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On the Reducibility of Relations: Variations on a Theme of Peirce

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Introduction. Throughout his philosophical career C.S.Peirce spent much effort on the logic of relations, which represents the mathematical aspect of his doctrine of categories. His main theses on this field were that, first, some of the fundamental relations which figure in human theoretical action are irreducibly triadic (in a sense to be explained below), the most important among them being the sign relation ("a means b for c"); and second, that tetradic, pentadic and higher relations could always be reduced to triadic ones, thereby corroborating the completeness of his list of categories.

Now a satisfying notion of reducibility of relations requires the full-blown formalism of predicate logic which Peirce did not have at his disposal [1)]. He allowed only the simplest possible compositions of relations, and in fact within his limitations the irreducibility thesis for ternary relations is rather obvious (see below), whereas the reducibility thesis for higher relations can easily be refuted by a counterexample, as was shown by Herzberger [2)]. Herzberger further showed that the irreducibility of triadic relations breaks down if one allows a slightly more general, but still natural composition procedure: all relations can then be reduced to binary ones if the domain of the relation is large enough, for example, infinite (prop.1). This elementary yet amazing result deserves to be more widely known than it seems to be. Another purpose of this note is to put the reducibility problem into its natural mathematical environment, which apparently has not been done so far. We give a formal proof that a reduction to unary relations (predicates) is generally impossible (prop.2). Next we show that, in contrast to the infinite case, relations on finite sets are "generically" irreducible, if irreducibility is given a practicable meaning (prop.3). Finally we adress the questions of criteria for reducibility and of graphical representations. As we go along, there arises a number of problems which may challenge the interested reader.

The formal setting. We work within a first order language $L_1(R)$ of relations, allowing sufficiently many relation symbols of arbitrary valency. (For traditional reasons we call the place number of a relation its *valency*. Peirce considered his "bonding" of relations as an analogue to the bonding of molecules as described in chemistry.) Function symbols are not required, and constants are superfluous since distinguished elements can always be thought of as incorporated in the relations. As usual, we allow for countably many variables. Thus the formulas of our language are precisely those relations which can be expressed by the given ones, making unlimited use of junctors and quantifiers. The valency of a formula is the number of its free variables. Note that the language of set theory is contained in $L_1(R)$, so our formal setting is certainly wide enough. In fact this should be so, since, as already Hume recognized, all of human cognition consists in relations.

By the theorem on prenex normal form, every formula S of $L_1(R)$ is logically equivalent to a formula

(1) $$Q_1 \;.\;.\;.\;.\; Q_n \; R \quad ,$$

where every Q_i is a quantifier and R is a formula without quantifiers. To R we can apply the theorem on disjunctive normal form from propositional logic and, since there are no function symbols, we obtain that R is logically equivalent to a relation of the form

(2) $$\vee_i \, (\wedge_j \, R_{ij}) \; ,$$

where the R_{ij} are undefined relation symbols or their negates. Clearly, the valency of S is

(3) $$v(S) = v(R) - n \; ,$$

and v(R) is the number of variables occurring in the totality of the R_{ij}. We call S *reductive* if $v(S) > v(R_{ij})$ for all i,j.

The bonding of variables is effectuated by the quantifiers. Peirce seems to have considered only bonding by the existential quantifier; but there are natural examples which use the universal quantifier, e.g.

two persons x,y are of equal strength <-> for all weights w, x can lever up w if and only if y can.

Since S is reducible if and only its negate is, we can in fact dispense of the universal quantifier by means of the equivalence $(\forall x \; R) \; \text{<->} \; (\neg \, \exists \, x \, \neg \, R)$, a common practice in formal logic; however, the prenex normal form must then be suitably modified: every Q_i is an existential quantifier or the negation of one. But another restriction maintained by Peirce is of a decidedly non-formal character: he allowed only bonding in pairs, which means that a variable which is bound by a quantifier occurs exactly twice in the scopus of that quantifier. It does not seem that Peirce has given any convincing arguments for justifying this restriction, and in fact, the perfectly natural example of

(4) persons x,y,z are relates <-> they have a common ancestor,

to be formalized as

$$R(x,y,z) \; \text{<->} \; \exists w \;\; A(x,w) \wedge A(y,w) \wedge A(z,w) \; ,$$

suggests that there are no such arguments. If one accepts the restriction, and considers only existential bonding of two relations, the irreducibility thesis for ternary relations is obviously true, since bonding any number of binary relations produces a relation of valency 2. (This is an analogue to the fact that the real line is the only noncompact one-dimensional manifold.) In contrast, higher valencies can be obtained from ternary ones; for example, bonding two ternary relations along one variable gives a tetradic relation. More generally, if R is obtained from relations S,T by bonding along k variables, formula (3) becomes

(3a) $$v(R) = v(S) + v(T) - 2k ,$$

which again confirms Peirce's theses in this case; if $v(R) = 3$ and $k > 0$, at least one of $v(S)$ and $v(T)$ must be ≥ 3, whereas for example $4 = 3 + 3 - 2$. Formula (3a) was known to Peirce and apparently was his main *mathematical* reason for his two theses. But of course the equation does not show that *all* higher relations can be thus obtained; and in fact, as mentioned above, Herzberger has constructed a counterexample on a domain containing only 4 elements.

There is also the possibility of reduction without quantifiers, the simplest cases being

$$R(x,y) \leftrightarrow S(x) \wedge T(y) \quad \text{or} \quad R(x,y) \leftrightarrow S(x) \vee T(y) ,$$

where S,T are unary relations (predicates); set-theoretically, these correspond to

$$R = S \times T \quad \text{and} \quad R = (S \times M) \cup (M \times T) ,$$

where M is the domain. Such cases may be regarded as "degenerate", since there is actually no "proper relation" between the relata. But this cannot be said of

$$R(x,y,z) = S(x,y) \wedge T(y,z) \wedge U(z,x) ,$$

exemplified in the domain of finite line segments in the plane, with

$$R(x,y,z) \leftrightarrow x,y,z \text{ form a triangle or a star} ,$$

$$S(x,y) = T(x,y) = U(x,y) \leftrightarrow x,y \text{ have one common endpoint.}$$

Results on reducibility. We have defined reductivity on the syntactic level, but it is obvious how to transfer the definition to the semantics. A relation R on a set M is called *reducible* if there is an interpretation of $L_1(R)$ on M such that R becomes the interpretation of some reductive formula. We begin with the main result:

Proposition 1 : (Herzberger) *Let* R *be a relation of valency* $v > 2$ *on the set* M. *If* card R $\leq$ card M, *then* R *is reducible*.

Proof: from the hypothesis there is an injection f: R --> M. We use f to define v functions R --> M^2 by

$$g_i(\underline{x}) = (x_i , f(\underline{x})) , i = 1,...,v, \quad \text{where} \quad \underline{x} = (x_1,...,x_n)$$

and corresponding binary relations $G_i = g_i(R)$. In other words,

$$G_i(u,v) \ \text{<->}\ v = f(\underline{x}) \text{ for some } \underline{x}\in R \text{ and } u = x_i .$$

From these definitions it follows easily that

$$R(\underline{x}) \ \text{<->}\ \exists\, w\, (G_1(x_1,w) \wedge ... \wedge G_v(x_v,w)) . \tag{5}$$

Namely, if $R(\underline{x})$ holds, take $w = \underline{x}$. If the right-hand side holds, there is a *unique* $\underline{x}\in R$ with $w = f(\underline{x})$ and x_i = i-th component of $\underline{x}$, $i = 1,...,v$, since f is injective. Thus R can be reduced to v binary relations.

It is a well-known theorem of cardinal arithmetic that card M^V = card M for infinite M and any finite v, so that the hypothesis of proposition 1 is satisfied in this case (here the axiom of choice is used implicitly). Let us write, for a binary relation S,

$$S(x) = \{y \mid (x,y) \in S\} = p_2\, p_1^{-1}(x),$$

where the p_i are the obvious projections. Then we have from the proof if proposition 1

Proposition 1': *For any infinite set* M *and any relation* R *of valency* v > 2 *on* M *there exist binary relations* $S_1,...,S_v$ *on* M *such that*

$$R(x_1,...,x_v) \ \text{<->}\ S_1(x_1) \cap ... \cap S_v(x_v) \neq \emptyset .$$

Another aspect is pointed out by

Proposition 1'': *Any relation of valency > 2 becomes reducible after a suitable enlargement of the domain.*

In fact, in order to satisfy the hypothesis of prop.1 one only has to form the coproduct (disjoint union) of M with a set of cardinality card R, and for this the set R itself is the natural candidate.

Proposition 1 is surprising for various reasons. First, its proof is a striking example for the usefulness of "ideal elements" in mathematics. Second, given some "manifold" of mathematical objects, objects with "special properties" usually live in "submanifolds of strictly smaller dimension" (think of reducibility of polynomials). Third, there are relations in mathematics which seem to be "generically" of higher valency, a basic example being linear independency of vectors. There should be applications of the proposition in all branches of mathematics. Particularly near at hand is geometry: in prop. 1', take $M = \mathbb{R}$, $v = 3$. For "concrete" objects R, say a surface or a 3-manifold with boundary embedded in space, there should be "concrete" functions f. This seems quite delicate, since there are no homeomorphisms between open sets in different dimensions.

To obtain a feeling for the construction, it is easier to define R via given S_i ; for instance, if $S_1 = S_2 = S_3$ is $[0,1] \times$ any nonvoid set of real numbers, one obtains the unit cube. (Exercise: describe R in the case $S_1 = S_2 = S_3 =$ unit circle.) This illustrates that the S_i are by no means unique, which is also clear from the proof of prop.1.
Note that the proof remains valid for $v = 2$, and though statement (5) then becomes pointless from the point of view of reducibility, it is not trivial in itself.

Analyzing the proof, one also sees that there are other possibilities for a reduction to lower valencies; in fact, every partition of the set of variables into sets of cardinality at most n-2 provides a reductive formula universally applicable in all infinite domains.

In this sense Peirce's irreducibility thesis for ternary relations is not only false, but its "contrary" is true. However, this is a purely mathematical fact, and the question remains to which extent the thesis is true in the actual processes of human theoretical action which were Peirce's main concern. The following discussion points to a possible difference.

The idea of the proof of prop. 1 was destilled by Herzberger from an example used by Peirce to demonstrate the reducibility of tetradic relations: "Take the quadruple fact that a sells b to c for the price d. This is a compound of two facts: first, that a makes with c a certain transaction, which we may name e; and second, that this transaction e is a sale of b for the price d. Each of these two facts is a genuine triple fact, and their combination makes up [as] genuine [a] quadruple fact as can be found." [3)] Peirce uses a single "ideal element" e, the sale, to reduce a tetradic relation to two ternary ones; comparing the argument with the proof of proposition 1, one sees that one can equally well employ the same ideal element to obtain a reduction to four binary relations. The relations $S_1,..,S_4$ of the proof of the proposition would read as "a makes a sale", "b is the object of a sale", "c is the customer in a sale" and "d is the price payed in a sale". Note that the concrete relata a,b,c,d, two persons, an object and an amount of money, are connected by the abstract relatum "sale". But in the same way the sign relation "a means b to c" can be "formally" reduced according to the scheme (5), the ideal element being a "signification": "a is the sign in a signification", "b is the signified content of a signification ", "c is the recipient of a signification". This sort of reduction may be called "formal" because in order to achieve it the original domain of the relation must be enlarged as to contain "ideal elements". Having used the abstractum "sale" in his own example, Peirce would have been forced to accept the abstractum "signification" for the reduction of the sign relation; but he had banned triple bonding. Example (4), in contrast, contains no abstractum and therefore might be called a "material" reduction (in fact, it is the definition of R in this case). For another example of a "material reduction", consider the sign relation "enriched by a situation", namely "a means b to c in situation d", which is reducible to a binary relation plus the sign relation, "if c is in situation d, then a means b to c". If we want to capture the difference of formal and material reduction in mathematical terms, we will probably have to ask to which extent an injection $f: R \dashrightarrow M$ can be given "naturally". It is well known that for $M = \mathbb{N}$ a bijection $M^V \simeq M$ can be defined recursively, whereas in general we only have an existential statement.

To round off the discussion, we show that a further reduction, namely to unary relations, is impossible in general, thereby confirming the threshold role played by 2 in creating multitude and complexity.

<u>Proposition 2</u>: *On any infinite set, there exist binary relations which are not reducible to predicates.*

Proof: If one thinks of examples as the unit circle in the plane, this seems obvious; yet a rigorous general proof requires an argument. Suppose that the binary relation R is given by a formula (1) with all R_{ij} of valency 1. We can assume that only existential quantifiers are involved, with or without negations. We now use that the existential quantifier is distributive over disjunctions,

$$\exists\, x\ (A(x) \vee B(x)) <\!\!-\!\!> (\exists\, x\ A(x)) \vee (\exists\, x\ (B(x)) \ ,$$

plus some Boolean algebra and obtain that R is a disjunction of relations

$$S(x,y) <\!\!-\!\!> Q_1\, x_1 \ldots Q_n\, x_n\ \ A_1(x_1) \wedge \ldots \wedge A_n(x_n) \wedge A(x) \wedge B(y),$$

where each Q_i is an existential quantifier or the negation of one. In other words,

(6) $\quad S(x,y) <\!\!-\!\!> A(x) \wedge B(y)$ and certain sets, which do not depend on x or y, are empty/not empty.

Now a relation R which is an m-fold disjunction of relations (6) has the following property:

(7) if $(x_1,y_1),\ldots,(x_{m+1},y_{m+1})$ are m+1 pairs satisfying R, then there are at least two pairs $(x_i,y_i),(x_j,y_j)$ such that (x_i,y_j) and (x_j,y_i) also satisfy R.

But any infinite set M contains a copy N of the natural numbers, and the successor relation on N certainly does not possess property (7). This proves proposition 2. Note that we have implicitly used the axiom of choice in this proof.

Let us now turn to finite sets. Here the reducibility question becomes trivial, since every relation on M of valency v can be viewed as a subset of M^v, and therefore can be represented as a finite disjunction of v-fold conjunctions of unary relations, each of which is satisfied by exactly one element. (Together with the previous results, this implies the remarkable fact that only binary relations on infinite sets have a chance to be irreducible.) This "extensional" reduction, however, is devoid of any practical or philosophical value. To be realistic, we must confine ourselves to a finite supply S of "reduction schemes", i.e. formulas (1) which are reductive. We say that a relation R on the set M is S-*reducible*, if R coincides with some interpretation on M of some member of S. Proposition 1 shows that on an infinite set every relation of valency >2 is S-reducible, where S consists of the single formula exemplified by (5). Denote by R(v,m) the number of relations of valency v on an m-set; R(v,m) is equal to the cardinality of the power set of M^v , namely exp($m^v \times$

log 2) .

Proposition 3: *Let* S *be any finite set of reduction schemes. Then for every valency* v > 1, *the quotient*

(8) (number of S-reducible v-relations on an m-set) / R(v,m)

tends to zero for m *tending to infinity.*

In this sense relations on finite sets are "generically" irreducible.

The proof is a simple counting argument. Given a formula (1), there are at most R(v-1,m) possible choices for every R_{ij} . Thus the numerator in (8) is bounded from above by $R(v\text{-}1,m)^c$, where c depends on S, but not on m. Since

$$c \times m^{v-1} - m^v \quad \text{-->} \quad -\infty \quad \text{if} \quad m \text{-->} \infty ,$$

the result follows from the formula for R(v,m).

Criteria for reducibility. Given a relation R on a set M, we ask whether or not it is reducible. A naive attack on the problem would include quantification over all relations of lesser valency. This being impossible, we again ask for a "realistic" criterion. Such a criterion should consist in testing R on finitely many vectors $\underline{x}$ and comparing the results according to a fixed scheme. In other words, the test should consist in a sentence (closed formula) of $L_1(R)$, involving only one relation symbol. Of course such a test can work only for some specific valency, and can only give a necessary condition. In view of the above discussion, we only need a test for binary relations (and only on infinite domains), and such a test was obtained in the proof of prop.3, namely formula (7). This was due to the fact that composition of unary relations is of very limited complexity.

A meaningful question arises if we ask, given a reductive formula F, whether or not a given relation R is an interpretation of F. There can be no test for a formula such as (5) which allows reductions over all infinite domains. For by prop.1 the test formula would have to true for all relations of valency v > 2 on infinite domains. But then it would have to be tautological, since otherwise its negation would have a model, and that model could be enlarged to an infinite model, contradiction. So the test formula is tautological, but this contradicts prop.3.

But for other cases there do exist test formulas. As was noted above, a relation of type

(9) $$R(x,y) \text{<->} A(x) \wedge B(y)$$

satisfies the formula

(10) $$R(x,y) \wedge R(x',y') \ \text{->}\ R(x,y') \wedge R(x',y) .$$

But the converse also holds: let R(x,y) be any binary relation, and let A, B be the projections of R to the first and second coordinate. Then clearly $R(x,y) \text{ -> } A(x) \wedge B(y)$. Suppose that (10) holds, and that $x \in A$, $y \in B$. Then there are x',y' such that R(x,y') and R(x',y). From (10), one obtains R(x,y), hence $R = A \times B$.

Similarly, a ternary relation of type

(11) $$R(x,y,z) \text{ <-> } A(x,y) \wedge B(y,z) \wedge C(z,x)$$

satisfies the formula

(12) $$(R(x,y,z') \wedge R(x,y',z) \wedge R(x',y,z)) \text{ -> } R(x,y,z) ,$$

and it is easy to show that from (12) it follows that R is of type (11): let A, B, C be the projections of R to the (x,y), (y,z), (z,x) coordinates, respectively. Again the direction -> in (11) is clear. But if the right-hand side of (11) is true, we must have x',y',z' satisfying $R(x,y,z') \wedge R(x',y,z) \wedge R(x,y',z)$ from which we infer R(x,y,z) by (12). It looks plausible that such test formulas exist for all formulas (reductive or not) involving no quantifiers; but I have not pursued this in any further detail.

Formulas and graphs. Peirce as well his commentators have tried to visualize the diverse schemes of composition of relations by graphs [4)]. Here is the method which seems (to me) to be the correct one. We confine ourselves to the case of formulas F for which the kernel (2) contains only a single conjunction.

The associated graph G(F) is bipartite, containing vertices of type V, corresponding to all variables occurring in F, and of type R, corresponding to all relation symbols occurring in F. Every vertex representing a variable x is connected to every relation symbol R_{ij} in which x occurs. From these data, the quantifier-free kernel of F can be reconstructed (up to the order of variables in the single relation symbols, but this is a notational matter and structurally irrelevant). To take care of the quantifiers, it suffices to label each vertex of type V, which represents a bound variable, with a symbol representing the type of quantifier by which it is bound. The number of unlabelled V-vertices is the valency of F, and F is reductive if this number is larger than the order (in the graph theoretical sense) of any R-vertex. When picturing such a graph in the plane, it is convenient to write the free V-vertices in the top line, the R-vertices in a second and the bound V-vertices (the "mediating" variables) in a third line. (In the examples which follow, only the existential quantifier is used, so we omit the labelling).

The "line" obtained by bonding n binary relations, that is

$$R(x,y) \text{ <-> } \exists\, x_1 \ldots \exists\, x_{n-1}\ R_1(x,x_1) \wedge R_2(x_1,x_2) \wedge \ldots \wedge R_n(x_{n-1},y)$$

corresponds to

$$\begin{array}{ccccccc} x & & & & & & y \\ & R_1 & R_2 & \ldots & & R_n & \\ & & x_1 & & x_{n-1} & & \end{array}$$

The universal reductive formula (5) has the graph

$$\begin{array}{ccccc} x_1 & . \; . & & . \; . & x_v \\ R_1 & . \; . & & . \; . & R_v \\ & & w & & \end{array}$$

One easily sees

<u>Proposition 4</u>: *The graph* G(F) *is connected if and only if* F *is not* $\wedge$*-reducible, i.e. does not have the form* F <-> R $\wedge$ T, *where* R *and* T *have no common free variables.*

It would be interesting to know if there is a logical counterpart of simple connectivity. The simplest relation having a non-simply connected graph is the predicate

$$P(x) \leftrightarrow \exists w\, R(x,w) \wedge S(x,w).$$

Finally, we may try to approach the question of reducibility graph-theoretically. Peirce's own example of reduction, quoted above, would become the transition from

$$\begin{array}{cccc} a & c & b & d \\ & R & & \end{array} \quad \text{to} \quad \begin{array}{cccc} a & c & b & d \\ S & & T & \\ & e & & \end{array}$$

where R(a,c,b,d) <-> "a sells b to c for the price d" , S(a,c,e) <-> "a performs a sale e to c", T(b,d,e) <-> "b is the object of a sale e with price d". Graph-theoretically, a vertex of order 4 is replaced by two vertices of order 3, joined by the additional mediating vertex e. It is rather obvious how to generalize the procedure to vertices of arbitrary order, leaving only vertices with orders at most 3. A certain analogue can be found in highway systems having no crossings but only simple turn-offs. Clearly the orders cannot be further reduced in general, since a connected graph with vertices of orders at most 2 is a circle or a line; note that our universal reductive formula (5) creates a single mediating vertex of order v. But it is not clear to me whether the reduction process can be given a "canonical" form (so as to make it a functor on a suitable category of graphs) and which further graph-theoretical properties it has.

References

1) See B.Russell, The Principles of Mathematics, London 1937, section 212 ss. for a discussion of the reducibility question from the point of view of modern logic, which however is still informal rather than formal. The elementary argument in the proof of prop.2 can hardly be new; but I have not seen it before.

2) H.G.Herzberger, Peirce's Remarkable Theorem, in: Sumner/Slater/Wilson (eds.), Pragmatism and Purpose, Toronto 1981: Toronto University Press, p.41 ss.

3) C.S.Peirce, Collected Papers, 1.363, 1.371.

4) See C.Hookway, Peirce, London 1985: Routledge & Kegan, p.97 ss.