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## Forcing unbalanced complete bipartite minors


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**Abstract**

Myers conjectured that for every integer $s$ there exists a positive constant $C$ such that for all integers $t$ every graph of average degree at least $Ct$ contains a $K_{s,t}$ minor. We prove the following stronger result: for every $0 < \varepsilon < 10^{-16}$ there exists a number $t_0 = t_0(\varepsilon)$ such that for all integers $t \geq t_0$ and $s \leq \varepsilon^6 t/\log t$ every graph of average degree at least $(1+\varepsilon)t$ contains a $K_{s,t}$ minor. The bounds are essentially best possible. We also show that for fixed $s$ every graph as above even contains $K_s + \overline{K_t}$ as a minor.

## 1 Introduction

Let $d(s)$ be the smallest number such that every graph of average degree greater than $d(s)$ contains the complete graph $K_s$ as minor. The existence of $d(s)$ was first proved by Mader [4]. Kostochka [3] and Thomason [10] independently showed that the order of magnitude of $d(s)$ is $s\sqrt{\log s}$. Later, Thomason [11] was able to prove that $d(s) = (\alpha + o(1))s\sqrt{\log s}$, where $\alpha = 0.638\dots$ is an explicit constant. Here the lower bound on $d(s)$ is provided by random graphs. In fact, Myers [6] proved that all extremal graphs are essentially disjoint unions of pseudo-random graphs.

Recently, Myers and Thomason [8] extended the results of [11] from complete minors to $H$ minors for arbitrary dense (and large) graphs $H$. The extremal function has the same form as $d(s)$, except that $\alpha \leq 0.638\dots$ is now an explicit parameter depending on $H$ and $s$ is replaced by the order of $H$. They raised the question of what happens for sparse graphs $H$. One partial result in this direction was obtained by Myers [7]: he showed that every graph of average degree at least $t+1$ contains a $K_{2,t}$ minor. This is best possible as he observed that for all positive $\varepsilon$ there are infinitely many graphs of average degree at least $t+1-\varepsilon$ which do not contain a $K_{2,t}$ minor. (These examples also show that random graphs are not extremal in this case.) More generally, Myers [7] conjectured that for fixed $s$ the extremal function for a $K_{s,t}$ minor is linear in $t$:

**Conjecture 1 (Myers)** *Given $s \in \mathbb{N}$, there exists a positive constant $C$ such that for all $t \in \mathbb{N}$ every graph of average degree at least $Ct$ contains a $K_{s,t}$ minor.*

Here we prove the following strengthened version of this conjecture. (It implies that asymptotically the influence of the number of edges on the extremal function is negligible.)

**Theorem 2** *For every* $0 < \varepsilon < 10^{-16}$ *there exists a number* $t_0 = t_0(\varepsilon)$ *such that for all integers* $t \geq t_0$ *and* $s \leq \varepsilon^6 t/\log t$ *every graph of average degree at least* $(1+\varepsilon)t$ *contains a* $K_{s,t}$ *minor.*

Theorem 2 is essentially best possible in two ways. Firstly, the complete graph $K_{s+t-1}$ shows that up to the error term $\varepsilon t$ the bound on the average degree cannot be reduced. Secondly, as we will see in Proposition 9 (applied with $\alpha := 1/3$), the result breaks down if we try to set $s \geq 18t/\log t$. Moreover, Proposition 9 also implies that if $t/\log t = o(s)$ then even a linear average degree (as in Conjecture 1) no longer suffices to force a $K_{s,t}$ minor.

The case where $s = ct$ for some constant $0 < c \leq 1$ is covered by the results of Myers and Thomason [8]. The extremal function in this case is $(\alpha \frac{2\sqrt{c}}{1+c} + o(1))r\sqrt{\log r}$ where $\alpha = 0.638\ldots$ again and $r = s + t$.

For fixed $s$, we obtain the following strengthening of Theorem 2:

**Theorem 3** *For every* $\varepsilon > 0$ *and every integer* $s$ *there exists a number* $t_0 = t_0(\varepsilon, s)$ *such that for all integers* $t \geq t_0$ *every graph of average degree at least* $(1+\varepsilon)t$ *contains* $K_s + \overline{K}_t$ *as a minor.*

This note is organized as follows. We first prove Theorem 2 for graphs whose connectivity is linear in their order (Lemma 8). We then use ideas of Thomason [11] to extend the result to arbitrary graphs. The proof of Theorem 3 is almost the same as that of Theorem 2 and so we only sketch it.

## 2 Notation and tools

We write $e(G)$ for the number of edges of a graph $G$, $|G|$ for its order and $d(G) := 2e(G)/|G|$ for its average degree. We denote the degree of a vertex $x \in G$ by $d_G(x)$ and the set of its neighbours by $N_G(x)$. If $P = x_1 \ldots x_\ell$ is a path and $1 \leq i \leq j \leq \ell$, we write $x_i P x_j$ for its subpath $x_i \ldots x_j$.

We say that a graph $H$ is a *minor* of $G$ if for every vertex $h \in H$ there is set $C_h \subseteq V(G)$ such that all the $C_h$ are disjoint, each $G[C_h]$ is connected and $G$ contains a $C_h$-$C_{h'}$ edge whenever $hh'$ is an edge in $H$. $C_h$ is called the *branch set corresponding to* $h$.

We will use the following result of Mader [5].

**Theorem 4** *Every graph* $G$ *contains a* $\lceil d(G)/4 \rceil$*-connected subgraph.*

Given $k \in \mathbb{N}$, we say that a graph $G$ is *k-linked* if $|G| \geq 2k$ and for every $2k$ distinct vertices $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ of $G$ there exist disjoint paths $P_1, \ldots, P_k$ such that $P_i$ joins $x_i$ to $y_i$. Jung as well as Larman and Mani independently proved that every sufficiently highly connected graph is $k$-linked. Later, Bollobás and Thomason [2] showed that a connectivity linear in $k$ suffices. Simplifying the argument in [2], Thomas and Wollan [9] recently obtained an even better bound:

**Theorem 5** *Every* $16k$*-connected graph is* $k$*-linked.*

Similarly as in [11], given positive numbers $d$ and $k$, we shall consider the class $\mathcal{G}_{d,k}$ of graphs defined by

$$\mathcal{G}_{d,k} := \{G \,:\, |G| \geq d,\ e(G) > d|G| - kd\}.$$

We say that a graph $G$ is *minor-minimal in* $\mathcal{G}_{d,k}$ if $G$ belongs to $\mathcal{G}_{d,k}$ but no proper minor of $G$ does. The following lemma states some properties of the minor-minimal elements of $\mathcal{G}_{d,k}$. The proof is simple, its counterpart for digraphs can be found in [11, Section 2]. (The first property follows by counting the number of edges of the complete graph on $\lfloor(2-\varepsilon)d\rfloor$ vertices.)

**Lemma 6** *Given $0 < \varepsilon < 1/2$, $d \geq 2/\varepsilon$ and $1/d \leq k \leq \varepsilon d/2$, every minor-minimal graph in $\mathcal{G}_{d,k}$ satisfies the following properties:*

(i) $|G| \geq (2-\varepsilon)d$,

(ii) $e(G) \leq d|G| - kd + 1$,

(iii) *every edge of $G$ lies in more than $d-1$ triangles,*

(iv) *$G$ is $\lceil k\rceil$-connected.*

We will also use the following easy fact, see [11, Lemma 4.2] for a proof.

**Lemma 7** *Suppose that $x$ and $y$ are distinct vertices of a $k$-connected graph $G$. Then $G$ contains at least $k^2/4|G|$ internally disjoint $x$-$y$ paths of length at most $2|G|/k$.*

## 3 Proof of theorems

The strategy of the proof of Theorem 2 is as follows. It is easily seen that to prove Theorem 2 for all graphs of average degree at least $(1+\varepsilon)t =: d$, it suffices to consider only those graphs $G$ which are minor-minimal in the class $\mathcal{G}_{d/2,k}$ for some suitable $k$. In particular, together with Lemma 6 this implies that we only have to deal with $k$-connected graphs. If $d$ (and so also $k$) is linear in the order of $G$, then a simple probabilistic argument gives us the desired $K_{s,t}$ minor (Lemma 8). In the other case we use that by Lemma 6 each vertex of $G$ together with its neighbourhood induces a dense subgraph of $G$. We apply this to find 10 disjoint $K_{10s,\lceil d/9\rceil}$ minors which we combine to a $K_{s,t}$ minor.

**Lemma 8** *For all $0 < \varepsilon, c < 1$ there exists a number $k_0 = k_0(\varepsilon, c)$ such that for each integer $k \geq k_0$ every $k$-connected graph $G$ whose order $n$ satisfies $k \geq cn$ contains a $K_{s,t}$ minor where $t := \lceil(1-\varepsilon)n\rceil$ and $s := \lceil c^4\varepsilon n/(32\log n)\rceil$. Moreover, the branch sets corresponding to the vertices in the vertex class of the $K_{s,t}$ of size $t$ can be chosen to be singletons whereas all the other branch sets can be chosen to have size at most $8\log n/c^2$.*

**Proof.** Throughout the proof we assume that $k$ (and thus also $n$) is sufficiently large compared with both $\varepsilon$ and $c$ for our estimates to hold. Put $a := \lfloor 4\log s/c \rfloor$. Successively choose $as$ vertices of $G$ uniformly at random without repetitions. Let $C_1$ be the set of the first $a$ of these vertices, let $C_2$ be the set of the next $a$ vertices and so on up to $C_s$. Let $C$ be the union of all the $C_i$. Given $i \le s$, we call a vertex $x \in G - C$ *good for* $i$ if $x$ has at least one neighbour in $C_i$. Moreover, we say that $x$ is *good* if it is good for every $i \le s$. Thus

$$\mathbb{P}(x \text{ is not good for } i) \le \left(1 - \frac{d_G(x) - as}{n}\right)^a \le \mathrm{e}^{-a(k-as)/n} \le \mathrm{e}^{-ac/2}$$

and so $x$ is not good with probability at most $s\mathrm{e}^{-ac/2} < \varepsilon/2$. Therefore the expected number of good vertices outside $C$ is at least $(1-\varepsilon/2)|G-C|$. Hence there exists an outcome $C_1, \dots, C_s$ for which at least $(1-\varepsilon/2)|G-C|$ vertices in $G - C$ are good.

We now extend all these $C_i$ to disjoint connected subgraphs of $G$ as follows. Let us start with $C_1$. Fix a vertex $x_1 \in C_1$. For each $x \in C_1 \setminus \{x_1\}$ in turn we apply Lemma 7 to find an $x$-$x_1$ path of length at most $2n/k \le 2/c$ which is internally disjoint from all the paths chosen previously and which avoids $C_2 \cup \dots \cup C_s$. Since Lemma 7 guarantees at least $k^2/4n \ge as \cdot 2/c$ short paths between a given pair of vertices, we are able to extend each $C_i$ in turn to a connected subgraph in this fashion. Denote the graphs thus obtained from $C_1, \dots, C_s$ by $G_1, \dots, G_s$. Thus all the $G_i$ are disjoint.

Note that at most $2as/c$ good vertices lie in some $G_i$. Thus at least $(1-\varepsilon/2)|G-C| - 2as/c \ge (1-\varepsilon)n$ good vertices avoid all the $G_i$. Hence $G$ contains a $K_{s,t}$ minor as required. (The good vertices avoiding all the $G_i$ correspond to the vertices of the $K_{s,t}$ in the vertex class of size $t$. The branch sets corresponding to the vertices of the $K_{s,t}$ in the vertex class of size $s$ are the vertex sets of $G_1, \dots, G_s$.) □

**Proof of Theorem 2.** Let $d := (1+\varepsilon)t$ and $s := \lfloor \varepsilon^6 d/\log d \rfloor$. Throughout the proof we assume that $t$ (and thus also $d$) is sufficiently large compared with $\varepsilon$ for our estimates to hold. We have to show that every graph of average degree at least $d$ contains a $K_{s,t}$ minor. Put $k := \lceil \varepsilon d/4 \rceil$. Since $\mathcal{G}_{d/2,k}$ contains all graphs of average degree at least $d$, it suffices to show that every graph $G$ which is minor-minimal in $\mathcal{G}_{d/2,k}$ contains a $K_{s,t}$ minor. Let $n := |G|$. As is easily seen, (i) and (iv) of Lemma 6 together with Lemma 8 imply that we may assume that $d \le n/600$. (Lemma 8 is applied with $c := \varepsilon/2400$ and with $\varepsilon$ replaced by $\varepsilon/3$.) Let $X$ be the set of all those vertices of $G$ whose degree is at most $2d$. Since by Lemma 6 (ii) the average degree of $G$ is at most $d$, it follows that $|X| \ge n/2$. Let us first prove the following claim.

> *Either $G$ contains a $K_{s,t}$ minor or $G$ contains $10$ disjoint $\lceil 3d/25 \rceil$-connected subgraphs $G_1, \dots, G_{10}$ such that $3d/25 \le |G_i| \le 3d$ for each $i \le 10$.*

Choose a vertex $x_1 \in X$ and let $G_1'$ denote the subgraph of $G$ induced by $x_1$ and its neighbourhood. Then $|G_1'| = d_G(x_1) + 1 \le 2d + 1$. Since by Lemma 6 (iii) each edge between $x_1$ and $N_G(x_1)$ lies in at least $d/2-1$ triangles, it follows that the minimum degree of $G'_1$ is at least $d/2-1$. Thus Theorem 4 implies that $G'_1$ contains a $\lceil 3d/25\rceil$-connected subgraph. Take $G_1$ to be this subgraph. Put $X_1 := X \setminus V(G_1)$ and let $X'_1$ be the set of all those vertices in $X_1$ which have at least $d/500$ neighbours in $G_1$.

Suppose first that $|X'_1| \geq |X|/10$. In this case we will find a $K_{s,t}$ minor in $G$. Since the argument is similar to the proof of Lemma 8, we only sketch it. Set $a := \lfloor 10^4 \log s\rfloor$. This time, we choose the $a$-element sets $C_1, \dots, C_s$ randomly inside $V(G_1)$. Since every vertex in $X'_1$ has at least $d/500$ neighbours in $G_1$, the probability that the neighbourhood of a given vertex $x \in X'_1$ avoids some $C_i$ is at most $s\mathrm{e}^{-a/(3\cdot 10^3)} < \varepsilon$. So the expected number of such bad vertices in $X'_1$ is at most $\varepsilon|X'_1|$. Thus for some choice of $C_1, \dots, C_s$ there are at least $(1-\varepsilon)|X'_1| \geq (1-\varepsilon)n/20 \geq t$ vertices in $X'_1$ which have a neighbour in each $C_i$. Since the connectivity of $G_1$ is linear in its order, we may again apply Lemma 7 to make the $C_i$ into disjoint connected subgraphs of $G_1$ by adding suitable short paths from $G_1$. This shows that $G$ contains a $K_{s,t}$ minor.

Thus we may assume that at least $|X_1| - |X|/10 \geq 9|X|/10 - 3d > 0$ vertices in $X_1$ have at most $d/500$ neighbours in $G_1$. Choose such a vertex $x_2$. Let $G'_2$ be the subgraph of $G$ induced by $x_2$ and all its neighbours outside $G_1$. Since by Lemma 6 (iii) every edge of $G$ lies in at least $d/2-1$ triangles, it follows that the minimum degree of $G'_2$ is at least $d/2 - 1 - d/500 > 12d/25$. Again, we take $G_2$ to be a $\lceil 3d/25\rceil$-connected subgraph of $G'_2$ obtained by Theorem 4.

We now put $X_2 := X_1 \setminus (X'_1 \cup V(G_2))$ and define $X'_2$ to be the set of all those vertices in $X_2$ which have at least $d/500$ neighbours in $G_2$. If $|X'_2| \geq |X|/10$, then as before, we can find a $K_{s,t}$ minor in $G$. If $|X'_2| \leq |X|/10$ we define $G_3$ in a similar way as $G_2$. Continuing in this fashion proves the claim. (Note that when choosing $x_{10}$ we still have $|X_9| - |X|/10 \geq |X|/10 - 9\cdot 3d > 0$ vertices at our disposal since $n \geq 600d$.)

Apply Lemma 8 with $c := 1/25$ to each $G_i$ to find a $K_{10s,\lceil d/9\rceil}$ minor. Let $C^i_1, \dots, C^i_s, D^i_1, \dots, D^i_{9s}$ denote the branch sets corresponding to the vertices of the $K_{10s,\lceil d/9\rceil}$ in the vertex class of size $10s$. By Lemma 8 we may assume that all the $C^i_j$ and all the $D^i_j$ have size at most $8\cdot 25^2 \log|G_i| \leq 10^5 \log d$ and that all the branch sets corresponding to the remaining vertices of the $K_{10s,\lceil d/9\rceil}$ are singletons. Let $T^i \subseteq V(G_i)$ denote the union of all these singletons. Let $C$ be the union of all the $C^i_j$, let $D$ be the union of all the $D^i_j$ and let $T$ be the union of all the $T^i$.

We will now use these 10 $K_{10s,\lceil d/9\rceil}$ minors to form a $K_{s,t}$ minor in $G$. Recall that by Lemma 6 (iv) the graph $G$ is $\lceil \varepsilon d/4\rceil$-connected and so by Theorem 5 it is $\lfloor \varepsilon d/64\rfloor$-linked. Thus there exists a set $\mathcal{P}$ of $9s$ disjoint paths in $G$ such that for all $i \leq 9$ and all $j \leq s$ the set $C^i_j$ is joined to $C^{i+1}_j$ by one of these paths and such that no path from $\mathcal{P}$ contains an inner vertex in $C \cup D$. (To see this, use that $\varepsilon d/64 \geq 100s \cdot 10^5 \log d \geq |C \cup D|$.)

The paths in $\mathcal{P}$ can meet $T$ in many vertices. But we can reroute them such that every new path contains at most two vertices from each $T^i$. For every path $P \in \mathcal{P}$ in turn we will do this as follows. If $P$ meets $T^1$ in more than 2 vertices, let $t$ and $t'$ denote the first and the last vertex from $T^1$ on $P$. Choose some set

$D_j^1$ and replace the subpath $tPt'$ by some path between $t$ and $t'$ whose interior lies entirely in $G[D_j^1]$. (This is possible since $G[D_j^1]$ is connected and since both $t$ and $t'$ have a neighbour in $D_j^1$.) Proceed similarly if the path thus obtained still meets some other $T^i$. Then continue with the next path from $\mathcal{P}$. (The sets $D_j^i$ used for the rerouting are chosen to be distinct for different paths.) Note that the paths thus obtained are still disjoint since $D$ was avoided by all the paths in $\mathcal{P}$.

We now have found our $K_{s,t}$ minor. Each vertex lying in the vertex class of size $s$ of the $K_{s,t}$ corresponds to a set consisting of $C_j^1 \cup \cdots \cup C_j^{10}$ together with the (rerouted) paths joining these sets. For the remaining vertices of the $K_{s,t}$ we can take all the vertices in $T$ which are avoided by the (rerouted) paths. There are at least $t$ such vertices since these paths contain at most $20 \cdot 9s$ vertices from $T$ and $|T| - 180s \geq 10d/9 - 180s \geq t$. □

**Proof of Theorem 3 (Sketch).** Without loss of generality we may assume that $\varepsilon < 10^{-16}$. The proof of Theorem 3 is almost the same as that of Theorem 2. The only difference is that now we also apply Lemma 7 to find $\binom{s}{2}$ short paths connecting all the pairs of the $C_i$. This can be done at the point where we extend the $C_i$'s to connected subgraphs. □

The following proposition shows that the bound on $s$ in Theorem 2 is essentially best possible. Its proof is an adaption of a well-known argument of Bollobás, Catlin and Erdős [1].

**Proposition 9** *There exists an integer $n_0$ such that for each integer $n \geq n_0$ and each number $\alpha > 0$ there is a graph $G$ of order $n$ and with average degree at least $n/2$ which does not have a $K_{s,t}$ minor with $s := \lceil 2n/\alpha \log n \rceil$ and $t := \lceil \alpha n \rceil$.*

**Proof.** Let $p := 1 - 1/\mathrm{e}$. Throughout the proof we assume that $n$ is sufficiently large for our estimates to hold. Consider a random graph $G_p$ of order $n$ which is obtained by including each edge with probability $p$ independently from all other edges. We will show that with positive probability $G_p$ is as required in the proposition. Clearly, with probability $> 3/4$ the average degree of $G_p$ is at least $n/2$. Hence it suffices to show that with probability at most $1/2$ the graph $G_p$ will have the property that its vertex set $V(G_p)$ can be partitioned into disjoint sets $S_1, \ldots, S_s$ and $T_1, \ldots, T_t$ such that $G_p$ contains an edge between every pair $S_i, T_j$ ($1 \leq i \leq s$, $1 \leq j \leq t$). Call such a partition of $V(G_p)$ *admissible.* Thus we have to show that the probability that $G_p$ has an admissible partition is $\leq 1/2$.

Let us first estimate the probability that a given partition $\mathcal{P}$ is admissible:

$$\begin{aligned}\mathbb{P}(\mathcal{P} \text{ is admissible}) &= \prod_{i,j} \left(1-(1-p)^{|S_i||T_j|}\right) \le \exp\left(-\sum_{i,j}(1-p)^{|S_i||T_j|}\right) \\ &\le \exp\left(-st\prod_{i,j}(1-p)^{|S_i||T_j|(st)^{-1}}\right) \le \exp\left(-st(1-p)^{n^2(st)^{-1}}\right) \\ &\le \exp\left(-\frac{2n^2}{\log n}\cdot n^{-\frac{1}{2}}\right) \le \exp(-n^{\frac{4}{3}}).\end{aligned}$$

(The first expression in the second line follows since the arithmetic mean is at least as large as the geometric mean.) Since the number of possible partitions is at most $n^n$, it follows that the probability that $G_p$ has an admissible partition is at most $n^n \cdot \mathrm{e}^{-n^{4/3}} < 1/2$, as required. $\square$

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