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Eigenvalues of the Laplacian for Bianchi groups

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Abstract. We present results concerning the computation of Laplace eigenvalue spectra for certain arithmetic, discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ acting on the hyperbolic upper half space. These subgroups are $\mathrm{PSL}(2,\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D}\,)$. Special attention is devoted to the cases $D = 1, 2, 3, 7, 11$ having fundamental domains with one cusp. It is proved that the spectra are not simple. We sketch a method showing how such eigenvalues and associated eigenfunctions can be computed. Some of the eigenvalues are recognized as being lifts from the modular group $\mathrm{PSL}(2,\mathbb{Z})$. As an application to quantum chaos, we demonstrate that the spectra exhibit random fluctuations close to Poissonian. Samples of eigenvalues are listed.

Key words. Arithmetic quantum chaos, Bianchi groups, Picard group, modular group, hyperbolic 3-space, non-Euclidean Laplacian, Hecke operators, eigenvalues, multiplicities, liftings.

1. Introduction. One of the main focuses in quantum chaos has been the study of spectra of two-dimensional billiard systems. As early as 1924, Artin had introduced such a system [3] for which number theory can be viewed as the applied mathematics, both on the classical and on the quantum mechanical side.

He considered the classical motion of a point mass on the modular domain, that is on the fundamental domain of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ acting on the hyperbolic upper half plane. By introducing a symbolic dynamics via continued fractions he was able to show that the system is chaotic, namely what was called quasi-ergodic.

Studying the quantized version of the system amounts to the study of the spectrum of the associated non-Euclidean Laplacian $\Delta = y^2(\partial_x^2 + \partial_y^2)$ for $\mathrm{PSL}(2,\mathbb{Z})$ invariant functions.

According to the general picture, classically integrable systems show spectral fluctuations similar to a Poisson spectrum in which the distances of consecutive eigenvalues have a Poisson distribution, whereas the fluctuations of classically chaotic time-reversal invariant systems can be described by the Gaussian orthogonal ensemble (GOE) of random matrix theory [17].

Therefore it came as a surprise as one was able to show numerically that the discrete spectrum of $\mathrm{PSL}(2,\mathbb{Z})$ is close to a Poisson one [6,8,25]. This exceptional behaviour has its origin in the arithmetic structure of the underlying group. A survey of this situation of so-called arithmetic

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quantum chaos was given by Sarnak [24], see also [4,7].

In this article we study three dimensional analogues of the situation described. We want to gain some insight into the discrete spectra of the Laplacian for Bianchi groups $\mathrm{PSL}(2,\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of an imaginary quadratic number field. These are arithmetic groups acting discontinuously on the hyperbolic upper half space. The associated fundamental domains are not compact, but have a finite hyperbolic volume and may be interpreted as three-dimensional billiard domains with constant negative curvature on which the classical flow is completely chaotic [2].

The arithmetic structure gives rise to the existence of self-adjoint Hecke operators similar to those for the modular group. It will be shown that the Hecke operators do not commute with all of the symmetries of the fundamental domains which leads to multiple eigenvalues. Using these operators we describe how to construct, in the spirit of [26], a method to compute Laplace eigenvalues and eigenfunctions. The algorithm is applied to the number fields $\mathbb{Q}\left(\sqrt{-D}\right)$, $D = 1, 2, 3, 7, 11$, and as a result of the numerical experiments we are able to demonstrate that the Poissonian behaviour carries over to three dimensions. Since there are four different symmetry classes in each case, we have a sample of altogether twenty spectra containing between 100 and 900 levels each. There are certain eigenvalues occurring across the different number fields which will shown to be associated with eigenfunctions of the Laplacian for the modular group.

Previous calculations of small eigenvalues for $D = 1$ (Picard group) are due to Golovčanskiĭ & Smotrov [10], and Huntebrinker [12] presents finite element approximations of small eigenvalues for a whole bunch of domains in hyperbolic 3-space, among them $D = 1, 2, 3$.

2. Hyperbolic 3-space. Let $\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}^+$ denote the upper half space model of hyperbolic 3-space. Elements of $\mathbb{H}^3$ are written as $P = (z, r)$, $z = x + yi$, $r > 0$. Geometrically we think of r being the height and $\mathbb{C}$ placed horizontally. The metric on $\mathbb{H}^3$ is given by

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2} \, .$$

Hyperbolic planes are Euclidean half planes and half spheres, geodesics are half circles and half lines orthogonal to the lower boundary $\mathbb{C}$, respectively. With the metric comes the volume element

$$dv = \frac{dx\, dy\, dr}{r^3}$$

and the hyperbolic Laplacian

$$\Delta = r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r} \, .$$

The matrix group $\mathrm{SL}(2,\mathbb{C})$ operates as a transformation group on the upper half space. The action of an element of the group on $\mathbb{H}^3$ is defined by

$$(2.1)\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}(z,r) = \left(\frac{(az+b)\,\overline{(cz+d)} + a\bar{c}r^2}{|cz+d|^2 + |c|^2 r^2}, \frac{r}{|cz+d|^2+|c|^2r^2} \right).$$

Similarly, $\mathrm{GL}(2,\mathbb{C})$ operates on $\mathbb{H}^3$, the only difference to $\mathrm{SL}(2,\mathbb{C})$ being that in (2.1) one has to replace the last nominator r by $r\,|\det(M)|$, where $M \in \mathrm{GL}(2,\mathbb{C})$ is the given matrix.

The hyperbolic metric and hence the volume element and the Laplacian are invariant under the actions of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{GL}(2,\mathbb{C})$. The group of orientation preserving isometries $\mathrm{Iso}^+\left(\mathbb{H}^3\right)$ can be identified with $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})\,/\{\pm I\}$. For simplicity we will not distinguish between matrices of $\mathrm{SL}(2,\mathbb{C})$ and their pre-images in $\mathrm{PSL}(2,\mathbb{C})$.

3. Bianchi groups. Let $D > 0$ be a square-free rational integer, $\mathcal{O}$ the ring of integers of the imaginary quadratic number field $K = \mathbb{Q}\left(\sqrt{-D}\,\right)$, and $\mathcal{O}^\times$ its group of units. The elements of $\mathcal{O}$ are given by

$$\begin{array}{lll} a + b\sqrt{-D} & \text{for} & D \equiv 1,2 \bmod 4, \\[6pt] \dfrac{a+b\sqrt{-D}}{2} & \text{for} & D \equiv 3 \bmod 4, \quad a+b \text{ even}, \end{array}$$

where a, b are rational integers. $\mathcal{O}^\times$ is just $\{\pm 1\}$ except for the two cases $D = 1, 3$ where we have $\{\pm 1, \pm i\}$ and $\{\pm 1, \exp(\pm i\pi/3), \exp(\pm 2\pi i/3)\}$, respectively.

For any D the corresponding Bianchi group [5] is defined as $\Gamma = \mathrm{PSL}(2,\mathcal{O})$. Γ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and operates discontinuously on $\mathbb{H}^3$, that is for any $P \in \mathbb{H}^3$ the orbit ΓP does not contain accumulation points with respect to the hyperbolic metric. A fundamental domain $\mathcal{F} \cong \Gamma\backslash\,\mathbb{H}^3$, $\mathcal{F} \subset \mathbb{H}^3$, of Γ is a 'nice' set of representatives of the set of orbits $\Gamma\backslash\,\mathbb{H}^3$. It may be chosen as a hyperbolic polyhedron [28].

Bianchi groups are co-finite, the hyperbolic volume $\mathrm{vol}\,(\mathcal{F}) < \infty$, but not co-compact since the fundamental domains are not compact but have so-called cusps where $\mathcal{F}$ extends to the infinitely remote boundary $\mathbb{C}\cup\{\infty\}$. The number of Γ-inequivalent cusps is equal to the class number h of the number field K. In this article we concentrate on Bianchi groups having exactly one cusp. They come ninefold, namely for D = 1, 2, 3, 7, 11, 19, 43, 67, 163.

Obviously, $D = 1$ leads to the Gaussian number field and the corresponding Bianchi group $\mathrm{PSL}(2,\mathbb{Z}[i])$ is the Picard group [18].

4. Fundamental domains. We want to describe appropriate fundamental domains, well suited for our purposes. An extensive treatment is given by Swan [28]. Let $\omega \in \mathcal{O}$ be an integer and $\varepsilon \in \mathcal{O}^\times$ a unit, then $\mathrm{PSL}(2,\mathcal{O})$ contains at least the following elements and corresponding

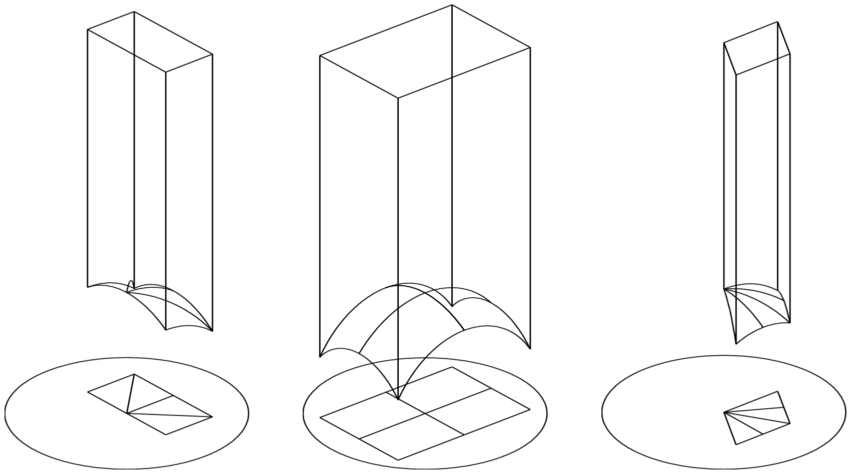

FIG. 4.1. *Fundamental domains for* $\mathrm{PSL}(2,\mathcal{O})$, $D=1,2,3$.

transformations

$$\begin{pmatrix}1 & \omega\\ 0 & 1\end{pmatrix} \quad : \quad (z,r)\longmapsto (z+\omega, r)\,, \tag{4.1}$$

$$\begin{pmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{pmatrix} \quad : \quad (z,r)\longmapsto (\varepsilon^2 z, r)\,, \tag{4.2}$$

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \quad : \quad (z,r)\longmapsto \left(\frac{-\overline{z}}{|z|^2+r^2}, \frac{r}{|z|^2+r^2}\right). \tag{4.3}$$

Since $\mathcal{O}$ forms a Euclidean lattice in $\mathbb{C}$ we see by (4.1) that there is always a (horizontal) translation in $\mathrm{PSL}(2,\mathcal{O})$ moving a given point P to $F\times\mathbb{R}^+$ where $F\subset\mathbb{C}$ is a fundamental cell of the lattice. Especially it is possible to chose F to be the Wigner-Seitz cell of $\mathcal{O}$ centered at the origin. That is it contains those points which are closer to zero than to any other integer.

For $D\neq 1,3$, (4.2) leads just to the identity; for the Picard group it gives the additional half turn $(z,r)\mapsto(-z,r)$, and for $D=3$ there are additional (horizontal) rotations with angles $\pm 2\pi/3$ since in the latter cases we have $\varepsilon^2=\pm1$ and $\varepsilon^2\in\{1,\exp(\pm 2\pi i/3)\}$, respectively. Above F may be reduced accordingly to one half or one third of its original shape.

Transformation (4.3) fixes the geodesic arc joining i and $-i$ and describes a rotation about π around this arc. It maps the unit half-sphere at the origin onto itself, its inside to the outside, and vice versa. Therefore one may chose $\mathcal{F}$ to be on either side of the sphere. We prefer the outside for placing the cusp at ∞.

These elements already generate the whole group for $D=1,2,3,7,11$,

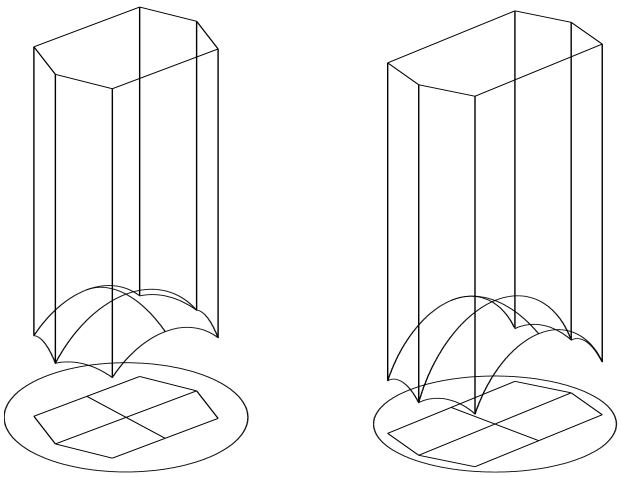

FIG. 4.2. *Fundamental domains for* $\mathrm{PSL}(2,\mathcal{O})$, $D = 7, 11$.

and writing $(\mathcal{O}^\times)^2 = \{\varepsilon^2 \mid \varepsilon \in \mathcal{O}^\times\}$ we have made plausible the following

LEMMA 4.1 (SWAN). *Let $F \subset \mathbb{C}$ be as above such that $(\mathcal{O}^\times)^2 F$ forms a Wigner-Seitz cell of $\mathcal{O}$ (the situation is drawn in Figure 4.3). Fundamental domains of $\mathrm{PSL}(2,\mathcal{O})$ for $D = 1, 2, 3, 7, 11$ are then given by (see Figures 4.1, 4.2)*

$$\mathcal{F} = \left\{(z,r) \in \mathbb{H}^3 \mid z \in F,\, |z|^2 + r^2 > 1\right\} . \tag{4.4}$$

The north eastern and nothern corners of F, and numerical values of the hyperbolic volumes of $\mathcal{F}$ are given by

	NE	*N*	$\mathrm{vol}(\mathcal{F})$
$D = 1$	$(1+i)/\,2$	—	0.305321864725740
2	$(1+i\sqrt{2})/\,2$	—	1.003841003341198
3	$(1+i/\sqrt{3})/\,2$	$i/\sqrt{3}$	0.169156934401609
7	$(1+3i/\sqrt{7})/\,2$	$2i/\sqrt{7}$	0.888914927816353
11	$(1+5i/\sqrt{11})/\,2$	$3i/\sqrt{11}$	1.382608307902647

In all other cases the situation is more involved and the fundamental domains are only subsets of (4.4).

Latter fact follows from the observation that the unit spheres around lattice points, the integers, do not cover $\mathbb{C}$ any more. The volumes have been obtained by evaluating $\int_{\mathcal{F}} dv$, partly numerically using Mathematica.

5. Spectrum of the Laplacian I. Basics. As before $\Gamma = \mathrm{PSL}(2,\mathcal{O})$. Let $L^2(\Gamma\backslash\mathbb{H}^3)$ be the space of Γ-invariant functions on $\mathbb{H}^3$, $f \circ \gamma = f$ for

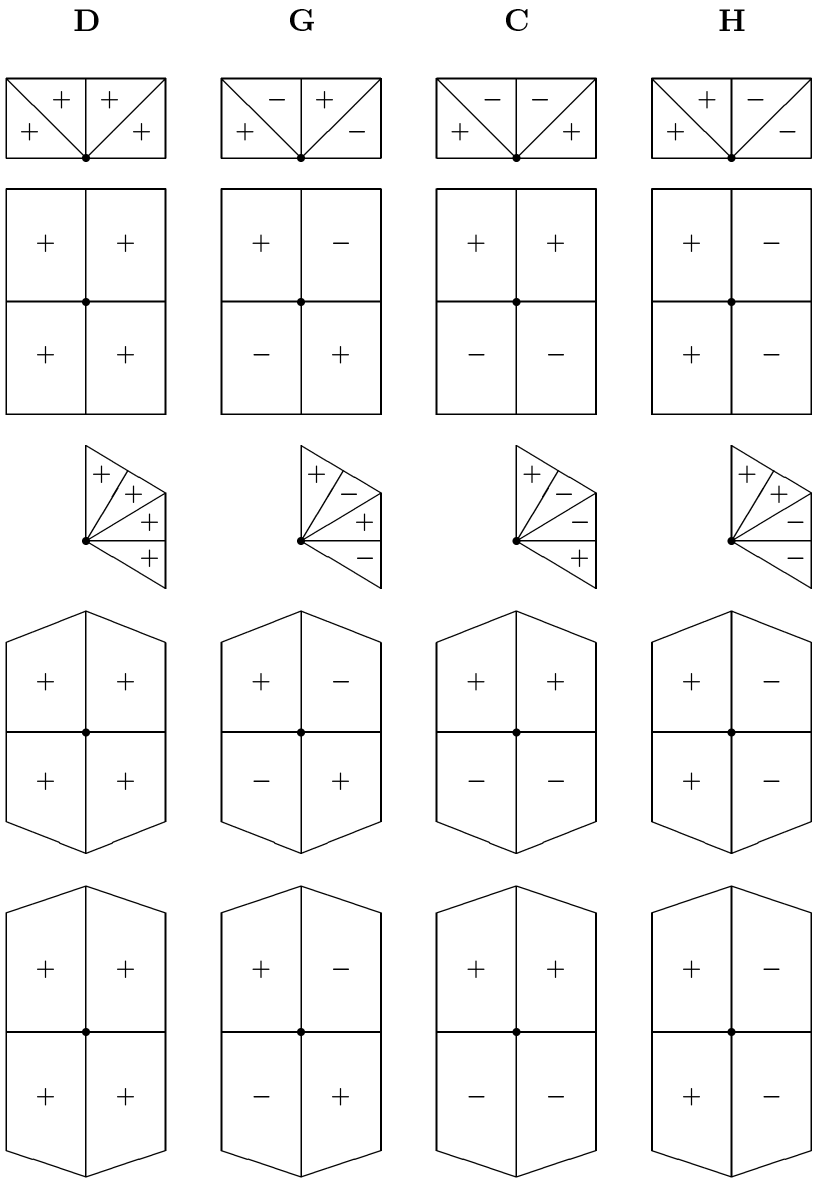

Fig. 4.3. *Symmetry classes and fundamental domains of* $\mathrm{PSL}(2,\mathcal{O})\setminus\mathbb{H}^3$, $D = 1,2,3,7,11$ *(from above), projected onto the lower boundary* $\mathbb{C}$. *Three-dimensional the domains sit above unit spheres centered at the origin which is marked by dots. The width of each domain is one except for* $D=3$ *where it is* $1/2$. *For* $D = 2,7,11$ *the shapes correspond to fundamental cells of the underlying lattice* $\mathcal{O}$; *for* $D = 1,3$ *there are additional rotations with angles* π *and* $2\pi/3$, *respectively.*

all $\gamma \in \Gamma$, which are square-summable with respect to the Petersson scalar product

$$(f, g) = \int_{\mathcal{F}} f\overline{g}\, dv\,.$$

It is known that $-\Delta$ is an essentially self-adjoint operator on a suitable domain in $L^2(\Gamma\backslash\mathbb{H}^3)$ and has a unique extension to a positive self-adjoint operator which we call $-\Delta$ as well (cf. [19]).

The spectrum[1] of $-\Delta$ on $L^2(\Gamma\backslash\mathbb{H}^3)$ comprises two parts. The continuous one consisting of the interval $[1, \infty)$ with multiplicity equal to the number of cusps h, and the discrete one

$$0 = \lambda_0 < \frac{3}{4} \le \mu_1 \le \ldots \le \mu_n < 1 \le \lambda_1 \le \lambda_2 \le \ldots$$

with $\lambda_j \to \infty$ as $j \to \infty$. λ_0 corresponds to the eigenspace of constant functions, and $\{\mu_j\}$ is the finite set of so-called small eigenvalues. It is conjectured that there are no small eigenvalues for any D which is indeed true for $D = 1, 2, 3, 7, 11$ [27]. The bound $3/4$ can be found in [22].

Let us call a function $f : \mathbb{H}^3 \to \mathbb{C}$ a cusp form if it is a Γ-invariant eigenfunction of the Laplacian

$$-\Delta f = (1 + R^2) f, \tag{5.1}$$

that vanishes at the cusps, and let the space of cusp forms be the subspace of $L^2(\Gamma\backslash\mathbb{H}^3)$ which is spanned by cusp forms. This corresponds essentially to the discrete part of the spectrum, with exception of the zero eigenvalue and the possible omission of a finite number of small eigenvalues. We write the eigenvalue $\lambda = 1 + R^2$, with $R > 0$ for $\lambda > 1$, and speak of R as eigenvalue, too.

Inspection of Lemma 4.1 and Figure 4.3 shows that the fundamental domains have rotational and reflectional symmetries with respect to the coordinate z, leaving r fixed. Furthermore, since for fixed r it looks essentially like the Euclidean one, the Laplacian commutes with these symmetries. Therefore, the space of cusp forms decomposes into an orthogonal sum

$$\mathbf{D} \oplus \mathbf{G} \oplus \mathbf{C} \oplus \mathbf{H}\,,$$

where the subspaces are spanned by cusp forms of the following symmetries, $\varepsilon \in \mathcal{O}^\times \backslash (\mathcal{O}^\times)^2$,

(5.2)

	$(z,r) \mapsto (\varepsilon z, r)$	$(z,r) \mapsto (-\overline{z}, r)$
D	even	even
G	even	odd
C	odd	even
H	odd	odd

[1] We do not care about the additional minus sign and mean by the spectrum of the Laplacian in effect the spectrum of $-\Delta$.

We call functions in $\mathbf{D} \oplus \mathbf{G}$ ε-even (or just even), functions in $\mathbf{C} \oplus \mathbf{H}$ ε-odd (or odd). The same symbols will also be used to denote the respective sets of eigenvalues. Figure 4.3 shows the situation for those D where we carry out numerical computations, the other ones are similar to $D = 2, 7, 11$.

6. Fourier expansion. Let f be a cusp form for $\mathrm{PSL}(2, \mathcal{O})$. Separation of variables in (5.1) and periodicity in the z coordinate leads to the Fourier expansion

$$\begin{aligned} f(z,r) &= \sum_{n\in\Lambda}{}' a(n)\, r\, K_{iR}\left(2\pi|n|r\right) e^{2\pi i <n,z>} \\ &= \sum_{n\in\mathcal{O}}{}' c(n)\, r\, K_{iR}\left(\frac{2\pi}{A}|n|r\right) e^{\frac{2\pi i}{A} <in,z>}, \end{aligned} \tag{6.1}$$

where $<w,z> = (w\overline{z} + \overline{w}z)/2$ is the Euclidean scalar product on $\mathbb{R}^2$, $\Lambda = \{z \in \mathbb{C} \mid <z,n> \in \mathbb{Z} \text{ for all } n \in \mathcal{O}\}$ the dual lattice to $\mathcal{O}$, the prime means skipping zero in the summation, A is the volume of a fundamental cell of $\mathcal{O}$ ($A = \sqrt{D}$ for $D \equiv 1, 2$ and $A = \sqrt{D}/2$ for $D \equiv 3 \bmod 4$), and $K_\nu(z)$ the modified Bessel function [15].

The first equation is the usual expansion, but we prefer (6.1) since it simplifies bookkeeping by indexing the coefficients just with the relevant integers.

We have the following connection between symmetry transformations of f and their impact on the coefficients.

Lemma 6.1. *Let $f(z,r)$ be a function on $\mathbb{H}^3$ with Fourier expansion (6.1) and coefficients $c(n)$, $\varepsilon \in \mathcal{O}^\times$. Then the following functions, as functions of z and r, have also Fourier expansions of form (6.1) with n-th Fourier coefficients given by*

$$\begin{aligned} f(\varepsilon z, r) &: c(\varepsilon n), \qquad & f(-z,r) &: c(-n), \\ f(\overline{z}, r) &: c(-\overline{n}), \qquad & f(-\overline{z}, r) &: c(\overline{n}). \end{aligned}$$

Proof. This follows from the observations that $<in,\overline{z}> = <-i\overline{n}, z>$, $<in, \varepsilon z> = <i\overline{\varepsilon} n, z>$, and $\overline{\varepsilon} = \varepsilon^{-1}$, after inserting them into the expansion of f evaluated at the transformed points. ∎

7. Hecke operators. Hecke operators for $\mathrm{PSL}(2, \mathcal{O})$ have been studied by Heitkamp [11]. In the simplest case of class number one they may be introduced quite explicitely in an analogous way to the modular group [13,14,29].

Let f be a Γ-invariant function on $\mathbb{H}^3$, then for any non-zero integer $m \in \mathcal{O}$ the action of the Hecke operator T_m is given by

$$\mathrm{T}_m f(z,r) = \frac{1}{|m|} \sum_{\substack{ad=m \\ b \bmod (d)}}^{\times} f \circ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} (z,r)\,, \tag{7.1}$$

where $a, b, d \in \mathcal{O}$, (d) denotes the principal ideal generated by d, and the crossed sum means d is taken modulo units, it runs through a complete set of non-associate divisors of m. If one inserts f in form of Fourier expansion (6.1) one is eventually led to the Fourier expansion of $\mathrm{T}_m f$ (cf. [10])

$$\text{(7.2)}\quad \mathrm{T}_m f(z,r) = \sum_{n\in\mathcal{O}}{}' \sum_{d|(\overline{m},n)}{}^{\times} c\left(\frac{\overline{m}n}{d^2}\right) r\, K_{iR}\left(\frac{2\pi}{A}|n|r\right) e^{\frac{2\pi i}{A}<in,z>},$$

where (m,n) denotes a greatest common divisor of m and n, and again d is taken modulo units.

The Hecke operators are self-adjoint operators on $L^2(\Gamma\backslash\mathbb{H}^3)$, map cusp forms to cusp forms, and commute with each other and with the Laplacian. Therefore one may choose in the space of cusp forms a basis, whose elements are cusp forms and simultaneously eigenfunctions of all the Hecke operators.

Suppose such an f is given, we may call it a Hecke eigenfunction, denote the (real) eigenvalues of T_m by t_m,

$$-\Delta f = (1+R^2)f, \qquad \mathrm{T}_m f = t_m f \quad \text{for all } m\in\mathcal{O}^*,$$

where $\mathcal{O}^* = \mathcal{O}\backslash\{0\}$, then by (7.2) one has $t_m c(n) = \sum^{\times}_{d|(\overline{m},n)} c\left(\frac{\overline{m}n}{d^2}\right)$, and specializing to $n=1$,

$$\text{(7.3)}\qquad t_m c(1) = c(\overline{m}).$$

Plugging this back into the previous equation we arrive at the so-called Hecke relations of the coefficients

$$\text{(7.4)}\qquad c(m)c(n) = c(1) \sum_{d|(m,n)}{}^{\times} c\left(\frac{mn}{d^2}\right),$$

which look like perfectly the same as those for the modular group.

The first Fourier coefficient is never zero since otherwise f vanishes identically due to equation (7.3). Thus we are allowed to define a normalized cusp form to be a Hecke eigenfunction whose first Fourier coefficient is chosen to be one, $c(1)=1$. By (7.3) all other coefficients will be real, too. Furthermore, we can forget about the t_m since

$$\text{(7.5)}\qquad \mathrm{T}_m f = c(\overline{m})f.$$

We also note that Hecke operators at units ε are just rotations; for any Γ-invariant f

$$\text{(7.6)}\qquad \mathrm{T}_\varepsilon f(z,r) = f(\varepsilon z, r),$$

especially $\mathrm{T}_{\varepsilon^2} f(z,r) = (\mathrm{T}_\varepsilon)^2 f(z,r) = f(\varepsilon^2 z, r) = f(z,r)$ reduces to the identity by (4.2). Moreover, suppose f is a normalized cusp form, the eigenvalues fulfill

$$\text{(7.7)}\qquad c(\varepsilon) = \pm 1,$$

and letting $n = p$ be a prime, (7.4) is simplified to

$$c(mp) = c(m)c(p) - c(m/p), \tag{7.8}$$

where the last term is zero unless p divides m. Applying the formula recursively, it is possible to compute all the coefficients if only the prime coefficients are known.

8. Spectrum of the Laplacian II. Symmetries. We want to gain further insight into the structure of the disrete spectrum and the fashion in which symmetries and Hecke operators are related. Also, a first portion of our numerical findings, in connection with lifts, will be presented in this section.

Recall the definition of symmetry classes (5.2) and let

$$\mathrm{P}f(z,r) = f(-\overline{z},r)$$

denote the reflection at the plane $x = 0$. The ε-rotation can be expressed by the appropriate Hecke operator T_ε with eigenvalue ± 1, see equations (7.6), (7.7).

THEOREM 8.1. *Let f be a Γ-invariant function on $\mathbb{H}^3$.*

a) Reflections and Hecke operators generally do not commute, rather it holds

$$\mathrm{T}_m\, \mathrm{P} f = \mathrm{P}\, \mathrm{T}_{\overline{m}}\, f.$$

b) If f is a Hecke eigenfunction with eigenvalues t_m, then $\mathrm{P}f$ is a Hecke eigenfunction with eigenvalues $t_{\overline{m}}$.

c) If f is a normalized cusp form with Fourier coefficients $c(m)$, then $\mathrm{P}f$ is a normalized cusp form with Fourier coefficients $c(\overline{m})$.

Proof. From the definition and the commutative property of the Hecke operators we deduce

$$\mathrm{T}_m\, \mathrm{P}f(z,r) = \sum_{\substack{ad=m \\ b \bmod (d)}}^{\times} \mathrm{P}f\left(\frac{az+b}{d}, \left|\frac{a}{d}\right| r\right) = \sum_{\substack{ad=m \\ b \bmod (d)}}^{\times} f\left(-\frac{\overline{az+b}}{\overline{d}}, \left|\frac{a}{d}\right| r\right)$$

$$= \sum_{\substack{ad=m \\ b \bmod (d)}}^{\times} \mathrm{T}_{-1}\, f\left(\frac{\overline{az}+\overline{b}}{\overline{d}}, \left|\frac{a}{d}\right| r\right) = \sum_{\substack{ad=\overline{m} \\ b \bmod (d)}}^{\times} \mathrm{T}_{-1}\, f\left(\frac{a\overline{z}+b}{d}, \left|\frac{a}{d}\right| r\right)$$

$$= \mathrm{T}_{\overline{m}}\, \mathrm{T}_{-1}\, f(\overline{z},r) = \mathrm{T}_{-1}\, \mathrm{T}_{\overline{m}}\, f(\overline{z},r) = \mathrm{T}_{\overline{m}}\, f(-\overline{z},r) = \mathrm{P}\, \mathrm{T}_{\overline{m}}\, f(z,r).$$

This proves a). Part b) is just an obvious application of a), and c) follows from b) and Lemma 6.1 since the first Fourier coefficient of $\mathrm{P}f$, $c\left(\overline{1}\right) = c(1) = 1$, is already normalized. ∎

THEOREM 8.2. *a) There exist bases of normalized cusp forms for $\mathbf{D}\oplus\mathbf{G}$ and for $\mathbf{C} \oplus \mathbf{H}$, but not for the individual symmetry classes separately.*

b) There are no reflection-odd ($z \mapsto -\overline{z}$) Hecke eigenfunctions.

c) For the set of eigenvalues of the Laplacian holds $\mathbf{G} \subset \mathbf{D}$, $\mathbf{H} \subset \mathbf{C}$.

Proof. a) The existence of the bases is a simple consequence of the fact that ε-rotations are identical to the application of the Hecke operator T_ε. That the statement cannot be pushed further to the individual symmetry classes follows from part a) of Theorem 8.1. For b), suppose $f \neq 0$ is odd with respect to the reflection $z \mapsto -\overline{z}$. Then, by Lemma 6.1, $c(n) = -c(\overline{n})$, especially $c(1) = -c(1)$, that is $c(1) = 0$, so $c(n) = 0$ for all n by (7.3), and hence $f = 0$ which is a contradiction. To prove c), suppose λ belongs to symmetry class $\mathbf{G}$ (or $\mathbf{H}$). Then there exists an ε-even (resp. ε-odd) Hecke eigenfunction f with $-\Delta f = \lambda f$. Write $f = f^+ + f^-$, where $\mathrm{P} f^+ = +f^+$ and $\mathrm{P} f^- = -f^-$. From part b) of the theorem $f^+ \neq 0$. But, by construction $f^+ \in \mathbf{D}$ (resp. $f^+ \in \mathbf{C}$), and $-\Delta f^+ = \lambda f^+$. □

In consideration of Theorem 8.2 a) the 'good' or natural notions for even and odd for $\mathrm{PSL}(2, \mathcal{O})$-invariant functions seem to be those introduced earlier as ε-even and ε-odd.

The situation described in the theorems is different from that in two dimensions, where the Hecke operators do commute with reflections. It seems remarkable that geometrical symmetries do not suffice to explain the multiplicities in the spectra, rather additional self-adjoint operators, the T_m, have to be taken into account.

In the supplement to his thesis [12], Huntebrinker introduces the notion of dual eigenvalues to characterize eigenvalues occuring simultaneously in two symmetry classes with opposite symmetries on each plane of symmetry, in order to describe what he observes from his computations. Theorem 8.2 gives an explanation of how these dual eigenvalues arise.

P-even Hecke eigenfunctions (symmetry classes $\mathbf{D}$, $\mathbf{C}$) are not ruled out, and indeed, we see them numerically. They can be identified by looking at their coefficients, one has $c(m) = c(\overline{m})$ for all $m \in \mathcal{O}^*$ in this case. This coincidence should not exist by chance, and one may suspect that these highly symmetrical functions are lifts from two dimensions. We have identified a part of them which will be described in the following conjecture. The notions referring to objects in two dimensions are defined similarly to their three-dimensional counterparts.

Conjecture 8.1. *Let g, given on the Poincaré upper half plane by*

$$g(x+iy) = \sum_{n \in \mathbb{Z}}{}' a(n) \sqrt{y}\, K_{ir}(2\pi |n| y)\, e^{2\pi i n x}, \quad y > 0,$$

be a cusp form for the modular group with Laplace eigenvalue $\lambda = \frac{1}{4} + r^2$. Assume g is eigenfunction of all the Hecke operators[2] *and normalized to $a(1) = 1$. Fix some K of class number one, let $R = 2r$, and define the following numbers,*

$$c(\varepsilon) = 1, \qquad \varepsilon \in \mathcal{O}^\times,$$

[2] For the sake of clarity $\mathrm{T}_m\, f(z) = m^{-1/2} \sum_{\substack{ad=m \\ b \bmod d}} f((az+b)/d)$, $m, d > 0$.

TABLE 8.1

The first (even + and odd -) eigenvalues of the Laplacian for PSL$(2,\mathbb{Z})$ *and their lifts in* PSL$(2,\mathcal{O})$. *For* PSL$(2,\mathbb{Z})$ *is given* $2R$ *where* $\lambda = 1/4 + R^2$ *is the true eigenvalue. The values, truncated after seven digits, are taken from previous higher precision computations [26]. The right part lists the corresponding* R $(\lambda = 1 + R^2)$ *for* PSL$(2,\mathcal{O})$ *as found independently in the current calculations. They all belong to symmetry class* **D**.

PSL$(2,\mathbb{Z})$	$D=1$	2	3	7	11
19.0673905 -	19.067391	19.067390	19.067391	19.067390	19.067391
24.3460166 -	24.346017	24.346017	24.346017	24.346017	24.346017
27.5595027 +	27.559503	27.559503	27.559503	27.559503	27.559503
28.7170190 -	28.717020	28.717019	28.717019	28.717019	28.717019
32.2761463 -	32.276146	32.276146	32.276147	32.276146	
33.2885184 -	33.288518	33.288519	33.288518	33.288519	
35.4771267 +	35.477126	35.477127	35.477127	35.477127	
36.3618356 -	36.361836	36.361836	36.361836	36.361835	
38.8469629 +	38.846963	38.846963	38.846963	38.846963	
38.9694277 -	38.969428	38.969428	38.969429	38.969428	
40.2133893 -	40.213389		40.213390		
42.6315918 +	42.631591		42.631592		
42.9581150 -	42.958115		42.958115		
44.3893479 -	44.389350		44.389348		
45.5718169 +	45.571817		45.571817		

for $p \in \mathcal{O}$ *a prime*

$$c(p) = \begin{cases} a(p\overline{p}), & \text{if } p \notin \mathbb{Z}, \\ a^2(p) - 2, & \text{if } p \in \mathbb{Z}, \end{cases}$$

and for general $n \in \mathcal{O}$, *compute* $c(n)$ *according to formula (7.8).*

Then the function $f(z,r)$, *given by Fourier expansion (6.1) with parameter* R *and coefficients* $c(n)$ *as defined above, is a normalized cusp form for* PSL$(2,\mathcal{O})$ *with Laplace eigenvalue* $\lambda = 1 + R^2$. *By construction it belongs to symmetry class* **D**.

Of course the statement ought to be a theorem. While we have not proved it, it pops not out of a clear sky and there is some evidence of its truth. That lifts between different dimensions occur is well established and can be seen, for example, in Rudnick & Sarnak [20] for a related situation of compact hyperbolic manifolds, which gives also the relation of the Laplace eigenvalues. The conversion of the coefficients mimics one which holds for lifts in the case of real number fields, see Saito [21], Corollary 2 of Theorem 3, and Doi & Naganuma [9].

From the computational point of view, the situation seems to be clear. We have found all of the lifts described in the conjecture which fall into the range of the conducted calculations. The eigenvalues and coefficients for PSL$(2,\mathcal{O})$ agree up to the numerical accuracy with those of PSL$(2,\mathbb{Z})$ which we know from previous higher accuracy computations [25,26]. Table

8.1 gives an account for the first eigenvalues. To avoid misconceptions, these lifts do not exhaust the Hecke eigenfunctions in **D**. The question remains where the other suspected lifts in **D** and those in **C** come from.

9. Method of computation. Throughout this section we fix some D of class number one and a symmetry class, ε-even or ε-odd. According to Theorem 8.2 a) we want to compute elements of a basis of normalized cusp forms for $\mathbf{D}\oplus\mathbf{G}$ respectively $\mathbf{C}\oplus\mathbf{H}$. That is to find both eigenvalues of the Laplacian and eigenvalues of the Hecke operators (= Fourier coefficients), which amounts to knowing the corresponding eigenfunctions. We shall derive from the eigenvalue equation of the Hecke operators a sytem of non-linear equations for the coefficients and outline how to find the sought-after cusp forms as its solutions.

9.1. Generic algorithm. We commence with rewriting Fourier expansion (6.1). Assume f is a Hecke eigenfunction. Since $c(m) = \pm c(\varepsilon m)$ for all $\varepsilon \in \mathcal{O}^\times$ it is sufficient to sum over the coefficients modulo units, so

$$\begin{aligned} f(z,r) &= \sum_{n\in\mathcal{O}/\mathcal{O}^\times}{}' c'(n)\, r\, K_{iR}\left(\frac{2\pi}{A}|n|r\right) \sum_{\varepsilon\in\mathcal{O}^\times} (\pm 1) e^{\frac{2\pi i}{A} <in,z>} \\ &= \sum_{n\in\mathcal{O}/\mathcal{O}^\times}{}' c(n)\, r\, K_{iR}\left(\frac{2\pi}{A}|n|r\right) cs(n,z), \end{aligned} \tag{9.1}$$

where the exponentials combine to simple sums of sines or cosines and the coefficients are rescaled to absorb the factors 2 respectively $2i$ which arise from building the trigonometric functions. For example for the Picard group

$$cs(n,z) = \cos(2\pi <in,z>) \pm \cos(2\pi <n,z>),$$

where the plus is for even and the minus sign for odd functions, respectively. Normalize again to $c(1) = 1$, and all the Hecke relations remain valid. The transformation has the intended side effect of making $f(z,r)$ real since the K-Bessel function is real for real argument and purely imaginary order, and $c(n)$ and $cs(n,z)$ are real, anyway.

Now let $p \in \mathcal{O}$ be a prime. The eigenvalue equation of the Hecke operators (7.5) and their definition (7.1) may be composed to get

$$c(\overline{p})f(z,r) - \frac{1}{|p|}\sum_{b \bmod (p)} f\left(\frac{z+b}{p}, \frac{r}{|p|}\right) - \frac{1}{|p|} f(pz,|p|r) = 0. \tag{9.2}$$

Since f is Γ-invariant one may pull its arguments back into the fundamental domain. We denote the $\mathrm{PSL}(2,\mathcal{O})$-equivalent counterparts by

$$(z_{p,b}, r_{p,b}) \equiv \left(\frac{z+b}{p}, \frac{r}{|p|}\right), \qquad (z_p, r_p) \equiv (pz, |p|r).$$

By introducing the transformed points and Fourier expansion (9.1) into (9.2), then exchanging summations we arrive at the central identity upon which the numerics is based,

$$\sum_{n\in\mathcal{O}/\mathcal{O}^\times}{}' c(n)\ \left[\ c(\overline{p})SKC(1,n;R) - SKC(p,n;R)\ \right] = 0, \tag{9.3}$$

where

$$\begin{aligned} SKC(1,n;R) &= r\,K_{iR}\left(\frac{2\pi}{A}|n|r\right) cs(n,z),\\ SKC(p,n;R) &= \frac{1}{|p|}\sum_{b \bmod (p)} r_{p,b}\,K_{iR}\left(\frac{2\pi}{A}|n|r_{p,b}\right) cs(n,z_{p,b}) + \\ &\quad + \frac{1}{|p|} r_p\,K_{iR}\left(\frac{2\pi}{A}|n|r_p\right) cs(n,z_p). \end{aligned}$$

Due to (7.8) and the remark thereafter one can think of $c(n)$ being a function, precisely a polynomial, of the prime coefficients $c(p)$ with $|p| \le |n|$. Consequently, the left hand side of (9.3) is a function of the prime coefficients $c(p)$, $p \in \mathcal{P}/\mathcal{O}^\times$, the eigenvalue parameter R, and the point (z,r); furthermore it depends upon the initial choice of $p \in \mathcal{P}$. Here we have introduced the symbol $\mathcal{P}$ to denote the set of primes in $\mathcal{O}$. From the derivation it is clear that for any Hecke eigenfunction, as expressed by its coefficients and (Laplace) eigenvalue, (9.3) is an identity for all $(z,r) \in \mathbb{H}^3$.

Now we want to turn the tables and read (9.3), after fixing some $(z,r) \in \mathbb{H}^3$, as an equation for the unknown prime coefficients which depends on the eigenvalue parameter R. Note that there is one such an equation for every p, giving an infinite dimensional system of equations for the collection of prime coefficients.

To be accessible to the computer we have to introduce a numerical cut-off. Recall the Bessel function $K_{iR}(x)$ is oscillatory for $x < R$ with an amplitude roughly equal to $(2\pi/R)^{1/2}e^{-\pi R/2}$ and dies off exponentially like $(\pi/2x)^{1/2}e^{-x}$ as $x \to \infty$. Furthermore, since we have moved all points with the possible exception of (z,r) into the fundamental domain, the r-coordinates occurring in (9.3) are uniformly bounded from below by some $r_{min} > 0$ and the left hand side of (9.3) is nicely convergent. Fixing some $M = M(R) > 0$ large enough makes the arguments of the Bessel functions in (9.3) bigger than $\frac{2\pi}{A}Mr_{min}$ for $|n| > M$ and the Bessel function terms arbitrarely small, and hence we may treat them as zero for practical purposes. $M(R)$ can be adjusted during the computations to achieve a desired accuracy. Thus we are led to the following finite system of non-linear equations with parameter R for the prime coefficients $c(p)$, $p \in \mathcal{P}/\mathcal{O}^\times$, $|p| < M(R)$, which completely determine the function f within numerical accuracy,

$$(9.4)\qquad \left(\sum_{\substack{n\in\mathcal{O}/\mathcal{O}^\times\\ |n|<M(R)}}{}' c(n)\ \left[\ c(p)SKC(1,n;R)-SKC(\overline{p},n;R)\ \right]=0\right)_{\substack{p\in\mathcal{P}/\mathcal{O}^\times\\ |p|<M(R)}}.$$

This is in fact a whole family of systems depending on the choice of (z,r) which is hidden in the *SKC* terms.

It is easily seen that f is normalized cusp form with eigenvalue $\lambda = 1+R^2$ if and only if it is (that is its prime coefficients) a Γ-invariant solution of (9.4) with parameter R. One direction is clear from the derivation. Conversely, suppose f is Γ-invariant with parameter R. Then, due to the Fourier ansatz, it is an eigenfunction of the Laplacian for Γ with eigenvalue $\lambda = 1+R^2$. Moreover its coefficients have the built-in property to fulfill the Hecke relations (7.4) with $c(1)=1$ which may be introduced in (7.2), showing that f is already a Hecke eigenfunction.

Hence we have to detect the Γ-invariant solutions of (9.4). The Fourier ansatz (9.1) assures that any solution is invariant under $\begin{pmatrix}1&\omega\\0&1\end{pmatrix}$ and $\begin{pmatrix}\varepsilon&0\\0&\varepsilon^{-1}\end{pmatrix}$, $\omega\in\mathcal{O}$, $\varepsilon\in\mathcal{O}^\times$, and the only thing possibly missing is the invariance with respect to the remaining generators of the group, which is for $D = 1,2,3,7,11$ just $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, confer (4.1) to (4.3).

Considering the eigenvalue parameter R as an additional unknown, (9.4) is an underdetermined system and its solutions are in general smooth curves in the space of R and the prime coefficients, call it $c(p)$-R-space. In summarizing the previous discussion we are led to the

Generic Algorithm 9.1. *Design some real-valued testfunction $\phi = \phi(\{c(p)\},R)$ which has zeros if and only if the function $f(z,r)$, which is determined by its Fourier expansion (9.1) with the given prime coefficients $\{c(p)\}$ and parameter R, is invariant under generators $\{g_i\}$ of* $\mathrm{PSL}(2,\mathcal{O})$ *such that $\left\{g_i,\begin{pmatrix}1&\omega\\0&1\end{pmatrix},\begin{pmatrix}\varepsilon&0\\0&\varepsilon^{-1}\end{pmatrix}\right\}$ generates the whole group. Find solutions of (9.4) and trace their paths in $c(p)$-R-space by some continuation method. Evaluate the testfunction ϕ along these curves and look for its roots. If such a zero is found then the associated function $f(z,r)$ is a normalized cusp form for* $\mathrm{PSL}(2,\mathcal{O})$ *with Laplace eigenvalue $\lambda = 1+R^2$ and Hecke eigenvalues $c(p)$.*

Finding roots of ϕ is a one-dimensional problem since the solution curves can be parametrized by the arclength, or some other suitable parameter.

Solutions of (9.4) suffice to determine in principle the whole Fourier expansion. Suppose $p\in\mathcal{P}$, $|p|\geq M(R)$, then $c(p) = \mathrm{T}_{\overline{p}}\,f(z,r)/f(z,r)$, where the right hand side can be evaluated in the same manner as the left hand side of one of the equations in (9.4) and thus depends only on prime coefficients with $|p|<M(R)$.

TABLE 9.1
Eigenvalues of the Laplacian for PSL$(2,\mathcal{O})$. *Symmetry classes* **D** *&* **G** *(ε-even). Obviously double eigenvalues belong to* **D** *and* **G**, *single ones to* **D**. *Listed is* R, *the true eigenvalue is* $\lambda = 1 + R^2$.

D&G	$D=1$	2	3	7	11
1	8.555250	5.242134	12.500100	7.419980	4.859365
2	11.108567	6.642690	15.009706	9.046284	6.640645
3	12.869911	7.886976	17.098656	9.085356	7.326654
4	14.079661	8.845529	18.824509	9.085358	7.982257
5	15.348279	9.736081	19.067391	10.563745	8.151646
6	15.891843	10.332226	21.078005	10.780342	8.151646
7	17.336404	10.506216	21.152230	11.308639	9.498483
8	17.451320	10.506216	22.610090	11.308639	9.845725
9	17.451320	10.909561	22.654855	12.601473	9.845725
10	17.776641	11.623077	22.654856	12.601473	10.475947
11	19.067391	11.935184	23.772885	12.674630	10.749183
12	19.222903	12.446448	24.324671	13.520752	11.430138
13	19.411192	12.852945	24.346017	13.520752	11.430138
14	20.007546	12.852946	25.768004	13.797744	11.661166
15	20.707989	13.161019	26.086666	13.983709	11.661167
16	20.707989	13.486900	26.268613	14.677219	11.712182
17	20.815269	13.826893	26.268613	14.818383	11.840603
18	21.428871	13.945952	27.380121	14.818383	12.507511
19	22.122303	13.945952	27.559503	15.332649	12.565192
20	22.630553	14.541733	27.994643	15.723998	13.176599
21	22.962301	14.622458	28.717019	15.723999	13.176600
22	22.962301	15.082414	29.074053	16.280519	13.365865
23	23.496177	15.166414	29.074053	16.280519	13.534807
24	23.527845	15.166415	29.080101	16.497276	13.534808
25	23.889784	15.623369	29.277193	16.497276	13.733958
26	23.889784	16.019564	30.279769	16.530034	14.049286
27	24.346017	16.070355	30.279769	16.931559	14.100186
28	24.575015	16.070355	30.761380	16.931559	14.580426
29	24.700459	16.122432	30.854759	16.967195	14.580426
30	25.470675	16.122432	31.518802	17.300300	14.692071
31	25.507246	16.277503	31.844339	17.949069	14.692071
32	25.723922	16.673004	31.975879	18.164015	14.854940
33	25.723922	16.802075	31.975879	18.200172	14.854941
34	25.918644	17.371611	32.276147	18.200172	15.202066
35	25.918644	17.566266	32.887287	18.210418	15.242453
36	26.426958	17.566266	33.268061	18.210418	15.769299
37	27.033261	17.676530	33.288518	18.219635	15.769300
38	27.142920	17.676530	33.622519	18.219635	15.769722
39	27.144985	17.727904	33.622519	18.895335	15.987444
40	27.144985	17.727904	33.710856	19.067390	15.987444

9.2. Specific implementation. Some remarks are in order to explain how we have translated the Generic Algorithm 9.1 into a concrete computational scheme.

As testfunction we used the obvious

$$\phi\left(\{c(p)\}, R\right) = f(z_0, r_0) - f\left(\frac{-\overline{z}_0}{|z_0|^2 + r_0^2}, \frac{r_0}{|z_0|^2 + r_0^2}\right),$$

where (z_0, r_0) is some point in hyperbolic 3-space such that neither r_0 nor $r_0 \big/ \left(|z_0|^2 + r_0^2\right)$ is too small (otherwise one were to compute too many Fourier coefficients). Invariance in a single point of course is not sufficient. Thus we took a couple of points and looked for common roots of the corresponding testfunctions. Finding the roots of ϕ is just the usual detection of changes of sign and subsequent application of some standard root finding algorithm like the secant method.

To track the solution curves of (9.4) we have fixed R as curve parameter and discretized the R-axis. The transition from one R^{old} to the next R^{new} consists of an Euler predictor step, that is one proceeds along the tangent of the curve at R^{old} in $c(p)$-R-space to R^{new}, and the subsequent correction of this initial guess for the coefficients at R^{new} by Newton's method, keeping R^{new} frozen. A typical width of the R-grid was one tenth of the mean distance of successive eigenvalues which is given by Weyl's law; confer the following Section 10. Observe that Newton's method is practicable since the derivative of the left hand side of (9.4) with respect to the coefficients can easily be computed. The Jacobian is

$$DH\left(\{c(p)\}\right) = \left(\sideset{}{'}\sum_{\substack{n \in \mathcal{O}/\mathcal{O}^\times \\ |n| < M(R)}} \left[\frac{\partial c(n)}{\partial c(q)} \left[c(p) SKC(1,n) - SKC(\overline{p}, n) \right] + \right.\right.$$
$$\left.\left. + \, \delta_{pq} c(n) SKC(1,n) \right] \right)_{\substack{p,q \in \mathcal{P}/\mathcal{O}^\times \\ |p|,|q| < M(R)}}$$

where δ_{pq} denotes the Kronecker symbol and the parameter R has been dropped. Writing $H\left(\{c(p)\}\right)$ for the left hand side of the system (9.4), the k-th corrector step in Newton's method consists of solving the linearized system

$$DH\left(\left\{c^k(p)\right\}\right) \cdot \left(\Delta c^k(p)\right) + H\left(\left\{c^k(p)\right\}\right) = 0 \tag{9.5}$$

for the vector of corrections $\left(\Delta c^k(p)\right)$, $p \in \mathcal{P}/\mathcal{O}^\times$, $|p| < M(R)$, and setting

$$c^{k+1}(p) = c^k(p) + \Delta c^k(p)$$

for the $(k+1)$-th iterate of the coefficients.

TABLE 9.2
Eigenvalues of the Laplacian for PSL$(2,\mathcal{O})$. *Symmetry classes* **C** *&* **H** *(ε-odd). Obviously double eigenvalues belong to* **C** *and* **H**, *single ones to* **C**. *Listed is* R, *the true eigenvalue is* $\lambda = 1 + R^2$.

C&H	$D=1$	2	3	7	11
1	6.622118	4.944483	7.072007	5.063246	3.873897
2	10.180799	7.356242	11.008113	6.107909	4.418109
3	12.115274	8.319914	13.293163	8.382473	6.471802
4	12.115275	8.744178	14.863598	8.401428	6.796408
5	12.879369	8.744178	16.137241	9.808216	7.594323
6	14.148331	10.164498	16.137242	9.808216	8.187225
7	14.952442	10.164499	17.396686	10.156948	8.990595
8	14.952443	10.319906	18.160139	11.057446	9.446480
9	16.207594	11.480404	19.350972	11.098375	9.446480
10	16.994969	11.480404	19.350972	12.097801	9.690009
11	16.994969	11.541693	20.178906	12.097801	9.795788
12	17.863057	12.218451	20.942241	12.759942	10.747470
13	17.863057	12.218451	21.933177	12.759942	10.747470
14	18.243911	12.648494	21.933177	12.781852	10.841192
15	18.832990	12.943369	22.261927	13.226412	11.050069
16	19.430543	13.724718	23.274886	13.743304	11.510077
17	19.430543	13.724719	23.274887	14.107445	12.025939
18	20.300307	13.761942	23.456049	14.332283	12.053292
19	20.300307	13.761943	24.503342	14.332284	12.249421
20	20.606867	14.153652	24.503342	14.774395	12.249421
21	21.379660	14.153653	25.245970	14.774395	12.346860
22	21.379661	14.507079	25.714833	14.787962	12.346860
23	21.442459	14.581454	26.154642	15.192280	12.877797
24	21.832490	14.581454	26.300680	15.192280	12.877797
25	21.832490	15.022883	26.300680	15.614105	13.096112
26	22.584753	15.463121	27.196168	15.739746	13.187336
27	22.584753	15.463122	27.196168	16.290857	13.539757
28	22.854291	15.785004	27.886485	16.290857	13.781386
29	23.497683	16.002271	28.235814	16.447517	14.074418
30	23.497683	16.002273	28.518240	16.447518	14.074418
31	23.842759	16.241653	28.518240	16.461553	14.480552
32	23.895157	16.241653	28.743224	16.815949	14.503818
33	23.895158	16.777388	28.743224	16.815949	14.503818
34	24.421338	16.777388	29.465364	17.486188	14.540694
35	24.421338	16.870407	29.652296	17.486189	14.694259
36	25.032781	16.870407	29.652297	17.593281	14.694259
37	25.032781	17.073613	30.439215	17.593281	14.845790
38	25.429054	17.143992	30.504975	17.682569	14.845790
39	25.775886	17.331710	30.504975	17.806385	15.187791
40	25.775886	17.331710	30.905564	17.835000	15.499480

In terms of continuation methods (see Allgower & Georg [1] for a monographical treatment) this is a somewhat crude proceeding which could certainly be improved since it does not make efforts to exploit much of the actual behaviour of the solution curves, that is the stepsize is fixed and R may not be a good parameter. Especially one can not pass over turning points of the curve in $c(p)$-R-space with respect to the coordinate R.

There are two principal time consuming things in the algorithm. First of all the set up of the constants *SKC* at a given R and secondly the repeated solving of the linear system (9.5) in each corrector step. Since the *SKC* terms depend exclusively on the parameter R they have to be established only once at each point of the R-grid and can henceforth be used for all corrector steps at the given R. This is a real pay-off for the rigid a priori fixing of the parameter R since we observe numerically that there are easily several dozen different solutions of the non-linear system (9.4) at a single R which have to be tracked simultaneously, and their number is generally increasing with increasing R since more and more prime coefficients get involved (the cut-off $M(R)$ is an increasing function of R).

Now that we have explained our path following strategy it remains the question how to find *any* solution which can be traced. To this end we have chosen, for every R in the grid typically a dozen, vectors of prime coefficients randomly to be used as starting points of the Newton iteration. In most of the computations we took $c(p) \in [-2.2, 2.2]$ equidistributed which is justified by the fact that for almost all prime coefficients $|c(p)| < 2$, see [23].

One can not expect to find all eigenvalues in a given R-interval within a single run. For example, one cannot assure to have found all solutions of the non-linear system, or a solution may slip through the test on changes of sign of ϕ, or an eigenvalue may be located too close to a turning point of the solution curve to be detected. Therefore we have done several runs with different choices of the point (z, r) which determines the *SKC* terms in (9.4). The completeness was checked by looking at the numerically given function $N_{\text{fluct}}(R)$ from the next Section 10 which oscillates nicely about zero if the spectrum is complete. There is no proof of completeness available but this seems to be a generally accepted test.

9.3. Results. The algorithm described in the preceding two paragraphs was applied to the number fields $D = 1, 2, 3, 7, 11$ and both symmetry classes, even and odd. We have not made efforts to attain the highest possible precision and computed eigenvalues and coefficients to an accuracy of six digits. In principle there is no problem to improve the accuracy by enlarging the cut-off M and including more coefficients, once an eigenfunction is known. There is a simple test for the obtained accuracy because of some redundancy in the computed data. Either a cusp form f is reflection-even, $\mathrm{P}f = f$, then we have $c(p) = c(\overline{p})$, or it has a partner $\mathrm{P}f$ with the same eigenvalue, then $c(p; f) = c(\overline{p}; \mathrm{P}f)$. Since all these objects are com-

Table 9.3
Number of computed Laplace eigenvalues for Bianchi groups.[4] *The last column displays the number of prime coefficients being used at the largest R.*

	D	**G**	**C**	**H**	total	$R <$	$\#c(p)$
$D = 1$	814	436	712	583	2545	80	114
2	335	173	294	230	1034	40	150
3	882	458	805	605	2750	100	95
7	274	172	279	184	909	40	84
11	189	104	197	109	599	30	189

puted independently one may cross-check if they agree up to the desired precision.

The number and ranges of the computed eigenvalues are displayed in Table 9.3, which also contains the number of prime coefficients for the largest eigenvalues indicating the size of system (9.4). As a reference we list the first forty even and odd eigenvalues for the five examples in Tables 9.1 and 9.2, respectively. The results in connection with lifts and further examples of (lifted) eigenvalues have already been reported in Section 8.

It should be noted that we have not found any other multiple eigenvalues among the sample of all eigenvalues for all D besides those expected from Theorem 8.2 c) and Conjecture 8.1.

All the computed prime coefficients obey the Ramanujan-Petersson bound $|c(p)| < 2$.

To get an impression of the expense of the numerical experiments we report on the following typical cpu-times on a MIPS R8000 processor. For $D = 2$, $R \in (0, 30)$, 20/15, 13.5h; $D = 2$, $R \in (30, 40)$, 10/15, 38h; $D = 7$, $R \in (0, 40)$, 15/10, 10.5h; $D = 11$, $R \in (20, 30)$, 15/15, 45h. Here, for example, 20/15 means that the R-grid has a width of a twentieth of the mean level distance, and at every R we make 15 random guesses for the coefficients. The last number is the cpu-time in hours. To find the eigenvalues for $D = 7$, for example, we did about half a dozen runs of this kind in both of the symmetry classes.

10. Spectrum of the Laplacian III. Statistics. Now we want to demonstrate that the discrete spectra of the Laplacian for Bianchi groups exhibit random fluctuations close to Poissonian. That is the exceptional behaviour of the spectra of arithmetic groups which has been observed in two dimensions is retained in three dimensions. We will consider the level spacing distribution.

To analyze the spectra we have to do the usual desymmetrization and unfolding procedures. Since the use of Hecke operators forces us to compute eigenvalues in **D** and **G**, resp. **C** and **H**, simultaneously we still have to

[4] There might be slight differences to the true number since no proof of completeness is available.

distribute them into the individual symmetry classes. This is an easy task if one looks at the Fourier coefficients of the associated eigenfunctions f. Either $c(m) = c(\overline{m})$ for all m, then f belongs to **D**, resp. **C**, or we have two functions, f and $g = \mathrm{P}f$, with the same eigenvalue and Fourier coefficients related by $c(m; f) = c(\overline{m}; g)$, then each of **D** and **G**, resp. **C** and **H**, inherits one of these eigenvalues.

Define for a given sequence of eigenvalues (R_i) the level counting function or spectral staircase

$$N(R) = \#\{i \mid R_i \leq R\}$$

and think of it as being built from two parts

$$N(R) = N_\infty(R) + N_{\mathrm{fluct}}(R),$$

where N_∞ is a smooth function describing the average increase in the number of levels and N_{fluct} describes the fluctuations around the mean, $\lim_{X\to\infty} \frac{1}{X}\int_1^X N_{\mathrm{fluct}}(x)\,dx = 0$. The unfolded spectrum is obtained by setting

$$x_i = N_\infty(R_i),$$

and the sequence of (nearest-neighbour) level spacings, defined by

$$s_i = x_{i+1} - x_i,$$

will have a mean value of one as $i \to \infty$.

To perform the unfolding for the numerically given eigenvalues we have used Weyl's law in the following form

$$N_\infty(R) = \frac{1}{4}\frac{\mathrm{vol}\,(\mathcal{F})}{6\pi^2}R^3 + a_1 R^2 + a_2 R\log(R) + a_3 R + a_4,$$

where the constants for the Picard group ($D = 1$) are borrowed from Matthies' [16] desymmetrized Weyl's laws, and for $D = 2, 3, 7, 11$ we have done least mean square fits of $N_\infty(R)$ to the true $N(R)$. The first summand is just the Weyl term in three dimensions.

The relevant probability densities for finding a spacing of given size are

$$\begin{aligned} P_{\mathrm{Poi}}(s) &= e^{-s}, \\ P_{\mathrm{GOE}}(s) &\simeq \frac{\pi}{2}se^{-\frac{\pi}{4}s^2}, \end{aligned}$$

where the Poisson distribution holds for a sequence of equidistributed random numbers, and GOE (Gaussian orthogonal ensemble) refers to the eigenvalues of large symmetric random matrices, see [17] for details. P_{GOE} is well approximated by Wigner's surmise, as displayed.

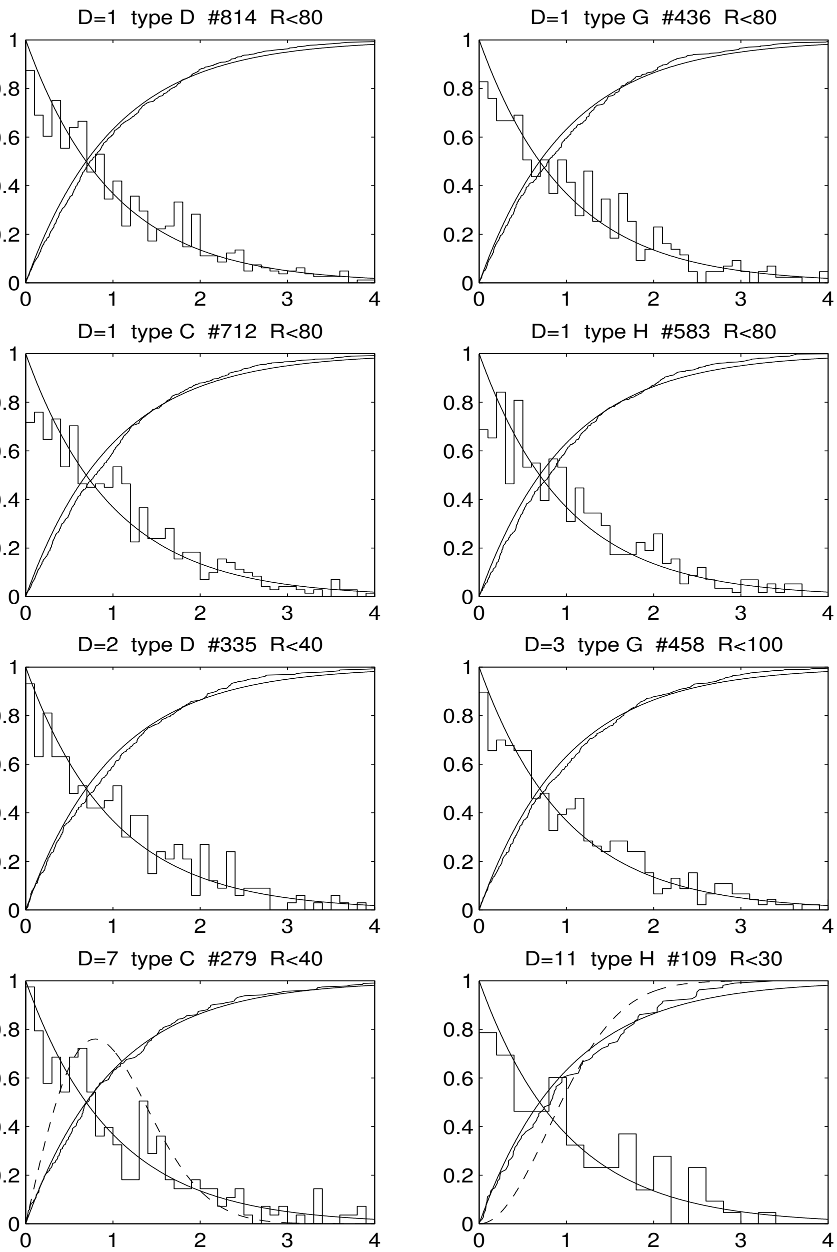

FIG. 10.1. *Level spacing distributions for* $\mathrm{PSL}(2, \mathcal{O})$. *The headings indicate the group, the symmetry class, the number of eigenvalues, and the eigenvalue range* ($\lambda = 1 + R^2$)*, respectively. The abscissae display the spacings* s. *The curves starting at the origin are the integrated distributions, showing the fraction of spacings up to a given length. For comparison, the smooth curves show the Poissonian and the dashed ones GOE.*

While generally the Poissonian is expected for integrable systems and GOE behaviour for time-reversal invariant chaotic systems, we recognize for the arithmetic Bianchi groups level spacing distributions close to Poissonian. Figure 10.1 displays this remarkable feature for eight of the twenty desymmetrized spectra under consideration. The figures of the remaining twelve look similarly. Observe that even for $D = 11$, symmetry $\mathbf{H}$, where we know only about a hundred eigenvalues, the agreement is reasonably well. Nevertheless a closer look at the figures shows there are some systematic deviations from the Poissonian. The number of observed small spacings is less, and the number of medium spacings bigger than expected. But we note that for any of these spectra the empirical distribution is not stationary with respect to the number of levels included: it comes the closer to the Poissonian the more levels are taken into account.

A similar behaviour has previously been observed in two dimensions for $\mathrm{PSL}(2,\mathbb{Z})$, where a sample of eigenvalues in the range of the 20000-th level is close to Poissonian whereas the low levels show notable deviations, and to a somewhat smaller extent for a compact arithmetical billiard, see [8] and [4]. We may close with the following

CONJECTURE 10.1. *The level spacing distributions of Laplace eigenvalues for Bianchi groups approach the Poisson distribution as the number of levels included goes to infinity.*

In view of [4,8], one is tempted to regard this as a universal feature of arithmetic systems.

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