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BEYOND MONOTONICITY IN REGULARIZATION METHODS FOR NONLINEAR COMPLEMENTARITY PROBLEMS*

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Abstract: Regularization methods for the solution of nonlinear complementarity problems are standard methods for the solution of monotone complementarity problems and possess strong convergence properties. In this paper, we replace the monotonicity assumption by a P_0-function condition. We show that many properties of regularization methods still hold for this larger class of problems. However, we also provide some counterexamples which indicate that not all results carry over from monotone to P_0-function complementarity problems.

Key Words: Nonlinear complementarity problem, regularization method, P_0-function, mountain pass theorem.

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1 Introduction

We consider the *nonlinear complementarity problem* which is to find a vector in $\mathbb{R}^n$ satisfying the conditions

$$x \geq 0, \ \ F(x) \geq 0, \ \ x^T F(x) = 0;$$

here all inequalities are taken componentwise and $F : \mathbb{R}^n \to \mathbb{R}^n$ is any given function which we assume to be continuously differentiable throughout this paper.

There exist several methods for the solution of the complementarity problem NCP(F), see, e.g., the recent paper [11]. The particular class of methods to be considered in this paper are the so-called *regularization methods*, which are designed to handle *ill-posed* problems. In fact, regularization-type methods have recently been used very successfully in order to improve the robustness of several complementarity solvers on difficult test problems, see [1, 2]. For a detailed discussion of ill-posedness in mathematical programming, we refer the reader to [8]. Very roughly speaking, an ill-posed problem may be difficult to solve since small errors in the computations can lead to a totally wrong solution.

Regularization methods try to circumvent this difficulty by substituting the solution of the original problem with the solution of a sequence of well-posed (i.e. nicely behaved) problems whose solutions form a trajectory converging to the solution of the original problem. In the context of complementarity problems, if we consider the so called *Tikhonov-regularization*, this scheme consists in solving a sequence of complementarity problems NCP(F_ε)

$$x \geq 0, \ \ F_\varepsilon(x) \geq 0, \ \ x^T F_\varepsilon(x) = 0,$$

where $F_\varepsilon(x) := F(x) + \varepsilon x$ and ε is a positive parameter converging to 0.

Regularization methods for complementarity problems have already been considered in the literature, see, e.g., [22] and [6, Theorem 5.6.2 (b)]. The basic results that can be established in the monotone case, and that parallel the classical ones for regularization methods for convex optimization problems, see [8] or [21], are:

(a) The regularized problem NCP(F_ε) has a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$.

(b) The trajectory $x(\varepsilon)$ is continuous for $\varepsilon > 0$.

(c) For $\varepsilon \to 0$, the trajectory $x(\varepsilon)$ converges to the least l_2-norm solution of NCP(F) if NCP(F) has a nonempty solution set, otherwise it diverges.

In this paper, we try to generalize as much as possible the above results to the larger class of P_0 nonlinear complementarity problems. Actually such a scheme has already been considered in the case of P_0 linear complementarity problems in [24] (see also [6]). These results will be discussed in Section 2 where we also show that their generalization to nonlinear problems is nontrivial. In Section 3, we then extend point (a) to the class of P_0-function complementarity problems, whereas Section 4 is devoted to the (partial) generalization of points (b) and (c). In Section 5 we investigate an algorithm which requires only an approximate solution of the perturbed problems. We conclude with some final remarks in Section 6.

2 Preliminaries

We first restate some basic definitions.

Definition 2.1 *A matrix $M \in \mathbb{R}^{n \times n}$ is called a*

(a) P_0-matrix *if, for every $x \in \mathbb{R}^n$ with $x \neq 0$, there is an index $i_0 = i_0(x)$ with*

$$x_{i_0} \neq 0 \quad \text{and} \quad x_{i_0}[Mx]_{i_0} \geq 0;$$

(b) P-matrix *if, for every $x \in \mathbb{R}^n$ with $x \neq 0$, it holds that*

$$\max_i x_i[Mx]_i > 0;$$

(c) R_0-matrix *if $x = 0$ is the only solution of NCP(F) for $F(x) := Mx$.*

We refer the reader to the excellent book [6] by Cottle, Pang and Stone for a discussion of several properties of these classes of matrices. Some nonlinear generalizations of these classes are defined in

Definition 2.2 *The function $F : \mathbb{R}^n \to \mathbb{R}^n$ is called a*

(a) P_0-function *if, for all $x, y \in \mathbb{R}^n$ with $x \neq y$, there is an index $i_0 = i_0(x, y)$ with*

$$x_{i_0} \neq y_{i_0} \quad \text{and} \quad (x_{i_0} - y_{i_0})[F_{i_0}(x) - F_{i_0}(y)] \geq 0;$$

(b) P-function *if, for all $x, y \in \mathbb{R}^n$ with $x \neq y$, it holds that*

$$\max_i (x_i - y_i)[F_i(x) - F_i(y)] > 0;$$

(c) uniform P-function *if there is a constant $\mu > 0$ such that*

$$\max_i (x_i - y_i)[F_i(x) - F_i(y)] \geq \mu \|x - y\|^2$$

holds for all $x, y \in \mathbb{R}^n$.

Obviously, every uniform P-function is a P-function and every P-function is a P_0-function. Moreover, an affine mapping $F(x) := Mx+q$ is a P_0-function (P-function) if and only if M is a P_0-matrix (P-matrix). Moreover, the class of P_0-functions includes the class of monotone functions. For further discussions, we refer the reader to Moré and Rheinboldt [19].

In the affine case, there are some known results for regularization methods which partially generalize the properties (a) and (c) illustrated in the introduction from monotone to P_0 problems. We summarize these results in the following theorem.

Theorem 2.3 *Assume that $F(x) = Mx + q$ with $M \in \mathbb{R}^{n\times n}$ being a P_0-matrix and $q \in \mathbb{R}^n$. Then*

(a) The regularized problem NCP(F_ε) has a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$.

(b) If M is also an R_0-matrix, then the sequence $x(\varepsilon)$ is bounded for $\varepsilon \to 0$, and every limit point is a solution of NCP(F).

A proof of these results can be found in [6, Theorem 5.6.2 (a)]. Note also that, in [24], point (b) is proved under an assumption which implies that the original problem has a unique solution. In the linear case the proof of statement (a) is quite simple because if M is a P_0-matrix, then $M + \varepsilon I$ is a P-matrix by Theorem 3.4.2 in [6], so that NCP(F_ε) has a unique solution by Theorem 3.3.7 in [6].

Therefore, in an attempt to extend the previous results from the linear to the nonlinear case, the following question seems very natural: Is F_ε a uniform P-function for every fixed $\varepsilon > 0$ if F itself is a P_0-function? If the answer to this question were in the affirmative, then point (a) above could readily be extended, since a complementarity problem with a uniform P-function has a unique solution ([17, Corollary 3.2]). Unfortunately, the following example shows that F_ε is not necessarily a uniform P-function over $\mathbb{R}^n_+$ when F is nonlinear.

Example 2.4 Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x) := F(x_1, x_2) := \begin{pmatrix} 0 \\ -e^{x_1} \end{pmatrix}.$$

Since the Jacobian

$$F'(x) = \begin{pmatrix} 0 & 0 \\ -e^{x_1} & 0 \end{pmatrix}$$

is obviously a P_0-matrix for all $x \in \mathbb{R}^2$, the function F itself is a P_0-function by Corollary 5.3 in [19]. Now let $\varepsilon > 0$ and define

$$F_\varepsilon(x) = F(x) + \varepsilon x = \begin{pmatrix} \varepsilon x_1 \\ \varepsilon x_2 - e^{x_1} \end{pmatrix}.$$

We want to show that F_ε is not a uniform P-function on $\mathbb{R}^n_+$. This means that we want to show that, *given a fixed value* ε, we can find, *for every fixed value* μ, two points in $\mathbb{R}^n_+$ (possibly depending on μ) for which the definition of uniform P-function is not satisfied with that μ.

We will actually show that F_ε is not a uniform P-function for every positive ε. So suppose that $\varepsilon > 0$ is fixed. Choose a positive μ. Consider the following point $x = (x_1, x_2)$:

$$x_1 = 1, \quad x_2 = \sqrt{\frac{\varepsilon}{\mu}}(c-1), \tag{1}$$

where c is a constant such that

$$\begin{aligned} c &\geq 2 & (2) \\ \frac{\varepsilon^2}{\mu}(c-1)^2 - \sqrt{\frac{\varepsilon}{\mu}}(e^c - e^1) &\leq \varepsilon. & (3) \end{aligned}$$

Note that it is always possible to choose c large enough so that (3) is satisfied; in fact the second term on the left hand side of (3) is negative and decreases exponentially with c and hence dominates the first term. Multiplying (3) by $(c-1)^2$, we also obtain

$$\frac{\varepsilon^2}{\mu}(c-1)^4 - \sqrt{\frac{\varepsilon}{\mu}}(c-1)^2(e^c - e^1) \leq \varepsilon(c-1)^2. \tag{4}$$

We also have, by (1),

$$\varepsilon(c-1)^2 < \mu + \varepsilon(c-1)^2 = \mu(1 + \frac{\varepsilon}{\mu}(c-1)^2) = \mu(x_1^2 + x_2^2). \tag{5}$$

Set $y = cx$. Then

$$\begin{aligned} & \max_{i \in \{1,2\}} (x_i - y_i)[F_{\varepsilon,i}(x) - F_{\varepsilon,i}(y)] \\ = \; & \max\left\{\varepsilon(x_1 - y_1)^2, \varepsilon(x_2 - y_2)^2 + (x_2 - y_2)(e^{y_1} - e^{x_1})\right\} \\ = \; & \max\left\{\varepsilon(c-1)^2 x_1^2, \varepsilon(c-1)^2 x_2^2 - (c-1)x_2(e^{cx_1} - e^{x_1})\right\} \\ \overset{(1)}{=} \; & \max\left\{\varepsilon(c-1)^2, \frac{\varepsilon^2}{\mu}(c-1)^4 - \sqrt{\frac{\varepsilon}{\mu}}(c-1)^2(e^c - e^1)\right\} \\ \overset{(4)}{=} \; & \varepsilon(c-1)^2 \\ \overset{(5)}{<} \; & \mu(x_1^2 + x_2^2) \\ = \; & \frac{\mu}{(c-1)^2}\|x - y\|_2^2 \\ \overset{(2)}{\leq} \; & \mu\|x - y\|_2^2. \end{aligned}$$

Hence F_ε is not a uniform P-function.

In the next section we shall show that, in spite of the fact that F_ε is not necessarily a uniform P-function, the regularized problems NCP(F_ε) have a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$. However, due to Example 2.4, the analysis is more complicated than one would expect.

3 Existence of Regularized Solutions

In this section, we want to prove that the regularized problem NCP(F_ε) has a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$. The main tool for proving this result is the (nonsmooth) function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\varphi(a,b) := \sqrt{a^2+b^2} - a - b.$$

This function was introduced by Fischer [12] and plays a central role in the design of several nonsmooth Newton-type methods for the solution of NCP(F), see, e.g., [10, 7]. Here, however, we use this function as a theoretical tool. To this end, let us introduce the equation-operator $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi(x) := \begin{pmatrix} \varphi(x_1, F_1(x)) \\ \vdots \\ \varphi(x_n, F_n(x)) \end{pmatrix}$$

as well as the corresponding merit function $\Psi : \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi(x) := \frac{1}{2}\Phi(x)^T\Phi(x) = \frac{1}{2}\|\Phi(x)\|^2.$$

We summarize some of the elementary properties of these functions in the following result.

Proposition 3.1 *The following statements hold:*

(a) $x^* \in \mathbb{R}^n$ solves NCP(F) if and only if x^* solves the nonlinear system of equations $\Phi(x) = 0$.

(b) The merit function Ψ is continuously differentiable on the whole space $\mathbb{R}^n$.

(c) If F is a P_0-function, then every stationary point of Ψ is a solution of NCP(F).

Proof. See, e.g., [13, 10, 7]. □

For the regularized problem, we define the corresponding equation-operator and the corresponding merit function similarly by

$$\Phi_\varepsilon(x) := \begin{pmatrix} \varphi(x_1, F_{\varepsilon,1}(x)) \\ \vdots \\ \varphi(x_n, F_{\varepsilon,n}(x)) \end{pmatrix}$$

and

$$\Psi_\varepsilon(x) := \frac{1}{2}\Phi_\varepsilon(x)^T\Phi_\varepsilon(x),$$

where $F_{\varepsilon,i}$ denotes the ith component function of F_ε. The main result of this section is based on the following three preliminary results.

Lemma 3.2 *Let $\varepsilon > 0$ be arbitrary. Then the Jacobian matrices $F'_\varepsilon(x)$ are P-matrices for all $x \in \mathbb{R}^n$. In particular, the function $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is a P-function.*

Proof. Since F is a P_0-function, the Jacobian matrices $F'(x)$ are P_0-matrices for all $x \in \mathbb{R}^n$ by Theorem 5.8 in [19]. In view of Theorem 3.4.2 in [6], the Jacobian matrices $F'_\varepsilon(x) = F'(x) + \varepsilon I$ are therefore P-matrices for all $x \in \mathbb{R}^n$. Hence F_ε is a P-function by Theorem 5.2 in [19]. $\square$

A proof of the following simple result can be found in [15].

Lemma 3.3 *Let $\{a^k\}, \{b^k\} \subseteq \mathbb{R}$ be any two sequences such that $a^k, b^k \to +\infty$ or $a^k \to -\infty$ or $b^k \to -\infty$. Then $|\varphi(a^k, b^k)| \to \infty$.*

The following Proposition contains the main step in order to prove the existence of a solution of the regularized problems NCP(F_ε).

Proposition 3.4 *Suppose that F is a P_0-function and $\varepsilon > 0$. Then the merit function Ψ_ε is coercive, i.e.,*

$$\lim_{\|x\| \to \infty} \Psi_\varepsilon(x) = +\infty$$

Proof. Suppose by contradiction that the theorem is false. Then we can find an unbounded sequence $\{x^k\}$ such that $\{\Psi_\varepsilon(x^k)\}$ is bounded. Since the sequence $\{x^k\}$ is unbounded, the index set $J := \{i \in \{1, \ldots, n\} | \{x_i^k\}$ is unbounded$\}$ is nonempty. Subsequencing if necessary, we can assume without loss of generality that $\{|x_j^k|\} \to +\infty$ for all $j \in J$. Let $\{y^k\}$ denote the bounded sequence defined in the following way:

$$y_i^k := \begin{cases} 0 & \text{if } i \in J \\ x_i^k & \text{if } i \notin J. \end{cases}$$

From the definition of $\{y^k\}$ and the assumption that F is a P_0-function we get

$$\begin{aligned} 0 &\leq \max\nolimits_{1 \leq i \leq n} (x_i^k - y_i^k)[F_i(x^k) - F_i(y^k)] \\ &= \max\nolimits_{i \in J} x_i^k [F_i(x^k) - F_i(y^k)] \\ &= x_j^k [F_j(x^k) - F_j(y^k)], \end{aligned} \tag{6}$$

where j is one of the indices for which the max is attained which we have, without loss of generality, assumed to be independent of k. Since $j \in J$, we have that

$$\{|x_j^k|\} \to \infty. \tag{7}$$

We now consider two cases.
Case 1: $x_j^k \to +\infty$.
In this case, since $F_j(y^k)$ is bounded by the continuity of F_j, (6) implies that $F_j(x^k)$ does not tend to $-\infty$. This in turn implies

$$\left\{ \sqrt{(x_j^k)^2 + (F_j(x^k) + \varepsilon(x_j^k))^2} - x_j^k - (F_j(x^k) + \varepsilon x_j^k) \right\} \to +\infty$$

by Lemma 3.3 since $F_j(x^k) + \varepsilon x_j^k$ tends to $+\infty$.
Case 2: $x_j^k \to -\infty$.
In this case it follows immediately from Lemma 3.3 that

$$\left\{\sqrt{(x_j^k)^2 + (F_j(x^k) + \varepsilon(x_j^k))^2} - x_j^k - (F_j(x^k) + \varepsilon x_j^k)\right\} \to +\infty$$

(both if $F_j(x^k) + \varepsilon x_j^k$ is unbounded or not).

In either case we get $\Psi_\varepsilon(x^k) \to +\infty$, thus contradicting the boundedness of the sequence $\{\Psi_\varepsilon(x^k)\}$. □

Note that Proposition 3.4 can also be stated in an equivalent way by saying that the level sets $\mathcal{L}_\varepsilon(c) := \{x \in \mathbb{R}^n \,|\, \Psi_\varepsilon(x) \leq c\}$ are compact for every $c \in \mathbb{R}^n$. We are now in a position to prove the following existence and uniqueness result.

Theorem 3.5 *Assume that F is a P_0-function. Then the regularized complementarity problem NCP(F_ε) has a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$. Then F_ε is a P-function by Lemma 3.2. Therefore NCP(F_ε) has at most one solution by Theorem 2.3 in [18].

In order to prove the existence of a solution, let $x^0 \in \mathbb{R}^n$ be arbitrary and define $c := \Psi_\varepsilon(x^0)$. Because of Proposition 3.4, the corresponding level set $\mathcal{L}_\varepsilon(c)$ is nonempty and compact. Hence the continuous function Ψ_ε attains a global minimum x_ε on $\mathcal{L}(c)$ which, in view of the definition of the level set, is also a global minimum of Ψ_ε on $\mathbb{R}^n$. Therefore x_ε is a stationary point of Ψ_ε. But F_ε is a P-function, in particular, F_ε itself is a P_0-function, so that x_ε must be a solution of NCP(F_ε) because of Proposition 3.1 (c). □

4 Behaviour of the Solution Path

The aim of this section is to study the properties of the solution path $\mathcal{P} := \{x(\varepsilon) \,|\, \varepsilon > 0\}$ and, in particular, conditions under which $x(\varepsilon)$ remains bounded when $\varepsilon \to 0$. The reason why we are interested in the boundedness of $x(\varepsilon)$ is because the following easily verifiable result holds.

Theorem 4.1 *Let $\{\varepsilon_k\}$ be a sequence of positive values converging to 0. If $\{x(\varepsilon_k)\}$ converges to a point $\bar{x}$, then $\bar{x}$ solves NCP(F).*

The first noteworthy property we can establish is the continuity of $x(\varepsilon)$.

Lemma 4.2 *Assume that F is a P_0-function. Then the mapping $\varepsilon \mapsto x(\varepsilon)$ is continuous at any $\varepsilon > 0$.*

Proof. By Lemma 3.2, the Jacobian matrix $F_\varepsilon'(x)$ is a P-matrix for every $\varepsilon > 0$ and every $x \in \mathbb{R}^n$; in particular, $M := F_\varepsilon'(x(\varepsilon))$ is a P-matrix. This immediately implies that every principal submatrix of M is again a P-matrix. Moreover, using the same technique of proof as for Lemma 2.3 in [3], it is easy to see that any Schur-complement of a P-matrix is also a P-matrix. Hence the assertion follows from Theorem 3.1 in Kyparisis [16]. $\square$

Note that Lemma 4.2 does not say anything about the continuity of the mapping $\varepsilon \mapsto x(\varepsilon)$ at $\varepsilon = 0$. Continuity at 0 is equivalent to convergence of the solution path $x(\varepsilon)$ when ε goes to 0. As discussed in the introduction, this result holds if F is monotone and the complementarity problem admits a solution. In the more general setting we are considering, we are no longer able to prove such a strong result. However, we can state the following result.

Theorem 4.3 *Let F be a P_0-function and assume that the solution set $\mathcal{S}$ of NCP(F) is nonempty and bounded. Then the path $\mathcal{P}_{\bar\varepsilon} = \{x(\varepsilon) \,|\, \varepsilon \in (0, \bar\varepsilon]\}$ is bounded for any positive $\bar\varepsilon$ and*

$$\lim_{\varepsilon \downarrow 0} \operatorname{dist}(x(\varepsilon)|\mathcal{S}) = 0.$$

We postpone the proof of this theorem until the next section, where it will follow from a more general result.

We next state two immediate consequences of Theorem 4.3.

Corollary 4.4 *Let F be a P_0-function and assume that NCP(F) has a unique solution $\bar x$. Then $\lim_{\varepsilon \downarrow 0} x(\varepsilon) = \bar x$.*

Due to a recent result in [9], the uniqueness of a solution of NCP(F) is, for P_0 complementarity problems, equivalent to the existence of an isolated solution of NCP(F). Hence, alternatively, we could have stated Corollary 4.4 under the assumption that NCP(F) has a locally isolated solution.

Corollary 4.5 *Let $F(x) = Mx + q$ be an affine mapping with $M \in \mathbb{R}^{n \times n}$ being a P_0- and R_0-matrix. Then the path $\mathcal{P}_{\bar\varepsilon}$ is bounded for any positive $\bar\varepsilon$ and*

$$\lim_{\varepsilon \downarrow 0} \operatorname{dist}(x(\varepsilon)|\mathcal{S}) = 0.$$

Proof. Since the solution set of NCP(F) is known to be nonempty and bounded under the stated assumptions (see [6]), the result follows immediately from Theorem 4.3. $\square$

Note that Corollary 4.5 is already known (see Theorem 5.6.2 (a) in [6], restated in Theorem 2.3 of this paper), however, our proof is completely different from the one given in [6]. Moreover, it is easy to see that Corollary 4.5 can easily be extended to nonlinear functions F if we assume that F is a P_0-function and an R_0-function. The definition of the latter class of functions as well as some of its properties are given in the recent paper [4], see also [23].

The following counterexample shows that it is not possible to remove the boundedness assumption of the solution set $\mathcal{S}$ in Theorem 4.3 without destroying the boundedness of the path $\mathcal{P}$. This contrasts sharply with what happens in the case of monotone complementarity problems, where we always have the boundedness of the trajectory if the solution set is nonempty.

Example 4.6 Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x) := Mx + q$, where

$$M := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q := \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Obviously, F is a P_0-function. The solution set $\mathcal{S}$ is given by

$$\mathcal{S} := \{(x_1, x_2) |\, (x_1, 1), x_1 \geq 0\} \cup \{(x_1, x_2) |\, (0, x_2), x_2 \geq 1\},$$

i.e., the solution set is unbounded. It is easy to see that $x(\varepsilon) := (1/\varepsilon, 0)$ is the unique solution of the corresponding regularized problem NCP(F_ε). Obviously, $x(\varepsilon)$ is neither convergent nor bounded for $\varepsilon \to 0$. Even worse, the distance of $x(\varepsilon)$ to the solution set $\mathcal{S}$ does not go to zero since $\operatorname{dist}(x(\varepsilon)|\mathcal{S}) = 1$ for every $\varepsilon > 0$.

5 Inexact Regularization Methods

In the previous section we have illustrated several properties of the trajectory $\mathcal{P}$ which suggest that the original problem NCP(F) can be solved by calculating the exact solutions of a sequence of regularized problems NCP(F_ε) for a sequence of parameters ε converging to 0. From a practical point of view, however, it is usually not possible to solve the regularized problems NCP(F_ε) exactly for each $\varepsilon > 0$. In the following, we therefore present an algorithm which only requires inexact solutions of these subproblems and which nevertheless preserves all the convergence properties of its exact counterpart.

Algorithm 5.1 *(Inexact Regularization Method)*

(S.0) Choose $\varepsilon_0 > 0, \alpha_0 \geq 0$, *and set* $k := 0$.

(S.1) Compute an approximate solution $x^k \in \mathbb{R}^n$ *of NCP(*F_{ε_k}*) such that*

$$\Psi_{\varepsilon_k}(x^k) \leq \alpha_k.$$

(S.2) Terminate the iteration if a suitable stopping criterion is satisfied.

(S.3) Choose $\varepsilon_{k+1} > 0, \alpha_{k+1} \geq 0$, *set* $k \leftarrow k + 1$, *and go to (S.1).*

Obviously, if we take $\alpha_k = 0$ at each iteration, we have $x^k = x(\varepsilon_k)$. Note that a point x^k satisfying $\Psi_{\varepsilon_k}(x^k) \leq \alpha_k$ can easily be obtained by, e.g., applying any unconstrained minimization technique to Ψ_{ε_k}. In fact, the level sets of Ψ_{ε_k} are compact and every stationary point $\bar{x}$ of Ψ_{ε_k} is such that $\Psi_{\varepsilon_k}(\bar{x}) = 0$. Therefore, every suitable minimization algorithm will produce a minimizing sequence and the point x^k can be surely determined in a finite number of steps. This situation reflects the fact that the perturbed problems are well-posed and this, in turn, is one of the main motivations for using regularization methods.

To establish a result generalizing Theorem 4.3 to Algorithm 5.1 we need some further technical results.

Lemma 5.2 *Let $C \subset \mathbb{R}^n$ be a compact set. Then, for every $\delta > 0$, there exists a $\bar{\varepsilon} > 0$ such that*

$$|\Psi_\varepsilon(x) - \Psi(x)| \leq \delta$$

for all $x \in C$ and all $\varepsilon \in [0, \bar{\varepsilon}]$.

Proof. The function $\Psi_\varepsilon(x)$ viewed as a function of both x and ε is continuous on the compact set $C \times [0, \bar{\varepsilon}]$. The lemma is then an immediate consequence of the fact that every continuous function on a compact set is uniformly continuous there. $\square$

Finally, we also restate a version of the famous Mountain Pass Theorem which is suitable for our purposes and which can easily be derived from standard statements of this theorem, see, e.g., Theorem 9.2.7 in [20].

Theorem 5.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and coercive. Let $C \subset \mathbb{R}^n$ be a nonempty and compact set and define m to be the least value of f on the (compact) boundary of C:*

$$m := \min_{x \in \partial C} f(x).$$

Assume further that there are two points $a \in C$ and $b \notin C$ such that $f(a) < m$ and $f(b) < m$. Then there exists a point $c \in \mathbb{R}^n$ such that $\nabla f(c) = 0$ and $f(c) \geq m$.

In the convergence analysis of Algorithm 5.1, we will implicitly assume that Algorithm 5.1 generates an infinite sequence so that the termination criterion in Step (S.2) is never active. The following result is our main convergence theorem for Algorithm 5.1. As known to the authors, this convergence theorem is new even for monotone complementarity problems.

Theorem 5.4 *Let F be a P_0-function and assume that the solution set $\mathcal{S}$ of NCP(F) is nonempty and bounded. Suppose that a sequence $\{x^k\}$ is generated according to Algorithm 5.1. If $\varepsilon_k \to 0$ and $\alpha_k \to 0$, then $\{x^k\}$ remains bounded, and every accumulation point of $\{x^k\}$ is a solution of NCP(F).*

Proof. We first note that it follows from a simple continuity argument that every accumulation point of the sequence $\{x^k\}$ is a solution of NCP(F). Hence it remains to be shown that $\{x^k\}$ is a bounded sequence. Assume the sequence $\{x^k\}$ is not bounded. Then, subsequencing if necessary, we have $\{\|x^k\|\} \to \infty$. Hence there exists a compact set $C \subset \mathbb{R}^n$ with $\mathcal{S} \subset \mathrm{int} C$ and

$$x^k \notin C \tag{8}$$

for all k sufficiently large. Let $a \in \mathcal{S}$ be an arbitrary solution of NCP(F). Then we have

$$\Psi(a) = 0.$$

Since

$$\bar{m} := \min_{x \in \partial C} \Psi(x) > 0,$$

we can apply Lemma 5.2 with $\delta := \bar{m}/4$ and conclude that

$$\Psi_{\varepsilon_k}(a) \leq \frac{1}{4}\bar{m} \tag{9}$$

and

$$m := \min_{x \in \partial C} \Psi_{\varepsilon_k}(x) \geq \frac{3}{4}\bar{m} \tag{10}$$

for all k sufficiently large. Since $\Psi_{\varepsilon_k}(x^k) \leq \alpha_k$ by Step (S.1) of Algorithm 5.1, we have

$$\Psi_{\varepsilon_k}(x^k) \leq \frac{1}{4}\bar{m} \tag{11}$$

for all k large enough since $\alpha_k \to 0$ by our assumption. Now let us fix an index k such that (8)–(11) hold. Applying the Mountain Pass Theorem 5.3 with $b := x^k$, we obtain the existence of a vector $c \in \mathbb{R}^n$ such that

$$\nabla \Psi_{\varepsilon_k}(c) = 0 \quad \text{and} \quad \Psi_{\varepsilon_k}(c) \geq \frac{3}{4}\bar{m} > 0.$$

In view of Proposition 3.1 (c), however, the stationary point c of Ψ_{ε_k} must be a global minimizer of Ψ_{ε_k} which gives us the desired contradiction. □

Obviously Theorem 4.3 follows from Theorem 5.4 by taking $\alpha_k = 0$ for all k and using Lemma 4.1. Also Corollaries 4.4 and 4.5 can easily be extended to the inexact framework.

Corollary 5.5 *Assume that F is a P_0-function and suppose that a sequence $\{x^k\}$ is generated according to Algorithm 5.1. Suppose that $\varepsilon_k \to 0$ and $\alpha_k \to 0$. Then, if NCP(F) has a unique solution $\bar{x}$, we have*

$$\lim_{\varepsilon_k \to 0} x^k = \bar{x}.$$

Corollary 5.6 *Let $F(x) = Mx + q$ be an affine mapping with $M \in \mathbb{R}^{n \times n}$ being a P_0- and R_0-matrix. Assume that $\{x^k\}$ is any sequence generated by Algorithm 5.1 such that $\varepsilon_k \to 0$ and $\alpha_k \to 0$. Then the sequence $\{x^k\}$ is bounded, and every accumulation point of the sequence $\{x^k\}$ is a solution of NCP(F).*

If F is a monotone function such that NCP(F) is strictly feasible (i.e., there exists a vector $\hat{x} \in \mathbb{R}^n$ such that $\hat{x} > 0$ and $F(\hat{x}) > 0$), then it is known ([14, Theorem 3.4]) that NCP(F) has a nonempty and bounded solution set. Hence we also obtain the following corollary from our main result 5.4 of this section.

Corollary 5.7 *Assume that F is a monotone function such that NCP(F) is strictly feasible. Suppose that $\varepsilon_k \to 0$ and $\alpha_k \to 0$. Then any sequence $\{x^k\}$ generated by Algorithm 5.1 remains bounded, and every accumulation point of $\{x^k\}$ is a solution of NCP(F).*

We finally stress that, as far as we know, the inexact regularization method 5.1 investigated in this Section is currently the only (implementable!) algorithm which guarantees that a solution of a P_0-function complementarity problem with a bounded and nonempty solution set can actually be computed.

6 Final Remarks

In this paper we have shown that, under appropriate assumptions, regularization methods can be successfully applied to P_0-complementarity problems. However, some properties which hold in the monotone case are lost. In particular, when the solution set of the problem is unbounded we can no longer guarantee that the trajectory generated by the regularization method is bounded. There is an open question which we think could be interesting to further investigate. When the solution trajectory $x(\varepsilon)$ is bounded, does it converge and, if it does converge, to which element? In the monotone case $x(\varepsilon)$ always converges to the least l_2-norm solution of NCP(F). In the P_0 case the least l_2-norm solution can even be not unique, since the solution set is not necessarily convex. Nevertheless, it would be interesting to characterize in some way the limit point(s) of $x(\varepsilon)$, when ε converges to 0.

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