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AN INEXACT QP-BASED METHOD FOR NONLINEAR COMPLEMENTARITY PROBLEMS

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Abstract. We consider a quadratic programming-based method for nonlinear complementarity problems which allows inexact solutions of the quadratic subproblems. The main features of this method are that all iterates stay in the feasible set and that the method has some strong global and local convergence properties. Numerical results for all complementarity problems from the MCPLIB test problem collection are also reported.

Key words. Complementarity problems, semismoothness, quadratic programs, global convergence, quadratic convergence.

1 Introduction

Let $F : X \to \mathbb{R}^n$ be a continuously differentiable function defined on an open set $X \subseteq \mathbb{R}^n$ containing the nonnegative orthant $\mathbb{R}^n_+$. The *nonlinear complementarity problem*, NCP(F) for short, is to find a solution of the following system of equations and inequalities:

$$x_i \geq 0, F_i(x) \geq 0, x_i F_i(x) = 0 \quad \forall i \in I := \{1, \ldots, n\}.$$

This problem arises frequently in many different areas like in operations research, in the engineering sciences or in economic equilibrium problems. The reader is referred to the recent survey paper [12] by Ferris and Pang for a brief description of many of these applications.

Most algorithms for the solution of the nonlinear complementarity problem are based on a suitable reformulation of this problem as a nonlinear system of equations, as a minimization problem or as a parametric problem, see, e.g., the overview [11] given by Ferris and Kanzow.

One of the most popular reformulations is based on the Fischer function

$$\varphi(a, b) := \sqrt{a^2 + b^2} - a - b$$

which was introduced in 1992 in the paper [15] by Fischer where it is attributed to a discussion with Burmeister. It is easily seen that this function has the following property:

$$\varphi(a, b) = 0 \Longleftrightarrow a \geq 0, b \geq 0, ab = 0.$$

Hence the nonlinear complementarity problem is equivalent to the constrained system of equations

$$\Phi(x) = 0, \;\; x \in X, \tag{1}$$

where $\Phi : X \to \mathbb{R}^n$ is defined by

$$\Phi(x) := \begin{pmatrix} \varphi(x_1, F_1(x)) \\ \vdots \\ \varphi(x_n, F_n(x)) \end{pmatrix},$$

or, alternatively, to the constrained minimization problem

$$\min \; \Psi(x) \quad \text{s.t.} \quad x \in X, \tag{2}$$

where $\Psi : X \to \mathbb{R}$ denotes the natural merit function

$$\Psi(x) := \frac{1}{2}\Phi(x)^T \Phi(x) = \frac{1}{2}\|\Phi(x)\|^2.$$

In the case where F is defined on the whole space $\mathbb{R}^n$, so that we can take $X = \mathbb{R}^n$, the reformulations (1) and (2) are actually unconstrained. In fact, for this case there exist a number of both efficient and robust algorithms for solving problem NCP(F) by using the Fischer function, see, e.g., [10, 5, 9, 22] and references therein.

However, in many applications the function F is only defined in a neighbourhood of the nonnegative orthant $\mathbb{R}^n_+$. Although many of these problems can still be solved by the unconstrained methods mentioned before just by incorporating some heuristic strategies (see, e.g., [7, 23]), they are obviously less robust, have no theoretical foundation and may run into numerical difficulties when applied to these kind of complementarity problems.

The aim of this paper is to describe an algorithm which avoids these difficulties by forcing all iterates to stay in the nonnegative orthant $\mathbb{R}^n_+$ and which, basically, has the same strong global and local convergence properties as the unconstrained methods based on the Fischer function. In contrast to these unconstrained methods, however, our subproblems are now quadratic programs instead of linear systems of equations. In order to improve the efficiency of the algorithm to be described in this paper, we allow inexact solutions of the quadratic subproblems.

The main motivation for this work comes from the earlier paper [8] by Facchinei et al. which investigates the theoretical convergence properties of a QP-based method for the solution of variational inequalities. Basically, the algorithm to be described here is a specialization of the one given in [8] to the framework of complementarity problems. However, in contrast to [8] and as said before, we allow inexact solutions of our QP-subproblems, with the inexactness being measured in a similar way as described by Gabriel and Pang [18] for their inexact version of the NE/SQP method from [31]. Therefore, the theoretical part of this paper exploits several times results from [8, 18]. In addition, we present extensive numerical results for our inexact QP-based method.

The paper is organized as follows: In Section 2, we present some background material on the equation operator Φ and the corresponding merit function Ψ. In Section 3, we then state our algorithm and show that it is well-defined as well as globally and locally fast convergent under mild assumptions. Some methods which can be used in order to solve our QP-subproblems inexactly are investigated in Section 4. In Section 5, we report our numerical results, and in Section 6 we conclude with some final remarks.

A few words about our notation: The Euclidian space is denoted by $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then $x \geq 0$ ($x > 0$) means that all components x_i of the vector x are nonnegative (positive). Similarly, expressions like $\min\{x, y\}$ or $\max\{0, x\}$ for vectors $x, y \in \mathbb{R}^n$ are defined componentwise. The nonnegative orthant $\{x \in \mathbb{R}^n \,|\, x \geq 0\}$ is abbreviated by $\mathbb{R}^n_+$. The positive orthant $\{x \in \mathbb{R}^n \,|\, x > 0\}$ is denoted by $\mathbb{R}^n_{++}$. We say that $G : X \to \mathbb{R}^n$ is a C^1 mapping (on $X \subseteq \mathbb{R}^n$) if G is continuously differentiable on X, and an LC^1 mapping if G is continuously differentiable with a locally Lipschitzian Jacobian. The Jacobian of a C^1 function G at a point $x \in \mathbb{R}^n$ will be denoted by $G'(x)$. Finally, we will make extensive use of the Landau symbols $o(\cdot)$ and $O(\cdot)$: If $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences of positive numbers, then $\alpha_k = O(\beta_k)$ if $\limsup_{k\to\infty} \alpha_k/\beta_k < +\infty$, and $\alpha_k = o(\beta_k)$ if $\lim_{k\to\infty} \alpha_k/\beta_k = 0$ for $\beta_k \to 0$.

2 Mathematical Background

In this section, we recall some properties of the functions Φ and Ψ which will be used in the convergence theory of our algorithm.

To this end, we first note that Φ is locally Lipschitz-continuous. Hence, by Rademacher's Theorem [35], it is differentiable almost everywhere. Let us denote by D_Φ the set of all points at which Φ is differentiable. Then Clarke's [3] *generalized Jacobian* of Φ at x, namely

$$\partial\Phi(x) := \mathrm{conv}\{H \,|\, \exists \{x^k\} \subseteq D_\Phi : x^k \to x \text{ and } \Phi'(x^k) \to H\},$$

where $\mathrm{conv}\mathcal{A}$ denotes the convex hull of a set $\mathcal{A}$, is a nonempty, convex and compact set [3].

Our first result is based on the fact that the operator Φ is (strongly) semismooth under certain assumptions. Since we will only need this result and not the definition of (strong) semismoothness, we do not restate this definition here. The interested reader is referred to the papers [27, 34, 33, 17] for some background material on (strong) semismoothness as well as to [10, 17] for a proof of the following Proposition.

Proposition 2.1 *The following statements hold:*

(a) It holds that

$$\|\Phi(x+d) - \Phi(x) - Hd\| = o(\|d\|) \quad \textit{for} \quad d \to 0 \textit{ and } H \in \partial\Phi(x+d).$$

(b) If F is an LC^1 mapping, then

$$\|\Phi(x+d) - \Phi(x) - Hd\| = O(\|d\|^2) \quad \textit{for} \quad d \to 0 \textit{ and } H \in \partial\Phi(x+d).$$

Now, let x^* be any given solution of $\mathrm{NCP}(F)$, and define the index sets

$$\begin{aligned} \alpha &:= \{i \in I \,|\, x_i^* > 0 = F_i(x^*)\}, \\ \beta &:= \{i \in I \,|\, x_i^* = 0 = F_i(x^*)\}, \\ \gamma &:= \{i \in I \,|\, x_i^* = 0 < F_i(x^*)\}. \end{aligned}$$

Then x^* is said to be an *R-regular* solution of $\mathrm{NCP}(F)$ if the submatrix $F'(x^*)_{\alpha\alpha}$ is nonsingular and the Schur-complement

$$F'(x^*)_{\beta\beta} - F'(x^*)_{\beta\alpha} F'(x^*)_{\alpha\alpha}^{-1} F'(x^*)_{\alpha\beta} \in \mathbb{R}^{|\beta|\times|\beta|}$$

is a P-matrix, i.e., has positive principal minors, see Robinson [36].

Using the notion of an R-regular solution, we have the following result whose proof follows from results in [10, 34, 32].

Proposition 2.2 *Let $x^* \in \mathbb{R}^n$ be an R-regular solution of NCP(F). Then the following statements hold:*

(a) *There are numbers* $c_1 > 0$ *and* $\delta_1 > 0$ *such that the matrices* $H \in \partial\Phi(x)$ *are nonsingular with*

$$\|H^{-1}\| \le c_1$$

for all x *such that* $\|x - x^*\| \le \delta_1$.

(b) *There are numbers* $c_2 > 0$ *and* $\delta_2 > 0$ *such that*

$$\|\Phi(x)\| \ge c_2\|x - x^*\|$$

for all x *with* $\|x - x^*\| \le \delta_2$.

We also recall an interesting result for the merit function Ψ. Its proof can be found in [10] (see also [20] for the differentiability statement).

Proposition 2.3 *The function* Ψ *is continuously differentiable everywhere with* $\nabla\Psi(x) = H^T\Phi(x)$ *for an arbitrary* $H \in \partial\Phi(x)$.

We finally recall a result which follows from the above-mentioned semismoothness of the operator Φ, from Proposition 2.2 (a), and from a general result by Facchinei and Soares [10].

Proposition 2.4 *Let* $x^* \in \mathbb{R}^n$ *be an R-regular solution of NCP(F). Suppose that there are two sequences* $\{x^k\}$ *and* $\{d^k\}$ *such that* $x^k \neq x^*$,

$$x^k \to x^*, \qquad \textit{and} \qquad \|x^k + d^k - x^*\| = o(\|x^k - x^*\|).$$

Then

$$\|\Phi(x^k + d^k)\| = o(\|\Phi(x^k)\|).$$

3 Inexact QP-based Algorithm

3.1 Statement of Algorithm

We want to solve the nonlinear complementarity problem NCP(F) by solving the constrained system of equations

$$\Phi(x) = 0, \quad x \ge 0, \tag{3}$$

or, equivalently, the constrained minimization problem

$$\min\ \Psi(x) \quad \text{s.t.} \quad x \ge 0. \tag{4}$$

The algorithm to be presented below is an iterative method which, basically, solves at each iteration a linearization of problem (3) in a least squares sense, i.e., the subproblem which has to be solved is basically the following one:

$$\min\ \frac{1}{2}\|\Phi(x^k) + H_k d\|^2 \quad \text{s.t.} \quad x^k + d \ge 0, \tag{5}$$

where $x^k \geq 0$ denotes the current iterate, $H_k \in \partial\Phi(x^k)$, and $d \in \mathbb{R}^n$ is the variable in which the minimization is carried out. Recalling that $\nabla\Psi(x^k) = H_k^T\Phi(x^k)$ for an arbitrary $H_k \in \partial\Phi(x^k)$ (cf. Proposition 2.3), it is easy to see that the objective function in (5) can be rewritten as follows:

$$\frac{1}{2}\|\Phi(x^k) + H_k d\|^2 = \Psi(x^k) + \nabla\Psi(x^k)^T d + \frac{1}{2} d^T H_k^T H_k d.$$

Therefore, neglecting the constant term $\Psi(x^k)$, the constrained linear least squares problem (5) is equivalent to

$$\min \ \nabla\Psi(x^k)^T d + \frac{1}{2} d^T H_k^T H_k d \quad \text{s.t.} \quad x^k + d \geq 0. \tag{6}$$

In order to prove a global convergence result, we replace this problem by a slightly perturbed one. To this end, we introduce a *forcing function* $\rho : \mathbb{R} \to \mathbb{R}_+$, i.e., a function with nonnegative values which is equal to 0 if and only if its argument is 0. We now replace (6) by the following quadratic programming problem which will be denoted by QP(x^k, H_k) throughout this paper:

$$\min \ \nabla\Psi(x^k)^T d + \frac{1}{2} d^T \left(H_k^T H_k + \rho(\Psi(x^k))I\right) d \quad \text{s.t.} \quad x^k + d \geq 0. \tag{7}$$

The basic idea of our algorithm is to iteratively solve this quadratic program and to globalize this procedure by using a simple line search in order to minimize the merit function Ψ. However, in contrast to the approach in [8], we allow inexact solutions of QP(x^k, H_k). This is highly important especially for large-scale problems. Moreover, inexact algorithms often have a better global convergence in practical computations than their exact counterparts.

In order to measure the inexactness of a solution of (7), we follow Gabriel and Pang [18] and reformulate the quadratic program QP(x^k, H_k) as a linear complementarity problem. To this end we first note that the (strictly) convex quadratic program (7) is equivalent to its KKT conditions. By setting

$$s := x^k + d, \tag{8}$$
$$M_k := H_k^T H_k + \rho(\Psi(x^k))I, \tag{9}$$
$$q^k := \nabla\Psi(x^k) - M_k x^k, \tag{10}$$

it is easy to see that these KKT conditions constitute exactly the linear complementarity problem

$$s \geq 0, M_k s + q^k \geq 0, s^T(M_k s + q^k) = 0.$$

Note that M_k is positive definite at least if x^k is not a solution of NCP(F) since $\rho(\Psi(x^k)) > 0$ in this case. The following result is a specialization of a more general Theorem by Pang [30] and shows that for the positive definite linear complementarity problem (and hence also for QP(x^k, H_k)), there is a nice and computable measure for the inexactness of an approximate solution of the underlying linear complementarity problem.

Lemma 3.1 *Let $M \in \mathbb{R}^{n\times n}$ be positive definite and $q \in \mathbb{R}^n$. Let $s^* \in \mathbb{R}^n$ denote the unique solution of LCP(q, M). Then there is a constant $c_3 > 0$ such that*

$$\|s - s^*\| \leq c_3 \| \min\{s, Ms + q\}\|$$

for all $s \in \mathbb{R}^n$.

We note that there are stronger results in the literature, see, e.g., Mangasarian and Ren [26], however, in our context we only need the above Lemma.

We are now in the position to give a precise description of our algorithm.

Algorithm 3.2 *(Inexact QP-based Semismooth Algorithm)*

(S.0) (Initial Data)
Choose $x^0 \geq 0, \sigma \in (0,1), \beta \in (0,1)$, and set $k := 0$.

(S.1) (Termination Criterion)
If x^k is a KKT point of (4): Stop.

(S.2) (Search Direction Calculation)
Choose $\varepsilon_k \geq 0$. Calculate a vector $d^k \in \mathbb{R}^n$ such that

(a) $\nabla \Psi(x^k)^T d^k < 0$ *(descent condition),*

(b) $x^k + d^k \geq 0$ *(feasibility condition),*

(c) $\| \min\{s^k, M_k s^k + q^k\}\| \leq \varepsilon_k$ *(inexactness condition),*

where $s^k := x^k + d^k$ and M_k, q^k are defined in (9) and (10), respectively.

(S.3) (Line Search)
Let $t_k := \max\{\beta^\ell \,|\, \ell = 0, 1, 2, \ldots\}$ such that

$$\Psi(x^k + t_k d^k) \leq (1 - \sigma t_k^2)\Psi(x^k).$$

(S.4) (Update)
Set $x^{k+1} := x^k + t_k d^k, k \leftarrow k + 1$, and go to (S.1).

Note that conditions (a)–(c) in Step (S.2) are very similar to those used by Gabriel and Pang [18] in their inexact NE/SQP method.

In the following convergence analysis, we will implicitly assume that Algorithm 3.2 generates an infinite sequence $\{x^k\}$, i.e., that there is no finite iteration index k such that x^k is a KKT point of (4).

The next two results show that Algorithm 3.2 is well-defined. We first prove that we can always compute a vector d^k such that conditions (a)–(c) of Step (S.2) in Algorithm 3.2 are satisfied.

Proposition 3.3 *As long as x^k is not a KKT point of (4), there is a direction vector $d^k \in \mathbb{R}^n$ satisfying conditions (a)–(c) of Step (S.2) in Algorithm 3.2.*

Proof. Let $\hat{d}^k \in \mathbb{R}^n$ be the exact solution of $\mathrm{QP}(x^k, H_k)$. Then, obviously, $x^k + \hat{d}^k \geq 0$ and $\|\min\{\hat{s}^k, M_k\hat{s}^k + q^k\}\| = 0$, where $\hat{s}^k := x^k + \hat{d}^k$. Moreover, it can be shown as in the proof of Theorem 4.4 in [8] that $\nabla\Psi(x^k)^T\hat{d}^k < 0$. Hence the vector $d^k := \hat{d}^k$ has all desired properties. □

We next show that it is also possible to find, at each iteration, a positive stepsize $t_k > 0$ such that the line search rule in Step (S.3) of Algorithm 3.2 is satisfied. We note that in the proof of this result, it is crucial that we use a squared stepsize on the right-hand side of the line search rule.

Proposition 3.4 *At each iteration k, there is a finite exponent $\ell_k \geq 0$ such that $t_k = \beta^{\ell_k} > 0$ satisfies the line search test in Step (S.3) of Algorithm 3.2.*

Proof. Assume there is an index k such that the test in Step (S.3) is not satisfied for any exponent $\ell \geq 0$, i.e.,

$$\frac{\Psi(x^k + \beta^\ell d^k) - \Psi(x^k)}{\beta^\ell} > -\sigma\beta^\ell\Psi(x^k)$$

for all $\ell \geq 0$. Taking the limit $\ell \to \infty$, we obtain from the continuous differentiability of the merit function Ψ (cf. Proposition 2.3) that

$$\nabla\Psi(x^k)^T d^k \geq 0.$$

This, however, contradicts the choice of d^k in Step (S.2) of Algorithm 3.2. □

From the previous two Propositions, it follows immediately that Algorithm 3.2 is well-defined. Moreover, in view of the very definition of Algorithm 3.2, we see that all iterates x^k are feasible for the complementarity problem $\mathrm{NCP}(F)$, i.e., $\{x^k\} \subseteq \mathbb{R}^n_+$.

3.2 Global Convergence

In this subsection, we will show that the Inexact QP-based Semismooth Algorithm 3.2 is globally convergent in the sense that every limit point x^* of a sequence generated by this algorithm is at least a KKT point of (4), i.e., satisfies the following conditions:

$$x^* \geq 0, \nabla\Psi(x^*) \geq 0, (x^*)^T\nabla\Psi(x^*) = 0. \tag{11}$$

A mild assumption which guarantees that such a KKT point is already a solution of $\mathrm{NCP}(F)$ is given in the following result which follows immediately from Fischer [16].

Theorem 3.5 *Assume that $x^* \in \mathbb{R}^n$ is a KKT point of (4) such that the Jacobian $F'(x^*)$ is a P_0-matrix, i.e., has nonnegative principal minors. Then x^* is a solution of NCP(F).*

In order to prove our global convergence theorem, we also restate the following result which is due to Gabriel and Pang [18]. Note that this result is a relatively simple consequence of Lemma 3.1.

Lemma 3.6 *Consider LCP(q^k, M_k) where each M_k is symmetric. Suppose that the sequence $\{M_k\}$ is bounded and uniformly positive definite, i.e.,*

$$x^T M_k x \geq \alpha x^T x$$

for some $\alpha > 0$ and for all $x \in \mathbb{R}^n$ and all k. Then there is a constant $c_4 > 0$ such that

$$\|s - \hat{s}^k\| \leq c_4 \| \min\{s, M_k s + q^k\}\|$$

for all k and for all $s \in \mathbb{R}^n$, where $\hat{s}^k$ denotes the unique solution of LCP(q^k, M_k).

We are now in the position to prove the main result of this subsection.

Theorem 3.7 *Suppose that $\{\varepsilon_k\} \to 0$. Then every limit point of the sequence $\{x^k\}$ generated by Algorithm 3.2 is a KKT point of (4).*

Proof. The sequence $\{\Psi(x^k)\}$ is obviously decreasing and bounded from below by 0. Hence it converges to a nonnegative value Ψ^*. If $\Psi^* = 0$, then, using a simple continuity argument, it follows that every limit point of the sequence $\{x^k\}$ is a global solution and therefore also a KKT point of (4).

So consider the case $\Psi^* > 0$. Let x^* denote a limit point of $\{x^k\}$, say $\{x^k\}_{K_1} \to x^*$. Let $\{d^k\}_{K_1}$ be the corresponding subsequence of directions generated by Algorithm 3.2, and let $\{\hat{d}^k\}_{K_1}$ be the sequence of exact solutions of the quadratic subproblems QP(x^k, H_k), $k \in K_1$. As in the proof of Theorem 5.2 in [8], it can be shown that there is a subsubsequence $\{\hat{d}^k\}_{K_2}, K_2 \subseteq K_1$, such that

— $\{\hat{d}^k\}_{K_2} \to d^*$ for some $d^* \in \mathbb{R}^n$,

— d^* is a solution of QP(x^*, H_*) for an element $H_* \in \partial\Phi(x^*)$,

— $d^* = 0$.

Now, in view of the upper semicontinuity of Clarke's generalized Jacobian ([3, Proposition 2.6.2 (c)]), the sequence of matrices $\{M_k\}_{K_2}$ is bounded, where M_k is the matrix defined in (9). Moreover, each M_k is obviously symmetric, and, since $\Psi^* > 0$, the sequence $\{M_k\}_{K_2}$ is also uniformly positive definite. Hence we can apply Lemma 3.6. To this end, let us write $s^k := x^k + d^k$ and $\hat{s}^k := x^k + \hat{d}^k$. Then $s^k, \hat{s}^k$ are inexact/exact solutions of the corresponding linear complementarity problem LCP(q^k, M_k), and we obtain from Lemma 3.6 as well as the inexactness condition (c) from Step (S.2) of Algorithm 3.2:

$$\|d^k - \hat{d}^k\| = \|s^k - \hat{s}^k\| \le c_4 \| \min\{s^k, M_k s^k + q^k\}\| \le c_4 \varepsilon_k \to 0. \tag{12}$$

Therefore, the sequence $\{d^k\}_{K_2}$ also converges to $d^* = 0$. Since, on the other hand, $d^* = 0$ is a solution of $\mathrm{QP}(x^*, H_*)$, it follows that $s^* := x^* + d^* = x^*$ is a solution of the corresponding linear complementarity problem

$$\begin{array}{rcl} s & \ge & 0, \\ (H_*^T H_* + \rho(\Psi^*) I)\, s + \nabla \Psi(x^*) - (H_*^T H_* + \rho(\Psi^*) I)\, x^* & \ge & 0, \\ s^T \left[(H_*^T H_* + \rho(\Psi^*) I)\, s + \nabla \Psi(x^*) - (H_*^T H_* + \rho(\Psi^*) I)\, x^* \right] & = & 0, \end{array}$$

so it follows that

$$x^* \ge 0, \nabla \Psi(x^*) \ge 0, (x^*)^T \nabla \Psi(x^*) = 0,$$

i.e., x^* is a KKT point of (4). □

3.3 Rate of Convergence

In this subsection, we show that, under the R-regularity condition and a suitable assumption on the sequence $\{\varepsilon_k\}$, the sequence $\{x^k\}$ generated by the Inexact QP-based Semismooth Algorithm 3.2 is locally superlinearly and even quadratically convergent.

In order to prove the main rate of convergence result, however, we need a couple of Lemmas.

Lemma 3.8 *Assume that $x^* \in \mathbb{R}^n$ is an R-regular solution of NCP(F), and suppose that $\varepsilon_k = O(\|\Phi(x^k)\|)$. Then there is a constant $c_5 > 0$ such that*

$$\|d^k\| \le c_5 \|\Phi(x^k)\|$$

for all $x^k \ge 0$ sufficiently close to x^*, where d^k denotes the direction vector generated in Step (S.2) of the Inexact QP-based Semismooth Algorithm 3.2.

Proof. Let $\hat{d}^k$ denote the exact solution of $\mathrm{QP}(x^k, H_k)$, where $H_k \in \partial \Phi(x^k)$. Since x^* is an R-regular solution of $\mathrm{NCP}(F)$, we can apply Proposition 2.2 and show in a similar way to Lemma 5.3 in [8] that, for all $x^k \ge 0$ sufficiently close to x^*,

$$\|\hat{d}^k\| \le \kappa_1 \|\Phi(x^k)\|$$

for some constant $\kappa_1 > 0$. On the other hand, by our assumption, there is a constant $\kappa_2 > 0$ such that

$$\varepsilon_k \le \kappa_2 \|\Phi(x^k)\|.$$

Taking into account the inequality (12), we therefore obtain

$$\|d^k\| \leq \|\hat{d}^k\| + \|d^k - \hat{d}^k\| \leq \kappa_1 \|\Phi(x^k)\| + c_4 \varepsilon_k \leq c_5 \|\Phi(x^k)\|$$

with $c_5 := \kappa_1 + c_4 \kappa_2$. □

In order to prove a sequential convergence result, we also need the following Proposition due to Moré and Sorensen [28].

Proposition 3.9 *Assume that $x^* \in \mathbb{R}^n$ is an isolated accumulation point of a sequence $\{x^k\} \subseteq \mathbb{R}^n$ such that $\{\|x^{k+1} - x^k\|\}_K \to 0$ for any subsequence $\{x^k\}_K$ converging to x^*. Then the whole sequence $\{x^k\}$ converges to x^*.*

Based on the previous two results, we are now in the position to prove a sequential convergence property.

Lemma 3.10 *Let $x^* \in \mathbb{R}^n$ be an R-regular solution of NCP(F), and assume that x^* is an accumulation point of a sequence $\{x^k\}$ generated by the Inexact QP-based Semismooth Algorithm 3.2. Then the entire sequence $\{x^k\}$ converges to x^*.*

Proof. It is well-known that any R-regular solution is a locally isolated solution of NCP(F) and hence necessarily an isolated accumulation point of the sequence $\{x^k\}$ (cf. the proof of Theorem 4.2 in [23]). So let $\{x^k\}_K$ be a subsequence converging to x^*. Then, by continuity, we have

$$\{\|\Phi(x^k)\|\}_K \to \|\Phi(x^*)\| = 0.$$

Therefore, Lemma 3.8 implies

$$\{\|d^k\|\}_K \to 0.$$

Since

$$\|x^{k+1} - x^k\| = t_k \|d^k\| \leq \|d^k\|$$

for all k, we thus obtain

$$\{\|x^{k+1} - x^k\|\}_K \to 0.$$

Hence the entire sequence $\{x^k\}$ converges to x^* because of Proposition 3.9. □

The next Lemma is crucial in order to prove a local superlinear or quadratic rate of convergence of a sequence generated by the Inexact QP-based Semismooth Algorithm 3.2.

Lemma 3.11 *Assume that $x^* \in \mathbb{R}^n$ is an R-regular solution of NCP(F), let $\{x^k\}$ be any sequence of iterates (not necessarily generated by Algorithm 3.2) converging to x^*, and let $\{d^k\}$ be the sequence of direction vectors satisfying conditions (a)–(c) of Step (S.2) in Algorithm 3.2. Then the following statements hold:*

(a) If $\varepsilon_k = o(\|\Phi(x^k)\|)$*, then*

$$\|x^k + d^k - x^*\| = o(\|x^k - x^*\|).$$

(b) If $\varepsilon_k = O(\|\Phi(x^k)\|^2), \rho(\tau) = O(\tau)$*, and if the Jacobian of* F *is locally Lipschitz continuous around* x^**, then*

$$\|x^k + d^k - x^*\| = O(\|x^k - x^*\|^2),$$

where ρ *denotes the forcing function used in* $QP(x^k, H_k)$*.*

Proof. First consider part (a). Again, we denote by $\hat{d}^k$ the exact solution of $\mathrm{QP}(x^k, H_k)$. Using Proposition 2.1, it can be shown in a similar way to Lemma 5.4 in [8] that

$$\|x^k + \hat{d}^k - x^*\| = o(\|x^k - x^*\|).$$

Hence it follows from (12) that

$$\begin{aligned}
\|x^k + d^k - x^*\| &\leq \|x^k + \hat{d}^k - x^*\| + \|d^k - \hat{d}^k\| \\
&= o(\|x^k - x^*\|) + c_4 \varepsilon_k \\
&= o(\|x^k - x^*\|) + o(\|\Phi(x^k)\|).
\end{aligned}$$

Since Φ is locally Lipschitz continuous, there is a constant $L > 0$ such that

$$\|\Phi(x^k)\| = \|\Phi(x^k) - \Phi(x^*)\| \leq L\|x^k - x^*\|$$

for all x^k sufficiently close to x^*. Combining the last two inequalities, we obtain

$$\|x^k + d^k - x^*\| = o(\|x^k - x^*\|),$$

which proves part (a).

Part (b) can be verified in a very similar way, so we omit the details here. □

The following result shows that the full stepsize will be taken whenever we are sufficiently close to an R-regular solution of $\mathrm{NCP}(F)$.

Lemma 3.12 *Assume that* $x^* \in \mathbb{R}^n$ *is an R-regular solution of NCP(F), let* $\{x^k\}$ *be any sequence of iterates (not necessarily generated by Algorithm 3.2) converging to* x^**, and let* $\{d^k\}$ *be the sequence of direction vectors satisfying conditions (a)–(c) of Step (S.2) in Algorithm 3.2 with* $\varepsilon_k = o(\|\Phi(x^k)\|)$*. Then the full step* $t_k = 1$ *will be accepted by the line search rule in Step (S.3) of the Inexact QP-based Semismooth Algorithm 3.2 for all* k *sufficiently large.*

Proof. In view of Proposition 2.4 and Lemma 3.11, we have

$$\frac{\|\Phi(x^k + d^k)\|}{\|\Phi(x^k)\|} \to 0$$

and therefore also

$$\frac{\Psi(x^k + d^k)}{\Psi(x^k)} \to 0$$

for $x^k \to x^*$. Hence the condition

$$\Psi(x^k + d^k) \leq (1 - \sigma)\Psi(x^k)$$

will be satisfied for all k sufficiently large, i.e., Algorithm 3.2 accepts $t_k = 1$ for all x^k sufficiently close to x^*. □

We now summarize our previous results and state our main local convergence Theorem for Algorithm 3.2.

Theorem 3.13 *Let $x^* \in \mathbb{R}^n$ be an accumulation point of a sequence $\{x^k\}$ generated by the Inexact QP-based Semismooth Algorithm 3.2, and let $\varepsilon_k = o(\|\Phi(x^k)\|)$. Assume that x^* is an R-regular solution of NCP(F). Then the following statements hold:*

(a) The entire sequence $\{x^k\}$ converges to x^*.

(b) Eventually the full stepsize $t_k = 1$ is accepted.

(c) The rate of convergence is Q-superlinear.

(d) The rate of convergence is Q-quadratic if, in addition, the assumptions of Lemma 3.11 (b) are satisfied.

Proof. Part (a) follows immediately from Lemma 3.10. Statement (b) is therefore a consequence of Lemma 3.12, so that the assertions about the rate of convergence follow directly from Lemma 3.11. □

4 Inexact Solution of QP-Subproblems

In this subsection, we address the problem of how to compute an inexact solution of the QP-subproblems, i.e., of how to compute a vector $d^k \in \mathbb{R}^n$ satisfying the three conditions (a)–(c) of Step (S.2) in the Inexact QP-based Semismooth Algorithm 3.2.

Since the quadratic program QP(x^k, H_k) is strictly convex or, equivalently, the corresponding linear complementarity problem LCP(q^k, M_k) is positive definite, there are many possible algorithms. However, since we are also interested in the solution of large-scale problems, we are looking for an iterative method whose iterations are relatively cheap to carry out. Moreover, the algorithm of choice has to guarantee the feasibility condition at each intermediate step.

We will discuss three possible methods for finding a direction $d^k \in \mathbb{R}^n$ satisfying conditions (a)–(c) of Step (S.2). These algorithms are based on the equivalent LCP-formulation of QP(x^k, H_k). The first method is a projection-type method, and the second algorithm is a splitting method. Both algorithms are known to be globally convergent to the unique solution of LCP(q^k, M_k) under our assumptions. Furthermore, it is known that these algorithms possess a *local* R-linear rate of convergence under certain assumptions, see, e.g., Tseng [37] as well as Luo and Tseng [25]. One of the main aims of this subsection is to show that these algorithms are actually *globally* Q-linearly convergent in our context.

The third method to be discussed here is an interior-point method which, in contrast to the projection-type and splitting methods, is also locally fast convergent at least in practice.

Throughout this Section, we assume that $M \in \mathbb{R}^{n\times n}$ is a given symmetric and positive definite matrix and that $q \in \mathbb{R}^n$ is an arbitrary vector.

4.1 Projection-type Methods

The algorithm we consider in this Subsection is the so-called *Extragradient Method*. This is a projection-type algorithm which can be applied to general monotone (even pseudomonotone) variational inequalities. Here we only give a precise description of this method when applied to the linear complementarity problem LCP(q, M). (The interested reader may consult the book [29] by Nagurney for some more details.)

Algorithm 4.1 *(Extragradient Method for LCP(q, M))*

(A) Choose $s^0 \geq 0, \gamma \in (0, 1/\|M\|)$, and set $i := 0$.

(B) If s^i satisfies a suitable termination criterion, stop.

(C) Compute

$$\begin{aligned} \bar{s}^i &:= \max\{0, s^i - \gamma(Ms^i + q)\}, \\ s^{i+1} &:= \max\{0, s^i - \gamma(M\bar{s}^i + q)\}. \end{aligned}$$

(D) Set $i \leftarrow i + 1$, and go to (B).

In the following convergence result, we implicitly assume that Algorithm 4.1 generates an infinite sequence. We show that this sequence converges globally Q-linearly to the unique solution of LCP(q, M), where, we recall, M is assumed to be positive definite.

Theorem 4.2 *The sequence $\{s^i\}$ generated by Algorithm 4.1 converges globally Q-linearly to the unique solution s^* of LCP(q, M).*

Proof. From, e.g., formula (2.39) in [29] it follows that there is a constant $\kappa_3 > 0$ such that the iterates $\{s^i\}, \{\bar{s}^i\}$ generated by the Extragradient Method 4.1 satisfy the inequality

$$\|s^{i+1} - s^*\|^2 \leq \|s^i - s^*\|^2 - \kappa_3 \|s^i - \bar{s}^i\|^2 \tag{13}$$

for all i. Since

$$\begin{aligned} \|s^i - \bar{s}^i\| &= \|s^i - \max\{0, s^i - \gamma(Ms^i + q)\}\| \\ &= \|\min\{s^i, \gamma(Ms^i + q)\}\| \\ &\geq \min\{1, \gamma\} \|\min\{s^i, Ms^i + q\}\|, \end{aligned}$$

we obtain from (13) that

$$\|s^{i+1} - s^*\|^2 \leq \|s^i - s^*\|^2 - \kappa_4 \|\min\{s^i, Ms^i + q\}\|^2$$

with $\kappa_4 := \kappa_3 \min^2\{1, \gamma\}$. Using the global error bound result from Lemma 3.1, we finally get

$$\begin{aligned} \|s^{i+1} - s^*\|^2 &\leq \|s^i - s^*\|^2 - \kappa_4 / c_3^2 \|s^i - s^*\|^2 \\ &= \kappa_5 \|s^i - s^*\|^2 \end{aligned}$$

for $\kappa_5 := 1 - \kappa_4/c_3^2 \in (0, 1)$ and all i. This shows that the sequence $\{s^i\}$ converges globally Q-linearly to s^*. $\square$

4.2 Splitting Methods

Splitting methods for the solution of linear complementarity problems are well-studied iterative methods, see, e.g., the book [4] by Cottle, Pang, and Stone. The basic idea of splitting methods for solving LCP(q, M) is to split the matrix M in the form

$$M = B + C$$

with $B, C \in \mathbb{R}^{n \times n}$, and to solve the possibly difficult problem LCP(q, M) by iteratively solving easier complementarity problems LCP$(\bar{q}^i, B)$ for certain vectors $\bar{q}^i, i = 1, 2, \ldots$. A detailed description of a general splitting method is as follows:

Algorithm 4.3 *(Splitting Method for LCP(q, M))*

(A) Let $M = B + C$ be a splitting of M, choose $s^0 \geq 0$, and set $i := 0$.

(B) If s^i satisfies a suitable termination criterion, stop.

(C) Set $\bar{q}^i := Cs^i + q$. Compute a solution s^{i+1} of the linear complementarity problem LCP($\bar{q}^i, B$).

(D) Set $i \leftarrow i+1$, and go to (B).

The following is a convergence theorem for a special class of splitting methods. Again, we assume implicitly that the splitting method generates an infinite sequence.

Theorem 4.4 *Suppose that $B - M$ is symmetric positive definite. Then the sequence $\{s^i\}$ generated by Algorithm 4.3 is well-defined and converges globally Q-linearly to the unique solution s^* of LCP(q, M).*

Proof. Since M and $B - M$ are (symmetric) positive definite, it follows that $B = B - M + M$ is also (symmetric) positive definite. Hence the linear complementarity problem LCP$(\bar{q}^i, B)$ in Step (C) of Algorithm 4.3 has a unique solution for each $i = 0, 1, 2, \ldots$, i.e., the Splitting Method 4.3 is well-defined.

Moreover, it is known that, under the stated assumptions, the sequence $\{s^i\}$ converges to the unique solution s^* of LCP(q, M), see, e.g., Theorem 5.6.1 in [4]. In particular, it is shown in the proof of this result that the inequality

$$\|s^{i+1} - s^*\|^2 \leq \|s^i - s^*\|^2 - \|s^{i+1} - s^i\|^2$$

holds for all i. From Lemma 5.1 in [37], it follows that there is a constant $\kappa_6 > 0$ such that

$$\|s^{i+1} - s^i\|^2 \geq \kappa_6 \| \min\{s^i, Ms^i + q\}\|^2.$$

Consequently, we have

$$\|s^{i+1} - s^*\|^2 \leq \|s^i - s^*\|^2 - \kappa_6 \| \min\{s^i, Ms^i + q\}\|^2$$

for all i. Applying Lemma 3.1, the global Q-linear convergence of the iterates $\{s^i\}$ to s^* can now be established in exactly the same way as in the proof of Theorem 4.2. □

4.3 Interior-Point Methods

In this Subsection, we restate an (infeasible) interior-point method for the solution of the linear complementarity problem LCP(q, M). This method is restated here because we use it in our numerical experiments as iterative solver for our QP-subproblems in Algorithm 3.2.

In contrast to the projection-type and splitting methods described in the previous two subsections, interior-point methods are, usually, globally Q-linearly *and* locally Q-superlinearly convergent. Moreover, all iterates stay in the positive orthant, so that also these methods fit well into our algorithmic framework.

In order to give a precise description, let us define the operator $G : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$G(s) := G(s_x, s_y) := \begin{pmatrix} Ms_x + q - s_y \\ S_x s_y \end{pmatrix},$$

where $S_x \in \mathbb{R}^{n \times n}$ is the diagonal matrix with the vector s_x on its diagonal. Further, let $\bar{e} \in \mathbb{R}^{2n}$ be a vector whose first n components are zeros and whose last n components are ones.

Algorithm 4.5 *(Interior-Point Method for LCP(q, M))*

(A) *Choose* $s^0 := (s_x^0, s_y^0) \in \mathbb{R}^{2n}_{++}, \tau \in (0, 1)$, *and set* $i := 0$.

(B) *If* s^i *satisfies a suitable termination criterion, stop.*

(C) *Set* $\mu_i := (s_x^i)^T s_y^i / n$, *choose* $\sigma_i \in [0, 1]$, *and find* $\Delta s^i = (\Delta s_x^i, \Delta s_y^i)$ *by solving the linear system*

$$G'(s^i)\Delta s = -G(s^i) + \sigma_i \mu_i \bar{e}.$$

(D) *Compute*

$$\hat{t}_i := \min \left\{ 2, \min_{\{j \mid [\Delta s_x^i]_j < 0\}} \left\{ \frac{-[\Delta s_x^i]_j}{[s_x^i]_j} \right\}, \min_{\{j \mid [\Delta s_y^i]_j < 0\}} \left\{ \frac{-[\Delta s_y^i]_j}{[s_y^i]_j} \right\} \right\}$$

and set $t_i := \min\{1, \tau \hat{t}_i\}$.

(E) *Set* $s^{i+1} := s^i + t_i \Delta s^i, i \leftarrow i + 1$, *and go to (B).*

Algorithm 4.5 is a typical interior-point method although not the most sophisticated one. The interested reader is referred to, e.g., the books [24] by Kojima et al. and [38] by Wright for some further details on interior-point methods. Here we only stress that the performance of the relatively simple Algorithm 4.5 is already quite good in our numerical experiments.

5 Numerical Results

We implemented Algorithm 3.2 in MATLAB and tested it on a SUN Sparc 20 station. As test problems, we used all complementarity problems from the MCPLIB collection by Dirkse and Ferris [6], see also Ferris and Rutherford [13]. However, we excluded the two von Thünen problems `pgvon105` and `pgvon106` since their implementation within the MCPLIB library seems to be somewhat crucial [14].

Our algorithm terminates if one of the following conditions is satisfied:

$$k > k_{\max}, \quad \Psi(x^k) < \varepsilon, \quad \text{or} \quad t_k < t_{\min},$$

where
$$k_{\max} = 400, \;\; \varepsilon = 10^{-12} \quad \text{and} \quad t_{\min} = 10^{-16}.$$
We further used the following settings for the parameters involved in Algorithm 3.2:
$$\beta = 0.3 \quad \text{and} \quad \sigma = 10^{-4}.$$
Moreover, the forcing function used in our implementation as a regularization of the QP-subproblems is
$$\rho(\tau) := \min\{10^{-6}, \tau\},$$
where $\tau = \Psi(x)$. Hence, only a minor perturbation of the QP-subproblems is used here. Finally, we took
$$\varepsilon_k := \max\{10^{-13}, \min\{1, \Psi(x^k)\}\}$$
as our measure of inexactness for the QP-subproblems, where the lower bound 10^{-13} plays the role of a safeguard. In view of Theorem 3.13 and these choices of our parameters, we can expect a local quadratic convergence of our inexact QP-based semismooth algorithm.

For the solution of our QP-subproblems, we implemented the infeasible interior-point method 4.5 using $\tau = 0.99995$ and, at each outer iteration k, the starting vector
$$s_x^0 := \max\{10e, x^k\}, \;\; s_y^0 := s_x^0$$
for the inner iteration, where $e \in \mathbb{R}^n$ denotes the vector of all ones. Moreover, the upper bound
$$i_{\max} = 30$$
is used in order to limit the number of inner iterations which are allowed at each step. If the algorithm is not able to solve a subproblem $\mathrm{QP}(x^k, H_k)$ within $i_{\max}$ inner iterations up to the desired accuracy, we switch to a projected gradient step, see Gafni and Bertsekas [19] for more details. We already stress at this point, however, that projected gradient steps had to be taken only for a very few test problems.

We report our numerical results in Table 5.1 using the following data:

problem:	name of the test problem in MCPLIB
n:	number of variables
SP:	number of starting point provided by MCPLIB
k:	number of (outer) iterations
F-ev.:	number of F-evaluations
F'-ev.:	number of Jacobian evaluations
Ψ:	final value of $\Psi(x)$
$\|\nabla\Psi\|$:	final value of $\|\nabla\Psi(x)\|$
ls:	cumulated number of linear system solves
P:	number of projected gradient steps taken.

Table 5.1: Numerical results for Algorithm 3.2.

problem	n	SP	k	F-ev.	F'-ev.	Ψ	$\|\nabla\Psi\|$	ls	P
bertsekas	15	1	321	642	322	1.3e-13	3.2e-5	354	0
bertsekas	15	2	319	647	320	1.2e-13	3.1e-5	363	0
bertsekas	15	3	344	699	345	1.9e-13	3.5e-5	416	0
billups	1	1	—	—	—	—	—	—	—
colvdual	20	1	13	23	14	5.8e-14	1.9e-5	71	0
colvdual	20	2	17	23	18	5.0e-14	2.4e-6	91	0
colvnlp	15	1	13	23	14	5.7e-14	1.7e-5	52	0
colvnlp	15	2	10	12	11	3.3e-21	8.7e-9	33	0
cycle	1	1	4	6	5	4.1e-19	9.1e-10	4	0
explcp	16	1	5	6	6	2.8e-18	2.4e-9	31	0
hanskoop	14	1	12	19	13	6.5e-18	4.0e-8	37	0
hanskoop	14	2	12	17	13	2.9e-15	2.6e-7	39	0
hanskoop	14	3	10	17	11	2.5e-15	2.4e-7	36	0
hanskoop	14	4	13	21	14	3.0e-15	2.6e-7	40	0
hanskoop	14	5	16	22	17	2.8e-15	2.6e-7	79	0
josephy	4	1	7	10	8	2.5e-14	8.0e-7	17	0
josephy	4	2	8	12	9	1.2e-23	1.7e-11	16	0
josephy	4	3	8	10	9	3.9e-23	3.6e-11	16	0
josephy	4	4	5	6	6	3.4e-20	8.6e-10	8	0
josephy	4	5	4	5	5	1.0e-13	2.1e-6	7	0
josephy	4	6	9	15	10	6.2e-18	2.7e-8	15	0
kojshin	4	1	8	13	9	3.3e-14	1.9e-6	18	0
kojshin	4	2	7	11	8	5.1e-17	1.3e-7	15	0
kojshin	4	3	9	13	10	3.0e-21	1.9e-9	24	0
kojshin	4	4	4	5	5	8.5e-14	1.1e-6	9	0
kojshin	4	5	5	6	6	1.7e-15	4.3e-7	10	0
kojshin	4	6	7	10	8	4.3e-15	5.2e-7	16	0
mathinum	3	1	6	9	7	1.3e-15	1.4e-7	8	0
mathinum	3	2	5	6	6	9.0e-13	3.6e-6	5	0
mathinum	3	3	11	15	12	3.0e-19	2.1e-9	16	0
mathinum	3	4	7	8	8	1.1e-16	4.0e-8	10	0
mathisum	4	1	6	8	7	6.7e-13	2.7e-6	8	0
mathisum	4	2	7	8	8	7.6e-24	4.2e-12	9	0
mathisum	4	3	11	14	12	4.6e-21	2.3e-10	16	0
mathisum	4	4	6	7	7	2.7e-21	1.7e-10	8	0
powell	16	1	7	8	8	2.5e-18	3.7e-8	45	1
powell	16	2	9	10	10	1.5e-18	2.9e-8	22	0
powell	16	3	10	14	11	1.3e-18	1.4e-8	56	1
powell	16	4	9	10	10	2.7e-16	2.2e-7	29	0

Table 5.1 (continued): Numerical results for Algorithm 3.2

problem	n	SP	k	F-ev.	F'-ev.	Ψ	$\|\nabla\Psi\|$	ls	P
scarfanum	13	1	7	8	8	6.9e-17	6.0e-7	24	0
scarfanum	13	2	9	11	10	2.9e-19	7.1e-9	38	0
scarfanum	13	3	8	12	9	8.4e-16	4.3e-7	33	0
scarfasum	14	1	8	11	9	5.5e-18	6.8e-8	27	0
scarfasum	14	2	9	11	10	4.8e-17	2.0e-7	32	0
scarfasum	14	3	13	23	14	5.1e-18	6.6e-8	58	0
scarfbnum	39	1	25	32	26	2.7e-13	2.6e-4	135	1
scarfbnum	39	2	53	108	54	3.3e-15	3.0e-5	401	6
scarfbsum	40	1	18	29	19	5.4e-21	1.5e-8	112	0
scarfbsum	40	2	23	24	24	5.4e-21	1.5e-8	180	3
sppe	27	1	8	9	9	2.1e-22	2.0e-11	18	0
sppe	27	2	6	7	7	2.8e-20	8.0e-10	16	0
tobin	42	1	8	10	9	7.3e-24	7.6e-12	21	0
tobin	42	2	9	13	10	4.8e-21	5.9e-10	29	0

Note that the cumulated number of linear system solves is equal to the cumulated number of inner iterations, so the column "ls" also provides information on the number of inner iterations.

In view of Table 5.1, the overall behaviour of Algorithm 3.2 is very good. The method was able to solve all but one complementarity problems from MCPLIB. The only exception is problem `billups` which is a very academic example designed by Billups [1] in order to make almost all state-of-the-art algorithms to fail on this problem, see also the comparison paper [2] by Billups, Dirkse and Ferris.

Some problems also occured for the `bertsekas` test example, where the number of outer iterations is quite large. The reason is that the algorithm stays for about 300 iterations close to a point where $\|\nabla\Psi(x^k)\| \approx 10^{-7}$, so the iterates are almost stationary points. However, after getting away from this point, the convergence was very fast.

All other problems were solved without any problems although some of these other test examples are sometimes said to be difficult like `colvdual`, `hanskoop`, `scarfbnum` and `scarfbsum`. So the algorithm seems to be very robust.

Of course, there is a loss of efficiency if compared to the unconstrained methods based on the Fischer function. Nevertheless, also the cumulated number of inner iterations needed by Algorithm 3.2 is still of a reasonable size. Basically, this is due to the moderate size of inaccuracy with which the QP-subproblems are solved. We also believe that the number of inner iterations can be further reduced by using a more sophisticated implementation of an infeasible interior-point method.

6 Concluding Remarks

We introduced an inexact QP-based method for the solution of nonlinear complementarity problems. It was shown to have some strong global and local convergence properties, both theoretically and numerically. In particular, the method seems to be more robust on the difficult test problems than some related unconstrained algorithms. Further improvements regarding the efficiency of the algorithm may be obtained by taking a closer look at the structure of the QP-subproblems and by using a more appropriate QP-solver. We leave this as a future research topic.

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