**HAMBURGER BEITRÄGE ZUR MATHEMATIK**

Heft 199

# Categories in Philosophy and Mathematics

Ernst Kleinert, Hamburg

September 2004

Categories in Philosophy and Mathematics

By Ernst Kleinert

1

It is well known that originally the relation between the mathematical and the various philosophical concepts of category was a very loose one, in fact, not far from a mere joke. To quote one of the protagonists: "Now the discovery of ideas as general as these is chiefly the willingness to make a brash or speculative abstraction, in this case supported by the pleasure of purloining words from the philosophers: "category" from Aristotle and Kant, "functor" from Carnap..." [1]. In spite of this semi-serious attitude is it clear a priori that there must be quite serious connections between the two. Philosophical categories were always intended to describe fundamentals of perception and theoretical activity of man (forms of intuition (*Anschauungsformen*) and concepts of understanding (*Verstandesbegriffe*), to use Kantian terms). Now mathematical categories were introduced when the relations between the various objects studied by mathematicians became so complicated (especially in algebra and topology) as to require a theory of relations, which thus appears as a sort of metatheory to the mathematical process. Hence if mathematics is something fundamental and of philosophical interest (and presumably no reader of these lines will deny this), the same must be true *a fortiori* of mathematical category theory. Consequently, given any philosophical system of categories deserving this name, it must be possible to localize mathematical categories within its framework, and in fact at a visible place. This essay is intended to provide a clearer picture of the situation and to propose an answer to the question of localization.

2

The difficulty with philosophical categories is that not only most philosophers, explicitly or implicitly, have set up different systems of categories but also have used different underlying concepts of category, and an overview over the different meanings of "category" would be bound to include more or less an overview over all philosophical systems emerged so far [2]. Nevertheless, we must try to get some ground under our feet; to this end, we concentrate on the two towering figures in the history of categories, which we have already met, Aristotle and Kant, thus leaving aside other notable attempts towards a philosophical system of categories [3]. Even with this restriction the relevant literature is formidable and we cannot do justice to it at this place. We adopt a naive point of view, trying to make sense of what we read in the most straightforward manner.

The greek verb κατηγορειν is originally of forensic usage, κατηγορειν τινα τινοσ, to accuse someone of something, lit. "to bring down to the αγορα ", the assembly or place where things are disputed in public; less specifically it simply means "to make a statement". As a philosophical terminus technicus "category" was introduced by Aristotle. To set the stage: any judgment (that what has a truth value) consists of expressions (having no truth values themselves). The single expressions denote the object and what we say about the object; the categories are abstracts of what is expressed by the most fundamental among our judgements. In chapter 4 of his book on categories Aristotle presents them as follows: "Expressions which are in no way composite signify substance (ουσια), quantity (ποσον), quality (ποιον), relation (προσ τι), place (που), time (ποτε), position (κεισθαι), state (εχειν), action (ποιειν) or affection (πασχειν). To sketch the meaning roughly, examples of substance are "man", or "the horse"; of quantity, such terms as "two cubits long" or "three cubits long"; of quality, such attributes as "white", "grammatical". "Double", "half", "greater" fall under the category of relation; "in the market place", "in the Lyceum", under that of place; "yesterday", "last year" under that of time. "Lying", "sitting" are terms indicating position, "shod", "armed", state; "to lance", "to cauterize", action, "to be lanced", "to be cauterized", affection." [4)]

The first category, substance, points to the thing itself (so to speak), the rest appears as a list of basic properties we can ascribe to the thing. The lack of organization in their enumeration has always been felt (a "rhapsodic snatching up", as Kant critized). Quality and relation are so universal as to encompass all the others; position and state seem to be mixtures of relation, place, and action; action and affection are just two aspects of the same processes. Certainly such fundamental things should make up a *system,* which reveals their unity, their hierarchy and mutual dependencies, and is free of redundancy. These questions, however, are not adressed; in the rest of the book (as far as we possess it) Aristotle discusses the single categories and their uses separately and is primarily concerned with the mechanisms of predication as such, seeking to clarify the different meanings and functions of "is" in a statement "a is b".

In fact we do not know exactly which function Aristotle consigned to his categories in the totality of human thought or how he came to sort out exactly these ten. In particular it has been the object of much discussion whether the categories served a logical or an ontological purpose. Now such a list of basic predications points to both sides by its very nature. It borders on tautology that the content of what we can say is limited by the form of our saying. This was expressed by Wittgenstein in a famous dictum, "The limits of my language are the limits of my world". But it really belongs to the fundamentals of Kant's transcendental philosophy, and a first stage of the idea is found already in one of the antique commentators of Aristotle, Olympiodoros: "It is impossible to speak of the one [the concepts] without speaking of the other [the things]."[5)] On the face of it, the emphasis seems to be on the material side - the categories as representing a series of questions such that every object is completely described when these questions are answered (as far as they make sense) [6)]. This may be called the "naive" interpretation of Aristotelian categories, capturing at least half of the intended meaning. As for completeness, it seems to be implied by the first sentence of the above quotation. In any case we should assume that Aristotle has given some thought to his choice of categories, regardless of its unorganized appearance. Considering the systematic character prevailing in all of his work, he must have considered a complete list of basic properties as being possible, and probably would have maintained that the one he devised for his logical purposes could serve as a first approximation. In later works, he seems to rely implicitly on the completeness of his list.

## 3

Trying (among other things) to remedy the deficiencies of Aristotle's list of categories, Kant arrived at what he himself called the "Copernican turn". He realized that the missing unity cannot be read off from the things but has to be found in the organization of human thought as revealing itself in the act of judgment; the same applies to completeness [7)]. Consequently he derived his "table of categories" from a "table of functions of judgments", which we consider first [8)]. He groups these functions under the four headings of quantity, quality, relation and modality. With respect to quantity, judgments can be universal (*allgemeine*, "all men are mortal"), particular (*besondere*, "some men are mortal") and singular (*einzelne*, "Socrates is mortal"); with respect to quality, affirmative (*bejahend*), negative (*verneinend*), and infinite (*unendlich*, exemplified by Kant with the sentence "The soul is not mortal"); with respect to relation, categorical, hypothetic and disjunctive; with respect to modality, problematic ("it might be the case"), assertoric ("it is the case") and apodictic ("it must be the case"); in the last two groups Kant uses the same terms of greek/latin origin. A few comments are in order.

With the modalities we can dispense most easily, since they are the same now as then, except that today we possess mathematical models of modal logic capable of arbitrary degrees of refinement [9)]. Among the qualities of judgments we will probably wonder about the special role attributed to the "infinite" judgments. In his own commentary (B 97/98) Kant tries to explain it, maintaining that whereas for "common" logic the other two suffice, in the "transcendental" logic the infinite type becomes necessary. But "a is not be" is nothing else than the negation of the affirmative judgment "a is b", and it is part of the nature of such negations that they place the object a among the infinitely many non-b's; there is no predicate "non-b" in addition to the negation of being b [10)]. Perhaps Kant let himself be seduced here by his affection for threefolds; one might also suspect a reminiscence of the tradition of "negative theology". Let us follow Schopenhauer's suggestion to just close this "dull window" (*blindes Fenster*)[11)]. Note also the difference between Kantian quality, which is a possible function of judgments, and Aristotelean quality, which refers to properties of things; this illustrates most clearly the "Copernican turn". The same applies to quantity, which in the Kantian sense refers to the role of quantifiers, in the Aristotelean sense to numerical quantity.

Passing to relations, we probably will ask what Kant means by "categorical" as naming a type of relation; what distinguishes "categorical" from "affirmative" or "assertoric"? This is explained later (B 128): the categorical relation is the one obtaining between a subject and its predicates, i.e. the one which is realized in every act of κατηγορειν. Here an objection is near at hand. Since the other two types of relation, implication and disjunction, belong to mere propositional logic, it seems that for Kant simple predicates (unary relations) suffice to arrive at full-scaled predicate logic, which is plainly false (at least from the point of view of modern formal logic). Now one cannot seriously assume that a thorough thinker such as Kant, having worked on the Critique for full ten years, should simply have neglected or even overlooked something as elementary as order relations [12)] or other fundamental relations, e.g. of geometric origin, as listed by Hume, whom he had studied [13)]. He must have been prepared to classify the statement "a is greater than b" with respect to relation, and since it is neither hypothetical nor disjunctive it must be categorical. On the other hand, Kant clearly knew the ancient distinction, discussed already by Platon and reflected in Aristotle's category of προσ τι, between predicates that are "inherent" in the subject ("the rose is red") and those which express a relation to another object ("a is greater than b"), not "inherent" in the subject. Indeed he is fully aware of the fundamental fact that the content of all cognition consists in relations (see section 6 below); one may even call it a central result of the Critique that it is the relations which constitute the relata, not vice versa as the Aristotelean tradition would have it [14)]. The problem is, therefore, *why* Kant calls "categoric judgements", expressing a relation of subject-predicate type, what we would call more generally "atomic formulas", thereby (apparently) contradicting himself. I cannot settle this question here [15)] (but see section 14); what matters for the moment is that Kant certainly did not overlook these judgments; at worst he called them by a wrong name.

Taking stock, then, we see that Kant's table contains, after some emendations, the usual quantifiers, negation, implication, disjunction and atomic formulas. Since conjunction can be expressed by negation and disjunction [16)], and functions can be expressed by relations, we discover, perhaps to our surprise, that Kant has designed, though in a non-formal and somewhat flawed manner, what we nowadays know as the standard building elements of formal predicate logic. The singular judgements are rendered in a natural way as formulas containing no variables, free or bound [17)]. This implies, incidentally, that Kant's claim of completeness of his table is better founded than is generally acknowledged.

## 4

Kant's categories correspond bijectively to the functions of judgment and therefore are listed under the same four headings which thus function as a sort of metacategories. The general scheme of the correspondence is described as follows: "The same function that gives unity to the different representations (*Vorstellungen*) in a judgment also gives unity to the mere synthesis of different representations in an intuition (*Anschauung*), which, expressed generally, is called the pure concept of understanding...In such a way there arise exactly as many pure concepts of the understanding, which apply to objects of intuition *a priori*, as there were logical functions of all possible judgments in the previous table" (B 104/105). The most basic "pure concepts of the understanding" are the categories. According to Kant, the categories of Quantity are Unity, Plurality, Totality; of Quality, Reality, Negation, Limitation; of Relation, Inherence and Subsistence (*substantia et accidens*), Causality and Dependence (cause and effect), Community (*Gemeinschaft* or *Wechselwirkung*); of Modality, Possibility - Impossibility, Existence - Non-existence, Necessity - Contingency.

Given that the categories are not merely names of the functions of judgments but have an independent meaning, we are faced with new problems. Probably the reader will find some difficulty (in spite of Kant's own explanation in B 110 ss) in understanding why the disjunctive judgments should correspond to the category of community or mutual action; the two members of a disjunction need not have to do anything with each other. It also looks strange that among just three categories of relation there should be distinct categories of causality and mutual action. We also wonder why the category of totality should correspond to the singular judgments, the category of unity to the universal ones, and not vice versa [18)]. However, what matters for us is the role which Kant ascribes to the categories (also called "ancestral concepts of pure understanding", *Stammbegriffe des Verstandes*) in human theoretical action. Given any object of perception, we can make statements about it. These statements (judgments) have a number of possible functions, as listed in the table. The categorical "software", inherent in our cognitive organization, regulates the application of these functions, creating unity in our perceptions and providing us with a coherent picture of the world. If we see a heap of green apples, we could also say "I see some green in the form of apples"; here it is the category of "inherence and subsistence" which decides on the role of subject and predicate. This is (slightly modified) the example by which Kants illustrates his own explanation of categories in B 128: "They are concepts of an object in general, by means of which its intuition is regarded as determined with regard to one of the logical functions of judgments". It is of the utmost importance (and is stressed by Kant on every occasion) to keep in mind that the categories are to applied exclusively to sense perception (*empirische Anschauung)*. They govern the process of bringing under concepts what we perceive, the translation of perceptions into statements. In contrast, Aristotelean categories are not confined to sense perception (note his example of "grammatical" as a possible quality).

## 5

Whether or not one accepts Kant's choice of categories (and it surely calls for modification and further elaboration), it seems natural to assume that a system of categories in the Kantian sense exists, and even if one denies this, it must be of interest to approach such a system as close as possible. However, for our present purpose of localization this is not quite enough. We are not satisfied by just spelling out what is the case; we want to (and often need to) change things in our favour by acting upon the world. The physical action, only rudimentary guided by instinct, must be preceded by theoretical action. By the *categorical system* of man I mean the totality of human faculties (*Vermögen,* to use a Kantian expression) to perceive the world, to bring it under concepts and to subject it to theoretical action. All of these activities take place within certains forms and are governed by certain laws. We have learned about the role of the categories. In his chapter on the "system of all basic principles of pure understanding" (*System aller Grundsätze des reinen Verstandes*) Kant has tried to capture some of the fundamental laws, and in his chapter on "regulative ideas" (to which we shall return) he has formulated guiding principles of theoretical action .

Assuming a philosophical doctrine of categories in the sense described above, it is not difficult to determine the place of mathematics. All of mathematics consists in isolating in systems of axioms the forms and laws which we find in the categorical system, and to derive consequences from these axioms. Our basic experiences of quantity (in the Aristotelean sense) include a notion of "more or less," mathematified in the axioms of partial order; our basic handling of quantities results in the axioms for a commutative semigroup and semiring. Basic experienc of space leads to a notion of neighborhood, the laws of which crystallize in the axioms of topological space; basic experiences of working with bodies in space are a source of elementary geometry [19)]. On the level of theoretical action, a basic process is the forming and handling of extensions of concepts, which leads to axiomatic set theory. This view of mathematics will be developed in more detail elsewhere [20)]; let us take it here as a background for the clarification of the relation between philosophical and mathematical categories. We now turn to the latter.

6

Though the reader is assumed to be familiar at least with the fundamentals of the theory, we must briefly review the axiomatics. A category consists of objects A, B,... and morphisms f, g, ... . To each morphism f there correspond two objects, its source S(f) and its target T(f). To each pair f,g of morphisms such that T(g) = S(f) there is associated a uniquely determined morphism h having S(h) = S(g) and T(h) = T(f); written h= fg. Thus, the undefined notions of category theory are two sorts (objects and morphisms), two unary functions (S and T) from the morphisms to the objects and one partially defined binary function (composition) from the morphisms to themselves. These are assumed to satisfy two axioms: the composition should be associative, i.e. (fg)h = f(gh) whenever T(h) = S(g) and T(g) = S(f), and for each object A there should exist an identity morphism $id_A$ satisfying $S(id_A) = T(id_A) = A$ and $f\, id_A = f$, $id_A\, g = g$ whenever S(f) = A = T(g).

Since the axioms contain postulates concerning the morphisms, but not the objects, it is clear that the emphasis is on the former; in fact the objects appear as mere indices to the morphisms, and it is even possible to replace them by "unit morphisms", thereby making the theory one-sorted [21]. Category theory cannot make statements about some "internal structure" of the objects, only about the network of relations which consists in the morphisms. But often the internal structure is reflected in this net of incoming and outgoing arrows, e.g. the structure of a group in a composition series, and in fact it turns out that many objects are completely determined by it, in particular all objects having some universal property [22]. In this way mathematical category theory reflects a fundamental property of all human cognition, which we have mentioned already, namely that it consists entirely in relations and not in some obscure intuition of pure substances. This has been seen since long. In Hume we read: "The idea of a substance as well as that of a mode, is nothing but a collection of simple ideas, that are united by the imagination, and have a particular name assigned to them"; and: "Our propension to confound identity with relation is so great, that we are apt to imagine something unknown and mysterious, connecting the parts, besides their relation" [23]. Here Kant is in full agreement: "The inner determinations of a *substantia phenomenon* in space...are nothing but relations, and it is itself entirely a sum total of mere relations". To ask for more is "entirely improper and irrational", because this would imply that "we have a faculty of cognition entirely distinct from the human not merely in degree but even in intuition and kind" (B 321, 333). We may even quote Nietzsche, whom we have not expected here, but who has brought to the point the ontological aspect: "The properties of things are effects on other things: if one thinks away other things, a thing has no properties; that is, there are no things without other things" [24]. We obtain a first approximation to the localization of mathematical category theory: it clearly belongs to the mathematics of relations.

## 7

When showing that a collection of objects and morphisms is indeed a category, the identities rarely cause any problems. Let us look for the origins of associativity. In constructions of categories from given ones (as in functor, quotient and slice categories) associativity is propagated from the given categories to the new ones; so we have to go back to what one could call "basic categories". In a poset viewed as category, associativity comes, so to speak, by "trivial coercion", since there is at most one morphism between any two given objects; this does not give us any insight into what associativity means. The most common "basic categories" are of type "set plus structure" and inherit associativity from the underlying category of sets. The associativity of composition of set maps is a special case of the associativity of composition of (binary) relations. Writing out the proof of this, one sees that associativity relies, in the last resort, on the associativity of conjunction in propositional logic. This must be considered as a basic law of theoretical activity [25]. The world, as the totality of "what is the case" (Wittgenstein) is sorrounding us as a conjunction of a plentitude of matters of fact which can be ordered and grouped together by theoretical action in many ways; and although certain orderings and groupings are often suggested to us, they can never be forced upon our thought in a strict sense. We are always free to change them *in thought* when reflecting one and the same status of the world and how we could possibly change it. In

Kantian terms again: sense perception plus the formative categorical processes provide us with a picture of the world, which we can subject to changes using productive imagination (*produktive Einbildungskraft*) in order to obtain a plan for our practical activitity [26].

Here it becomes evident that the associativity of conjunction is a purely theoretical feature and has no analogue in reality. If a,b,c denote actual processes and ab means that first a takes place and then b, then the expression a(bc) makes no sense, except if the product bc is given another meaning, which can only be "b followed by c *thought of* as a unity". If a bracket (...a)(b...) denotes an interruption between a and b, then all different bracketings of a product of three or more factors denote different processes; only their results may be equal or at least need not be distinguished. If a,b,c denote three parts of a climb, of which I can absolve only two in one day, then (ab)c and a(bc) will be two different processes, both resulting in that I have done the climb (but possibly with different results in more special respects). Similar remarks apply to commutativity. To see the difference between the two, suppose that a,b,c denote measurings on the quantum level. Then it is well known that ab and ba will have different results in general (this is the source of the uncertainty relation), whereas the expressions (ab)c and a(bc) cannot mean anything different from first a, then b, then c, and the equality (ab)c = a(bc) can only mean that it doesn't matter how we group the measurings *in thought* [27]. Thus we reach a second approximation to mathematical categories in terms of our quasi-Kantian categorical system: they contain mathematics of relations, as far as these occur in the theoretical action of man.

## 8

The role of mathematical categories for *mathematical* theory-building has been obvious from the moment of their invention. Let us see how they function outside of mathematics. In a series of papers, G.Reyes and others have revealed categorical structures in language and thought [28]. A very simple example occurs with the distinction of count nouns and mass nouns. Roughly speaking, the former refer to individuals ("dog") and usually take the plural, whereas the latter typically refer to a kind of substance ("metal") and do not take the plural [29]. Now both types of nouns can be ordered by subsumptions according to the ancient scheme of *arbor Porphyriana*, as in "a dog is an animal" and "iron is metal". Since subsumption is transitive (modus barbara), they can be regarded as forming categories, say CN and MN. There are transitions form CN to MN by "generalizing", e.g. forming the plural, and vice versa from MN to CN by "individualizing", e.g. by taking portions. These preserve the subsumptions ("a dog is an animal, therefore dogs are animals") and thus are functors, which are even adjoint to each other: "sheep are cattle" is equivalent with "a sheep is a piece of cattle". The unit and counit of the adjunction read as "portions of metal are metal" and "one dog is a number of dogs".

Another example occurs in the behaviour of predicates in subsumptions and syllogisms. As Plato already noted, though a thief is an man, a good thief is not necessarily a good man. Clearly the standard description of subsumptions by Euler (Venn) diagrams cannot render this phenomenon properly. Somehow the predicates should be connected with the situations (or respects) to which they apply. This can be accomplished by a formalism of categorical nature, which I try to describe roughly. One starts with a set S of "situations" or rather "descriptions of situations". S is partially orded by defining $U < V$ if U contains more information than V. Thus a downwards closed subset of S contains with every description all its possible "completions". Next, a *kind* is defined to be a set A together with a function from A to the set D(S) of downwards closed subsets of S. Think of the elements $a \in A$ as of "realizations" of CNs, and of the downward closed set associated to a as of the totality of situations in which a figures. The kinds form a category; morphisms are set functions f "compatible" with associated situations, meaning that in every situation containing a qua A there also occurs $f(a) = a$ qua B. The morphisms represent the subsumptions in CN. Finally a *predicate* of a kind A is function $p: A \dashrightarrow D(S)$ such that p(a) is a subset of the set associated to a by the kind A. Thus, a predicate of a is represented by the totality of situations in which it can be attributed to a (a form of extensionality), and there are no predicates "separated" from kinds. One then obtains a category of "kinds with predicates", which reveals what happens in examples like the one we started with: a subsumption can be a morphism of kinds without extending to a morphism of predicated kinds.

9

In everyday's planning there occur limits and colimits; most widely used are products. In the naive interpretation of Aristotelean categories these are a list of basic respects, such that every object is completely determined when determined in all of these respects. In this way every object corresponds to an element of a product the factors of which are the possible determinations in the various respects (though there will be many elements in the product having no object corresponding to them). The determinations in the basic respects may be viewed as a sort of coordinates, and if the basic respects can be chosen in such a way that these determinations can be represented by real numbers, one arrives at the concept of phase or configuration space used so successfully in modern physics, the most elementary example being the description of space by three coordinates. On a more mundane level, we use the naive Aristotelean scheme in many contexts, often looking to projections to subproducts formed by those respects which are momentarily relevant, e.g. if we only ask for material, size and colour when bying clothes. (Sometimes it takes a while to find the appropriate projection.)

Once one has developed a sense for categorical structures one sees them everywhere. Every time we use an expression "a pair of objects A,B" we form a coproduct inasmuch every operation performed with both A and B can be said to be performed with the pair. A society can function as a biproduct of its members: instead of sending the same letter to each member, one can send one letter to the society, and likewise the other way round. When we look for persons having eyes and hair of the same colour we form an equalizer. Every time we are given different groups of objects and gather those having the same weight or height etc. we form a (multiple) pullback. If in a population we form pairs of parents we employ a coequalizer (namely of the two maps which associate with every person its father and mother). A pushout is in the play if after buying two collections of stamps we sell the doublets. And the topological operation of glueing (which is a case of pushout) has even nontheoretical realizations.

10

On a more advanced level of theoretical action we find the theory of systems. Without undergoing the risk of defining what a system is, we can safely state that very often the relevant structures of systems are represented as graphs and therefore can be categorified naturally. In fact, there has been formulated a "Categorical Manifesto", expressed in five "basic doctrines" which extend the original role of categories from mathematics to systems: any species of structure should constitute a category; any "natural" construction of objects from other objects should be functorial; any "natural" relation between two objects arising from the same object by different constructions should be expressible as a natural transformation; any construction of a "complex" object from a pattern of "simple" objects should be a limit or colimit; and any "enrichment of structure" within one species should be realizable as a slice category [30)]. In accordance with these doctrines, Ehresmann and Vanbremeersch have shown *in extenso* how concepts like "control centers", "agents" and "approximation of the system's state" can be modelled in categorical terms [31)]. A basic case is the theory of input-output systems. Such a system consists in a set A of inputs, a set S of "inner states" and a set B of outputs together with a state transition function $A \times S \dashrightarrow S$ and an output function $S \dashrightarrow B$; like all such arrangements of sets and set maps these systems form a category SYS. The development of a system over time can be described by a functor T --> SYS, where T denotes a totally ordered set representing time. The behaviour of a system viewed from "outside", i.e. without knowledge of S, consists in the sets A and B and the function from the free semigroup with alphabet A to B produced by the functions of the system together with some initial state. There is a category BEH of "behaviours" and an intuitively obvious functor SYS --> BEH, which has a left adjoint: every behaviour can be realized in a "universal" way.
Although the systems themselves consist in sets and set maps, the role they play is best described in categorical terms; as was the case with the kinds and predicates in section 8.

11

Finally, at the highest level of theoretical action we find the simplest of universal objects, namely initial and terminal objects. It is a universal property of universal properties that they determine the objects having them uniquely (up to canonical isomorphisms). This has always been inherent in philosophical and theological speculation, but was rarely, if ever, made explicit. When we speak of a universe containing everything then it is obvious that there can be but one such universe since two of them would have to be contained in each other. Note that the argument is the same as the one showing uniqueness (up to isomorphism) of terminal objects. Likewise the origin of everything, if existent, can only be thought of as being unique, similarly the god who has created everything. Other instances of the same figure of thought occur in the attempts to prove the existence of god, by concepts like the "unmoved mover", the "highest being" or the "summum bonum" of medieval theology.

Another example of philosophical speculation turned into arrows is the "coincidence of opposites", *coincidentia oppositorum*, to which Nicolaus Cusanus attributed an important role in his epistemology and which later recurs, elaborated formidably, in Hegelian dialectic. Cusanus believed (in contrast to Hegel) that although the human mind is uncapable of really grasping this unity, we can approach it through mathematics [32]. His own illustrations probably will create doubt in the modern reader [33], but category theory presents veritable and realistic models. Following Lawvere [34] we realize the *coincidentia* by a functor having both a left and a right adjoint; it follows from the mechanisms of adjoint functors that the images of one object under the left and the right adjoint are in a certain sense opposites. The simplest example is given by the trivial functor from a category $\mathcal{A}$ into the trivial category (one object with its identity); the left (right) adjoint, if it exists, maps the object of the trivial category to an initial (terminal) object of $\mathcal{A}$. The forgetful functor from topological spaces to sets has as adjoints the functors of discrete and coarse topology. It is intuitively obvious that, given a set M, the discrete space $M_d$ and the coarse (codiscrete) space $M_c$ are opposites. In fact they represent the terminal and the initial object in the lattice of topologies on M; every map $M_d$ --> $M_c$ is a morphism, but only the constant maps $M_c$ --> $M_d$ are morphisms. Nevertheless they have the same origin in the more general category. Another example are the existential and universal quantifier, which can be realized, for a given set map f: M --> N, as adjoints to the functor P(f): P(N) --> P(M) on the power sets [35].

12

These last examples bring us closer to an answer to our localization problem. In his chapter "on the regulative use of ideas" (B 670 ss) Kant lays down the most general guiding principles of human theory-building, resuming them under the titles of unity, manifold and affinity. We understand things inasmuch as we are able to view them as being united under some general concept or principle; this requires sufficiently strong affinities between the things to be united; vice versa, given any general concept, we ask for all its possible specifications. As his examples show, Kant is thinking of the natural sciences of his time; but we may equally well substitute modern mathematics, which has introduced more and more general structures and at the same time works on the classification of the objects falling under them. The two processes depend on each other: "Reason thus prepares the field for the understanding 1. by a principle of sameness of kind in the manifold under higher genera, 2. by a principle of the variety of what is same in kind under lower species; and in order to complete the systematic unity it adds 3. still another law of the affinity of all concepts" (B 685); on the other hand, "Reason presupposes those cognitions of the understanding (*Verstandeserkenntnisse*) which are first applied to experience, and seeks the unity of those cognitions in accordance with ideas that go much further than experience can reach" (B 690). There must be basic principles of establishing unity which are prior to experience since the unity we eventually arrive at (say, in physics), is obviously not presented to us by experience itself [36]. Kant does not try to classify these basic principles, as he did in the case of his categories. Indeed one may doubt that such a classification be ever possible. Already the laws governing our sense perceptions were easy to grasp only in the case of quantity. In the case of shape, the mathematics of which is unfolded by geometry (nowadays the theory of manifolds which is comparatively young), this process required much more time; just consider how recent the elementary concept of simple connectivity is. The laws of theoretical action, by which we attempt to systematize our perceptions, are much more difficult to discover than the laws governing the perceptions themselves. Of the latter we become aware by pure phemomenology, of the former only by analyzing our thoughts; they have no direct empirical correlates [37], but prove themselves at best indirectly, by the success of the pictures of the world to which they lead us. And just as there are sense perceptions not regularly available to us there might be concept-ordering principles which simply have not emerged up to now; recall that the principle of mathematical induction seems to have been unknown before the 17. century. In any case a certain supply of such principles belongs to the categorical system; and it is obvious by now that mathematical category theory is nothing else than the mathematics emerging from this part of the system. This is precisely the role for which categories were introduced by their originators, and in which we found them in the applications. Both Kantian and mathematical categories function as creating unity on subsequent levels of theoretical activity: the former bring under concepts what we perceive, the latter are (part of) the mathematification of principles by which we order the conceptualized perceptions, as well as, on higher levels, the concepts of concepts. In fact, what can be closer to a regulative idea in the Kantian sense than a universal object, say a limit? The limit provides an object which (in general) cannot be found among the given objects, but which is suggested by these objects in a certain sense [38].

## 13

It might appear from the above examples that mathematical category theory, developed properly, in fact exhausts the mathematics of theoretical action. This view is not quite adequate. Theoretical action is performed using concepts. The *ratio essendi* of concepts is that there are diverse objects falling under them, making up their extension. The operations with extensions from which the set axioms emerge are no less fundamental in our theoretical activities, and they contribute no less to the order of concepts. We have listed categorical structures in theoretical action; it is hardly necessary to point to the uses of set operations in the same processes. They are by no means confined to the ubiquitious Boolean operations. When we speak of the contents of a bag, we form a set by comprehension; when speaking of the thoughts of men, we form a set by replacement; if we ask for the first three of the marathon, we ask for the extension of a set defined intentionally (by comprehension); custom officials taking a sample from each cargo realize a choice function. In many cases the set theoretical view is more appropriate than the arrow theoretical one, because the universal mapping properties which are the *ratio essendi* of categorical limits do not figure in the theoretical process. This is seen to be the case in the above example of the two collections of stamps: we simply form a union and subtract the intersection. The same applies to all unions when the coproduct property plays no role. If we group children of a class by their height, we form the fibres of a set map, and the categorical background (the epi-mono factorization in the category of sets) does not come in. Thus it depends on the circumstances (and sometimes also on habits and taste) whether to use sets or arrows.

14

To clarify the roles we must go back to the categorical organization of our theoretical activity. Though Kant agrees with Hume in that cognition consists in relations he insists, in contrast to Hume, that the concept of substance is indispensable for human thought, as belonging to its organization. *We* need the relata *before* we can speak of relations; the relations must obtain between objects given in advance. In words of Kant: "The understanding...demands first that something be given (at least in the concept) in order to be able to determine it in a certain way. Hence in the concept of pure understanding matter precedes form..."(B 322/323). Perhaps it is here that we must look for the reason why Kant calls categorical what we would view as genuine relational judgments. Category theory (formulated with one or two sorts) makes no exception; it cannot do without objects; morphisms may represent a transition, but only from one fixed state to another [39)]. The situation is clearly reflected in the structure of modern formal languages: to build up terms and formulas, we need relation symbols as well as variables or constants; this applies to categories as well as to sets. But it is the variables and constants which represent the underlying substance, the Aristotelean υποκειμενον, the *subiectum.* Now in which sense one may call set theory a mathematic of substance? It is easy to see that in mathematical practice sets indeed function as a raw material, as substances for mathematical thought, inasmuch as the same set can be equipped with various structures of algebraic, combinatorial or topological nature; just as on the physical level one may use the same brick at different places of the wall. In this respect categorical mathematics is less flexible, in that every category represents some specific sort of structure; it always takes a functor to pass from one to the other. The concept of "abstract set", made up of "*lauter Einsen*", as Cantor put it, is a fiction since it violates the principle of the *identitas indiscernibilium* valid in all experience; but exactly by stripping the elements of all individuality it clears the road for the speculative activity of thought. To be a member of some class of objects is the most general predicate which can be attributed to a single object; more precisely, it is the scheme of all predication. Thus set theory, abstracting from all specific properties objects may have, provides us with the most general notion of "entity as such", which indeed can (and does) play the role of the υποκειμενον in mathematical thought.

15

Thus we find that the appropriate picture is that of a natural division of labour rather than some sort of rivalry. The operations of forming extensions or collections and relating them are always intertwined and depend on each other, and the nature of this complementarity certainly deserves to be explored in more detail. Objects dissolve into nets of relations [40], but on the next higher level of theoretical action, relations and morphisms become new objects. Set mathematics leads to categories inevitably, but what would category theory be without set-valued functors? There are also technical aspects of the complementarity. Each side is the mathematification of some basic faculty of theoretical action, and each has difficulties in reconstructing what is basic for the other. To associate objects to given ones is such a basic faculty. Set mathematics reconstructs it in a rather clumsy way, first defining ordered pairs and products by the Zorn-Kuratowski trick, then functions in the extensional way as subsets of products having a uniqueness property; and it takes even longer before we can render intentionality, by defining functions in terms of explicit formulae, which always requires some sort of structure on the domain and codomain. On the other hand, in arrow mathematics one has to do real work in order to realize the elementary concept of subobjects of given objects. And though it is easy to define elements of an object X as arrows 1 --> X (1 denoting a terminal object), this suffices just for substitution into maps (for a: 1 --> X, f: X --> Y we can define $f(a) = f \circ a$), and this "local" elementhood does not allow us to express the everyday phenomenon that an element of a collection is at the same time an element of another collection (simply because arrows have a *unique* codomain), or that objects having elements are themselves elements of "higher" objects. It is only after climbing up to the heights of topos theory that these elementary situations become available in terms of arrows. We also should not forget the vast amount of combinatorial mathematics stubbornly refusing (up to now) *any* sort of conceptualization.

16

Having pointed to the necessity of some sort of "substance" and to the mutual dependence of objects and morphisms, we should not go too far and loose sight of the natural predominance of the latter, which results from the organization of human cognition, as described by Hume and Kant. It is unfolded systematically by category theory, but naturally reveals itself also in set mathematics. Set theory remains silent as to what the "nature" or "essence" of the objects might be; elements of sets are sets [41]. Set

mathematics can build structures only in the form of constellations of sets, in this respect not different from category theory, which characterizes objects by the network of morphisms connected with them. One may think of the set underlying a group as of its "substance"; but the "structure" (that what is mathematically relevant) resides in the multiplication table which is the graph of the multiplication map [42].
Historically, it was the birthday of modern mathematics and marked the emancipation from its antique precursors when the concept of function (which does not explicitly occur in Greek mathematics [43]) was made the dominating paradigm. As we all know, the theory of real functions flourished long before there was a satisfying theory of real numbers; even more paradoxically, such a theory would hardly have emerged without the refined study of functions which in fact enforced it. And we are all aware of the countless functions of functions. We use them in mathematics to describe algebraic operations, to represent objects in other objects, to define manifolds (by coordinate maps) or subsets (by characteristic functions), to describe distributions and colourings; we use them in applications to describe temporal developments, to express causal dependences, to identify things and persons and so on ad libitum. Often mathematical objects are identified with maps, as in all cases of representable functors; a process systematized by the Yoneda embedding of any small category into a functor category [44].

## 17

This brings us back to mathematical category theory, and in fact to an aspect which we have neglected so far. Originally categories were intended to serve as a metatheory to mathematical standard theories. But the history of the theory is at the same time the history of its emancipation from the original auxiliary role. During the 60's and 70's it became increasingly clear (chiefly by the work of F.W.Lawvere) that category theory was capable of providing a starting point for all of mathematics (metaphorically, to provide the whole building, not just its roof). The possibility of interpreting formal languages in any topos renders the topos in what has been called a "universe of mathematical discourse". More specifically, a well-pointed Boolean topos satisfying the axiom of choice and possessing a natural numbers object allows all the constructions which lead from the natural to the real numbers; thus it possesses a "real number object" of its own, satisfying all the L1-sentences which hold for the "standard" reals [45]. For instance, in the topos of sheaves over a topological space, the real numbers object is the sheaf of continuous real functions [46]; if the space shrinks to a point, we get back the standard set universe. Even more astonishing, to any topos there is associated a language of "local sets", of which the topos itself provides an interpretation; in fact, there is an equivalence of topos theory and local set theory [47]. In this way the instrument of theoretical activity originally conceived of as a metatheory to the mathematical standard universe has proved capable of reconstructing this whole universe internally, purely by its own means. In a sense, this is a triumph of idealism. But the same can be said of set mathematics; the reconstruction of the universe in terms of arrows is no more remarkable than its previous construction in terms of extensions of concepts, using (apart from equality) only a single undefined binary relation, i.e. by the simplest possible extension of a mere logic of identities. We certainly need both "ways of worldmaking" [48].

18

Given the dominating role of functions and relations in all of cognition, and given that there is no logical obstacle to founding mathematics on arrows, one should expect this to happen sooner or later in full seriousness, including all levels of mathematical teaching[49]. Why does this appear so unlikely? Of course we can only speculate on this point, but I can think of two reasons. The first is an analogy with physics. The world of microparticles is properly described by quantum mechanics, of which classical mechanics is a limiting case. Nevertheless, according to the Copenhagen interpretation, the measuring apparatus, and in particular the results of measurements, must be described classically. Similarly, topos theory is a generalization of set theory (as described above), and there are important structures having their natural habitat in the more general framework. But it seems plausible that everything which is to be applied outside of mathematics must first be formulated in "classical" terms. Both are special cases of the "fundamental theorem of transcendental philosophy": whatever enters our experience must be conform with the categorical organization of this experience [50]. It is only in indirect ways that we can come to terms with objects and processes not fitting this organization.

The second reason is, briefly, the time factor. In the above discussion we were concerned with the *results* of the cognitive process; but for the present question we have to take into account the processual character itself. It belongs to the *conditio humana* that we are faced with objects which we have to handle somehow, either by finding out the relations by which they are connected, or by creating new relations, and in the majority of cases the objects are captured naturally as extensions of concepts, i.e. as sets. The theoretical process creates relations rather than objects, for which there often holds some sort of "conservation law"; in many cases a recombination of the given objects is all we need. Of course the given objects, when looked at more closely, reveal internal relational structures (the mutual dependence of objects and relations), but they enter our thought as units. In most cases we perceive the objects instantaneously, but the relations only gradually. Thus at the beginning of the theoretical process, there is an asymmetry of objects and relations in favour of the objects, which in the result is turned into its opposite. To reverse this order would mean to replace the *intellectus ectypus*, who has to build up knowledge using discursive thought, comparing and relating given objects, with the *intellectus archetypus*, for whom cognition and creation coincide.

Let us conclude in an even more speculative tune, by mentioning an ontological aspect. The topos of sheaves over a space may be viewed as a universe of "variable" sets, which become "static" if the space is cut down to a point [50]. It is tempting to associate set mathematics to an ontology favouring substances, categorical mathematics on the other hand to "process metaphysics", as initiated in the Western tradition by Herakleitos and developed *more geometrico* by Whitehead [52)]. In fact, the former postulates a universe of objects ("being") and constructs (or reconstructs) the relations between them, which are objects themselves, whereas the latter postulates a network of morphisms, which may be thought of as transitions ("becoming"), and distinguishes objects by systems of morphisms connected with them. But of course we can escape the fundamental organization of human cognition in no way. One cannot really exchange Being for Becoming just by renaming the variables; every concept implies, qua concept, a negation of becoming inasmuch it suggests the possibility of some permanent order of things. Only a logical formalism making no use of variables or constants would be a real alternative. But this would obviously require a faculty of theoretical action different from ours and belonging to beings that, as Kant says, " we cannot even say are possible, let alone how they are constituted" (B 333).

References and Remarks

(1) S.Mac Lane, Categories for the Working Mathematician (2nd ed.), Springer-Verlag 1998. p.29. In fact Mac Lane had to pass an exam in philosophy when being promoted at Göttingen. (I owe this information to the referee.)

(2) Good overviews are in Aristoteles, Kategorien, Darmstadt 1984 by K.Oehler, and in the article "Kategorien" by H.M.Baumgartner, in: Handbuch Philosophischer Grundbegriffe, München 1973.

(3) In particular by C.S.Peirce, A.N.Whitehead, N.Hartmann, and (of more recent vintage) R. Chisholm, A Realistic Theory of Categories, Cambridge 1996.

(4) Translation by E.M.Edghill, The Work of Aristotle translated into English, Vol.1, Oxford 1928.

(5) Corpus Aristotelicum Graecum, Vol. XII, 19,9 ss.

(6) That is, determined in every respect, not just identified; for this, much less information suffices, e.g. location in spacetime.

(7) See "Prolegomena to Any Future Metaphysics", § 39, for Kant's own description of the train of thoughts which led him from Aristotle's categories to his tables.

(8) The table appears on page 95 of the Critique of Pure Reason, 2nd ed., customarily quoted with B+page number. We cite the Critique from the Cambridge edition 1997, translated and edited by P.Guyer and A.W.Wood.

(9) G.Reyes/H.Zolfaghari, Bi-Heyting Algebras, Toposes and Modalities, Journal of Phil.Logic 25 (1996), 25-43.

(10) The point was clarified by Frege in his essay "Die Verneinung" (in: G.Frege, Logische Untersuchungen, ed. G. Patzig, Göttingen 1966). Though Frege does not mention Kant there, one has the strong impression that his arguments are aimed precisely at this passage of the Critique.

(11) A.Schopenhauer, "Die Welt als Wille und Vorstellung", 2.Teilband, Kritik der Kantischen Philosophie.

(12) As suggested by G.Patzig (see his article on Kant in: Grundprobleme der großen Philosophen, Neuzeit 2, ed. J.Speck, Göttingen 1986.)

(13) Treatise Book I, pt. I, 6, On relations, contains a list of relations regarded as fundamental by Hume.

(14) See P.Schulthess, Relation und Funktion, Kantstudien (Ergänzungshefte) 113, Berlin 1981, in particular p.168 ss.

(15) In spite of the importance of this question, I have not found anything substantial concerning it in the standard literature on the table (K.Reich, Die Vollständigkeit der kantischen Urteilstafel, Hamburg 1986, M.Wolff, Die Vollständigkeit der kantischen Urteilstafel; mit einem Essay über Freges Begriffsschrift, Frankfurt 1995 ). In defence of Kant, we might point to another circumstance (of which however Kant was obviously unaware). If in "a is b" we take both a and b as variables, we obtain a *binary* relation; in fact this is adequate since equally often we try to find objects with given properties and to find out properties of given objects. The membership predicate "a is an element of b" is just a specification; and set-theoretical mathematics has shown how to derive the whole world of possible relations from this. One may also recall that in the extensional form (as set theory gives it) a relational statement "aRb" indeed becomes a predicate, namely $(a,b)\in R$. By making the ordered pair of the *relata* the subject of the statement, one can express the relation obtaining between them by a predicate. Examples for this can also be found in ordinary language, e.g. in "a and b are neighbours", mostly if the relation is symmetric. - The proper logic (or rather mathematic) of relations was developed only in the 19.century. In the philosophy of C.S.Peirce the distinction of predicates, binary and ternary relations becomes a central issue, unfolded in his doctrine of categories, which he simply named by the place numbers (Firstness, Secondness, Thirdness). According to Peirce, all relations with more places can be reduced to ternary ones; however, this holds only under some restrictions (see C.Hookway, Peirce, London 1985, p.97 ss). Here is a formal proof that not every binary relation can be reduced to predicates: let A,B,... denote predicate symbols. By the theorem on disjunctive normal form a binary relation defined by these predicates is a disjunction of relations of the form

$$R(x,y) <-> E(x)\wedge E'(x)\wedge..\wedge F(y)\wedge F'(y)\wedge... ,$$

where every E and F is some A or its negation, which we may abbreviate as $R(x,y) <-> P(x) \wedge Q(y)$. Now a relation $S(x,y)$ which is an m-fold disjunction of relations of this type has the following property:

If $(x_1,y_1),\ldots,(x_{m+1},y_{m+1})$ are m+1 pairs satisfying S, then there are at least two pairs $(x_i,y_i)$, $(x_j,y_j)$ such that $(x_i,y_j)$ and $(x_j,y_i)$ also satisfy S.

(This follows from the pigeonhole principle). But the successor relation of the natural numbers clearly does not have this property. Note also that every relation on a finite set M can be reduced extensionally to predicates $x = a$, where a runs over the names of the members of M. On the philosophical level the problem leads to he distinction of "internal" and "external" relations (Russell). This is a basic philosophical question and remains unsettled up to now, as most basic questions. Cf. the article on relations by G.Patzig in: Handbuch philosophischer Grundbegriffe, ed. H.Krings e.a., München 1974.

(16) Probably Kant was aware of this fact; one of his reflections seems to contain an instance of the de Morgan laws. Of course one may still ask whether this is a sufficient reason to exclude conjunction from the table. In Lambert, whom Kant knew, one finds a passage which perhaps served as a justification. See Schulthess (14), p. 280.

(17) Schopenhauer wanted to eliminate them too, subsuming them under the universal judgments ("All that is Socrates is mortal"). But this seems unnatural, and Kant's distinction adequate; the constants in formal languages serve exactly this purpose.

(18) See O.Höffe, Kants Kritik der Reinen Vernunft, München 2003, p.123 ss.

(19) The axioms of Euclidean geometry can be deduced from basic rules for working with physical matter; see R.Inhetveen, Konstruktive Geometrie, Zürich 1983.

(20) See my "Mathematik für Philosophen", Hamburger Beiträge zur Mathematik 187 (2003) (to appear in Leipziger Universitätsverlag). It should be added that this view of mathematics is quite different from Kant's own one.

(21) See Mac Lane, Categories for the Working Mathematician, p. 9.

(22) For a precise formulation of this as well as a proof in some important cases see P.Freyd, Abelian Categories, New York 1964, p.28 ss. In a sense this means that every object is a universal object.

(23) Treatise Book I, I.6, IV 6.

(24) F.Nietzsche, Nachgelassene Fragmente 1885-1887, ed. Colli/Montinari, Berlin 1988 (vol. 2, p.85, my translation).

(25) One will object that associativity of conjunction can be proved by the method of truth tables. But we would hardly accept the method if it would fail to yield the associativity which therefore has priority; so the proof simply shows that the method is sound. The basic laws of propositional logic are the touchstone of the method.

(26) This freedom of recombination in thought has a mathematical counterpart: whereas a randomly given binary operation on a set is unlikely to be associative, we usually can construct associative "environments" rather effortlessly, by using free modules or groups or simply writing symbols in a row. The Poincaré-Birkhoff-Witt theorem is a nontrivial example of such an associative environment.

(27) For more on associativity and commutativity, see my essay in Math.Sem.Ber. 2003.

(28) For the overall programm, see the volume "The Logical foundations of Cognition", ed. J.Macnamara/G.Reyes, Oxford UP 1994. The material in this section is taken from M.La Palme Reyes/J.Macnamara/G.Reyes/H.Zolfaghari, Count Nouns, Mass Nouns and their Transformations: a Unified Category-theoretic Semantics, in "Language, Logic and Conceptual Representation", ed.RJackendoff/P.Bloom/K.Wynn, MIT Press.

(29) It must be noted that the distinction is not always clear and varies among different languages.

(30) J.Goguen, A Categorical Manifesto, Mathematical Structures in Computer Science, I (1991). See also C.Rattray, Abstractly Modelling Complex Systems, in: Systems: Theory and Practice, ed. R.Albrecht, Springer-Verlag 1998.

(31) A.C. Ehresmann/J-P.Vanbremeersch, Outils mathematiques pour modeliser les systemes complexes, Cahiers de topologie et geometrie differentielle categoriques, XXXIII, 2.

(32) Nicolaus Cusanus, De Docta Ignorantia, Book I, 4, 11 ss.

(33) For example, he argues that the "infinite triangle" and the "infinite circle" coincide, by which it is demonstrated that the unique god, as the most perfect being represented by a circle, can at the same time be a trinity.

(34) F.W.Lawvere, Unity and Identity of Opposites in Calculus and Physics, Applied Categorical structures 4 (1996), 167-174. Lawvere requires further that the given functor F composes with its adjoints R and L to identity. But in any case one has canonical maps LF(A) -> A -> RF(A) for all objects A, and in many examples the extreme terms are opposites in an obvious sense even if the additional condition is not met.

(35) Cf. Mac Lane/Moerdijk, Sheaves in Geometry and Logic, I. 9

(36) The book of nature, said Galileo, is written in the language of mathematics. More correctly: we have to write this book ourselves in that language.

(37) For this reason Kant does not try to classify them, saying that there is no "transcendental deduction" for them, as for the categories.

(38) A very simple example is given by an infinite sequence $M_1$ --> $M_2$ --> ... of strictly increasing sets with successive embeddings. The "regulative idea" arising here is that of a "largest" set, which is mathematically realized as the limit of the system (the union, if the maps are inclusions of subsets). A nice and less trivial example are the ends of a (noncompact) topological space T (the projective limit of the connected components of all T-K, where K is compact, ordered by inclusion). The regulative idea here is "what the space looks like at infinity".

(39) Lawvere (Como Proceedings, Springer Lecture Notes 1488) put it thus: a bus net without stops makes no sense.

(40) There might be a physical analogue in a strong sense: it seem not impossible that the process of subdividing matter continues ad infinitum (think of string theory), thereby giving credit to an idea of Leibniz: every monad contains and is contained in infinitely many others.

(41) I take set theory in the pure sense, i.e. without urelements.

(42) Note however that the cardinality (a property of the "mere substance") can have very strong influence on the structure (in particular in the finite case).
(43) Perhaps we must seek here the reason for the fact that the Greek theory of the continuum, Eudoxos' doctrine of proportions, remained mathematically fruitless in spite of its ingenuity. The Greek did not care for functions defined on proportions, they did not even view them as numbers admitting algebraic operations.

(44) For more on the theme, see the essay "Vom Vorrang der Morphismen über die Objekte" in my "Beiträge zu einer Philosophie der Mathematik", Leipziger Universitätsverlag, 2003.

(45) An arrow theoretic version of the axiom of choice is that all epis split. The arrowfication of the natural numbers relies on their characterization by the Dedekind recursion theorem. Consider the category whose objects are triples (N,n,f), where N is a set, n is an element of N and f an endomap N --> N. A morphism of such triples is a set map "compatible" with the given structures, i.e. preserving the distinguished elements and commuting with the endomaps. The Peano axioms for such a triple T are equivalent

to the statement that T is an initial object in the category of triples, with the distinguished element serving as 1 and the endomap as the successor function. All of this can be formulated arrow-theoretically, the distinguished elements being represented by arrows 1 --> N.

(46) See S.Mac Lane/I.Moerdijk, Sheaves in Geometry and Logic, Springer-Verlag 1992, VI, 8. Lawvere has shown which topos axioms are required to make a topos equivalent to the topos of sets; see F.W.Lawvere, An elementary theory of the category of sets, Proc.Nat.Acad.Sci.U.S.A., 52 (1964), 1506-1511.

(47) See J.L.Bell, Toposes and Local Set Theories, Oxford 1988.

(48) I cannot agree with the view that topos axioms are more accessible than set axioms (see e.g. M.Barr/C.Wells,Toposes, Triples and Theories, online version Nov.2002, p.96). The latter express elementary operations of theoretical action (see section 13); it is only their formalization that requires some non-obvious logical devices. It is true that the general category axioms are easier, but to understand topos axioms, one has to understand arrow-theoretic exponentiation, adjoint and representable functors. One cannot seriously claim that any of these concepts is of an elementary nature. (Take any book on category theory and look where they are introduced, if at all).

(49) In fact very little has been done so far to realize what is theoretically possible. An example is the recent book by F.W.Lawvere/R.Rosebrugh, Sets for Mathematics, Cambridge 2003, which contains (under a somewhat misleading title) a purely categorical introduction to a set universe.

(50) One may note here that both quantum and topos logic are (in general) intuitionistic.

(51) Cf. the last chapter of the book by J.L.Bell (43).

(52) See N.Rescher, Process Metaphysics, New York 1996.