Hamburger Beiträge
zur Angewandten Mathematik

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Reihe A
Preprint 141
May 1999

Hamburger Beiträge zur Angewandten Mathematik

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A Note on the Order on the Set of Finite Subsets

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May 12, 1999

Abstract

Let T be a well-ordered set with an injective order-homomorphism $\varphi : T \to \mathbb{Q}$. In this note we construct a simple extension ϕ of φ to $\mathcal{P}_{fin}(T)$, the set of all finite subsets of T, such that ϕ is an injective order-homomorphism *wrt.* the induced well-order on $\mathcal{P}_{fin}(T)$. We apply this construction to lexicographic orders on $\mathbb{N}^n$.

Introduction

Let $(T, \leq)$ be a well-ordered set and let there be given a function $\varphi : T \to \mathbb{Q}$ such that

$$a < b \Longrightarrow \varphi(a) < \varphi(b)$$

for all $a, b \in T$, which implies that φ is an order-isomorphism from T onto $\varphi(T) \subset \mathbb{Q}$. Here, $<$ is the strict part of $\leq$.

Now it is well-known that $\mathcal{P}_{fin}(T)$, the set of all finite subsets of T, is also a well-ordered set by means of the following definition of the induced order on $\mathcal{P}_{fin}(T)$:

Definition 0.1. *For $\emptyset \neq A \in \mathcal{P}_{fin}(T)$, let $max(A)$ denote the unique maximal element of A. For $A, B \in \mathcal{P}_{fin}(T)$, $A \leq' B$ is defined by recursion as follows:*

(i) If $A = \emptyset$, then $A \leq' B$

(ii) If $A \neq \emptyset$, then $A \leq' B$ iff $B \neq \emptyset$ and the following condition holds:

$$\max(A) < \max(B), \text{ or}$$

$$\max(A) = \max(B) \text{ and } A \setminus \{\max(A)\} \leq' B \setminus \{\max(B)\} \ .$$

Using Cantor's "second" diagonal argument (cf. [1], p. 171) one can prove that $(\mathcal{P}_{fin}(T), \leq')$ is a well-ordered set.

The definition above shows that $\leq'$ is an extension of $\leq$ in the following sense: For all $a, b \in T$,

$$a \leq b \Longleftrightarrow \{a\} \leq' \{b\} \,.$$

Thus there is no need for different notations, and we will use $\leq$ also for the well-order on $\mathcal{P}_{fin}(T)$.

Our aim in this note is to construct an extension $\phi : \mathcal{P}_{fin}(T) \longrightarrow \mathbb{Q}$ of $\varphi : T \longrightarrow \mathbb{Q}$ such that $A < B \Longrightarrow \phi(A) < \phi(B)$, which implies that ϕ is an order-isomorphism from $\mathcal{P}_{fin}(T)$ onto its image under ϕ.

Let us look at the simple example $T = \mathbb{N}$ where $\varphi : \mathbb{N} \longrightarrow \mathbb{Q}$ is the natural embedding. Here is the first section of $\mathcal{P}_{fin}(T)$:

$$\begin{aligned} &\{1\} \leq \{2\} \leq \{1,2\} \leq \{3\} \leq \{1,3\} \leq \{2,3\} \leq \\ &\qquad \leq \{1,2,3\} \leq \{4\} \leq \{1,4\} \leq \{2,4\} \leq \ldots\ldots \end{aligned} \tag{0.1}$$

We define ϕ as follows:

$$\begin{aligned} \phi(\emptyset) &= 0\,, \\ \phi(A) &= \max(A) + \frac{1}{\max(A)}\,\phi(A \setminus \max(A))\,, \end{aligned} \tag{0.2}$$

and by an easy calculation we obtain, for the first part of the sequence in (0.1), the numbers $\phi(\{1\}), \phi(\{2\}), \phi(\{1,2\}), \ldots$:

$$1 \leq 2 \leq \frac{5}{2} \leq 3 \leq \frac{10}{3} \leq \frac{11}{3} \leq \frac{23}{6} \leq 4 \leq \frac{17}{4} \leq \frac{18}{4} \leq \ldots$$

(0.2) is a special case of a general construction which we shall perform in the next section.

1 Construction of ϕ in the general case

For $a \in T$, let a^+ be the unique successor of a, i.e., a^+ is the smallest element of T which is greater than $a : a < a^+ = \min\{b : a < b\}$. Let us define, for all $a \in T$,

$$\Delta(a) = \varphi(a^+) - \varphi(a) > 0 \quad .$$

Now we can define the function ϕ in a way similar to (0.2).

Definition 1.1. *For $\phi \neq A \in \mathcal{P}_{fin}(T)$:*

$$\begin{aligned} \phi(A) &= \varphi(\max(A)) + \frac{\Delta(\max(A))}{\varphi(\max(A))}\,\phi(A \setminus \max(A))\,; \\ \phi(\emptyset) &= 0\,. \end{aligned} \tag{1.1}$$

In the special case $T = \mathbb{N}$, we have $\varphi = id$, $\Delta(a) \equiv 1$, therefore (1.1) specializes to (0.2).

We are going to use a somewhat more concrete notation which helps us in understanding the mechanism of the evaluation of ϕ. Let $M = \{m_1, m_2, \ldots, m_k\} \in \mathcal{P}_{fin}(T)$ $k \geq 1$ be a given set; we always regard M as an ordered set with

$$m_1 > m_2 > \ldots > m_k$$

We also define

$$\Delta_i^M = \varphi(m_i^+) - \varphi(m_i) \quad \text{for} \quad i = 1, \ldots, k$$

Now, ϕ can be written as

$$\begin{aligned} \phi(\{m_1, \ldots, m_k\}) &= \varphi(m_1) + \frac{\Delta_1^M}{\varphi(m_1)} \cdot \phi(\{m_2, \ldots, m_k\}) \\ &= \varphi(m_1) + \frac{\Delta_1^M}{\varphi(m_1)} \left(\varphi(m_2) + \frac{\Delta_2^M}{\varphi(m_2)} \cdot \varphi(\{m_3, \ldots, m_k\}) \ldots \right. \end{aligned}$$

Defining the affin-linear functions $\psi_i^M : \mathbb{Q} \to \mathbb{Q}$ by

$$\psi_i^M(x) = \varphi(m_i) + \frac{\Delta_i^M}{\varphi(m_i)}\,x\,, \quad i = 1, 2, \ldots, k\,, \tag{1.2}$$

it is possible to write our definition of ϕ as follows:

$$\begin{aligned} \phi(\emptyset) &= 0\,, \\ \phi(\{m_1, \ldots, m_k\}) &= \psi_1^M(\phi(\{m_2, \ldots, m_k\}))\,. \end{aligned}$$

By recursion, this implies

$$\phi(\{m_1,\dots,m_k\}) = \psi_1^M \circ \psi_2^M \circ \dots \circ \psi_k^M(0)$$

Lemma 1.2. *For all $k \in \mathbb{N}$ and any finite set $M = \{m_1,\dots,m_k\}$ such that*

$$m_1 > m_2 > \dots > m_k$$

we have

$$\phi(\{m_1\}) \leq \phi(\{m_1,\dots,m_k\}) < \phi(\{m_1^+\}) \ . \tag{1.3}$$

Proof: ψ_i^M is a strictly isotone function with

$$\begin{aligned} \psi_i^M(\varphi(m_i)) &= \varphi(m_i) + \tfrac{\varphi(m_i^+)-\varphi(m_i)}{\varphi(m_i)}\,\varphi(m_i) \\ &= \varphi(m_i^+)\ , \end{aligned}$$

which implies

$$\varphi(m_i) \leq \psi_i^M(x) < \varphi(m_i^+)\ , \quad i = 1,2,\dots,k, \quad 0 \leq x < \varphi(m_i)\ . \tag{1.4}$$

For $i = k, x = 0$ this specializes to

$$\varphi(m_k) \leq \psi_k^M(0) \leq \varphi(m_k^+)\ ,$$

and, as $m_k^+ \leq m_{k+1}$ and $\varphi(m_k^+) \leq \varphi(m_{k+1})$,

$$\varphi(m_k) \leq \psi_k^M(0) < \varphi(m_{k+1})\ .$$

We now set $x = \psi_k^M(0), i = k-1$ in (1.4) and obtain

$$\varphi(m_{k-1}) \leq \psi_{k-1}^M \circ \psi_k^M(0) < \varphi(m_{k-1}^+)\ .$$

By finite induction from $i = k$ down to $i = 1$ this leads to

$$\varphi(m_1) \leq \psi_1^M \circ \psi_2^M \circ \dots \circ \psi_k^M(0) < \varphi(m_1^+)\ ,$$

which proves Lemma 1 because of the obvious equality $\phi(\{m\}) = \varphi(m)$ for all $m \in T$ ☐

Theorem 1.3. *For all $M, N \in \mathcal{P}_{fin}(T)$ such that $M < N$ in the induced order on $\mathcal{P}_{fin}(T)$, it follows that $\phi(M) < \phi(N)$.*

Proof: Let $M\{m_1, \dots, m_k\}$, $\quad k \geq 1$, and

$$N\{n_1, \dots, n_l\}, \quad l \geq 1,$$

be finite ordered subsets of T such that $M < N$. There are two cases: Either,

$$m_1 > m_2 > \dots > m_{p-1} > m_p > \dots > m_k$$

and

$$n_1 > n_2 > \dots > n_{p-1} > n_p > \dots > n_l \qquad ,$$

where $1 \leq p \leq \min(k, l)$ is the unique number with

$$m_i = n_i \qquad \text{for} \quad 1 \leq i \leq p-1 \text{ and} \qquad m_p < n_p \quad . \tag{1.5}$$

or, $m_i = n_i$ for $1 \leq i \leq k$ and $l > k$. Let us consider the first case.

Using the notation defined in (1.2), we get

$$\begin{aligned} \phi(M) &= (\psi_1^M \circ \cdots \circ \psi_{p-1}^M) \circ (\psi_p^M \circ \psi_{p+1}^M \circ \cdots \circ \psi_k^M(0)) \\ \phi(N) &= (\psi_1^N \circ \cdots \circ \psi_{p-1}^N) \circ (\psi_p^N \circ \psi_{p+1}^N \circ \cdots \circ \psi_l^N(0)) \,. \end{aligned}$$

Now, by Lemma 1.2 and 1.5,

$$\begin{aligned} x := \psi_p^M \circ \cdots \circ \psi_k^M(0) &< \phi(\{m_p^+\}) \\ &\leq \phi(\{n_p\}) \\ &\leq \phi(\{n_p, \dots, n_l\}) \\ &= \psi_p^N \circ \cdots \circ \psi_l^N(0) =: y \end{aligned} \tag{1.6}$$

Again by (1.5),

$$\psi_i^M = \psi_i^N \qquad \text{for} \quad 1 \leq i \leq p-1\,,$$

which implies

$$\begin{aligned} \phi(M) &= (\psi_1^M \circ \cdots \circ \psi_{p-1}^M)(x) \\ \phi(N) &= (\psi_1^M \circ \cdots \circ \psi_{p-1}^M)(y) \quad , \end{aligned}$$

where $x, y \in \mathbb{Q}$ are defined in (1.6). Now, each ψ_1^M is a strictly increasing function, and thus $\phi(M) < \phi(N)$. — The case $m_i = n_i$ for $1 \leq i \leq k$, $l < k$ can be treated similarily. □

Corollary 1.4. *ϕ is an order-isomorphism onto $\phi(\mathcal{P}_{fin}(T))$.*

2 Application

In this final part we show how to construct an injective order-homomorphism $\phi : \mathcal{P}_{fin}(\mathbb{N}^n) \longrightarrow \mathbb{Q}$ for all $n \geq 1$.

The first task is to define an injection

$$\varphi : \mathbb{N}^n \longrightarrow \mathbb{Q}$$

which is an order-homomorphism. Let us order $\mathbb{N}^n$ by the inverse lexicographical order:

$$(k_1, \ldots, k_n) \leq (m_1, \ldots, m_n)$$

iff the following condition holds: either $(k_1, \ldots, k_n) = (m_1, \ldots, m_n)$, or there exists $1 \leq i \leq n$ with $k_j = m_j$ for $i+1 \leq j \leq n$ and $k_i < m_i$.

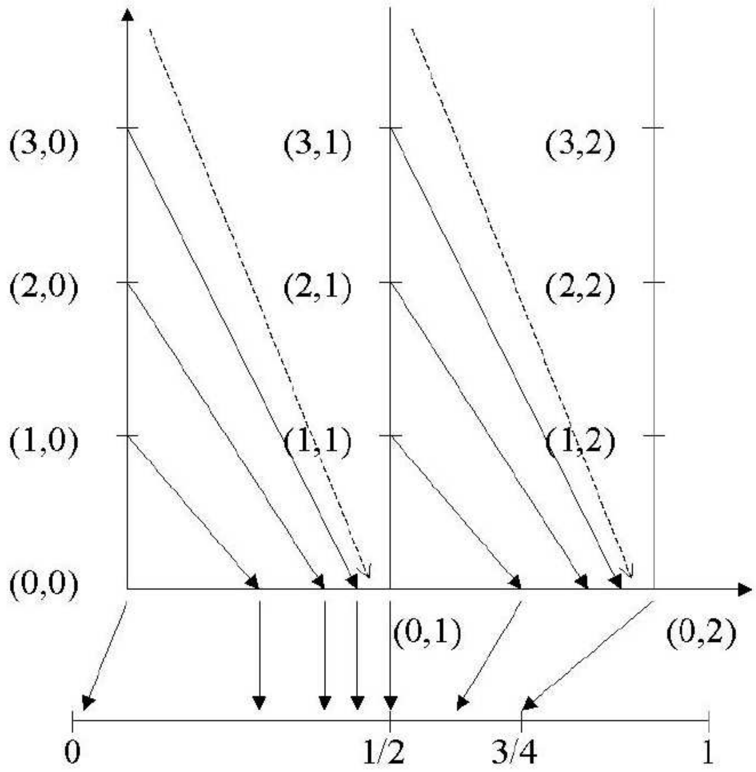

Figure 1

Definition 2.1. *Let $0 < \alpha \leq 1/2$ denote a fixed number. $\varphi : \mathbb{N}^n \longrightarrow \mathbb{Q}$ is defined by recursion as follows:*

$$\begin{aligned} \varphi(k_1) &= 1 - \alpha^{k_1} \qquad\qquad (k_1 \in \mathbb{N}) \\ \varphi(k_1, \ldots, k_j) &= 1 - \alpha^{-k_j} + \alpha^{-(k_j+1)}\,\varphi(k_1, \ldots, k_{j-1}) \end{aligned}$$

The idea underlying this construction is illustrated in Fig. 1 ($n = 2$).

Lemma 2.2.

(i) $0 \leq \varphi(k_1, \ldots, k_j) < 1$, $(k_1, \ldots, k_j) \in \mathbb{N}^j$, $\quad j \geq 1$.

(ii) $\varphi(k_j) \leq \varphi(k_1, \ldots, k_j) < \varphi(k_j + 1)$, $(k_1, \ldots, k_n) \in \mathbb{N}^n$.

Proof: (i) is, by definition, correct for $n = 1$ and follows easily by induction on $n \in \mathbb{N}$. The left inequality of (ii) is a direct implication of the definition above. The right-hand inequality is true, because

$$\begin{aligned}
\varphi(k_1, \ldots, k_j) &= 1 - \alpha^{-k_j} + \alpha^{-(k_j+1)} \varphi(k_1, \ldots, k_{j-1}) \\
&< 1 - \alpha^{-k_j} + \alpha^{-(k_j+1)} \\
&= 1 - \alpha^{-(k_j+1)} (\tfrac{1}{\alpha} - 1) \\
&\leq 1 - \alpha^{-(k_j+1)} \qquad \text{for } 0 < \alpha \leq \tfrac{1}{2} .
\end{aligned}$$

□

Corollary 2.3. *φ is an injective order-homomorphism from $\mathbb{N}^n$, equipped with the inverse lexicographical order, into $\mathbb{Q}$.*

Proof: Assume $(k_1, \ldots, k_n) \leq (m_1, \ldots, m_n)$. In case of inequality here is nothing left to prove. Therefore, let there exist $1 \leq i \leq n$ such that $k_j = m_j$ for $i + 1 \leq j \leq n$ and $k_i < m_i$. This implies by Lemma 2.5 (ii)

$$\varphi(k_1, \ldots, k_i) < \varphi(k_i{+}1) \leq \varphi(m_i) \leq \varphi(m_1, \ldots, m_i) .$$

Now, $k_j = m_j$ for $j = i + 1, \ldots, m$, which results in

$$\varphi(k_1, \ldots, k_{i+1}) < \varphi(m_1, \ldots, m_{i+1})$$

by Definition 2.5 and — by finite induction —

$$\varphi(k_1, \ldots, k_n) < \varphi(m_1, \ldots, m_n)$$

□

Now that we have defined $\varphi : \mathbb{N}^n \longrightarrow \mathbb{Q}$, the function $\phi : \mathcal{P}_{fin}(\mathbb{N}^n) \longrightarrow \mathbb{Q}$ can be easily constructed by means of Definition 1.1.

References

[1] Becker, Th., and Weispfennig, V. (1993) *Gröbner Basis. A Computational Approach to Commutative Algebra.* Springer Verlag, Second printing, 1998.