HAMBURGER BEITRÄGE
ZUR MATHEMATIK
Heft 200

Combinatorics of Quantifiers

Ernst Kleinert, Hamburg

September 2004

Combinatorics of quantifiers

by Ernst Kleinert

1. Introduction. It was an essential step (due to F.W.Lawvere) in the development of categorical logic to view quantifiers as operators (more precisely, as functors). Adopting this view and considering relations of valency v on an nonempty set M, one obtains a semigroupoid of operators on the lattice of power sets $P(M^k)$, $1 \leq k \leq v$, where the lattice structure is provided by the various projections, according to the Boolean structure of $P(\{1,...,v\})$. If all variables are bound, the values lie in the object $P(M^0) = P(1) = \{T,F\}$ of truth values (the subobject classifier) of the category of sets. In addition to this semigroupoid structure, which is rather trivial, the quantificational expressions carry a structure of logical dependence, which constitutes a preorder. The main interest lies in the "full" expressions (those which bind all v variables). Dividing out logical equivalence one obtains a poset, involving in its structure the ordered partitions of v and possessing numerical invariants which look quite curious. Further there is a Galois correspondence (a model theory en miniature) between sets of quantificational expressions and sets of relations on M. The main purpose of this paper is to present these elementary, yet nontrivial structures which seem to have been unnoticed up to now.

2. Quantifiers as operators. First we recall the generalities; more details can be found in [MM], Ch.I.9. Let f: M --> N be a set map. The contravariant powerset functor then provides a morphism of Boolean algebras P(f): P(N) --> P(M) by taking preimages. Also associated with f are two maps P(M) --> P(N), namely

$$(\exists f)(S) = f(S), \quad (\forall f)(S) = \{n \mid f^{-1}(n) \subset S\},\ S \subset M,$$

evidently satisfying, for $S \subset M$, $T \subset N$,

$$(P(f)(T) \subset S) \leftrightarrow (T \subset (\forall f)(S)\ ,$$
$$(S \subset P(f)(T)) \leftrightarrow ((\exists f)(S) \subset T)\ .$$

These equivalences say that, when P(f) is viewed as a functor, ∃f (∀f) are left (right) adjoint to P(f). Note that these maps are in general only morphisms of posets, not of Boolean algebras. If f is onto, one checks

$$(2.1) \qquad (\forall f) \circ P(f) = (\exists f) \circ P(f) = id_{P(N)}\ .$$

Now we specialize to projections. Let M be a nonempty set and $v \geq 2$ a natural number. For i = 1,...,v there are projections

$$p^i : M^v \text{-->} M^{v-1} \ ,$$

omitting the i-th coordinate. Let us write e_i and a_i for the left and right adjoints of P(i): = P(p^i). Thus by definition we have, for $S \subset M^v$,

$$e_i\,(S) = \{(m_1,..,m_i,...,m_v) \mid \exists m_i\ (m_1,..,m_i,...,m_v) \in S\ \}\ ,$$

(2.2)

$$a_i\,(S) = \{(m_1,..,m_i,...,m_v) \mid \forall m_i\ (m_1,..,m_i,...,m_v) \in S\}\ .$$

For v = 1 we define e, a : P(M) --> P(1) = {T,F} by

(2.3) e(S) = T iff S is nonempty, a(S) = T iff S = M.

These definitions do not yet permit a composition of the various e's and a's. Allowing for a major abuse of notation, we denote by p^i the operation of omitting the i-th variable, *as long as this operation makes sense*, i.e. as long as there is an i-th variable and at least one more, but regardless of which variables other than i are still in the play. Analogously we extend the definition of the operators e_i and a_i. For them the additional variable is not necessary, by definition 2.3. In this way we obtain a semigroupoid of operations, acting on the lattice of the various P(M^k), k = 1,...,v. The e_i and a_i satisfy the relations

$$e_i\, e_j \ = \ e_j\, e_i \ , \ a_i\, a_j \ = \ a_j\, a_i \quad \text{if } i \neq j\ ,$$

as is easily verified, but none other in general (unless M = 1), since, as everyone knows, one must not interchange the order of different quantifiers.

We shall also need the P(i). Since projections are onto, we have from (2.1) that

(2.4) $$e_i\, P(i) \ = \ a_i\, P(i) \ = \ \mathrm{id}$$

is the identity on the powerset of every partial product of M^v not containing the i-th coordinate. It is also straightforward to show that

(2.5) $$e_i\, P(j) \ = \ P(j)\, e_i\ , \quad a_i\, P(j) \ = \ P(j)\, a_i \quad \text{if } i \neq j\ ;$$

these maps go from every partial product containing the i-th, but not containing the j-th coordinate to the corresponding partial product containing the j-th and not containing the i-th coordinate.

In the formal language, the P(i) serve to "blow up" relations by introducing "ghost variables". For example, if R = R(x,y) is a binary relation, then P(z)R is the ternary relation S defined by

$$S(x,y,z) \text{ <-> } R(x,y)$$

(a "degenerate" relation in the terminology of Peirce).

We now free ourselves from the set M and consider an abstract semigroupoid $Q_0(v)$ on the alphabet $\{e_1, \ldots, e_v, a_1, \ldots, a_v\}$, consisting of words in which every index i occurs at most once. In $Q_0(v)$ we concentrate on the subset $Q_1(v)$ consisting of those words in which all indices occur, equivalently words of length v. In terms of quantifiers, these are the "full" quantifying expressions, binding all variables. In the sequel, we shall denote elements of $Q_1(v)$ by letters p,q,.. . For every nonempty set M and every $R \subset M^v$, p(R) is an element of {T,F}. We still must convince ourselves that these notations are in accordance with the usual meanings of "true" and "false" for quantificational statements. In order to make this precise, we consider a first order language with variables $x_1,\ldots,x_v$ and a single n-ary relation symbol R. Let us again write a_i and e_i for the quantifiers. For every $p \in Q_1(v)$, the expression pR is then a closed formula of our language. An interpretation of our language in a nonempty set M is given by specifying an arbitrary $R^M \subset M^v$. Then we indeed have

$$p(R^M) = T \iff M \models pR .$$

This simply states that our formalism is nothing else than a translation of the usual (Tarski) semantics into a language of operators. It is easy to check this directly; for a more general statement, see [MM], Ch.V.6.

3. The deductive structure. From our language we obtain a deductive system, which we may call (v-Rel), by adopting the usual rules of inference plus the additional axiom (eR), where $e = e_1 \ldots e_v$. This says that we concern ourselves only with nonempty relations. Logically, it allows us to derive existential statements from universal statements.

Now on the set $Q_1(v)$ there is a preorder, given by deduction. We write $p \to q$ if the formula $pR \to qR$ is a theorem of (v-Rel). From Gödel's completeness theorem we have that $p \to q$ if and only if, for every set M and nonempty $R \subset M^v$, p(R) = T implies q(R) = T. It follows from the axiom (eR) that $a \to p$ and $p \to e$ for all p, where $a = a_1 \ldots a_v$. Thus, a is the "largest" and e is the "smallest" element of $Q_1(v)$.

The symmetric group S(v) operates on $Q_1(v)$ by permuting the indices; let us write p^σ for the element of $Q_1(v)$ obtained from p by applying σ to the indices. Since renaming the variables does not affect logical dependence, we clearly have $p \to q$ if and only if $p^\sigma \to q^\sigma$.

We are now going to study the deductive structure of $Q_1(v)$ in more detail. The key is the following observation which is most plausible intuitively, but of course requires a formal proof.

3.1 Proposition. (1) *Suppose* $p = urw$, $q = usw$, *where* u,r,s,w *are elements of* $Q_0(v)$, *with* u,w *possibly empty. Then* $p \to q$ *if and only if* $r \to s$.

(2) *For any proper subset* $S \subset \{1,...,v\}$ *and* $p \in Q_1(v)$, *let* p_S *denote the element of* $Q_0(v)$ *obtained from* p *by eliminating all quantifiers with indices not in* S. *Then* $p \to q$ *implies* $p_S \to q_S$.

Proof. (1) Note that the "only if" - part follows from (2). For the "if"-part it suffices, by an induction on word length, to show the following: if $r \to s$, then also $a_i r \to a_i s$ and $ra_i \to sa_i$, where i is an index not occurring in r or s, and the same with a_i replaced by e_i. Arguing semantically, we assume $a_i r(R) = T$. Let $m \in M$ be arbitrary and put R' = the elements of R having m as i-th coordinate. From the hypothesis, we have $r(R') = T$. From $r \to s$, it follows that $s(R') = T$. Since m was arbitrary, we also have $a_i s(R) = T$. The argument for e_i is analogous. The proof of $r a_i \to s a_i$ is almost obvious: if $(ra_i)(R) = T$, then also $r(a_i(T)) = T$ and hence $s(a_i(R)) = (sa_i)(R) = T$. The argument for e_i is the same.

(2) It suffices to eliminate one variable at a time. Using the operation of $S(v)$, we may assume that the eliminated variable x_i occurs at the rightmost place of p, say $p = p'a_i$. Write q' for the word obtained from q by eliminating the i-th variable. Now suppose $p'(R) = T$. Using 2.4 and the hypothesis, we have

$$T = p'(a_i P(i))(R) = p(P(i))(R) = q(P(i))(R) .$$

In $qP(i)$ we may, using 2.5, shift $P(i)$ to the left until we reach the variable i. Using 2.4 once more, we see that $q'(R) = T$. Again the argument for e_i is the same.

Proposition 3.1 reduces the analysis of $Q_1(v)$ to a purely combinatorial matter. Taking S to contain first one, then two elements, we see: given $p \in Q_1(v)$, one obtains q with $p \to q$ by each of the following operations:

(3.2) (a) Replace an a_i by e_i ;

(b) Replace a subword $e_i a_j$ by $a_j e_i$;

(c) Replace a subword $a_i a_j$ ($e_i e_j$) by $a_j a_i$ ($e_j e_i$).

Note that (a) requires the axiom (eR). In the next paragraph the following notation will be useful: we write $p(i) = a$ (e) if p contains a_i (e_i).

Clearly the operations (c) are reversible. To every p there is associated an ordered partition $[b_1 ,.., b_r]$ of v, $b_1 + ... + b_r = v$, by collecting subsequent quantifiers of the same kind into one block. We may call p and q *block equivalent*, if they give rise to the same partition and differ only by permutations of the quantifiers within the single blocks. By (c), block equivalence implies logical equivalence. We shall see shortly that the converse is also true. This will follow from

3.3 Proposition. *Given* p, *all* q *with* p -> q *can be produced from* p *by repeated application of the operations* 3.2 (a) - (c).

Proof: suppose p -> q. Applying 3.1 (2) we see:

(3.4) If p(i) = e, then also q(i) = e ; if p contains .. a_i ... e_j .. (a_i occurs on the left of e_j), then q must contain .. a_i ... e_j .. or .. e_i ... e_j .. .

We show that (3.4) implies the deducibility of q from p by operations 3.2 (a) - (c). Let r be the word obtained from p by operations (a) for every i such that q(i) = e. Then r(i) = e (a) if and only if q(i) = e (a), and r still satisfies the second clause of 3.4 in a sharper form: if r contains .. a_i ... e_j .. , then so does q . But from this it is evident that q can by obtained from r by pushing to the right some of the e's, that is, by operations (b) and (c). This proves the proposition.

Corollary 1. *Two words are logically equivalent if and only if they are block equivalent.*

This is now evident because the operations (c) are the only reversible ones among those listed in 3.2.

We now identify logically equivalent words and write Q(v) for the set of equivalence classes. Clearly Q(v) is an ordered set by logical dependence.

Corollary 2. *The immediate successors of* p *in* Q(v) *are obtained by operation*s 3.2 (a), *if* a_i *occurs in the first block of* p, *and* 3.2 (b).

Proof: it follows from 3.3 that the immediate successors are to be found among those q obtained by a single application of one of the operations. Clearly (c) has no effect in Q(v). If a_j is preceded by an e_i, then $e_i\, a_j$ -> $e_i\, e_j$ can be factored as $e_i\, a_j$ -> $a_j\, e_i$ -> $e_j\, e_i = e_i\, e_j$. It is evident that operations (b) cannot be factored.

Let S be the operation which permutes e_i and a_i , for all i.

Corollary 3. S *is an antiautomorphism of* Q(v); *that is*, p -> q *implies* S(q) -> S(p).

This follows from an inspection of operations 3.2 (a) - (c).

The proof of 3.3 also yields the following explicit and purely combinatorial criterion for logical dependence:

Corollary 4. *Condition* 3.4 *is necessary and sufficient for* p -> q.

Note the equivalent "dual" version

(3.4 bis) If q(i) = a, then also p(i) = a ; if q contains .. e_i ... a_j .. , then p must contain .. e_i ... a_j .. or .. a_i ... a_j .. .

It is also a remarkable consequence of the above discussion that we can handle quantificational expressions by looking at only two variables at a time. Given the limitations of human intelligence, this is lucky.

4. The graph Q(v) and examples. We can view Q(v) as a graph in the usual way: the vertices are the elements of Q(v), are p and q are joined by an edge if and only if q is an immediate successor of p. Thus our graph is oriented; we let logical dependence point downwards. For v = 2 one obtains a hexagon,

a

e_1a_2 $\quad$ e_2a_1

a_2e_1 $\quad$ a_1e_2

e

For v = 3, the picture is

a

* * *

* * * * * *

* * *

* * *

* * * * * *

* * *

e

According to corollary 2, the first row contains all words of type eaa, belonging to the partition [1,2] ; the next row has all aea, belonging to [1,1,1] and so on. This reveals the most general scheme which applies to all the Q(v). Q(v) is made up of layers, which correspond to the ordered partitions of v; it is well known there are 2^{v-1} of them. On each layer, the symmetric group S(v) operates transitively. There are no deductive relations within the single layers. The antiautomorphism S maps the "upper hemisphere" to the "lower", commuting with the S(v)-operation. Each partition occurs twice, once beginning with an a-block, once with an e-block, the one in the upper, the other in the lower hemisphere; so there are 2^v layers.

The next case, Q(4), is so complicated that a complete picture is uninstructive. Here is the scheme of deductive depence between the layers, in other words the graph Q(4)/S(4). The layers are denoted by the partitions, and the symbol a or e following the partition indicates whether the words begin with an a-block or an e-block. Some of the layers are listed parallel, since there is (in general) no natural ordering among the immediate successors.

[4]a

[1,3]e

[1,1,2]a

[2,1,1]a   [2,2]e

[3,1]a   [1,1,1,1]e

[1,2,1]e

[1,2,1]a

[1,1,1,1]a   [3,1]e

[2,2]a   [2,1,1]e

[1,1,2]e

[1,3]a

[4]e

One sees that the deductive structure has very little to do with the lattice structure on the partitions induced by the coarser/finer relation; this destroys the hope for some "natural" construction which relates Q(v) to the lattice of partitions.

One can check that Q(3) is a lattice, whereas Q(4) is not. To see this, consider the diagram

$e_1\ a_2\ e_3\ a_4$ $e_3\ a_4\ e_1\ a_2$

$e_2\ e_3\ a_4\ e_1$ $e_4\ e_1\ a_2\ e_3$

of logical dependencies. If there were an inf r for the upper two, there would be a diagram

$$\begin{array}{ccc} e_1\, a_2\, e_3\, a_4 & & e_3\, a_4\, e_1\, a_2 \\ & r & \\ e_2\, e_3\, a_4\, e_1 & & e_4\, e_1\, a_2\, e_3 \end{array}$$

From 3.4 it follows that $r(4) = r(2) = a$, $r(1) = r(3) = e$, and that r must contain $.. e_3 ... a_4 ..$, $.. e_1 ... a_2 ..$, $.. a_2 ... e_3 ..$, and $.. a_4 ... e_1 ..$; but these requirements are inconsistent. More generally, one sees from 3.4: if the inf r of p and q exists, it must satisfy

> $r(i) = a$ if and only if $p(i) = a$ and $q(i) = a$; r contains $.. e_i ... a_j ..$
> if and only p and q do so.

This is in general an overdetermination. Of course some infs and sups do exist, for example

$$\inf(p, S(p)) = e, \sup(p, S(p)) = a$$

holds for all p. Thus Q(v) contains "complements", produced by S.

Note that Q(v) can be embedded into Q(v+1) by mapping each p to pe_{v+1} (in fact, one has v+1 such embeddings). Thus for $v \geq 4$ no Q(v) is a lattice.

Note also that the graph Q(3) is planar; it is easy to visualize Q(3) wrapped around a sphere, with a as the north and e as the south pole. I have not yet managed to decide on the planarity of Q(4), but it looks improbable.

5. Some numerology. Clearly $Q_1(v)$ has $2^v\, v!$ elements. To obtain the cardinality of Q(v), one has to sum up the number of words p belonging to each partition $b = [b_1 ,..., b_r]$. Since the indices may be permuted arbitrarily within the single blocks, we see that there are $b! := v!/b_1! ... b_r!$ such words. Since p may begin with an a or an e, we have

$$|Q(v)| = 2 \sum b!$$

where the sum extends over all ordered partitions b of v. To count the number of edges of Q(v), it suffices to count the number of immediate successors of each p, which follows from corollary 2. If p belongs to $[b_1 ,..., b_r]$, the number of immediate successors of p is

$$(5.1)\qquad \begin{array}{ll} b_1 + b_2 b_3 + ... & \text{if p begins with an a,} \\ b_1 b_2 + b_3 b_4 + ... & \text{if p begins with an e;} \end{array}$$

here it is understood that $b_i = 0$ if $i > r$. For v = 3, one obtains 26 vertices and 42 edges, for $v = 4$, there are 150 vertices and 344 edges. The most fundamental invariant of a connected graph is its Betti number (the free rank of the fundamental group or, equivalently, the free rank of the first homology). By a standard procedure (see e.g. Spanier [S], p. 141) we obtain 17 for $v = 3$ and $195 = 3.5.13$ for $v = 4$. One would like to understand how these crude numbers find their way into the seemingly "canonical" scheme of quantifications.

Let us have a closer look at immediate successors and predecessors. Suppose that there are immediate transitions between a layer corresponding to $b = [b_1, ..., b_r]$ and one corresponding to $c = [c_1, ..., c_s]$. Consider the bipartite subgraph of $Q(v)$ consisting of these layers and the successor arrows from b to c. From the transitive operation of $S(v)$ on both layers in follows that from every word in b there starts the same number of arrows to c, and at every word of c there arrives the same number of arrows from b. Thus we have an equation

$$\#\{\text{arrows starting from each word in b}\}\ b! = \#\{\text{arrows arriving at each word in c}\}\ c!$$

(Note that there may be other arrows from b to a layer different from c, and c may receive arrows from a layer different from b.) Given $b = [b_1, ..., b_r]$, the possibilities for c are as follows:

(5.2) (i) $b \to [1, b_1-1, b_2, ..., b_r]$ if the b-layer begins with a and $b_1 > 1$;

(ii) $b \to [b_2+1, b_3, ..., b_r]$ if the b-layer begins with a and $b_1 = 1$.

These are the successors reached by operations 3.2a; in both cases, there are b_1 arrows. Operations 3.2b are applicable to every a-block preceded by an e-block. We only list the cases where the e-block is the first block of p:

(iii) $b \to [b_1-1, 1, 1, b_2-1, b_3 ..., b_r]$ if $b_1, b_2 > 1$;

(iv) $b \to [1, 1, b_2-1, ..., b_r]$ if $b_1 = 1, b_2 > 1$;

(v) $b \to [b_1-1, 1, b_3+1, ..., b_r]$ if $b_1 > 1, b_2 = 1$;

(vi) $b \to [1, b_3+1, ..., b_r]$ if $b_1 = b_2 = 1$.

In all cases, there are $b_1 b_2$ arrows. The same distinctions have to be made when 3.2b is applied in the interior of b. One notes that b! divides c! or vice versa except in case (v).

6. A Galois correspondence. Let M be nonempty. With every subset $A \subset Q(v)$ we associate the set

$$\mathcal{R}(A) = \{R \subset M^v \mid p(R) = T \text{ for all } p \in A\}$$

of all relations on M satisfying all expressions of A. If $R \subset S$ and $p(R) = T$, then also $p(S) = T$ (recall that the relational statement $R(x_1,...,x_v)$ means $(x_1,...,x_v) \in R$). Thus $\mathcal{R}(A)$ is an upwards closed set of relations. It is obvious that $A \subset B$ implies $\mathcal{R}(B) \subset \mathcal{R}(A)$.

In the other direction we associate with every set C of v-relations on M the subset

$$\mathcal{Q}(C) = \{p \in Q(v) \mid p(R) = T \text{ for all } R \in C\}$$

of quantificational expressions satisfied by all R in C. This is a downwards closed subset of $Q_1(v)$, and again it is obvious that $C \subset D$ implies $\mathcal{Q}(D) \subset \mathcal{Q}(C)$. In total, we have a Galois correspondence between the set of downwards closed subsets of Q(v), ordered by inclusion, and the set of upwards closed subsets of $P(M^v)$, also ordered by inclusion.

As in all cases of Galois correspondences, it is essential to ask (in categorical terms) for the unit and counit of the adjunction; we concentrate on one-element subsets and start with the syntactic side. Since $p(R) = T$ and $R \subset S$ implies $p(S) = T$, there are minimal subrelations of R satisfying p. Such minimal relations can be written down explicitly, as can be grasped from the simplest nontrivial examples. From now on, we assume that $|M| > 1$ and denote by 1 a distinguished element of M.

For $p = a_1 e_2$ we set

$$R(p,f) = \{(m, f(m)) \mid m \in M\} ,$$

where f is any function M --> M. For $p = a_2 e_1$ we take

$$R(p,f) = \{(f(m), m) \mid m \in M\}$$

and further

$$R(e_1 a_2) = 1 \times M, \quad R(e_2 a_1) = M \times 1 .$$

It is immediate that these are indeed minimal relations satisfying the respective words, and that they are the most general ones, if we vary f and the distinguished element 1. One should also note that

$$a_2 e_1 (R(e_1 a_2)) = a_2 (M) = T$$

as it should be, whereas

(6.1) $$e_1 a_2 (R(a_2 e_1)) = e_1 (\{n \mid \forall m \, (n,m) \in R\} = e(\emptyset) = F ,$$

unless f is constant. Generalizing this scheme one can construct minimal relations satisfying a given p as follows. Let p belong to the partition $[b_1 ,..., b_r]$, and let $b_i \, b_{i+1}$ be an a-block of p followed by an e-block. At all indices corresponding to the variables occurring in b_i, insert a factor M at the i-th position in M^v. For every variable occurring in b_{i+1}, choose a function from the previously inserted product into M and insert the values at the index of that variable, together with the arguments. In this way one has to treat all pairs of a-e-blocks. If p begins with an e-block, insert a factor 1 at all indices occurring in this block; if p ends with an a-block, insert factors M at all indices occurring in this block. For example, if $p = e_2\, a_4\, e_1\, e_3\, a_5$, one obtains by this recipe the relation

$$R(p,f,g) = \{(f(m),1, g(m), m, n) \mid m,n \in M\} .$$

Again one sees that, varying all functions and the distinguished elements (the latter independently at all indices), one obtains all minimal relations satisfying p. If all functions are identically 1, one obtains relations of the form

$$R = M_1 \times ... \times M_v$$

where each M_i is = 1 or = M. For such R one has

$$a_i(R) = p^i(R), \text{ if } M_i = M, \quad a_i(R) = \emptyset , \text{ if } M_i = 1.$$

From this it is easy to see: if $q \in \mathcal{Q}\mathcal{R}(\{p\})$, then q(i) = a implies p(i) = a. Further, by essentially the same calculation as in 6.1 for the case v = 2, one sees: if q contains .. e_i ... a_j .. , then p must not contain .. a_j ... e_i .. . From corollary 4 above, we have p -> q. This shows

<u>6.2 Proposition</u>: *For any* $p \in Q(v)$, $\mathcal{Q}\mathcal{R}(\{p\})$ *consists of all* q *with* p -> q.

Remark: it follows from the Completeness theorem that, if $\neg$(p -> q), there exists some set M and v-relation R on M such that p(R) = T, q(R) = F. From the above discussion one gets such examples explicitly. We also see that a 2-element set M suffices.

In the other direction there is not much we can do, since v-relations on an arbitrary set cannot be systematically generated like the quantificational expressions, and there is no "canonical" procedure for associating with given relations expressions satisfying them. To get an impression of what happens, consider the diagonal relation

$$D = \{(m ,..., m) \mid m \in M\} .$$

Clearly p(D) = T for all p of type [1, v-1], beginning with a, but D does not satisfy any larger q (consider the immediate predecessors of these p). Thus $\mathcal{R}\mathcal{Q}(\{D\})$ consists of all R such that $p_i(R) = M$, i = 1,...,v, where now p_i denotes projection to the i-th coordinate. Among them D is minimal, and the other minimals are obtained by letting the symmetric group S(M) operate independently on all coordinates; in other words, these minimals make up one orbit of S(M) $\times$ ... $\times$ S(M). There is certainly some structure in these matters; one may ask, for example, which relations satisfy the dual of 6.2. But we shall not pursue these things any further here.

References

[MM] S.Mac Lane / I.Moerdijk, Sheaves in Geometry and Logic, Springer 1992.

[S] E.H.Spanier, Algebraic Topology, Springer 1966 (or any other book on algebraic topology).