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NUMERICAL COMPUTATION OF DEGENERATE HOPF BIFURCATION POINTS

HAIHONG XU[†§], VLADIMíR JANOVSKÝ[‡] AND BODO WERNER[†]

Abstract. In this paper a numerical method for the detection and computation of degenerate Hopf bifurcation points is presented. The degeneracies are classified and defining equations characterizing each of the equivalence classes are constructed by means of a generalized Liapunov-Schmidt reduction. The numerical computation of the sign of the first Liapunov coefficient which determines the stability of the bifurcating periodic orbits is discussed as well. Numerical experiments are performed for the clamped Hodgkin-Huxley equations.

Key words. Degenerate Hopf bifurcation, bordered matrices, generalized Liapunov-Schmidt reduction, test function, stability

AMS subject classifications. 58F14, 65H17

1. Introduction. We consider the dynamical system

$$\dot{u} = f(u, \lambda, \alpha), \tag{1}$$

where $f : \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^k \to \mathbb{R}^N$ is a sufficiently smooth mapping, $u \in \mathbb{R}^N$ is a vector of state variables, $\lambda \in \mathbb{R}^1$ is a distinguished bifurcation parameter, $\alpha = (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$ is a vector of additional auxiliary parameters and $k \geq 0$ is an integer. For $k = 0$, i.e., in the case of no auxiliary parameters, we denote $f(u, \lambda, \alpha)$ simply as $f(u, \lambda)$. The main aim is the numerical detection and computation of degenerate Hopf bifurcation points and the determination of stability of the bifurcating periodic orbits of (1).

First we quickly recall the hypotheses of classical Hopf bifurcations [6], [11], [5]. Assume that $(u_H, \lambda_H) \in \mathbb{R}^N \times \mathbb{R}^1$ is an equilibrium point, i.e., $f(u_H, \lambda_H) = 0$.

(H1) The Jacobian $f_u(u_H, \lambda_H)$ has algebraically simple eigenvalues $\pm i\omega_H \neq 0$ and no other eigenvalues are lying on the imaginary axis.

(H2) $\sigma'(\lambda_H) \neq 0$, where $\sigma(\lambda) + i\omega(\lambda)$ is an eigenvalue of $f_u(u(\lambda), \lambda)$ along the equilibrium solution manifold $(u(\lambda), \lambda)$ defined in a neighbourhood of (u_H, λ_H) and satisfying $\sigma(\lambda_H) = 0,\ \omega(\lambda_H) = \omega_H$.

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Provided that (H1) and (H2) are satisfied, there exists a unique branch of periodic orbits of the dynamical system (1) bifurcating from (u_H, λ_H), which can be parameterized by certain $x \in \mathbb{R}$ which can be interpreted as an amplitude of the periodic orbits. (u_H, λ_H) is called a Hopf bifurcation point. Sometimes we also call $(u_H, \lambda_H, \omega_H)$ a Hopf bifurcation point. A Hopf bifurcation point $(u_H, \lambda_H, \alpha_H) \in \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^k$ of (1) with auxiliary parameters is a Hopf bifurcation point (u_H, λ_H) of

$$f(\cdot, \cdot, \alpha_H) : \mathbb{R}^N \times \mathbb{R}^1 \to \mathbb{R}^N$$

in the above sense for the specified value α_H. Moreover along the branch of periodic orbits λ is an even function of the amplitude x given by

$$\lambda = \mu_2 x^2 + \mu_4 x^4 + \cdots .$$

Under the condition

(H3) $\mu_2 \neq 0$,

this bifurcation is supercritical (i.e. the branch of bifurcating periodic orbits appears for $\lambda > \lambda_H$) or subcritical ($\lambda < \lambda_H$) in accordance with μ_2 being positive or negative respectively. If $\mu_2 = 0$ and $\mu_4 \neq 0$, then the super- or subcriticality of the Hopf bifurcation is determined by the sign of μ_4, and so on.

If f depends on auxiliary parameters (a rather typical situation in applications), then changes of the auxiliary parameters may lead to a failure of some of the hypotheses for Hopf bifurcation. By degenerate Hopf bifurcation we mean a Hopf bifurcation where at least one of the above assumptions (H1)–(H3) fails. In this paper we shall consider those degeneracies where (H1) holds, but (H2) or/and (H3) fails. In [2] a classification of degenerate Hopf bifurcations up to codimension three is presented. The qualitative analysis is based on the contact equivalence of bifurcation problems. We shall present a method for the numerical computation of degenerate Hopf bifurcation points based on a generalized Liapunov-Schmidt reduction following [7] which allows the use in large systems.

Our approach to the numerical detection and computation of degenerate Hopf bifurcation points is related to the construction of defining equations which characterize the particular degeneracy and to the construction of test functions used to detect Hopf bifurcation points. The degenerate Hopf bifurcation point is computed as an isolated regular root of a minimally extended system consisting of the steady-state equation $f(u, \lambda, \alpha) = 0$ and the relevant defining equations (see e.g. [1] for minimally extended systems). No computation of eigenvalues and eigenvectors is involved. Especially for the classical Hopf bifurcation points which are called Hopf bifurcation points of Type 1 (see Table 1 and 2), the computation of the test function is only a by-product of the continuation of the equilibria.

The bordering vectors involved in the generalized Liapunov-Schmidt reduction are choosen in a special way such that the regularity of the bordered system is guaranted and the analysis and computations are simplified.

The degenerate Hopf bifurcation point of Type 3 (see Table 1 and 2) is very important in applications since it separates those Hopf bifurcation points with stable bifurcating periodic orbits from those with unstable ones. The sign of the test function for the detection of Hopf bifurcation points of Type 3 determines whether the bifurcating periodic orbits are stable or not — we need not to compute Floquet-multipliers. The super- and subcriticality of the Hopf bifurcations can also be determined in this way.

In §2 we shall resume the classification results for the equivalence classes of degenerate Hopf bifurcations up to codimension three, we essentially follow [2]. In §3 we introduce the generalized Liapunov-Schmidt reduction on the space of periodic functions as presented in [7]. In §4 we present the extended systems for the determination of degenerate Hopf bifurcation points in accordance with the classification from §2. In §5 we discuss the stability of periodic orbits bifurcating from Hopf bifurcation points. Numerical experiments concerning the detection and computation of degenerate Hopf bifurcation points of the clamped Hodgkin-Huxley equations are reported in §6.

2. Classification of degenerate Hopf bifurcations. In this section we temporarily assume that no auxiliary parameters are presented (k=0). Hypothesis (H1) implies that the linearization of (1) at $(u_H, \lambda_H) \in \mathbb{R}^N \times \mathbb{R}^1$ has $2\pi/\omega_H$-periodic solutions. Therefore one seeks periodic solutions of (1) near (u_H, λ_H) with period $2\pi/\omega$ for ω near ω_H. If $v = v(t) \in \mathbb{R}^N$ is a $2\pi/\omega$-periodic solution to (1), then, rescaling time by setting

$$s = \omega t, \qquad u(s) = v(t),$$

the solution v corresponds to a 2π-periodic solution $u = u(s)$ of a two-parameter dependent system $\mathcal{N}(u, \lambda, \omega) = 0$, where

$$\mathcal{N}(u, \lambda, \omega) := -\omega \dot{u} + f(u, \lambda),$$

$\mathcal{N} : U \times \mathbb{R}^1 \times \mathbb{R}^1 \to V, U := C^1(\mathrm{S}^1, \mathbb{R}^N), V := C^0(\mathrm{S}^1, \mathbb{R}^N)$, and S^1 denotes the unit circle, i.e., $\mathrm{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Hypotheses (H1)-(H2) imply that $(u_H, \lambda_H, \omega_H)$ is a singular point of

$\mathcal{N}$ with corank=2, see [3], namely,

$$\mathcal{N}(u_H, \lambda_H, \omega_H) = 0, \qquad \dim \mathrm{Ker}\mathcal{N}_u(u_H, \lambda_H, \omega_H) = 2.$$

The standard Liapunov-Schmidt reduction identifies, locally, solutions $(u, \lambda, \omega) \in U \times \mathbb{R}^1 \times \mathbb{R}^1$ of

$$\mathcal{N}(u_H + u, \lambda_H + \lambda, \omega_H + \omega) = 0$$

with solutions $(z, \lambda, \omega) \in \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$ of certain bifurcation equation

$$g(z, \lambda, \omega) = 0,$$

where $g : \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{C}^1$ is defined in a neighbourhood of the origin. Note that we have identified $\mathbb{R}^2$ with $\mathbb{C}^1$. The phase-shift equivariance of $\mathcal{N}$ implies that g, restricted to $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$, must have the form

$$g(x, \lambda, \omega) = x(p(x^2, \lambda, \omega) + iq(x^2, \lambda, \omega)),$$

where $p = p(r, \lambda, \omega)$ and $q = q(r, \lambda, \omega)$ are germs of smooth mappings $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$. Note that because of the phase-shift equivariance of $\mathcal{N}$, it suffices to restrict our discussion to $\mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$. Solutions of

$$g(x, \lambda, \omega) = 0$$

corresponding to periodic solutions of (1) are characterized by $x \neq 0$ and

$$\begin{aligned} p(x^2, \lambda, \omega) &= 0, \\ q(x^2, \lambda, \omega) &= 0. \end{aligned} \tag{2}$$

The fact that $\pm i\omega_H$ are algebraically simple eigenvalues makes it possible to reduce (2) to a single scalar equation

$$G(x, \lambda) := a(x^2, \lambda)x = 0, \tag{3}$$

where $a(x^2, \lambda) := p(x^2, \lambda, \omega(x^2, \lambda))$, and $\omega = \omega(x^2, \lambda)$ is implicitly defined by the last equation in (2).

One should observe that (3) has a $\mathbf{Z}_2$-symmetry in the sense that $G(-x, \lambda) = -G(x, \lambda)$. Let $\mathcal{E}(\mathbf{Z}_2)$ be the space of germs of smooth odd mappings $G : \mathbb{R}^2 \to \mathbb{R}^1$.

In [2] there is a complete classification of all $\mathbf{Z}_2$–equivariant $G(x, \lambda) = a(x^2, \lambda)x$ up to codimension 3 by showing that every G with codimension ≤ 3 is $\mathbf{Z}_2$–equivalent to one of the eleven normal forms. In this paper we shall only concentrate on four cases of these $\mathbf{Z}_2$–equivalence classes with the normal forms given in Table 1 though the techniques presented here can also be carried over to other singularities.

TABLE 1

Normal forms of $\mathbf{Z}_2$-equivalence classes on $\mathcal{E}(\mathbf{Z}_2)$.

Type	Normal form	codim	Parameters
1	$(\epsilon r + \delta\lambda)x$	0	$\epsilon = \mathrm{sgn}(a_r), \delta = \mathrm{sgn}(a_\lambda)$
2	$(\epsilon r + \delta\lambda^2)x$	1	$\epsilon = \mathrm{sgn}(a_r), \delta = \mathrm{sgn}(a_{\lambda\lambda})$
3	$(\epsilon r^2 + \delta\lambda)x$	1	$\epsilon = \mathrm{sgn}(a_{rr}), \delta = \mathrm{sgn}(a_\lambda)$
4	$(\epsilon r^2 + 2m\lambda r + \delta\lambda^2)x$	3(2)	$\epsilon = \mathrm{sgn}(a_{rr}), \delta = \mathrm{sgn}(a_{\lambda\lambda})$, $m = a_{r\lambda}/\sqrt{\lvert a_{rr}a_{\lambda\lambda}\rvert}, m^2 \neq \epsilon\delta$

TABLE 2

Defining equations of $\mathbf{Z}_2$-equivalence classes on $\mathcal{E}(\mathbf{Z}_2)$.

Type	Defining equations	Nondegeneracy conditions
1	$a = 0$	$a_r \neq 0, a_\lambda \neq 0$
2	$a = a_\lambda = 0$	$a_r \neq 0, a_{\lambda\lambda} \neq 0$
3	$a = a_r = 0$	$a_{rr} \neq 0, a_\lambda \neq 0$
4	$a = a_r = a_\lambda = 0$	$a_{rr} \neq 0, a_{\lambda\lambda} \neq 0, a_{rr}a_{\lambda\lambda} - a_{r\lambda}^2 \neq 0$

REMARK 2.1. *In Table 1 and 2, all derivatives are evaluated at the origin. The classification in Table 2 means that $G(x,\lambda) = a(r,\lambda)x$ with $r = x^2$ is equivalent the corresponding normal form in Table 1, if and only if $a(r,\lambda)$ satisfies the related defining equations and nondegeneracy conditions in Table 2.*

REMARK 2.2. *For Type 8, m is a modal parameter and can assume any real value. Apart from the exceptional values $m = 0, \pm\sqrt{\epsilon\delta}, \pm\infty$, this singularity has ($C^\infty$) codimension three and topological codimension two.*

3. The generalized Liapunov-Schmidt reduction. The above Liapunov-Schmidt reduction requires a precise knowledge of $\mathcal{N}_u$ at a singular point in advance, which is right an object of our computation. Generalized Liapunov-Schmidt reductions are less restrictive and computationally more convenient. It results in a reduced equation defined in a neighbourhood of the singular point and moreover this reduced equation is contact equivalent to the standard reduced bifurcation equation at the singular point[8]. Hence it can be used to characterize the singular point.

In this section, we introduce a version of the generalized Liapunov-Schmidt reduction on the space of periodic functions following [7]. The aim is to construct test functions and defining equations for the numerical computation of singularities classified in §2.

Let $L, M \in \mathbb{C}^N$ be complex vectors. We define operators $\mathbf{L} : U \to \mathbb{C}^1,\ \ \mathbf{M} : \mathbb{C}^1 \to V$ as

$$\begin{aligned} \mathbf{L}u &:= \frac{1}{2\pi}\int_0^{2\pi} (L^T u(s))e^{-is}ds, \qquad u \in U, && (4) \\ \mathbf{M}z &:= ze^{is}M + \bar{z}e^{-is}\bar{M}, \qquad\qquad z \in \mathbb{C}^1. && (5) \end{aligned}$$

The reduction will be concerned with periodic solutions $u \in U$ near a stationary function $u^0(t) \equiv u^0 \in \mathbb{R}^N$. Given a basepoint $(u^0, \lambda^0, \alpha^0, \omega^0) \in \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ of the reduction and $(z, \lambda, \alpha, \omega) \in \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$, we want to solve

$$\begin{aligned} \mathcal{N}(u^0 + u, \lambda^0 + \lambda, \alpha^0 + \alpha, \omega^0 + \omega) + \mathbf{M}\phi &= f(u^0, \lambda^0, \alpha^0), \\ \mathbf{L}u &= z \end{aligned} \qquad (6)$$

for $(u, \phi) \in U \times \mathbb{C}^1$.

If the linearization of (6) at $(u^0, \lambda^0, \alpha^0, \omega^0)$,

$$\begin{aligned} -\omega^0\dot{v} + f_u(u^0, \lambda^0, \alpha^0)v + \mathbf{M}\psi &= w, \\ \mathbf{L}v &= c, \end{aligned} \qquad (7)$$

is continuously invertible in the sense that for each $(w, c) \in V \times \mathbb{C}^1$ there exists a unique solution $(v, \psi) \in U \times \mathbb{C}^1$ depending continuously on (w, c), then (6) defines, locally, a unique pair of smooth mappings $u = u(z, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$ and $\phi = \phi(z, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$.

For a given $(w, c) \in V \times \mathbb{C}^1$ it is proposed in [7] to solve (7) by using Fourier analysis. Denote the k-th Fourier coefficient of $w = w(s) \in V$ by

$$[w]_k := \frac{1}{2\pi}\int_0^{2\pi} w(s)e^{-iks}ds, \quad k \in \mathbb{Z},$$

Since $[w]_k \in \mathbb{C}^N$ is complex conjugate to $[w]_{-k}$ for a real valued w, then we can develop v and w in (7) as Fourier series,

$$v = [v]_0 + \sum_{k=1}^{+\infty}\left([v]_k e^{iks} + [v]_{-k}e^{-iks}\right), \qquad w = [w]_0 + \sum_{k=1}^{+\infty}\left([w]_k e^{iks} + [w]_{-k}e^{-iks}\right). \quad (8)$$

Substituting (8) into (7) yields the following equations of the Fourier coefficients,

$$\begin{aligned} \left(f_u(u^0, \lambda^0, \alpha^0) - ik\omega^0\right)[v]_k &= [w]_k, \quad k \in \mathbb{Z}_+, k \neq 1, \\ \mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [v]_1 \\ \psi \end{pmatrix} &= \begin{pmatrix} [w]_1 \\ c \end{pmatrix}, \end{aligned}$$

where

$$\mathcal{B}(u^0,\lambda^0,\alpha^0,\omega^0) := \begin{pmatrix} f_u(u^0,\lambda^0,\alpha^0) - i\omega^0 & M \\ L^T & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^{N+1},\mathbb{C}^{N+1})$$

is calles a bordered matrix. Hence it is natural to require that $\det\mathcal{B}(u^0,\lambda^0,\alpha^0,\omega^0) \neq 0$ and $f_u(u^0,\lambda^0,\alpha^0) - ik\omega^0 \in \mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)$ are regular for $k \in \mathbb{Z}_+, k \neq 1$. We denote the set of all these $(u^0,\lambda^0,\alpha^0,\omega^0)$ as $\mathcal{D}$.

PROPOSITION 3.1.([7]) *Assume that $(u^0,\lambda^0,\alpha^0,\omega^0) \in \mathcal{D}$. Then for $(z,\lambda,\alpha,\omega) \in \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ in a neighbourhood of the origin, (6) defines a unique couple*

$$\begin{aligned} u &= u(z,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0), \\ \phi &= \phi(z,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0) \end{aligned}$$

as germs of smooth mappings

$$\begin{aligned} u &: \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1 \times \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1 \to \mathbb{R}^N, \\ \phi &: \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1 \times \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1 \to \mathbb{C}^1 \end{aligned}$$

with $u(0;u^0,\lambda^0,\alpha^0,\omega^0) = 0$, $\phi(0;u^0,\lambda^0,\alpha^0,\omega^0) = 0$.

We call ϕ a *generalized reduction* of $\mathcal{N}$ at the point $(u^0,\lambda^0,\alpha^0,\omega^0)$.

Since we are concerned only with the behaviour of u and ϕ for $(u^0,\lambda^0,\alpha^0,\omega^0)$ close to a Hopf bifurcation point $(u_H,\lambda_H,\alpha_H,\omega_H)$, it suffices to require that $(u_H,\lambda_H,\alpha_H,\omega_H) \in \mathcal{D}$. Hypothesis (H1) guarantees that $f_u(u_H,\lambda_H,\alpha_H) - ik\omega_H (k \in \mathbb{Z}_+, k \neq 1)$ are regular. We consider now the choice of L and M in order that $\mathcal{B}(u_H,\lambda_H,\alpha_H,\omega_H)$ is regular. The simplicity of the eigenvalues $\pm i\omega_H$ implies that the corresponding eigenvectors $\xi_H,\eta_H \in \mathbb{C}^N$ can be chosen such that

$$\begin{gathered} (f_u(u_H,\lambda_H,\alpha_H) - i\omega_H)\xi_H = 0, \quad (f_u^T(u_H,\lambda_H,\alpha_H) + i\omega_H)\eta_H = 0, \\ \bar{\xi}_H^T\xi_H = 1, \quad \bar{\eta}_H^T\xi_H = 1, \quad \eta_H^T\xi_H = 0. \end{gathered} \tag{9}$$

Thus, choosing $L, M \in \mathbb{C}^N$ such that

$$L^T\xi_H \neq 0, \qquad \bar{\eta}_H^T M \neq 0, \tag{10}$$

the Keller lemma guarantees that $\mathcal{B}(u_H,\lambda_H,\alpha_H,\omega_H)$ is regular.

In accordance with Proposition 3.1, the implicit definition (6) of u and ϕ implies that solutions $(u,\lambda,\alpha,\omega) \in U \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ of

$$\mathcal{N}(u^0+u,\lambda^0+\lambda,\alpha^0+\alpha,\omega^0+\omega) = f(u^0,\lambda^0,\alpha^0) \tag{11}$$

are, locally, in one-to-one correspondence with solutions $(z, \lambda, \alpha, \omega) \in \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ of

$$\phi(z, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) = 0. \tag{12}$$

The isomorphism that takes $(u, \lambda, \alpha, \omega)$ to $(z, \lambda, \alpha, \omega)$ and vice versa is given by $z = \mathbf{L}u$ and $u = u(z, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$ respectively.

It is easy to prove that ϕ is S^1-equivariant, i.e.,

$$e^{i\theta}\phi(z, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) = \phi(e^{i\theta}z, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0), \quad \theta \in \mathrm{S}^1.$$

This implies that restricting z to $x = \mathrm{Re}z \in \mathbb{R}^1$, ϕ has the form

$$\begin{aligned} \phi(x, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) \;&=\; x(p(x^2, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) \\ &\quad + iq(x^2, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)), \end{aligned} \tag{13}$$

where $p = p(r, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$ and $q = q(r, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$ are germs of smooth mappings $p(\cdot; u^0, \lambda^0, \alpha^0, \omega^0), q(\cdot; u^0, \lambda^0, \alpha^0, \omega^0) : \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1 \to \mathbb{R}^1$.

For each solution $(z, \lambda, \alpha, \omega) \in \mathbb{C}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ of (12), there is $\theta \in \mathrm{S}^1$ such that $x = e^{i\theta}z \in \mathbb{R}^1$. Hence we have

$$\phi(x, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) = 0. \tag{14}$$

Conversely, if $(x, \lambda, \alpha, \omega) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ is a solution of (14), then for any $\theta \in \mathrm{S}^1$ and $z = e^{i\theta}x$, $(z, \lambda, \alpha, \omega)$ is a solution of (12). Hence in order to solve (12), it suffices to solve (14). Obviously (14) is always satisfied for $x = 0$. From the local uniqueness of the solutions of (6) it follows that u is time-independent if $\mathbf{L}u = z = 0$. This implies that the solutions of (14) with $x = 0$ correspond with the time-independent solutions of (11). Periodic solutions of (11) are therefore related to nontrivial solutions $x \neq 0$ of (14). Because of (14) they are given by

$$\begin{aligned} p(x^2, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) = 0, \\ q(x^2, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0) = 0, \end{aligned} \qquad x \neq 0. \tag{15}$$

LEMMA 3.2. *Assume that* $(u_H, \lambda_H, \alpha_H, \omega_H)$ *is a Hopf bifurcation point. Then* $q_\omega(0; u_H, \lambda_H, \alpha_H, \omega_H) \neq 0$*, and* $p_\omega(0; u_H, \lambda_H, \alpha_H, \omega_H) = 0$ *for* $L = \bar{\xi}_H, M = \xi_H$ *from (9).*

Proof. See Appendix 7.1.

Now the implicit function theorem guarantees that the last equation in (15) has a unique solution $\omega = \omega(x^2, \lambda, \alpha; u^0, \lambda^0, \alpha^0, \omega^0)$ for $(u^0, \lambda^0, \alpha^0, \omega^0)$ in a neighbourhood of

$(u_H, \lambda_H, \alpha_H, \omega_H)$ and for (x, λ, α) in a neighbourhood of the origin. Substituting this information into the former equation in (15) yields a single scalar equation

$$a(x^2, \lambda, \alpha; u^0, \lambda^0, \alpha^0, \omega^0) := p(x^2, \lambda, \alpha, \omega(x^2, \cdots); u^0, \lambda^0, \alpha^0, \omega^0) = 0. \tag{16}$$

Note that (16) has numerical advantages because it is defined for $(u^0, \lambda^0, \alpha^0, \omega^0)$ in a neighbourhood of a Hopf bifurcation point. We shall use this equation to construct the defining equations and test functions for the numerical computation of degenerate Hopf bifurcation points in accordance with Table 2. We remark that the classification in Table 1 and 2 can be carried over to $G(x, \lambda, \alpha) := xa(x^2, \lambda, \alpha; u^0, \lambda^0, \alpha^0, \omega^0)$, where α can be treated as unfolding parameters. Observe that $a(0, 0, 0; u^0, \lambda^0, \alpha^0, \omega^0) = 0$ if and only if the Hopf condition (H1) is fulfilled for $(u_H, \lambda_H, \alpha_H, \omega_H) = (u^0, \lambda^0, \alpha^0, \omega^0)$.

4. Defining equations and test functions. The aim is to find numerically convenient scalar defining equations $f(u^0, \lambda^0, \alpha^0, \omega^0) = 0$ for the determination of Hopf points of Type 1–4 in acordance with Table 1 and 2 of §2. These *test functions* $f : \mathcal{D} \to \mathbb{R}^1$ are based on the complex reduction $\phi(x, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$ in §3 and its derivatives with respect to the first four variables evaluated at $(x, \lambda, \alpha, \omega) = 0$. They are defined in a neighbourhood of the Hopf bifurcation point and can be computed conveniently.

In order to construct defining equations, we proceed as follows: instead of the defining equation $a = 0$ in Table 2, we choose

$$f^{(1)}(u^0, \lambda^0, \alpha^0, \omega^0) := p(0; u^0, \lambda^0, \alpha^0, \omega^0) = 0,$$
$$f^{(2)}(u^0, \lambda^0, \alpha^0, \omega^0) := q(0; u^0, \lambda^0, \alpha^0, \omega^0) = 0$$

as defining equations for Hopf bifurcations, see (13). For degeneracies we first choose a singularity type from Table 1 and consider the relevant defining equations from Table 2. We evaluate the required derivatives of a and express them by means of the derivatives of ϕ with respect to $x \in \mathbb{R}^1$ and $\lambda \in \mathbb{R}^1$ at $(x, \lambda, \alpha, \omega) = 0$. The procedure yields *codim+2* defining equations, the number of degeneracy conditions is related to the codimension *codim* in Table 1. We explain the idea by an example, the other cases can be treated analogously.

Let us consider the Hopf bifurcation point of Type 4 and assume that $m \neq 0$. Then the topological codimension should be two. Assume $(u^0, \lambda^0, \alpha^0, \omega^0) \in \mathcal{D}$. In accordance with Table 2, we construct defining equations $f^{(i)} : \mathcal{D} \to \mathbb{R}^1, i = 3, 4$ by

$$f^{(3)}(u^0, \lambda^0, \alpha^0, \omega^0) := \frac{\partial}{\partial r} a(r, 0, 0; u^0, \lambda^0, \alpha^0, \omega^0) \mid_{r=0} = 0,$$
$$f^{(4)}(u^0, \lambda^0, \alpha^0, \omega^0) := \frac{\partial}{\partial \lambda} a(0, \lambda, 0; u^0, \lambda^0, \alpha^0, \omega^0) \mid_{\lambda=0} = 0.$$

Now we introduce $\mathcal{F} : \mathcal{D} \to \mathbb{R}^{N+4}$,

$$\mathcal{F}(u, \lambda, \alpha, \omega) := \begin{pmatrix} f(u, \lambda, \alpha) \\ f^{(1)}(u, \lambda, \alpha, \omega) \\ f^{(2)}(u, \lambda, \alpha, \omega) \\ f^{(3)}(u, \lambda, \alpha, \omega) \\ f^{(4)}(u, \lambda, \alpha, \omega) \end{pmatrix}, \tag{17}$$

and call a point $(u_H, \lambda_H, \alpha_H, \omega_H) \in \mathcal{D}$ *a degenerate Hopf bifurcation point of Type 4*, if it is a root of

$$\mathcal{F}(u, \lambda, \alpha, \omega) = 0 \in \mathbb{R}^{N+4}, \tag{18}$$

and if the relevant nondegeneracy conditons in Table 2, namely

$$a_{rr} \neq 0, \quad a_{\lambda\lambda} \neq 0, \quad a_{rr}a_{\lambda\lambda} - a_{r\lambda}^2 \neq 0,$$

are satisfied at this point.

The partial derivatives of a can be expressed by those of ϕ,

$$\begin{aligned}
f^{(1)}(u^0, \lambda^0, \alpha^0, \omega^0) &= \mathrm{Re}\phi_x(0; u^0, \lambda^0, \alpha^0, \omega^0), \\
f^{(2)}(u^0, \lambda^0, \alpha^0, \omega^0) &= \mathrm{Im}\phi_x(0; u^0, \lambda^0, \alpha^0, \omega^0), \\
f^{(3)}(u^0, \lambda^0, \alpha^0, \omega^0) &= \frac{1}{6}\Big(\mathrm{Re}\phi_{xxx}(0; u^0, \lambda^0, \alpha^0, \omega^0) \\
&\quad -\frac{\mathrm{Re}\phi_{x\omega}(0; u^0, \lambda^0, \alpha^0, \omega^0)}{\mathrm{Im}\phi_{x\omega}(0; u^0, \lambda^0, \alpha^0, \omega^0)}\mathrm{Im}\phi_{xxx}(0; u^0, \lambda^0, \alpha^0, \omega^0)\Big), \\
f^{(4)}(u^0, \lambda^0, \alpha^0, \omega^0) &= \mathrm{Re}\phi_{x\lambda}(0; u^0, \lambda^0, \alpha^0, \omega^0) \\
&\quad -\frac{\mathrm{Re}\phi_{x\omega}(0; u^0, \lambda^0, \alpha^0, \omega^0)}{\mathrm{Im}\phi_{x\omega}(0; u^0, \lambda^0, \alpha^0, \omega^0)}\mathrm{Im}\phi_{x\lambda}(0; u^0, \lambda^0, \alpha^0, \omega^0).
\end{aligned}$$

From Lemma 3.2 we know that the special choice $L = \bar{\xi}_H$ and $M = \xi_H$ will simplify the last two defining equations,

$$\begin{aligned}
f^{(3)}(u^0, \lambda^0, \alpha^0, \omega^0) &= \mathrm{Re}\phi_{xxx}(0; u^0, \lambda^0, \alpha^0, \omega^0), \\
f^{(4)}(u^0, \lambda^0, \alpha^0, \omega^0) &= \mathrm{Re}\phi_{x\lambda}(0; u^0, \lambda^0, \alpha^0, \omega^0).
\end{aligned}$$

To find systematically such a highly degenerate Hopf point, we proceed as follows for the case with two (equal to the codimension) auxiliary parameters $\alpha \in \mathbb{R}^2$.

Step 1. Fixing the auxiliary parameters we apply a pseudo-arclength continuation to the system

$$\mathcal{F}(u,\lambda,\alpha,\omega) := \begin{pmatrix} f(u,\lambda,\alpha) \\ f^{(2)}(u,\lambda,\alpha,\omega) \end{pmatrix} = 0 \tag{19}$$

to follow a path of equilibrium points, $(u(\tau),\lambda(\tau),\omega(\tau))$, where τ is a pseudo-arclength parameter. Note that during the continuation of equilibrium points L and M need only to satisfy that the bordered matrix $\mathcal{B}$ is regular and need not to satisfy (9)-(10). During the continuation we monitor the test function

$$T_1(\tau) := \mathrm{Re}\phi_x(0; u(\tau),\lambda(\tau),\alpha,\omega(\tau)) \tag{20}$$

to detect Hopf bifurcation points, see [7] and [12]. A sign change of $T_1(\tau)$ indicates that the branch passes possibly through a Hopf bifurcation point of Type 1, **HP1**, which is defined as a root of the system

$$\mathcal{F}(u,\lambda,\alpha,\omega) := \begin{pmatrix} f(u,\lambda,\alpha) \\ f^{(1)}(u,\lambda,\alpha,\omega) \\ f^{(2)}(u,\lambda,\alpha,\omega) \end{pmatrix} = 0, \tag{21}$$

and at which the relevant nondegeneracy conditions in Table 2 are satisfied. A Hopf bifurcation point of Type 1 is just a regular root of (21), i.e., we can employ (21) to locate this Hopf bifurcation point, see Theorem 4.1 below. But we should not suppress that we have to choose the right $\omega(\tau)$–branch to be able to detect a specific Hopf point.

Step 2. Release one of the auxiliary parameter, e.g. α_1, and apply a pseudo-arclength continuation to the system (21) to follow the path of Hopf bifurcation points $(u(\tau),\lambda(\tau),\alpha_1(\tau),\alpha_2,\omega(\tau))$ through the point **HP1**. During the continuation we monitor the test functions

$$T_2(\tau) := \mathrm{Re}\phi_{x\lambda}(0; u(\tau),\lambda(\tau),\alpha_1(\tau),\alpha_2,\omega(\tau)) \tag{22}$$

and

$$T_3(\tau) := \mathrm{Re}\phi_{xxx}(0; u(\tau),\lambda(\tau),\alpha_1(\tau),\alpha_2,\omega(\tau)). \tag{23}$$

A sign change of $T_2(\tau)$ indicates that the branch passes possibly through a degenerate

Hopf bifurcation point of Type 2, **HP2**, which is defined as a root of the system

$$\mathcal{F}(u,\lambda,\alpha,\omega) := \begin{pmatrix} f(u,\lambda,\alpha) \\ f^{(1)}(u,\lambda,\alpha,\omega) \\ f^{(2)}(u,\lambda,\alpha,\omega) \\ f^{(4)}(u,\lambda,\alpha,\omega) \end{pmatrix} = 0, \tag{24}$$

and at which the relevant nondegeneracy conditions in Table 2 are satisfied. Under some unfolding conditions a Hopf bifurcation point of Type 2 is a regular root of (24), i.e., we can employ (24) to locate this point, see Theorem 4.1 below.

In a complete analogy, a sign change of $T_3(\tau)$ indicates that the branch passes possibly through a degenerate Hopf bifurcation point of Type 3, **HP3**, which is defined as a root of the system

$$\mathcal{F}(u,\lambda,\alpha,\omega) := \begin{pmatrix} f(u,\lambda,\alpha) \\ f^{(1)}(u,\lambda,\alpha,\omega) \\ f^{(2)}(u,\lambda,\alpha,\omega) \\ f^{(3)}(u,\lambda,\alpha,\omega) \end{pmatrix} = 0, \tag{25}$$

and at which the relevant nondegeneracy conditions in Table 2 are satisfied.

REMARK 4.1. *During the continuation of Hopf bifurcation points, we choose L and M as in Lemma 3.2. That is the reason why we have the simple test functions in (22) and in (23) for the relevant degeneracies. Therefore L and M must change with the parameters during the continuation. The computation of ξ_H is simple, since the purely imaginary eigenvalue $i\omega_H$ has been obtained during the contiuation of Hopf bifurcation points and we have only to determine a null vector of the matrix $f_u(u_H,\lambda_H,\alpha_H) - i\omega_H$.*

Step 3. Release the second auxiliary parameter α_2 and apply a pseudo-arclength continuation to the system (24), starting from the point **HP2**, or to the analogous system (25), starting from the point **HP3**, to follow a path of degenerate Hopf bifurcation points $(u(\tau),\lambda(\tau),\alpha(\tau),\omega(\tau))$. During the continuation we monitor the test function $T_3(\tau)$, or correspondingly the test function $T_2(\tau))$, with $\alpha_2 = \alpha_2(\tau)$ to detect degenerate Hopf bifurcation points of Type 4. Then we employ the system (17) to locate them.

The following theorems guarantee that the above procedure can be performed by Newton-like methods.

THEOREM 4.1. *Assume that $(u_H,\lambda_H,\alpha_H,\omega_H) \in \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ is a Hopf bifurcation point.*

(i) *If* $(u_H, \lambda_H, \alpha_H, \omega_H)$ *is a Hopf bifurcation point of Type 1, then for fixed* $\alpha = \alpha_H$*, there exists a unique branch of regular solutions of (19) in a neighbourhood of* $(u_H, \lambda_H, \alpha_H, \omega_H)$ *passing through* $(u_H, \lambda_H, \alpha_H, \omega_H)$*.*

(ii) *A Hopf bifurcation point of Type 1 is a regular root of (21).*

(iii) *A Hopf bifurcation point of Type 2 is a regular root of (24) if and only if*

$$\mathrm{Re}\phi_{x\alpha_1} \neq 0.$$

(iv) *A Hopf bifurcation point of Type 3 is a regular root of (25) if and only if*

$$\mathrm{Re}\phi_{xxx\alpha_1} \neq 0.$$

(v) *A Hopf bifurcation point of Type 4 is a regular root of (17)-(18) if and only if*

$$\det \begin{pmatrix} 0 & \mathrm{Re}\phi_{x\alpha_1} & \mathrm{Re}\phi_{x\alpha_2} \\ \mathrm{Re}\phi_{xxx\lambda} & \mathrm{Re}\phi_{xxx\alpha_1} & \mathrm{Re}\phi_{xxx\alpha_2} \\ \mathrm{Re}\phi_{x\lambda\lambda} & \mathrm{Re}\phi_{x\lambda\alpha_1} & \mathrm{Re}\phi_{x\lambda\alpha_2} \end{pmatrix} \neq 0.$$

THEOREM 4.2. **(i)** *Along the solution branch of (19), at a Hopf bifurcation point of Type 1 we have*

$$T_1(\tau) = 0, \qquad \frac{dT_1(\tau)}{d\tau} \neq 0.$$

(ii) *Along the Hopf curve defined by (21), at a Hopf bifurcation point of Type 2 we have*

$$T_2(\tau) = 0, \qquad \frac{dT_2(\tau)}{d\tau} \neq 0.$$

A corresponding result holds for a Hopf bifurcation point of Type 3 with T_2 *being replaced by* T_3*.*

(iii) *Along the degenerate Hopf curve defined by (24), at a Hopf bifurcation point of Type 4 we have*

$$T_2(\tau) = 0, \qquad \frac{dT_2(\tau)}{d\tau} \neq 0$$

for T_2 *in (22) with* $\alpha_2 = \alpha_2(\tau)$*. A corresponding result holds, if (24) and* T_2 *are replaced by (25) and* T_3 *respectively.*

Before we prove Theorem 4.1 and Theorem 4.2, we make the following observation.

LEMMA 4.3. *For $(x,\lambda,\alpha,\omega) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^k \times \mathbb{R}^1$ in a neighbourhood of the origin and for $(u^0,\lambda^0,\alpha^0,\omega^0) \in \mathcal{D}$ in a neighbourhood of a Hopf bifurcation point, we have*

$$\begin{aligned}
u(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0) &= u(x,\lambda,\alpha,0;u^0,\lambda^0,\alpha^0,\omega^0+\omega),\\
\phi(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0) &= \phi(x,\lambda,\alpha,0;u^0,\lambda^0,\alpha^0,\omega^0+\omega),\\
u_\lambda(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0) &= u_{\lambda^0}(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0)\\
&\quad +(I+u_{u^0}(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0))[u_\lambda]_0,\\
\phi_\lambda(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0) &= \phi_{\lambda^0}(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0)\\
&\quad +\phi_{u^0}(x,\lambda,\alpha,\omega;u^0,\lambda^0,\alpha^0,\omega^0)[u_\lambda]_0,
\end{aligned}$$

where $[u_\lambda]_0 = -(f_u(u^0,\lambda^0,\alpha^0))^{-1} f_\lambda(u^0,\lambda^0,\alpha^0)$ and I is the identity operator. The corresponding equations also hold by replacing the subscript λ by α and λ^0 by α^0 respectively.

Proof. See Appendix 7.2.

Sketch of a proof of Theorem 4.1. **(i)** We need only to prove that the Jacobian of system (19) at $(u_H,\lambda_H,\alpha_H,\omega_H)$, namely

$$\begin{pmatrix} f_u & f_\lambda & 0 \\ \mathrm{Im}\phi_{xu^0} & \mathrm{Im}\phi_{x\lambda^0} & \mathrm{Im}\phi_{x\omega^0} \end{pmatrix}, \tag{26}$$

has maximal rank $N+1$. Note that $f_u(u_H,\lambda_H,\alpha_H)$ is regular (Hypothesis H1). Then, by using some block elimination we see that the rank of (26) equals the rank of the matrix

$$\begin{pmatrix} f_u & f_\lambda & 0 \\ 0 & \mathrm{Im}\phi_{x\lambda^0} - \mathrm{Im}\phi_{xu^0}(f_u)^{-1} f_\lambda & \mathrm{Im}\phi_{x\omega^0} \end{pmatrix}. \tag{27}$$

From Lemma 4.3 we have $\mathrm{Im}\phi_{x\lambda^0} - \mathrm{Im}\phi_{xu^0}(f_u)^{-1} f_\lambda = \mathrm{Im}\phi_{x\lambda}$, $\mathrm{Im}\phi_{x\omega^0} = \mathrm{Im}\phi_{x\omega}$. Hence (27) equals

$$\begin{pmatrix} f_u & f_\lambda & 0 \\ 0 & \mathrm{Im}\phi_{x\lambda} & \mathrm{Im}\phi_{x\omega} \end{pmatrix}.$$

From Lemma 3.2 we know that $\mathrm{Im}\phi_{x\omega} = q_\omega \neq 0$. Hence the Jacobian has maximal rank $N+1$.

(ii) Analogously we see that the regularity of the Jacobian of (21) at the Hopf bifurcation point is equivalent to the regularity of the matrix

$$\begin{pmatrix} f_u & f_\lambda & 0 \\ 0 & \mathrm{Re}\phi_{x\lambda} & \mathrm{Re}\phi_{x\omega} \\ 0 & \mathrm{Im}\phi_{x\lambda} & \mathrm{Im}\phi_{x\omega} \end{pmatrix}.$$

Its submatrix

$$\mathbf{B} = \begin{pmatrix} \mathrm{Re}\phi_{x\lambda} & \mathrm{Re}\phi_{x\omega} \\ \mathrm{Im}\phi_{x\lambda} & \mathrm{Im}\phi_{x\omega} \end{pmatrix}$$

is regular, because $\det\mathbf{B} = q_\omega a_\lambda$, $q_\omega \neq 0$ (Lemma 3.2) and $a_\lambda \neq 0$ (the nondegeneracy condition). Hence the Jacobian of (21) is regular.

(iii) Note that the auxiliary parameter α_1 is released now. The regularity of the Jacobian of (24) is equivalent to the regularity of the matrix

$$\mathbf{B} = \begin{pmatrix} \mathrm{Re}\phi_{x\lambda} & \mathrm{Re}\phi_{x\alpha_1} & \mathrm{Re}\phi_{x\omega} \\ \mathrm{Im}\phi_{x\lambda} & \mathrm{Im}\phi_{x\alpha_1} & \mathrm{Im}\phi_{x\omega} \\ \mathrm{Re}\phi_{x\lambda\lambda} & \mathrm{Re}\phi_{x\lambda\alpha_1} & \mathrm{Re}\phi_{x\lambda\omega} \end{pmatrix}.$$

Now $\mathrm{Re}\phi_{x\lambda} = f^{(4)} = 0$; From Lemma 3.2 we get $\mathrm{Im}\phi_{x\omega} \neq 0$ and $\mathrm{Re}\phi_{x\omega} = 0$. Then we have

$$a_{\lambda\lambda} = \frac{1}{\mathrm{Im}\phi_{x\omega}}(\mathrm{Re}\phi_{x\lambda\omega}\mathrm{Im}\phi_{x\lambda} - \mathrm{Re}\phi_{x\lambda\lambda}\mathrm{Im}\phi_{x\omega})$$

and

$$\begin{aligned} \det\mathbf{B} &= -\mathrm{Re}\phi_{x\alpha_1}(\mathrm{Re}\phi_{x\lambda\omega}\mathrm{Im}\phi_{x\lambda} - \mathrm{Re}\phi_{x\lambda\lambda}\mathrm{Im}\phi_{x\omega}) \\ &= -\mathrm{Re}\phi_{x\alpha_1}\mathrm{Im}\phi_{x\omega}a_{\lambda\lambda}. \end{aligned}$$

Therefore the nondegeneracy condition $a_{\lambda\lambda} \neq 0$ implies that the regularity of $\mathbf{B}$ is equivalent to $\mathrm{Re}\phi_{x\alpha_1} \neq 0$.

Similarly we can prove the cases **(iv)** and **(v)**.

The proof of Theorem 4.2 is simple and therefore omitted.

5. Stability of the bifurcating periodic orbits. In applications of Hopf bifurcation theorems to systems modeling natural phenomena, the determination of the stability and the branching directions of the periodic solutions bifurcating from a Hopf bifurcation point is important. We shall obtain this information from the sign of the test functions as a by-product.

In this section we do not consider auxiliary parameters and assume that $(u_H, \lambda_H, \omega_H) \in \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^1$ is a Hopf bifurcation point of Type 1. For simplicity, we assume that the eigenvalues of $f_u(u_H, \lambda_H)$, except for $\pm i\omega_H$, have strictly negative real parts.

The periodic solutions bifurcating from $(u_H, \lambda_H, \omega_H)$ are related to the solutions $(x, \lambda, \omega) \in \mathbb{R}^1 \times \mathbb{R}^1 \times \mathbb{R}^1$ of the system

$$\begin{aligned} p(r, \lambda, \omega; u_H, \lambda_H, \omega_H) = 0, \\ q(r, \lambda, \omega; u_H, \lambda_H, \omega_H) = 0, \end{aligned} \tag{28}$$

where $r = x^2 > 0$. If we choose $L = \bar{\xi}_H, M = \xi_H$, then from Lemma 3.2 we have $p_\omega(0; u_H, \lambda_H, \omega_H) = 0, q_\omega(0; u_H, \lambda_H, \omega_H) = 1$. From (16) we have $a_\lambda = p_\lambda - \frac{p_\omega}{q_\omega} q_\lambda$. Hence due to the nondegeneracy conditions we have $p_\lambda(0; u_H, \lambda_H, \omega_H) = a_\lambda(0; u_H, \lambda_H, \omega_H) \neq 0$. Thus the matrix

$$\begin{pmatrix} p_\lambda & p_\omega \\ q_\lambda & q_\omega \end{pmatrix}$$

is regular. The implicit function theorem guarantees that there exists a unique solution

$$\lambda = \lambda(r; u_H, \lambda_H, \omega_H),$$
$$\omega = \omega(r; u_H, \lambda_H, \omega_H)$$

of (28). At the Hopf bifurcation point we have

$$\lambda_r = -\frac{p_r}{p_\lambda} = -\frac{\frac{1}{6}\mathrm{Re}\phi_{xxx}}{\mathrm{Re}\phi_{x\lambda}}. \tag{29}$$

Now we are going to show that $\mathrm{Re}\phi_{x\lambda} = -\sigma'(\lambda_H)$, where $\sigma(\lambda)$ is defined by the eigenvalue problem along the equilibrium solution branch,

$$(f_u(u(\lambda), \lambda) - (\sigma(\lambda) + i\omega(\lambda)))c(\lambda) = 0.$$

Differentiating with respect to λ, evaluating at the Hopf bifurcation point and multiplying from the left with $\bar{\eta}_H^T$, we obtain

$$\bar{\eta}_H^T(f_{uu}(u_H, \lambda_H)u_\lambda + f_{u\lambda}(u_H, \lambda_H) - (\sigma'(\lambda_H) + i\omega'(\lambda_H)))\xi_H = 0.$$

From the table in the appendix and from $Z(\omega_H) = \bar{\eta}^T$, $[u_x]_1 = \xi_H$, we get

$$\begin{aligned} \phi_{x\lambda} &= -Z(\omega_H)(f_{uu}[u_\lambda]_0 + f_{u\lambda})[u_x]_1 \\ &= -\bar{\eta}_H^T(f_{uu}[u_\lambda]_0 + f_{u\lambda})\xi_H \\ &= -(\sigma'(\lambda_H) + i\omega'(\lambda_H))\bar{\eta}_H^T\xi_H \\ &= -(\sigma'(\lambda_H) + i\omega'(\lambda_H)). \end{aligned}$$

Hence

$$\lambda_r = \frac{\mathrm{Re}\phi_{xxx}}{6\sigma'(\lambda_H)}.$$

$\lambda_r > 0$ ($\lambda_r < 0$) implies that the Hopf bifurcation is supercritical (subcritical), while $\sigma'(\lambda_H) > 0$ ($\sigma'(\lambda_H) < 0$) implies that the branch of equilibrium solutions loses the stability for $\lambda > \lambda_H$ ($\lambda < \lambda_H$). Therefore we have

THEOREM 5.1. *Assume that* $(u_H, \lambda_H, \omega_H) \in \mathbb{R}^N \times \mathbb{R}^1 \times \mathbb{R}^1$ *is a Hopf bifurcation point. If* $\mathrm{Re}\phi_{xxx} > 0$*, then the branch of periodic solutions is stable. If* $\mathrm{Re}\phi_{xxx} < 0$*, then the branch of periodic solutions is unstable.*

In fact, direct calculations yield

$$\mathrm{Re}\phi_{xxx}(0; u_H, \lambda_H, \alpha_H, \omega_H) = -6\omega_H l_1(0), \tag{30}$$

where $l_1(0)$ is the first Liapunov coefficient, for its formula see e.g. [9] p. 163. Hence we obtain a convenient approach to the numerical computation of the first Liapunov coefficient.

From (30) we see that the degenerate Hopf bifurcations with $a_r = \mathrm{Re}\phi_{xxx} = 0$ (i.e. Type 3 and 4) are essentially zero Liapunov coefficient Hopf bifurcations. If $\mathrm{Re}\phi_{xxx} = 0$ and $\mathrm{Re}\phi_{xxxxx} \neq 0$ (we still need $a_\lambda \neq 0$), then

$$\lambda_{rr} = -\frac{p_{rr}}{p_\lambda} = \frac{\mathrm{Re}\phi_{xxxxx}}{60\sigma'(\lambda_H)}.$$

Thus, if $\mathrm{Re}\phi_{xxxxx} > 0$, then the branch of periodic solutions is stable. If $\mathrm{Re}\phi_{xxxxx} < 0$, then the branch of periodic solutions is unstable.

For the degenerate Hopf bifurcations with $a_\lambda = \mathrm{Re}\phi_{x\lambda} = 0$, there may be bifurcating periodic solutions in a full neighbourhood of the Hopf bifurcation point and no exchange of the stability[5], [3]. We shall not consider these cases here.

6. Numerical example. We consider the Hodgkin-Huxley equations that model electrical activity in the giant axon of a squid under certain controlled experimental conditions, see [3], [5], [10] and [4]. These equations exhibit periodic solutions that arise from Hopf bifurcations. The equations also contain several auxiliary parameters, and the dependence of the periodic solutions on the various parameters is rather involved. The equations are fourfold:

$$\begin{aligned}
\frac{dV}{dt} &= \frac{1}{C}[-G(V, M, N, H) + I],\\
\frac{dM}{dt} &= \Phi(T)[(1-M)\alpha_M(V) - M\beta_M(V)],\\
\frac{dN}{dt} &= \Phi(T)[(1-N)\alpha_N(V) - N\beta_N(V)],\\
\frac{dH}{dt} &= \Phi(T)[(1-H)\alpha_H(V) - H\beta_H(V)],
\end{aligned} \tag{31}$$

where t is the time, V is the potential difference across the membrane, M is the sodium activation, N is the potassium activation, H is the sodium inactivation, $\Phi(T) =$

$3^{(T-6.3)/10}$ is a temperature compensation factor and T is the temperature, C is the membrane capacitance per unit area, here $C = 1.0$, I is the external current stimulus. The functions in the equations are given by

$$G(V, M, N, H) = \bar{g}_{Na} M^3 H (V - V_{Na}) + \bar{g}_K N^4 (V - V_K) + \bar{g}_L (V - V_L),$$

$$\begin{aligned} \alpha_M(V) &= \Psi(\tfrac{25-V}{10}), & \beta_M(V) &= 4e^{(-V/18)}, \\ \alpha_N(V) &= 0.1\Psi(\tfrac{10-V}{10}), & \beta_N(V) &= 0.125e^{(-V/80)}, \\ \alpha_H(V) &= 0.07e^{(-V/20)}, & \beta_H(V) &= (1 + e^{(30-V)/10})^{-1}, \end{aligned}$$

where $\Psi(x) = x/(e^x - 1)$ for $x \neq 0$, and $\Psi(0) = 1$. The constants

$$\begin{aligned} \bar{g}_{Na} &= 120, & \bar{g}_K &= 36, & \bar{g}_L &= 0.3, \\ V_{Na} &= 115, & V_K &= -12, & V_L &= 10.599 \end{aligned}$$

were obtained from experimental data. All constants and variables are dimensionless, and the variables M, N and H lie in the interval $[0, 1]$.

In what follows we shall consider the external current stimulus I as the bifurcation parameter. By the consideration of codimension one bifurcations, we treat the temperature T as an auxiliary parameter. For higher codimension we shall vary $\bar{g}_{Na}$ and V_{Na} respectively.

All computations were performed on a SUN in double precision. All complex linear systems were solved in complex arithmetics by means of the standard LU decomposition from LINPACK.

For the continuation of the equilibrium solutions using system (19), we choose the initial point as below. For any choice of the parameters I and T, an equilibrium solution (V_*, M_*, N_*, H_*) of (31) must satisfy

$$J_* := J(V_*) = \frac{\alpha_J(V_*)}{\alpha_J(V_*) + \beta_J(V_*)}, \quad J = M,\ N,\ H \tag{32}$$

and

$$G(V_*, M_*, N_*, H_*) + I = 0. \tag{33}$$

For $T = 6.3$ we choose an arbitrary value of V_*, say $V_* = 5.0$. Then we obtain M_*, N_*, H_* from (32). Substituting (V_*, M_*, N_*, H_*) into (33), we obtain the corresponding value of the parameter I for which we have an equilibrium solution. The Jacobian of the right hand side of (31) at this equilibrium point has a pair of conjugate complex eigenvalues.

We choose the imaginary part of the complex eigenvalues as the initial value of ω_*. Figure 1 shows the branch of the equilibrium solutions.

The choice of the bordering vectors L and M has to ensure the regularity of the bordered matrix $\mathcal{B}$. Since this is fulfilled generically, we choose $L = M = (1, 0, \cdots, 0)^T \in \mathbb{C}^N$ for the continuation of (19). For the continuation of Hopf curves by the systems (21), (24), (25), etc., we choose $\bar{L} = M = \xi_H$, which varies during the continuation, see Remark 4.1.

During the continuation of equilibrium solutions, monitoring the test function (20), two Hopf bifurcation points of Type 1, A and B, were found, see Table 3, in which "Tp" denote the Type of the Hopf bifurcation points. For the computation of the test function (20), [12] and [7] suggested a nonlinear elimination. Here, the computation of the test function (20) is only a by-product of the continuation procedure in form of a Newton predictor-corrector-algorithm. At the predictor step of a current point we need to solve system (38) and (39) in order to compute the Jacobian of (19). We are computing $T_1(\tau) = \mathrm{Re}\phi_x(0; u(\tau), \lambda(\tau), \alpha, \omega(\tau))$ simultaneously with $f^{(2)}(u(\tau), \lambda(\tau), \alpha, \omega(\tau)) = \mathrm{Im}\phi_x(0; u(\tau), \lambda(\tau), \alpha, \omega(\tau))$. For the details of the Newton-like predictor-corrector on the extended systems, see [7].

The continuation procedure of Hopf bifurcation points of Type 1 using (21) shows that the two points A and B are connected by a Hopf curve, see Figure 2. Two Hopf bifurcation points of codimension 1, called C and D, were found during this continuation. Point C is a Hopf bifurcation point of Type 3. At this point the sign of the test function $T_3(\tau)$ changes from negative to positive, i.e., the stability of the bifurcating periodic orbits changes from unstable to stable and the bifurcation changes from subcritical to supercritical (now the sign of the test function $T_2(\tau)$ is negative), see (29). Point D is a Hopf bifurcation point of Type 2. At this point the sign of the test function $T_2(\tau)$ changes from negative to positive, i.e., the bifurcation changes again from supercritical to subcritical, but the stability of the bifurcating periodic orbits does not change. In fact, Point D is just a turning point of the branch of Hopf bifurcation points of Type 1 with respect to the parameter T.

We have computed the bifurcating periodic orbits from different Hopf bifurcation points by using AUTO94 of Doedel in order to compare the results about the stability exchange. The periodic solutions bifurcating from the two Hopf bifurcation points A and B are connected to each other globally by the variation of I. They change stability at a turning point. When the temperature is increased the loops of the periodic solutions shrink. Using AUTO94 we failed to switch from a Hopf bifurcation point close to the point C or D to bifurcating periodic solutions. An alternative approach to the approximation of the periodic solutions is presented in [7].

We have performed the continuations of Hopf bifurcation points of Type 2 and Type 3 for different choices of the continuation parameter, see Figures 3-6. Points E, F, G, H are Hopf bifurcation points of Type 4. With increasing codimension of the Hopf bifurcation points the computation becomes more complicated since we have to compute higher order derivatives. We have used the formulas in the appendix and also numerical differentiation respectively to compute the Jacobian of (17). The results are the same, but the first method is much more efficient.

TABLE 3

Hopf bifurcation points of the clamped Hodgkin-Huxley equations.

Pt	Tp	V	M	N	H	I	T	$\bar{g}_{N_a}$	V_{N_a}
A	1	5.345856	.0972574	.4017841	.4062275	9.779638	6.3	120.0	115.0
B	1	21.94191	.4196774	.6432495	.0703554	154.5266	6.3	120.0	115.0
C	3	16.16858	.2764980	.5674827	.1346130	73.09019	28.85250	120.0	115.0
D	2	16.35405	.2807007	.5700900	.1318332	75.02891	28.85794	120.0	115.0
E	4	15.98082	.2722777	.5648324	.1374842	74.08889	26.37718	109.3667	115.0
F	4	14.68948	.2442262	.5463174	.1588431	65.99559	18.04756	89.70577	115.0
G	4	15.76536	.2674780	.5617778	.1408503	72.26422	25.77825	120.0	104.9327
H	4	14.64019	.2431904	.5456011	.1597160	64.61148	19.79890	120.0	92.94621

7. Appendix.

7.1. Proof of Lemma 3.2. In accordance with the relevant formulas in §7.3, we have

$$\phi_{x\omega}(0; u_H, \lambda_H, \alpha_H, \omega_H) = iZ(\omega_H)[u_x]_1.$$

Note that

$$\phi_x(0; u_H, \lambda_H, \alpha_H, \omega_H) = p(0; u_H, \lambda_H, \alpha_H, \omega_H) + iq(0; u_H, \lambda_H, \alpha_H, \omega_H) = 0.$$

Therefore, referring to (38) and (39), we have

$$(f_u(u_H, \lambda_H, \alpha_H) - i\omega_H)[u_x]_1 = 0, \qquad L[u_x]_1 = 1,$$
$$Z(\omega_H)(f_u(u_H, \lambda_H, \alpha_H) - i\omega_H) = 0, \quad Z(\omega_H)M = 1.$$

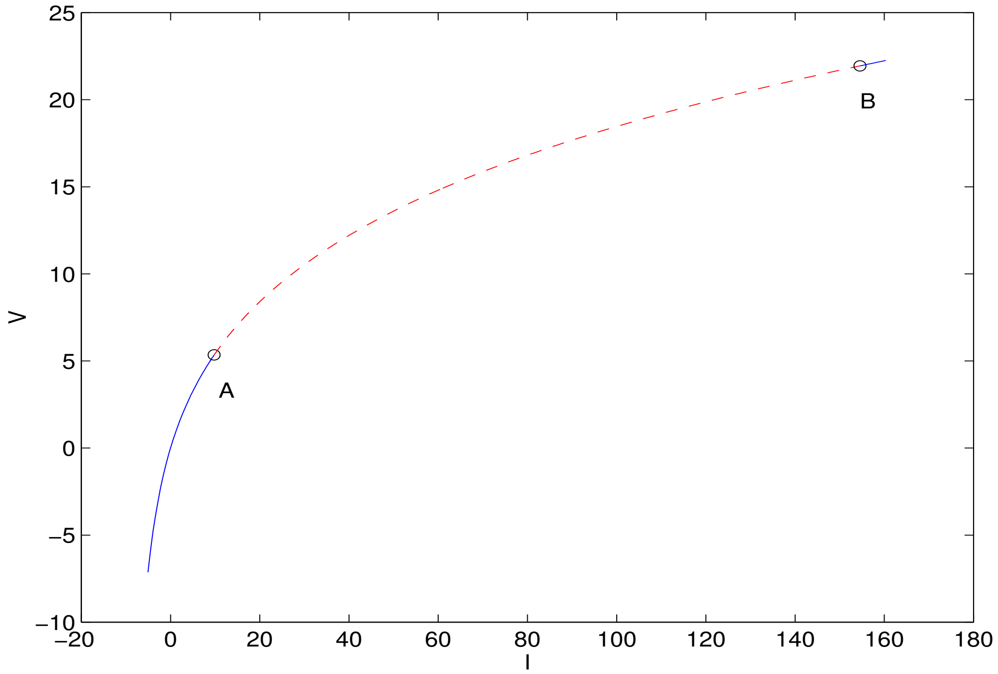

FIG. 1. branch of equilibrium points.

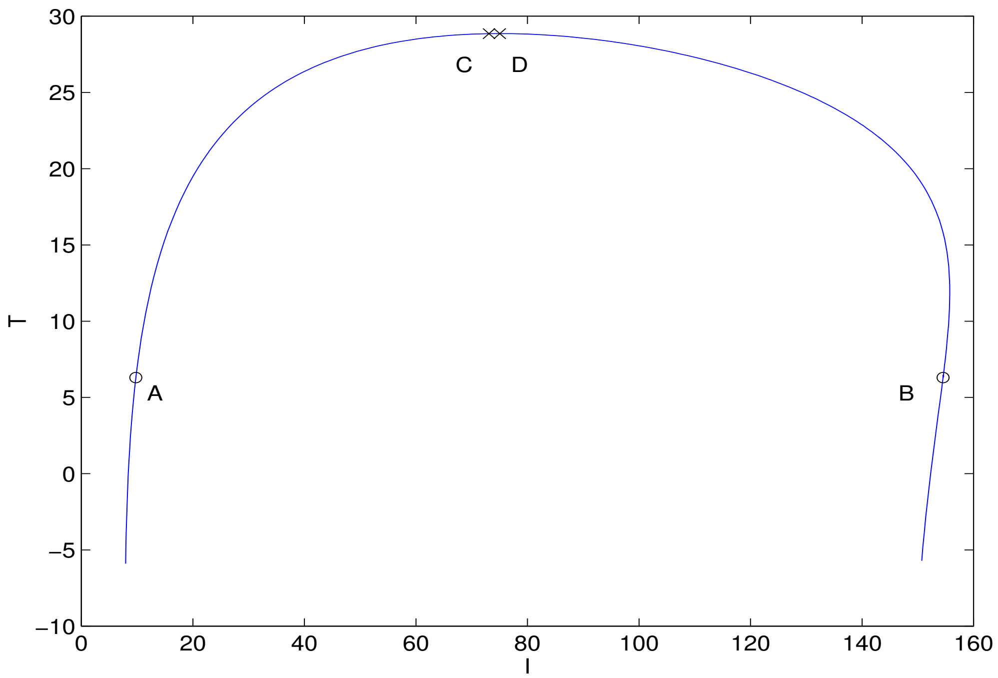

FIG. 2. branch of HBP of Type 1.

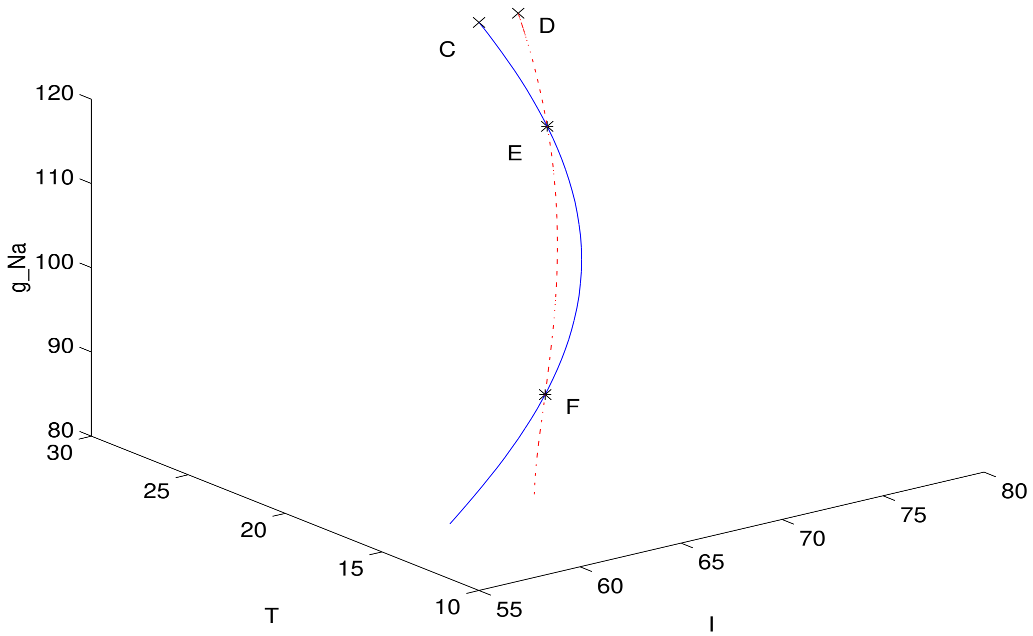

FIG. 3. branches of HBP of Type 2 and Type 3.

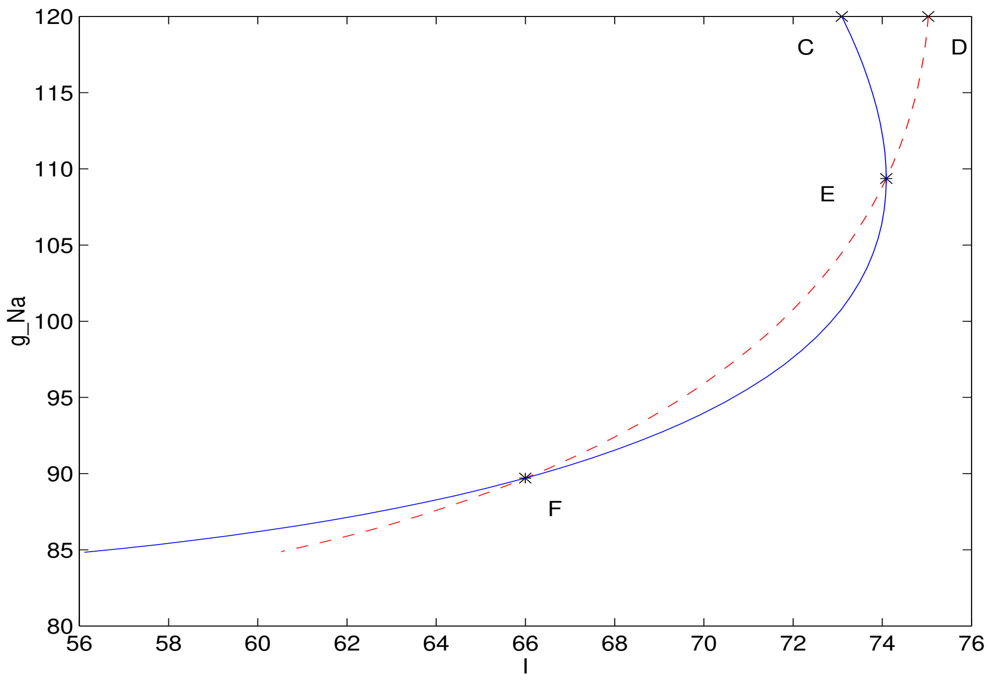

FIG. 4. branches of HBP of Type 2 and Type 3.

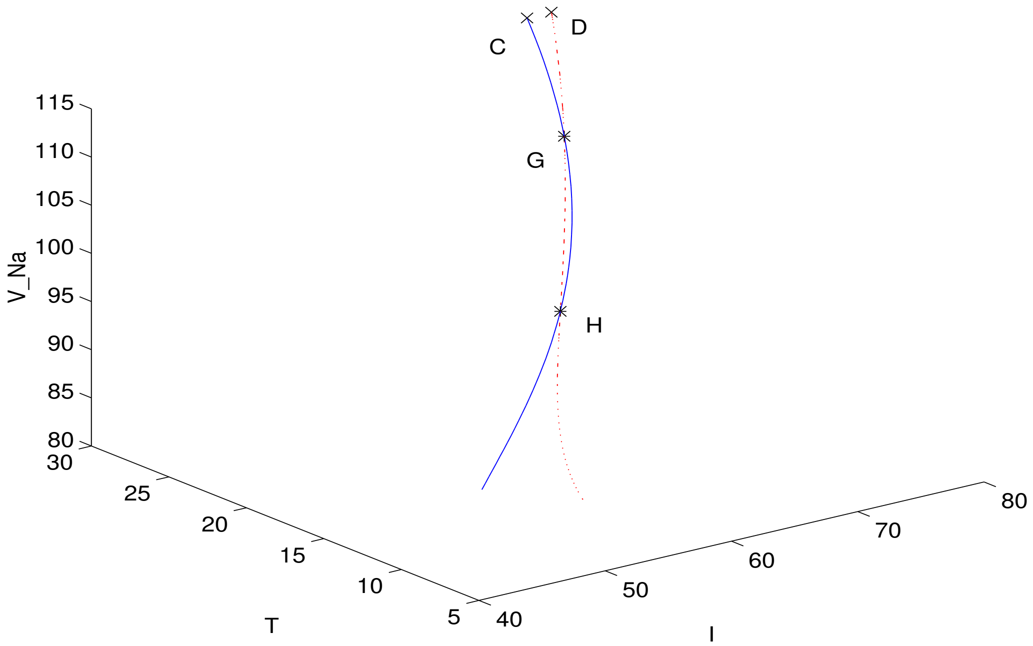

FIG. 5. branches of HBP of Type 2 and Type 3.

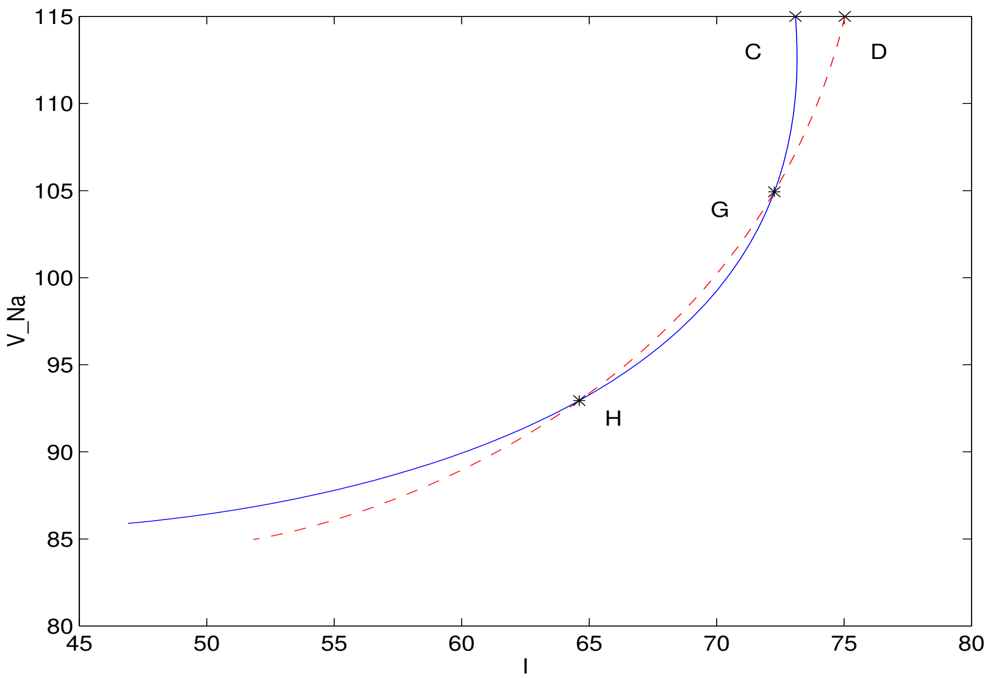

FIG. 6. branches of HBP of Type 2 and Type 3.

Due to (9), $[u_x]_1$ and $Z(\omega_H)$ are nontrivial multipules of ξ_H and $\bar{\eta}_H^T$ respectively, i.e., there exist two constants $c_1, c_2 \in \mathbb{C}^1$ such that $[u_x]_1 = c_1\xi_H, Z(\omega_H) = c_2\bar{\eta}_H^T$. For the choice $L = \bar{\xi}_H$, $M = \xi_H$, we have $c_1 = c_2 = 1$, thus $Z(\omega_H)[u_x]_1 = c_1c_2\bar{\eta}_H^T\xi_H = c_1c_2 = 1$. Thus we have

$$q_\omega(0; u_H, \lambda_H, \alpha_H, \omega_H) = \mathrm{Im}(\phi_{x\omega}(0; u_H, \lambda_H, \alpha_H, \omega_H)) = 1,$$
$$p_\omega(0; u_H, \lambda_H, \alpha_H, \omega_H) = \mathrm{Re}(\phi_{x\omega}(0; u_H, \lambda_H, \alpha_H, \omega_H)) = 0,$$

which proves the assertion of Lemma 3.2.

7.2. Proof of Lemma 4.3. Due to the local uniqueness of the solutions of (6), the former two equalities can easily be obtained. Differentiating (6) with respect to λ and evaluating at $(u, \lambda, \alpha, \omega) = 0$, we obtain

$$\begin{aligned} -\omega^0\dot{u}_\lambda + f_u^0 u_\lambda + f_\lambda^0 + \mathbf{M}\phi_\lambda &= 0, \\ \mathbf{L}u_\lambda &= 0, \end{aligned} \tag{34}$$

where $f_u^0 := f_u(u^0, \lambda^0, \alpha^0)$ and $f_\lambda^0 := f_\lambda(u^0, \lambda^0, \alpha^0)$. Analogously we have

$$\begin{aligned} -\omega^0\dot{u}_{\lambda^0} + f_u^0 u_{\lambda^0} + \mathbf{M}\phi_{\lambda^0} &= 0, \\ \mathbf{L}u_{\lambda^0} &= 0, \end{aligned} \tag{35}$$

and

$$\begin{aligned} -\omega^0\dot{u}_{u^0} + f_u^0(I + u_{u^0}) + \mathbf{M}\phi_{u^0} &= f_u^0, \\ \mathbf{L}u_{u^0} &= 0. \end{aligned} \tag{36}$$

Multiplying (36) from the right by $[u_\lambda]_0 = -(f_u^0)^{-1}f_\lambda^0$ and adding to (35), we obtain

$$\begin{aligned} -\omega^0\dot{X} + f_u^0 X + \mathbf{M}Y &= -f_\lambda^0, \\ \mathbf{L}X &= 0, \end{aligned} \tag{37}$$

where $X = u_{\lambda^0} + (I + u_{u^0})[u_\lambda]_0$ and $Y = \phi_{\lambda^0} + \phi_{u^0}[u_\lambda]_0$.

Comparing (37) to (34) and using the uniqueness of the solutions we obtain Lemma 4.3.

7.3. Formulas for the partial derivatives of u and ϕ. We consider

$$u = u(x, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0),$$
$$\phi = \phi(x, \lambda, \alpha, \omega; u^0, \lambda^0, \alpha^0, \omega^0)$$

as functions of $(x, \lambda, \alpha, \omega)$ in a neighbourhood of the origin and of $(u^0, \lambda^0, \alpha^0, \omega^0)$ in a neighbourhood of a Hopf bifurcation point. We list the formulas for some partial derivatives of u and ϕ evaluated at $(x, \lambda, \alpha, \omega) = 0$. It is understood that f and its partial derivatives in the formulas are evaluated at $(u^0, \lambda^0, \alpha^0)$. Some of the formulas can be found in [7]. We remark that the remaining seven types of degenerate Hopf bifurcation points which are not contained in Table 1 and 2 can be computed in analogy to the types 1–4.

We set

$$A(\kappa) := f_u(u^0, \lambda^0, \alpha^0) - i\kappa.$$

Hypothesis (H1) guarantees that there exists the inverse $A^{-1}(k\omega^0)$ for $k \in \mathbb{Z}, k \neq \pm 1$ and for $(u^0, \lambda^0, \alpha^0, \omega^0)$ close enough to a Hopf bifurcation point.

We only list the partial derivatives with respect to ω, since they coincide with those with respect to ω^0, see Lemma 4.3. Derivatives of u and ϕ with respect to α can be obtained by replacing the subscript λ by α. Derivatives of u and ϕ with respect to α^0 can be obtained by replacing the subscript λ by α and λ^0 by α^0 respectively. We remark that the computations need the solution of linear systems with the same coefficient matrix $\mathcal{B}$ (or its transpose) and those of the linear systems involving $A(\kappa)$.

$$\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [u_x]_1 \\ \phi_x \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{38}$$

$$(Z(\omega^0),\ \phi_x)\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0) = (0,\ 1)\,. \tag{39}$$

$$\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [u_{x\lambda}]_1 \\ \phi_{x\lambda} \end{pmatrix} = \begin{pmatrix} -(f_{uu}[u_\lambda]_0 + f_{u\lambda})[u_x]_1 \\ 0 \end{pmatrix}.$$

$$\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [u_{x\omega}]_1 \\ \phi_{x\omega} \end{pmatrix} = \begin{pmatrix} i[u_x]_1 \\ 0 \end{pmatrix}.$$

$$\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [u_{xu^0}]_1 \\ \phi_{xu^0} \end{pmatrix} = \begin{pmatrix} -f_{uu}[u_x]_1 \\ 0 \end{pmatrix}.$$

$$\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [u_{x\lambda^0}]_1 \\ \phi_{x\lambda^0} \end{pmatrix} = \begin{pmatrix} -f_{u\lambda}[u_x]_1 \\ 0 \end{pmatrix}.$$

$$\mathcal{B}(u^0, \lambda^0, \alpha^0, \omega^0)\begin{pmatrix} [u_{xxx}]_1 \\ \phi_{xxx} \end{pmatrix} = \begin{pmatrix} r_2 \\ 0 \end{pmatrix},$$

where $r_2 = -3(f_{uuu}[u_x]_1^2[u_x]_{-1} + f_{uu}([u_x]_1[u_{xx}]_0 + [u_x]_{-1}[u_{xx}]_2))$.

ϕ_{xx}	0
$\phi_{x\lambda}$	$-Z(\omega^0)(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_x]_1$
$\phi_{x\omega}$	$iZ(\omega^0)[u_x]_1$
ϕ_{xu^0}	$-Z(\omega^0)f_{uu}[u_x]_1$
$\phi_{x\lambda^0}$	$-Z(\omega^0)f_{u\lambda}[u_x]_1$
$\phi_{xx\lambda}$	0
$\phi_{x\lambda\lambda}$	$-Z(\omega^0)(((f_{uuu}[u_\lambda]_0+f_{uu\lambda})[u_\lambda]_0+f_{uu}[u_{\lambda\lambda}]_0+(f_{uu\lambda}[u_\lambda]_0+f_{u\lambda\lambda}))[u_x]_1$ $+2(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{x\lambda}]_1)$
$\phi_{x\lambda\alpha}$	$-Z(\omega^0)(((f_{uuu}[u_\alpha]_0+f_{uu\alpha})[u_\lambda]_0+f_{uu}[u_{\lambda\alpha}]_0+(f_{uu\lambda}[u_\alpha]_0+f_{u\lambda\alpha}))[u_x]_1$ $+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{x\alpha}]_1)+(f_{uu}[u_\alpha]_0+f_{u\alpha})[u_{x\lambda}]_1)$
$\phi_{x\lambda\omega}$	$Z(\omega^0)(i[u_{x\lambda}]_1-(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{x\omega}]_1)$
$\phi_{x\lambda u^0}$	$-Z(\omega^0)((f_{uuu}[u_\lambda]_0+f_{uu}[u_{\lambda u^0}]_0+f_{uu\lambda})[u_x]_1+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{xu^0}]_1+f_{uu}[u_{x\lambda}]_1)$
$\phi_{x\lambda\lambda^0}$	$-Z(\omega^0)((f_{uu\lambda}[u_\lambda]_0+f_{uu}[u_{\lambda\lambda^0}]_0+f_{u\lambda\lambda})[u_x]_1+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{x\lambda^0}]_1+f_{u\lambda}[u_{x\lambda}]_1)$
$\phi_{x\lambda\alpha^0}$	$-Z(\omega^0)((f_{uu\alpha}[u_\lambda]_0+f_{uu}[u_{\lambda\alpha^0}]_0+f_{u\lambda\alpha})[u_x]_1+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{x\alpha^0}]_1+f_{u\alpha}[u_{x\lambda}]_1)$
ϕ_{xxx}	$-3Z(\omega^0)(f_{uuu}[u_x]_1^2[u_x]_{-1}+f_{uu}([u_x]_1[u_{xx}]_0+[u_x]_{-1}[u_{xx}]_2))$
ϕ_{xxxx}	0
$\phi_{xxx\lambda}$	$-Z(\omega^0)(3(f_{uuuu}[u_\lambda]_0+f_{uuu\lambda})[u_x]_1^2[u_x]_{-1}+3f_{uuu}(2[u_x]_1[u_x]_{-1}[u_{x\lambda}]_1+[u_x]_1^2[u_{x\lambda}]_{-1})$ $+3(f_{uuu}[u_\lambda]_0+f_{uu\lambda})([u_x]_1[u_{xx}]_0+[u_x]_{-1}[u_{xx}]_2)+3f_{uu}([u_{x\lambda}]_1[u_{xx}]_0+[u_{x\lambda}]_{-1}[u_{xx}]_2)$ $+3f_{uu}([u_x]_1[u_{xx\lambda}]_0+[u_x]_{-1}[u_{xx\lambda}]_2)+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{xxx}]_1)$
$\phi_{xxx\omega}$	$Z(\omega^0)(i[u_{xxx}]_1-3f_{uuu}(2[u_x]_1[u_x]_{-1}[u_{x\omega}]_1+[u_x]_1^2[u_{x\omega}]_{-1})$ $-3f_{uu}([u_{x\omega}]_1[u_{xx}]_0+[u_{x\omega}]_{-1}[u_{xx}]_2+[u_x]_1[u_{xx\omega}]_0+[u_x]_{-1}[u_{xx\omega}]_2))$
ϕ_{xxxu^0}	$-Z(\omega^0)(3f_{uuuu}[u_x]_1^2[u_x]_{-1}+3f_{uuu}(2[u_x]_1[u_x]_{-1}[u_{xu^0}]_1+[u_x]_1^2[u_{xu^0}]_{-1})$ $+3f_{uuu}([u_x]_1[u_{xx}]_0+[u_x]_{-1}[u_{xx}]_2)+3f_{uu}([u_{xu^0}]_1[u_{xx}]_0+[u_{xu^0}]_{-1}[u_{xx}]_2)$ $+3f_{uu}([u_x]_1[u_{xxu^0}]_0+[u_x]_{-1}[u_{xxu^0}]_2)+f_{uu}[u_{xxx}]_1)$
$\phi_{xxx\lambda^0}$	$-Z(\omega^0)(3f_{uuu\lambda}[u_x]_1^2[u_x]_{-1}+3f_{uuu}(2[u_x]_1[u_x]_{-1}[u_{x\lambda^0}]_1+[u_x]_1^2[u_{x\lambda^0}]_{-1})$ $+3f_{uu\lambda}([u_x]_1[u_{xx}]_0+[u_x]_{-1}[u_{xx}]_2)+3f_{uu}([u_{x\lambda^0}]_1[u_{xx}]_0+[u_{x\lambda^0}]_{-1}[u_{xx}]_2)$ $+3f_{uu}([u_x]_1[u_{xx\lambda^0}]_0+[u_x]_{-1}[u_{xx\lambda^0}]_2)+f_{u\lambda}[u_{xxx}]_1)$
ϕ_{xxxxx}	$-Z(\omega^0)(10f_{uuuuu}[u_x]_1^3[u_x]_{-1}^2$ $+10f_{uuuu}([u_x]_1^3[u_{xx}]_{-2}+3[u_x]_1^2[u_x]_{-1}[u_{xx}]_0+3[u_x]_1[u_x]_{-1}^2[u_{xx}]_2)$ $+15f_{uuu}([u_x]_1[u_{xx}]_0^2+2[u_x]_1[u_{xx}]_2[u_{xx}]_{-2}+2[u_x]_{-1}[u_{xx}]_0[u_{xx}]_2)$ $+10f_{uuu}([u_x]^2[u_{xxx}]_{-1}+2[u_x]_1[u_x]_{-1}[u_{xxx}]_1+[u_x]_{-1}^2[u_{xxx}]_3)$ $+10f_{uu}([u_{xx}]_0[u_{xxx}]_1+[u_{xx}]_2[u_{xxx}]_{-1}+[u_{xx}]_{-2}[u_{xxx}]_3)$ $+5f_{uu}([u_x]_1[u_{xxxx}]_0+[u_x]_{-1}[u_{xxxx}]_2))$

$[u_\lambda]_0$	$-A^{-1}(0)f_\lambda$
$[u_{\lambda\lambda}]_0$	$-A^{-1}(0)((f_{uu}[u_\lambda]_0+f_{u\lambda})[u_\lambda]_0+f_{u\lambda}[u_\lambda]_0+f_{\lambda\lambda})$
$[u_{\lambda\alpha}]_0$	$-A^{-1}(0)((f_{uu}[u_\alpha]_0+f_{u\alpha})[u_\lambda]_0+f_{u\lambda}[u_\alpha]_0+f_{\lambda\alpha})$
$[u_{\lambda u^0}]_0$	$-A^{-1}(0)(f_{uu}[u_\lambda]_0+f_{u\lambda})$
$[u_{\lambda\lambda^0}]_0$	$-A^{-1}(0)(f_{u\lambda}[u_\lambda]_0+f_{\lambda\lambda})$
$[u_{\lambda\alpha^0}]_0$	$-A^{-1}(0)(f_{u\alpha}[u_\lambda]_0+f_{\lambda\alpha})$
$[u_{xx}]_0$	$-2A^{-1}(0)f_{uu}[u_x]_1[u_x]_{-1}$
$[u_{xx\lambda}]_0$	$-A^{-1}(0)(2(f_{uuu}[u_\lambda]_0+f_{uu\lambda})[u_x]_1[u_x]_{-1}+2f_{uu}([u_x]_{-1}[u_{x\lambda}]_1+[u_x]_1[u_{x\lambda}]_{-1})$ $+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{xx}]_0)$
$[u_{xx\omega}]_0$	$-2A^{-1}(0)f_{uu}([u_x]_{-1}[u_{x\omega}]_1+[u_x]_1[u_{x\omega}]_{-1})$
$[u_{xxu^0}]_0$	$-A^{-1}(0)(2f_{uuu}[u_x]_1[u_x]_{-1}+2f_{uu}([u_x]_{-1}[u_{xu^0}]_1+[u_x]_1[u_{xu^0}]_{-1})+f_{uu}[u_{xx}]_0)$
$[u_{xx\lambda^0}]_0$	$-A^{-1}(0)(2f_{uu\lambda}[u_x]_1[u_x]_{-1}+2f_{uu}([u_x]_{-1}[u_{x\lambda^0}]_1+[u_x]_1[u_{x\lambda^0}]_{-1})+f_{u\lambda}[u_{xx}]_0)$
$[u_{xxxx}]_0$	$-A^{-1}(0)(6f_{uuuu}[u_x]_1^2[u_x]_{-1}^2$ $+6f_{uuu}([u_x]_1^2[u_{xx}]_{-2}+2[u_x]_1[u_x]_{-1}[u_{xx}]_0+[u_x]_{-1}^2[u_{xx}]_2)$ $+3f_{uu}([u_{xx}]_0^2+2[u_{xx}]_2[u_{xx}]_{-2})+4f_{uu}([u_x]_1[u_{xxx}]_{-1}+[u_x]_{-1}[u_{xxx}]_1))$

$[u_{xx}]_2$	$-A^{-1}(2\omega^0)f_{uu}[u_x]_1[u_x]_1$
$[u_{xx\lambda}]_2$	$-A^{-1}(2\omega^0)((f_{uuu}[u_\lambda]_0+f_{uu\lambda})[u_x]_1^2+2f_{uu}[u_x]_1[u_{x\lambda}]_1+(f_{uu}[u_\lambda]_0+f_{u\lambda})[u_{xx}]_2)$
$[u_{xx\omega}]_2$	$A^{-1}(2\omega^0)(2i[u_{xx}]_2-2f_{uu}[u_x]_1[u_{x\omega}]_1)$
$[u_{xxu^0}]_2$	$-A^{-1}(2\omega^0)(f_{uuu}[u_x]_1^2+2f_{uu}[u_x]_1[u_{xu^0}]_1+f_{uu}[u_{xx}]_2)$
$[u_{xx\lambda^0}]_2$	$-A^{-1}(2\omega^0)(f_{uu\lambda}[u_x]_1^2+2f_{uu}[u_x]_1[u_{x\lambda^0}]_1+f_{u\lambda}[u_{xx}]_2)$
$[u_{xxx}]_3$	$-A^{-1}(3\omega^0)(f_{uuu}[u_x]_1^3+3f_{uu}[u_x]_1[u_{xx}]_2)$
$[u_{xxxx}]_2$	$-A^{-1}(2\omega^0)(4f_{uuuu}[u_x]_1^3[u_x]_{-1}+6f_{uuu}([u_x]_1^2[u_{xx}]_0+2[u_x]_1[u_x]_{-1}[u_{xx}]_2)$ $+6f_{uu}[u_{xx}]_0[u_{xx}]_2+4f_{uu}([u_x]_1[u_{xxx}]_1+[u_x]_{-1}[u_{xxx}]_3))$

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