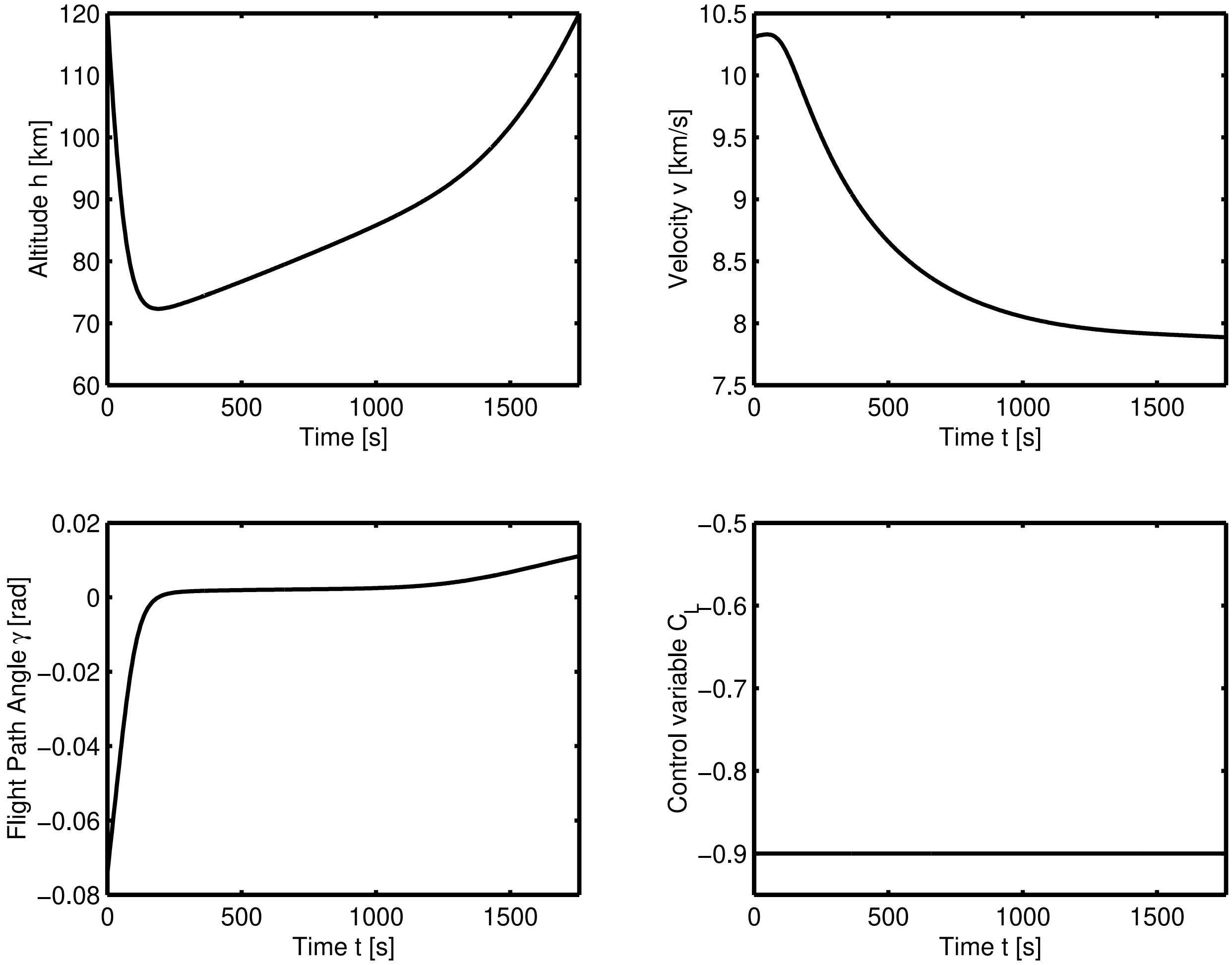

Fig. 1 Problem P1: State Variables and Control Function for Control Structure A.

First step, the state variables are determined solving a BVP which consists of the ode's given in Eq.(20), a trivial ode with respect to the free final time and the boundary conditions given in Eq.(21). The solution obtained by multiple shooting techniques is shown in Fig.1. It widely corresponds to the solution given in Ref. 8.

In order to check the necessary optimality conditions, the BVP is enlarged by the adjoint variables and the corresponding natural boundary conditions. The numerical result is shown in Fig.2. It turns out that the adjoint variable λ_γ is positive during the whole maneuver, whereas λ_v has an isolated zero at $\tau_1 \approx 384.5526\,\mathrm{s}$ (rescaled). So the necessary conditions given in Eq.(26a) are satisfied on the interval $[0, \tau_1[$. On the remaining subinterval $[\tau_1, t_1]$ the necessary condition Eq.(26c) is satisfied for $\eta^* \equiv \eta_{\min}$ if and only if $\partial H/\partial\eta > 0$. This condition however holds only in a subinterval $]\tau_1, \tau_2]$, where $\tau_2 \approx 1448.1360\,\mathrm{s}$. For $t > \tau_2$ the function $\partial H/\partial\eta$ is negative. Thus the solution obtained does not satisfy the minimum principle.

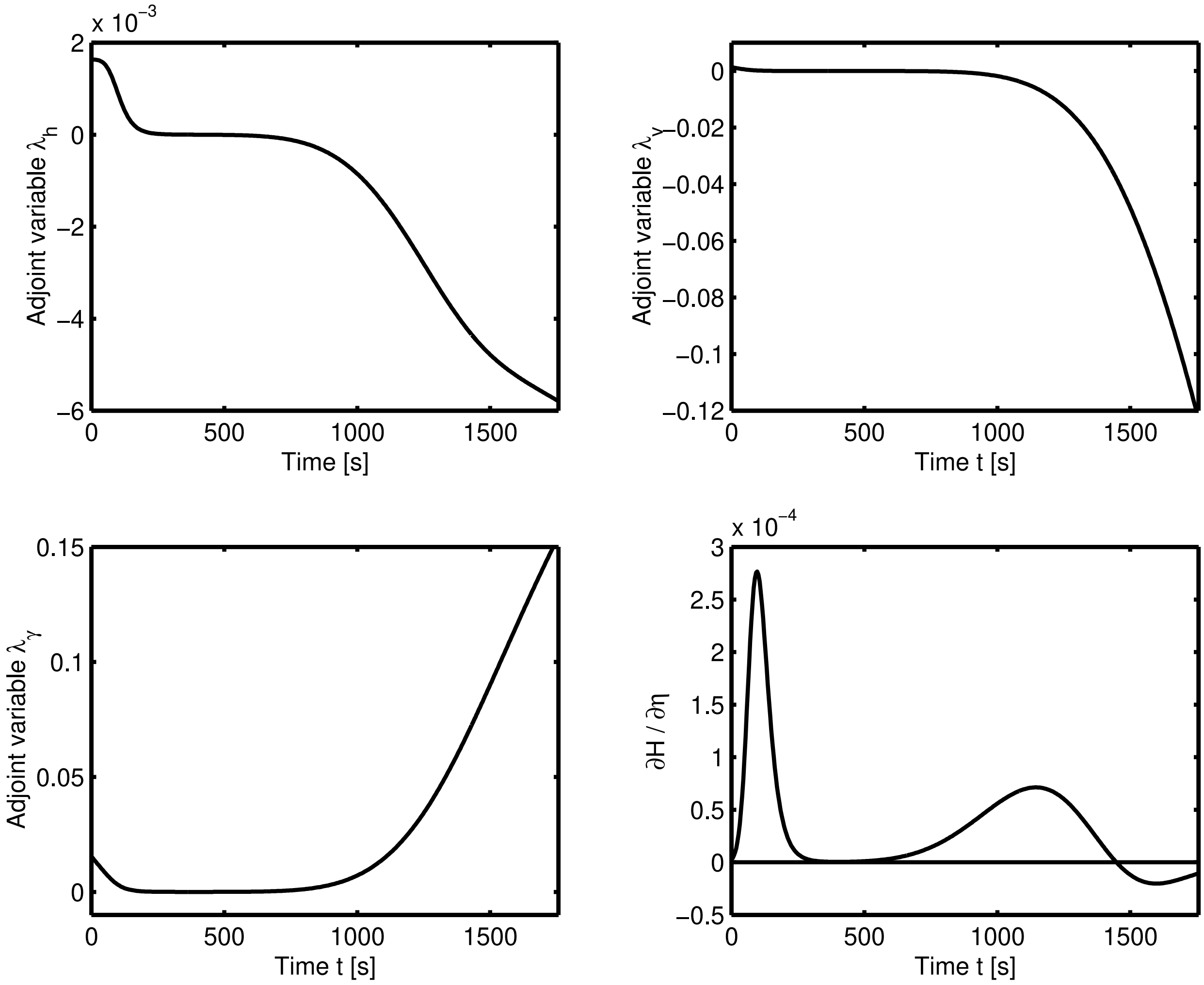

Fig. 2 Problem P1: Adjoint Variables and $\partial H/\partial \eta$ for Control Structure A.

From Fig.2 it is obvious how to change the control structure such that the minimum principle is satisfied:

$$\textbf{Structure B:} \qquad \eta(t) = \begin{cases} \eta_{\min} \quad , & \text{if} \quad 0 \le t \le \tau, \\ \eta_{\text{free}} \quad , & \text{if} \quad \tau < t \le t_1 \end{cases} \tag{31}$$

Here, τ is a switching point for which τ_2 may serve as initial estimation. The corresponding switching condition is given by Eq.(29). Starting with the solution for structure A, the numerical solution for structure B is obtained in a few iteration steps. Qualitativly the solutions behave similar, however the minimum principle is satisfied only for the control structure B. The state variables and the control history are shown in Fig. 3.

In Table 2 the characteristic quantities of both solutions are summarized. One observes that for practical purposes the difference between the two trajectories may be neglectable. However, the influence of the free control subarc grows if one decreases the HEO and the LEO radii. So for a rather extreme case of an outer altitude for HEO of $h_{00} = 10,000$ km and a final altitude for LEO of $h_{11} = 130$ km, the switching to the free control subarcs occurs before half of the maneuver time and we found the characteristic velocity of $\Delta V = 1.233691\,\mathrm{km/s}$ for the control structure A and $\Delta V = 1.232203\,\mathrm{km/s}$ for the control structure B.

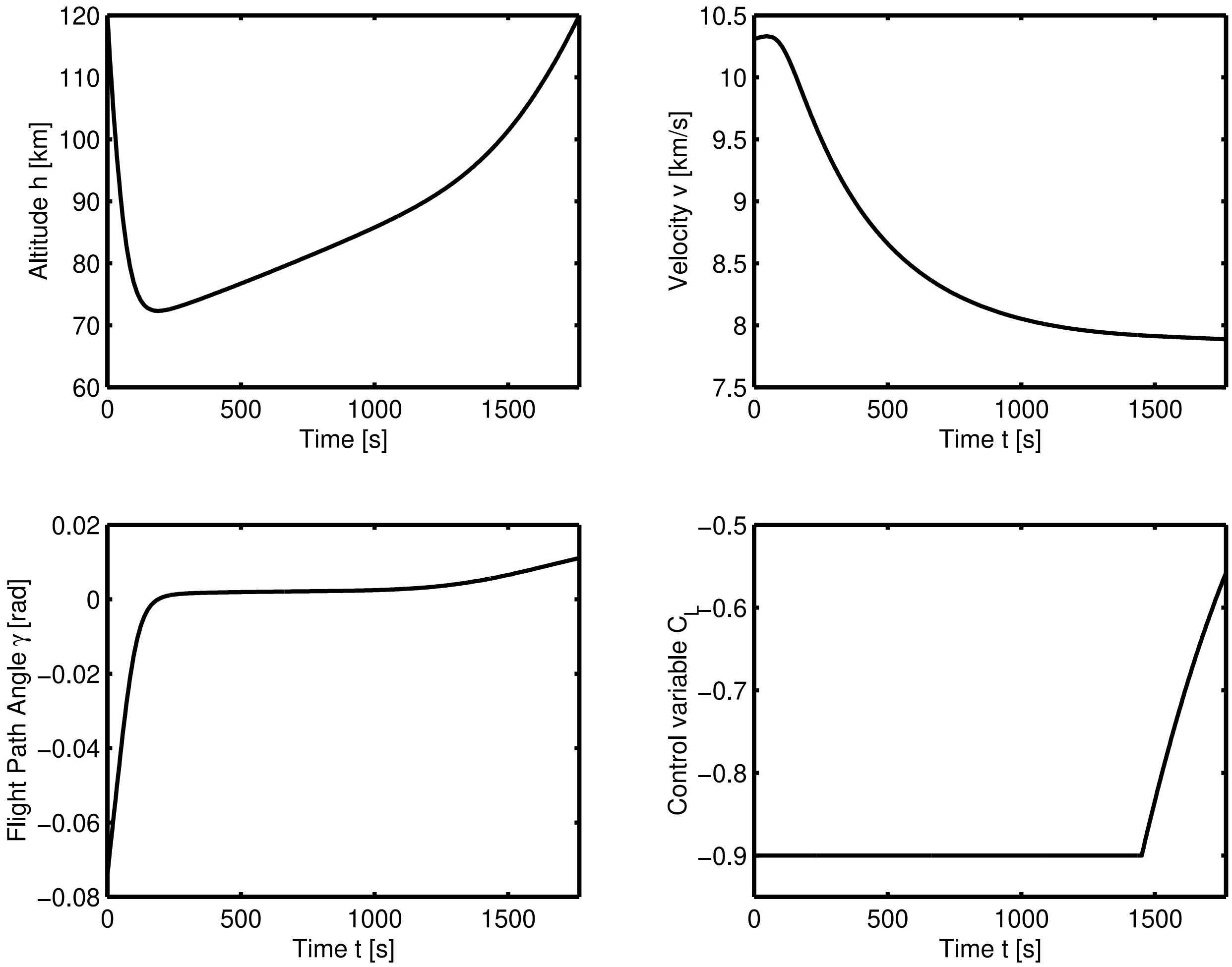

Fig. 3 Problem P1: State Variables and Control Function for Control Structure B.

A detailed investigation of the control structure depending on the radii of HEO and LEO can be found in Baumann (Ref. 16).

6. Modeling the Aeroassisted Orbital Transfer by Finite Thrust Subarcs

In this section we consider the problem of fuel–optimal coplanar aeroassisted orbital transfer without the assumption of impulsive velocity changes. Therefore, we consider again the whole maneuver including the nonatmospheric subarcs. We introduce the magnitude of the thrust vector T and the angle ψ of the thrust vector with respect to the local horizon as new control variables and the mass of the vehicle m as a new state variable.

Table 2. Problem P1, characteristic quantities of two trajectories.

	Control Structure A	Control Structure B
Characteristic velocity ΔV [km/s]	1.55937609	1.55936115
Final time t_1 [s]	1757.33513	1766.26952
Switching time τ [s]	–	1450.12860
$v(0)$, $v(t_1)$ [km/s]	10.3095772 7.88671278	10.3095772 7.88672725
$\gamma(0)$, $\gamma(t_1)$ [rad]	-0.073981285 0.011123104	-0.073981285 0.011112244
Adjoint Variables $\lambda(0)$ (scaled)	$0.16356463 \times 10^{-2}$ $0.13806978 \times 10^{-2}$ $0.15353768 \times 10^{-1}$	$0.16356463 \times 10^{-2}$ $0.13806977 \times 10^{-2}$ $0.15353768 \times 10^{-1}$

In this terms the optimal control problem can be stated as follows.

Problem P2. Determine the free transfer time t_{11} and the control variables $T(t), \psi(t)$, and $C_L(t)$, $t_{00} \le t \le t_{11}$, such that the functional

$$I = -m(t_{11}) \tag{32}$$

is minimized subject to the state equations

$$dh/dt = v \sin\gamma\,, \tag{33a}$$

$$dv/dt = (T/m)\cos\psi - D/m - \mu/r^2 \sin\gamma\,, \tag{33b}$$

$$d\gamma/dt = (T/(mv))\sin\psi + L/(mv) + (v/r - \mu/(r^2 v))\cos\gamma\,, \tag{33c}$$

$$dm/dt = -T/v_c\,, \tag{33d}$$

the boundary conditions

$$h(0) = h_{00}\,, \qquad h(t_{11}) = h_{11}\,, \tag{34a}$$

$$v(0) = v_{00}\,, \qquad v(t_{11}) = v_{11}\,, \tag{34b}$$

$$\gamma(0) = 0\,, \qquad \gamma(t_{11}) = 0\,, \tag{34c}$$

$$m(0) = m_{00}\,, \tag{34d}$$

and the control constraints

$$0 \;\leq\; T \;\leq T_{\max}\,, \tag{35a}$$

$$C_{L\min} \;\leq C_L \;\leq\; C_{L\max}\,. \tag{35b}$$

The approximations of drag and lift are chosen as before, cf. Eqs.(5–7). Again the atmospheric density ρ is set to zero outside the atmosphere, i.e. for $h > h_a$. The data of Problem 2 are chosen as in the Table 1. The additional parameters can be found in Table 3. The values for the initial mass m_{00} and the reference surface area S are chosen such that at the exit point of the initial thrust subarc, the ratio m/S corresponds to the value given in Table 1. Note that the value for the equivalent exit velocity v_c corresponds to a specific impulse of 420 s.

Table 3. Additional parameters for Problem P2

m_{00}	$15,651.09\,\mathrm{kg}$	S	$36.2874\,\mathrm{m}^2$
$T_{\max}$	$5,000\,\mathrm{N}$	v_c	$4.115452135 \times 10^3\,\mathrm{m/s}$

For numerical purposes we again use a scaled version of Problem P2. To this end, in addition to the Eqs. (15–17), we define

$$\overline{m}(\overline{t}) \;:=\; m(a\overline{t})/m_{00}\,, \quad \overline{\zeta}(\overline{t}) \;:=\; a^2 T(a\overline{t})/(r_a m_{00})\,, \quad \overline{\psi}(\overline{t}) \;:=\; \psi(a\overline{t})\,. \tag{36}$$

Further we use the abbreviations given in Eq.(18), where m has to be substituted by the initial mass m_{00}, and in addition, the following ones

$$A_3 \;:=\; r_a/(a\,v_c), \qquad \zeta_{\max} \;:=\; a^2\,T_{\max}/(r_a\,m_{00})\,, \tag{37a}$$

$$\overline{h}_{00} \;:=\; h_{00}/h_a, \qquad \overline{h}_{11} \;:=\; h_{11}/h_a\,, \tag{37b}$$

$$\overline{v}_{00} \;:=\; a\,v_{00}/r_a, \qquad \overline{v}_{11} \;:=\; a\,v_{11}/r_a\,. \tag{37c}$$

As before, we choose $t_{00} := 0$ and we denote with t_f the scaled final time, $t_f := t_{11}/a$. With this definitions we can rewrite Problem 2 as follows

Problem P2'. Determine the free final time t_f, and the control variables $\zeta(t)$, $\psi(t)$, and $\eta(t)$, $0 \leq t \leq t_f$, such that

$$I \;=\; -\,m(t_f) \tag{38}$$

is minimized subject to the state equations

$$dh/dt \;=\; b\,v\,\sin\gamma\,, \tag{39a}$$

$$dv/dt \;=\; (\zeta/m)\cos\psi \;-\; A_1\,(\delta/m)\,(1+\eta^2)\,v^2 \;-\; (b/(b-1+h))^2\,\sin\gamma\,, \tag{39b}$$

$$d\gamma/dt \;=\; (\zeta/(mv))\sin\psi \;+\; A_2\,(\delta/m)\,\eta\,v \tag{39c}$$

$$+\;\; \Big[(b/(b-1+h))v \;-\; (b/(b-1+h))^2/v\Big]\;\cos\gamma\,, \tag{39d}$$

$$dm/dt \;=\; -A_3\,\zeta\,, \tag{39e}$$

the two-point-boundary conditions

$$h(0) \;=\; \overline{h}_{00}\,, \qquad h(t_f) \;=\; \overline{h}_{11}\,, \tag{40a}$$

$$v(0) \;=\; \overline{v}_{00}\,, \qquad v(t_f) \;=\; \overline{v}_{11}\,, \tag{40b}$$

$$\gamma(0) \;=\; 0\,, \qquad \gamma(t_f) \;=\; 0\,, \tag{40c}$$

$$m(0) \;=\; 1\,, \tag{40d}$$

and the control constraint

$$0 \;\leq\; \zeta(t) \;\leq\; \zeta_{\max}\,, \tag{41a}$$

$$\eta_{\min} \;\leq\; \eta(t) \;\leq\; \eta_{\max}\,. \tag{41b}$$

It may be noticed that for Problem P2, one can formulate a transformed version analogous to Problem (P1') as well, where the coast subarcs ($T \equiv 0$) outside the atmosphere are omitted. This can be done in the same way as before by applying the conservation of energy and angular momentum. Thus, one obtains an optimal control problem with merely piecewise continuous state variables. The discontinuities occur at the switching points at the end of the first and at the beginning of the second thrust subarcs. At these points one has well-defined jumps in the state variables v and γ and a preassigned values for $h(\tau^+)$ or $h(\tau^-)$, respectively. Also for this kind of optimal control problem one can apply necessary conditions. We found however that the resulting BVP is not significant better suited for the numerical computation than the one obtained directly. Further, in this formulation one would get no information about eventually occurring additional thrust subarcs outside of the atmosphere. So we decided to stay with the original formulation of Problem (P1) or its scaled version, Problem (P1'), respectively.

In the following the necessary conditions for Problem 2' briefly are summarized. Let λ_h, λ_v, λ_γ, and λ_m denote the adjoint variables with respect to the states. Then, with the Hamiltonian

$$H \;=\; \lambda_h\,dh/dt \;+\; \lambda_v\,dv/dt \;+\; \lambda_\gamma\,d\gamma/dt \;+\; \lambda_m\,dm/dt \tag{42}$$

the following adjoint differential equations are obtained

$$d\lambda_h/dt \;=\; \lambda_v\,\Big[A_1(\delta'(h)/m)\,(1+\eta^2)v^2 \;-\; 2\,(b^2/c^3)\,\sin\gamma\Big]$$

$$+\;\; \lambda_\gamma\Big[-A_2(\delta'(h)/m)\,\eta v \;+\; (bv/c^2 \;-\; 2b^2/(c^3 v))\,\cos\gamma\Big]\,, \tag{43a}$$

$$
\begin{aligned}
d\lambda_v/dt &= -\lambda_h b \sin\gamma \;+\; 2\lambda_v A_1(\delta(h)/m)\,(1+\eta^2)v \\
&- \;\lambda_\gamma \left[-\zeta/(mv^2)\sin\psi \;+\; A_2(\delta(h)/m)\,\eta \;+\; (b/c + b^2/(c^2v^2))\;\cos\gamma\right], \qquad (43b)
\end{aligned}
$$

$$
d\lambda_\gamma/dt = -\lambda_h bv\cos\gamma \;+\; \lambda_v (b/c)^2\cos\gamma \;+\; \lambda_\gamma \left[(b/c)v \;-\; b^2/(c^2v)\right]\sin\gamma, \qquad (43c)
$$

$$
\begin{aligned}
d\lambda_m/dt &= -\zeta/(m^2v)\sqrt{\lambda_\gamma^2 \,+\, (v\lambda_v)^2} \;-\; \lambda_v A_1(\delta/m^2)\,(1+\eta^2)v^2 \\
&+ \;\lambda_\gamma A_2(\delta/m^2)\,\eta v\,. \qquad (43d)
\end{aligned}
$$

Again the abbreviation $c := b-1+h$ is used. The optimal control functions are determined via the minimum principle. One obtains the following bang-bang control law for the thrust magnitude

$$
\zeta \;=\; \begin{cases} \zeta_{\max} \;, & \text{if } \; S \,<\, 0, \\ 0 \;, & \text{if } \; S \,>\, 0, \end{cases} \qquad (44)
$$

where S denotes the switching function

$$
S \;=\; -\,1/(mv)\sqrt{\lambda_\gamma^2 \,+\, (v\lambda_v)^2} \;-\; A_3\,\lambda_m\,. \qquad (45)
$$

The thrust angle ψ is determined by the relations

$$
\sin\psi \;=\; -\,\lambda_\gamma \,/\, \sqrt{\lambda_\gamma^2 \,+\, (v\lambda_v)^2}\,, \quad \cos\psi \;=\; -\,v\lambda_v \,/\, \sqrt{\lambda_\gamma^2 \,+\, (v\lambda_v)^2}\,. \qquad (46)
$$

Finally, the control variable η corresponding to the lift coefficient is given as in the impulsive case by the Eqs. (25), (26a–c).

The necessary conditions are completed by the natural boundary conditions. With respect to the free final mass to be maximized, and with respect to the the free final time the natural boundary conditions are given by

$$
\lambda_m(t_f) \;=\; -\,1\,, \quad H[t_f] \;=\; 0\,. \qquad (47)
$$

For a proper numerical treatment of the resulting multi-point boundary value problem we take into account the discontinuity of the air density at the begin and at the end of the atmospheric subarc $[t_0, t_1]$. So the necessary conditions for optimal control problems with a piecewise defined right-hand side of the state equations have to be applied, cf. Chudej (Ref. 17).

In the special case considered, the discontinuities occur at the (variable) switching points t_0 and t_1 (entry - and exit point of the atmospheric subarc), which are fixed by the switching conditions

$$
h(t_0) \;=\; h(t_1) \;=\; 0\,. \qquad (48)
$$

The necessary conditions state that there occur jump discontinuities in the adjoint variable λ_h :

$$
\lambda_h(t_0^+) \;=\; \lambda_h(t_0^-) \,+\, \nu_0\,, \quad \lambda_h(t_1^+) \;=\; \lambda_h(t_1^-) \,+\, \nu_1\,, \qquad (49)
$$

where the Lagrange parameter ν_0 and ν_1 are determined such that the Hamiltonian is continuous at t_0 and t_1 :

$$H[t_0^+] \;=\; H[t_0^-]\,, \quad H[t_1^+] \;=\; H[t_1^-]\,, \tag{50}$$

For the numerical computation we assume the following control structure with respect to the thrust:

$$T: \qquad T_{\max} \quad - \quad 0 \quad - \quad T_{\max}\,. \tag{51}$$

Further, we assume that the thrust subarcs $[0,\tau_1]$ and $[\tau_3,t_f]$ do not intersect the atmospheric subarc $[t_0,t_1]$, i.e. $\tau_1 < t_0 < \tau_2 < t_1 < \tau_3$, where τ_2 denotes the switching point with respect to the control variable η.

7. Numerical Results for Problem 2

By the necessary conditions derived in Section 6, and by an estimated switching structure with respect to the control variables ζ and η, we obtain a multi-point BVP for the numerical solution via multiple shooting techniques.

The variables of this BVP are the state (h,v,γ,m) and the corresponding adjoint variables $(\lambda_h,\lambda_v,\lambda_\gamma,\lambda_m)$. Additional parameters are the final time t_f, the five switching points $(\tau_1,t_0,\tau_2,t_2,\tau_3)$, and the Lagrange parameters (ν_1,ν_2).

The differential equations are given by the Eqs. (39a–e), (43a–e), where the control variables are determined by Eq. (46) and by

$$\zeta(t) \;=\; \begin{cases} \zeta_{\max}, & \text{if} \quad t \in [0,\tau_1] \cup [\tau_3,t_f], \\ 0, & \text{else}, \end{cases} \tag{52}$$

$$\eta(t) \;=\; \begin{cases} \eta_{\min}, & \text{if} \quad t \in [t_0,\tau_2], \\ \eta_{\text{free}}, & \text{if} \quad t \in \,]\tau_2,t_1], \\ 0, & \text{else}. \end{cases} \tag{53}$$

The boundary conditions are given in Eqs. (40a–d), (47).

Further, we have jump conditions, given in Eq.(49), and switching conditions, given in Eqs. (48), (50).

Finally, due to the estimated switching structure, we obtain the additional switching conditions

$$S[\tau_1] \;=\; S[\tau_3] \;=\; 0 :, \qquad \eta_{\text{free}}(\tau_2) \;=\; \eta_{\max}\,. \tag{54}$$

For a successful application of multiple shooting technique to this rather complicated BVP the knowledge of a suitable initial estimation of the solution trajectory is necessary. This was obtained in several steps.

First, the solution of Problem 1 is completed to a trajectory which starts at HEO and ends at LEO. This trajectory can be found easily by integration of the state and adjoint variables of Problem (P1") to the left (up to HEO) and to the right (up to LEO).

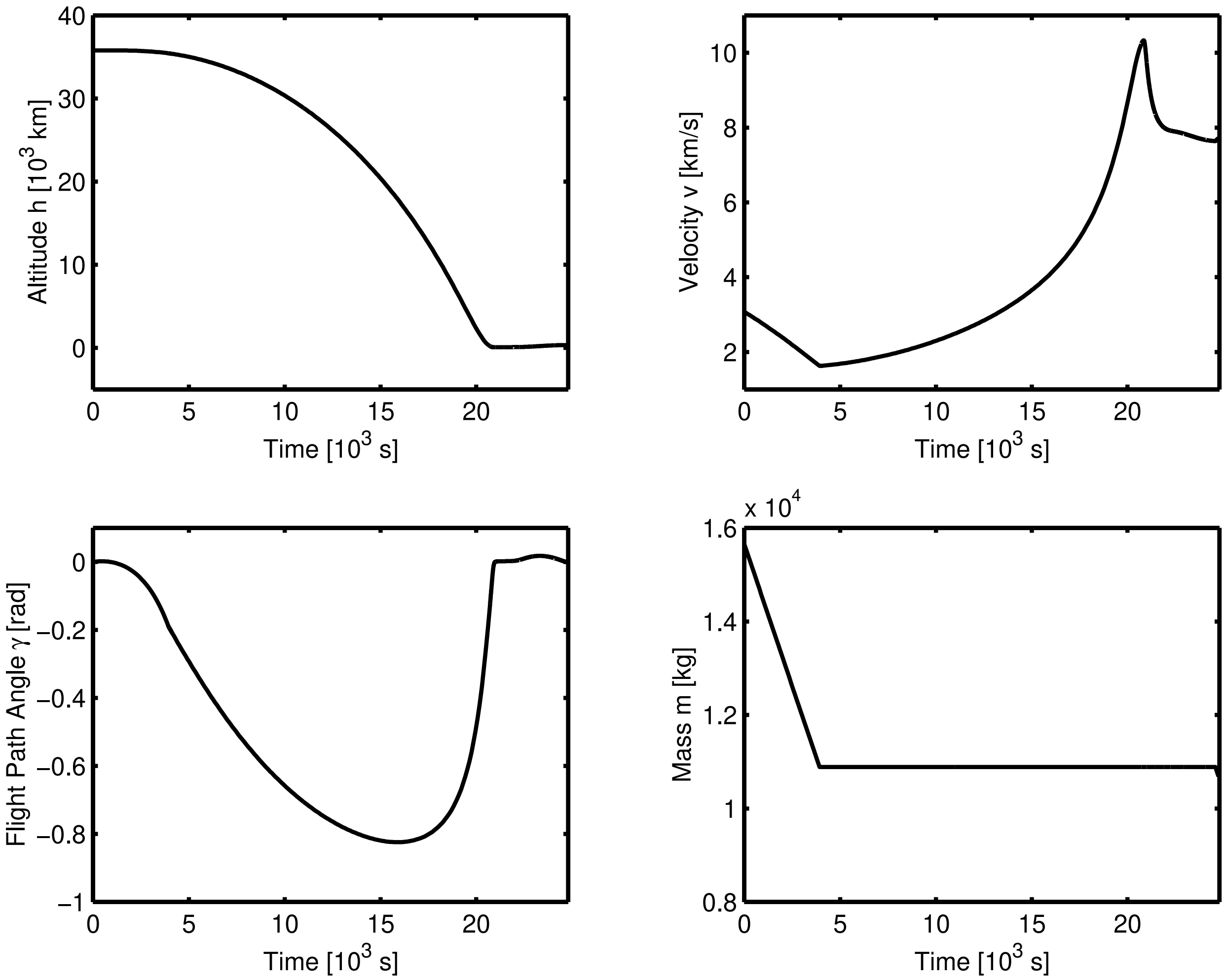

Fig. 4 Problem P2: State Variables.

It results a solution candidate of the impulsive AOT which minimizes the functional (2). Thus the adjoint variables of this solution satisfy the corresponding natural boundary conditions

$$\lambda_v(0) \;=\; 1, \qquad \lambda_v(t_f) \;=\; -1\,. \tag{55}$$

In a second step, the state m, the adjoint λ_m and small finite thrust subarcs at the begin and the end of the trajectory as well are build in. In order to stay in the neighborhood of the solution of Problem P1, we choose a rather high value of maximal thrust, $T_{\max} = 1,270,000\,\mathrm{N}$. Applying continuation techniques, in a third step the natural boundary conditions are adapted, starting with the conditions (55) and ending with the boundary conditions (47). Finally in a last step, the value of $T_{\max}$ is reduced via continuation techniques up to the desired value given in Table 3.

The solution obtained by multiple shooting techniques is shown in the Figures 4–6. In Fig.4 the time behavior of the state variables is shown, Fig.5 gives the time behavior of the adjoints and in the Fig. 6 the optimal control functions and the switching function as well are shown. The solution satisfies the necessary conditions, especially the minimum principle. The behavior of the switching function shows that the estimation of control structure has been correct.

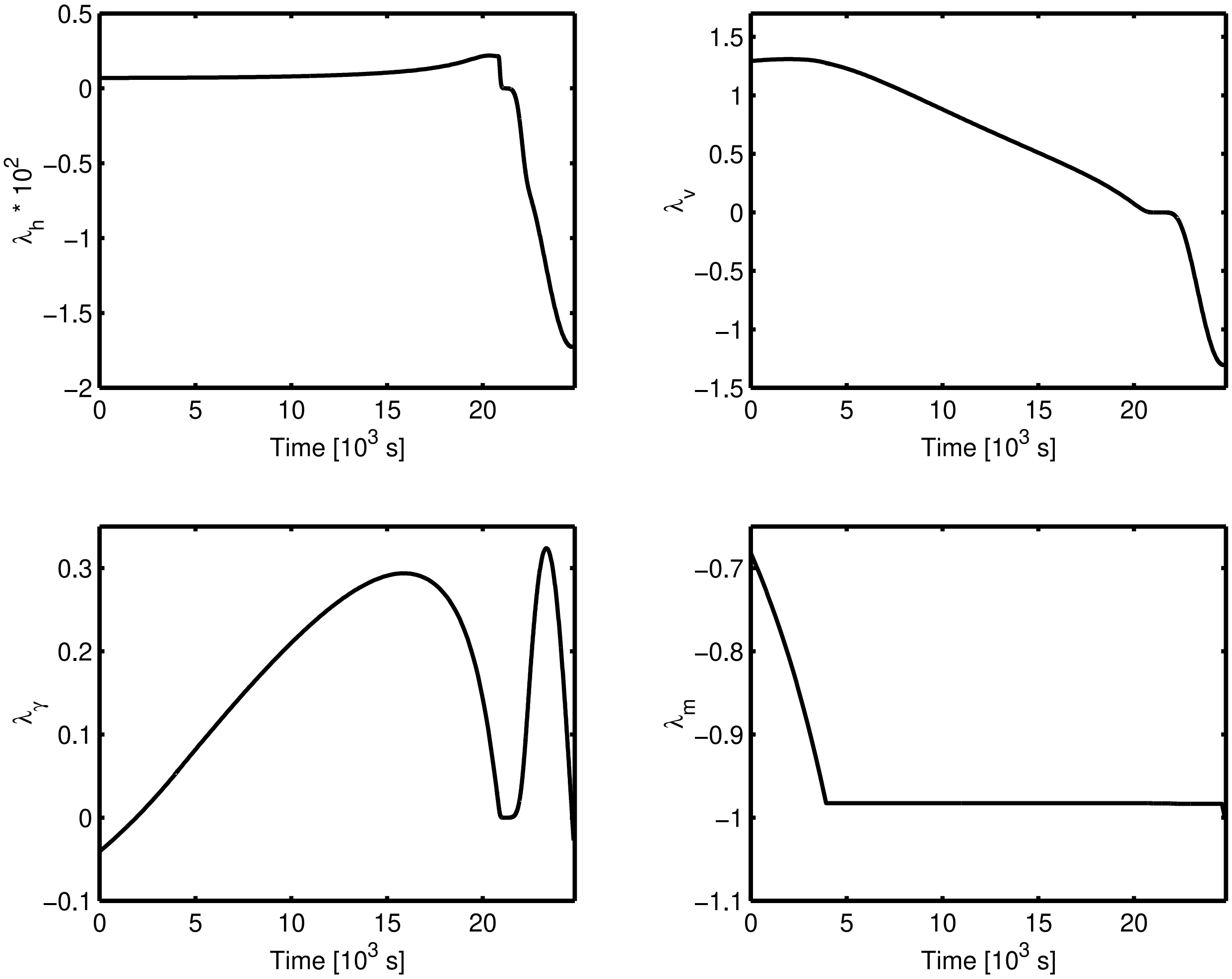

Fig. 5 Problem P2: Adjoint Variables.

On the atmospheric subarc, which is given by the entry-time $t_1 \approx 20,796$ s and the exit-time $t_2 \approx 22,562$ s, the qualitative behavior of the state and adjoint variables is similar to that of the impulsive thrust solution shown in the Figs. 1–2. Also the time spent in the atmospheric subarc mainly remains unchanged.

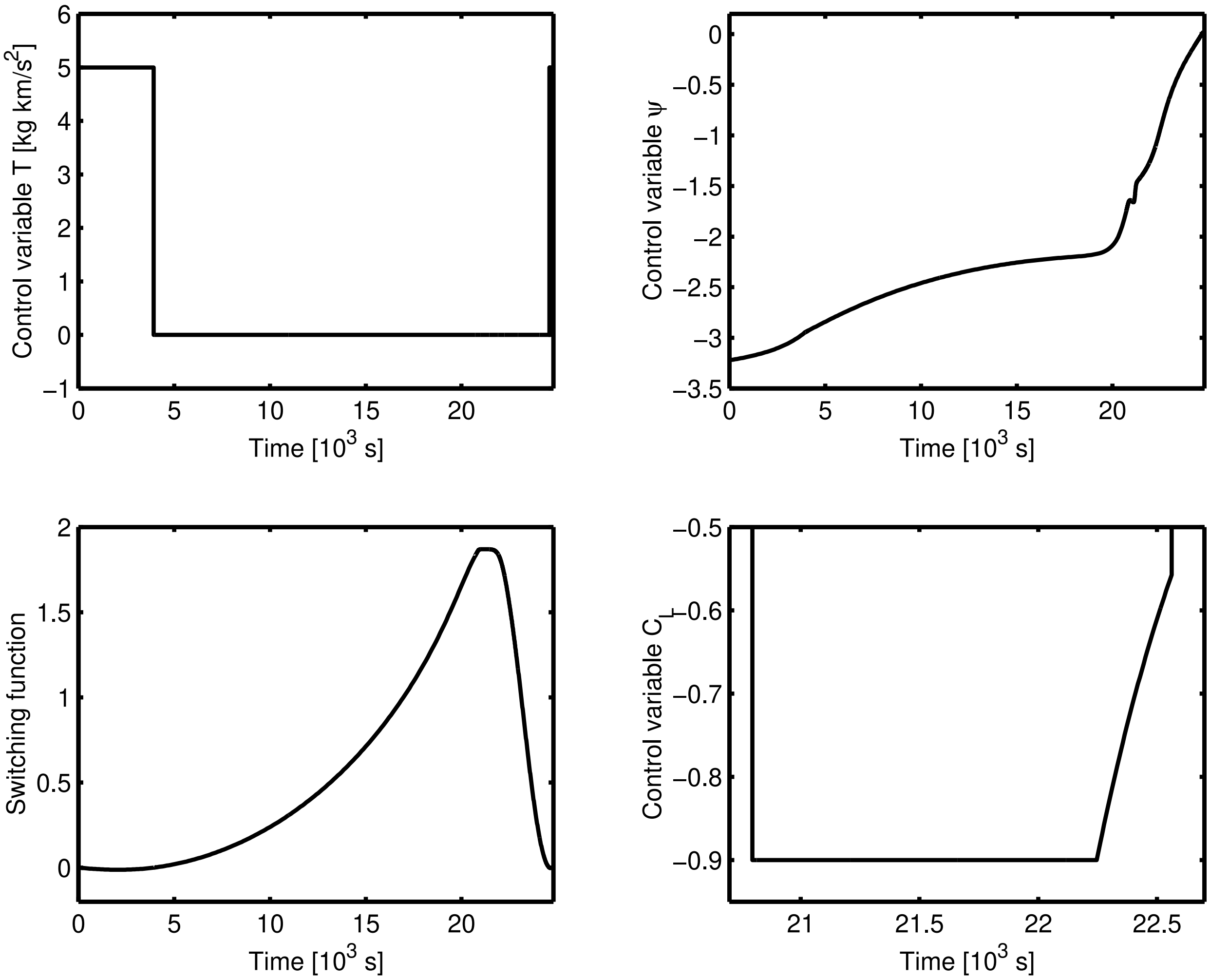

Fig. 6 Problem P2: Control Variables and Switching Function.

The behavior of the trajectory near Earth's atmosphere is shown in the Fig. 7. The trajectory starts at the point $(r_{00}, 0)$, outside of the window shown in the Figure and marked by the horizontal dashed line. The points of entry into the atmosphere and exit from the atmosphere are marked, the second thrust subarc as well. The dotted line gives the upper bound of atmosphere. The unit used in the Figure is 1000 km.

The characteristic physical quantities of the solution are summarized in Table 4. For reasons of comparison the characteristic quantities of a solution with a large value of $T_{\max}$ are added. These later results are very close to the results obtain for the impulsive thrust solution of Problem P1 which are summarized in Table 2. The total maneuver time is slightly enlarged to about 6h 52min against a transfer time of about 6h 18min for the impulsive thrust solution. About 1h 6min of the transfer time is spent on the thrust subarcs at the begin and at the end of the transfer.

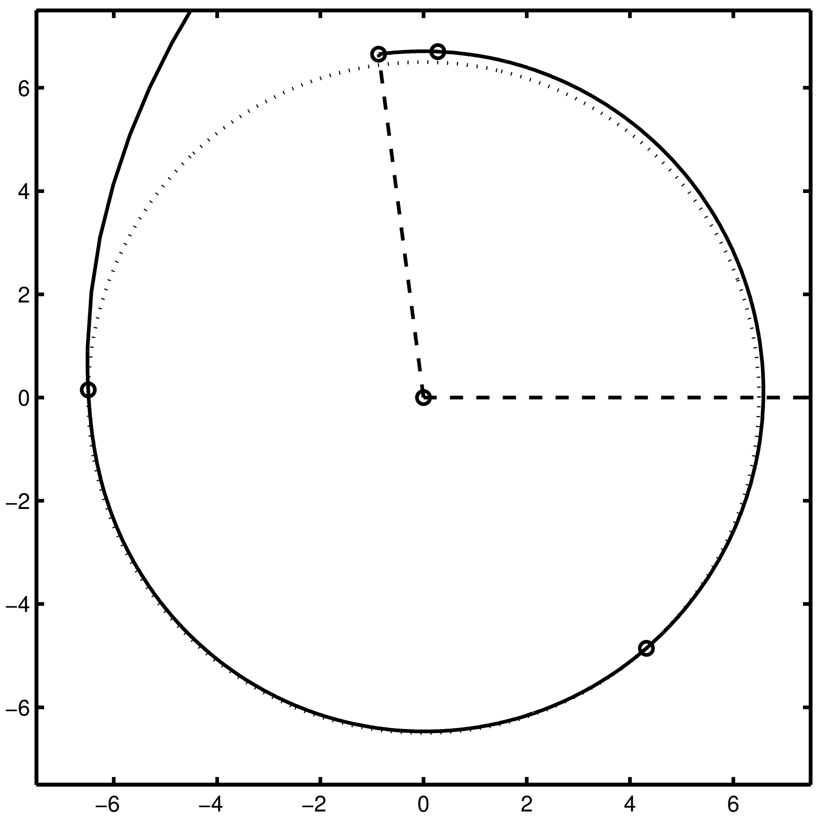

Fig. 7 Problem P2: Trajectory near Earth' atmosphere.

Table 4. Problem P2, characteristic quantities of two trajectories.

	$T_{\max} = 5 \times 10^3\text{N}$	$T_{\max} = 1200 \times 10^3\text{N}$
Charact. velocity ΔV [km/s]	1.51700452	1.55936311
Total time t_f [s]	$24,739.50$	$22,629.43$
Length of thrust subarcs [s]	3921.92 149.453	16.5957 0.62301
Fuel consumption [kg]	4946.44	4936.18
Length of the atmospheric subarc [s]	1766.380	1766.336

For reasons of completeness and in order to enable numerical comparisons, the essential numerical quantities including the (scaled with respect to $[0,1]$) switching points are listed in Table 5.

Table 5. Problem P2, numerical characteristic quantities of the solution trajectories.

Switching points with respect to T	τ_1	=	0.15852853
	τ_3	=	0.99395891
Switching points with respect to the atmospheric part	t_0	=	0.84059043
	τ_2	=	0.89920336
	t_1	=	0.91198962
Adjoint variables	$\lambda_h(0)$	=	$0.69020288 \times 10^{-3}$
	$\lambda_v(0)$	=	0.12940999×10^{1}
	$\lambda_\gamma(0)$	=	$-0.40572594 \times 10^{-1}$
	$\lambda_m(0)$	=	-0.68216143
Final time (scaled)	t_f	=	0.29818789×10^{2}
Lagrange parameter	ν_0	=	$-0.19031375 \times 10^{-5}$
	ν_1	=	$-0.24609427 \times 10^{-4}$

8. Conclusions

In this paper the problem of fuel-optimal coplanar aeroassisted orbital transfer has been considered in two different formulations. First, the classical model has been reconsidered. Therein, the transfer starts with an impulsive burn on the geosynchronous orbit and ends with a second impulsive burn on a low Earth orbit. The magnitude of the impulsive burns have been measured by the characteristic velocities. Therefore the performance index is chosen as the sum of both characteristic velocities. In a standard way, the problem could be reduced to the atmospheric subarc. The bounded lift coefficient figured as control variable.

Necessary conditions have been derived and a boundary value problem including the adjoint equations has been solved by multiple shooting techniques. It turned out that the optimal control consists of one boundary and one free subarc.

In the second part, a new and more realistic formulation of the problem has been considered, where the impulsive burns have been replaced by finite thrust subarcs with variable thrust direction and a bounded magnitude of thrust. These functions have been introduced as additional control variables. Further, the mass of the vehicle is considered as new state variable allowing to choose the final mass as performance index. Now for the optimization the complete transfer trajectory has been computed.

Again for the numerical solution, a multi-point boundary value problem has been derived using the necessary conditions of optimal control theory. The optimal control structure for the lift coefficient consists of one boundary and one free subarc, and, for the magnitude of thrust, it consists of two maximal thrust subarcs at the begin and at the end of the trajectory and a minimal subarc in between.

Comparisons between the two models show that the quantities inside the atmosphere do not differ significantly. However, they differ with resect to the total transfer time and with respect to the thrust control outside of the atmosphere.

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