Mathematisches Seminar

Hamburger Beiträge zur Mathematik

Heft 131

On the extension of involutorial Bol loops

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ABSTRACT. The group theoretical problem of the existence of system of representatives T of the subgroup H of G such that T consists of conjugacy classes of involutions leads to the theory of Bol loops of exponent 2.

In this paper, we develop a theory of extensions of such loops and give two applications of the theory. First, we classify all Bol loops of exponent 2 of order 16; second, we classify all (left) Bol loops of exponent 2 whose right nucleus has index 2. In particular, we give a class of examples of non-nilpotent such Bol loops.

Introduction

Consider the following group theoretical problem. Let G be a finite group, H a subgroup of G and T a system of representatives of the left cosets of H such that $T = \{1\} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s$ where $\mathcal{C}_1, \ldots, \mathcal{C}_s$ are conjugacy classes of involutions of G and T generates G. Is it then true that G is a 2-group?

Define the natural map $\sigma : G \to T$ by $\sigma(x)H = xH$ for all $x \in G$, and define the binary operation $x * y = \sigma(xy)$ on T. We have

$$x * (y * (x * z)) = \sigma(xyxz) = (xyx) * z = (x * (y * x)) * z$$

and $x * x = 1$ for all $x, y, z \in T$. It follows that $(T, *)$ is a *Bol loop* in the sense of [**2**]. Following [**7**], a Bol loop with $x * x = 1$ is called an *involutorial Bol loop.*

In this paper we discuss a method that allows to "double" involutorial Bol loops L, i.e., we characterize extensions E of L such that L has index 2 in E.

After some preliminaries, where we recall some properties of Bol loops and related structures in a leisurely style, we give our characterization of extensions in section 2.

Section 3 is devoted to isomorphisms between such extensions. In the section 4 we describe how to find all involutorial Bol loops of order 16 using a computer.

In the final section 5 we characterize all involutorial Bol loops L with large right nucleus $\mathcal{N}_r = \{n \in L;\ \forall x, y \in L : x(yn) = (xy)n\}$, i.e., $\mathcal{N}_r$ is of index 2 in L. Beginning with order 16, there exist examples of such loops with trivial center $\mathcal{Z}$. These are therefore not centrally nilpotent.

The second author was supported by the "János Bolyai" Fellowship, the Blaschke Stiftung and the OTKA grants F030737, T029849.

1. Preliminaries

A set L with a binary operation "$\cdot$" is called a *groupoid* if there is a unit element 1, i.e., $1 \cdot a = a \cdot 1 = a$ for all $a \in L$. A *left loop* is a groupoid such that the equation $ax = b$ has a unique solution $x \in L$ for all $a, b \in L$. If in addition, the equation $ya = b$ has a unique solution $y \in L$ then L is called a *loop*.

In a left loop the *left translations* $\lambda_a : L \to L;\ x \mapsto ax$ are bijections for all $a \in L$. Therefore $\lambda(L) = \{\lambda_a;\ a \in L\}$ generates a subgroup $\mathcal{M}_\ell$ of $\mathrm{Sym}(L)$. The stabilizer of 1 inside $\mathcal{M}_\ell$ is denoted by $\mathcal{D}$ and is generated by the *precessions* $\delta_{a,b} := \lambda_{ab}^{-1}\lambda_a\lambda_b$, $a, b \in L$, see [**4**, (2.6.1)].

A left loop L is called A_ℓ*-left-loop* if $\mathcal{D}$ is a subgroup of the automorphism group of L. In fact it suffices to assume that all the precessions are automorphisms.

Let G be a group with subgroup Ω. We say that the set $T \subseteq G$ is a *transversal* of G/Ω if it is a system of representatives of the left cosets of Ω in G with $1 \in T$. In particular, if L is a left loop, then $\lambda(L)$ is a transversal of $\mathcal{M}_\ell/\mathcal{D}$, see [**4**, (2.6.2)].

Conversely, every transversal T of G/Ω carries a left loop structure in a natural way: For any elements $s, t \in T$ there are unique elements $s * t \in T$ and $d_{s,t} \in \Omega$ with $st = (s * t)d_{s,t}$, and $(T, *)$ is a left loop. For use in section 5, we also define the subgroup

$$D = D(L) = \langle d_{s,t};\ s, t \in L\rangle \quad \text{of } \Omega.$$

Notice that the involutorial Bol loop from the introduction was constructed this way. For more on the relation between loops and transversals see [**1**] or [**4**].

If a groupoid L satisfies the *Bol identity*

$$a(b \cdot ac) = (a \cdot ba)c \quad \text{for all } a, b, c \in L$$

and every element has a left inverse, then L is a Bol loop [**4**, (3.10)]. Bol loops are *left alternative*, i.e., $a \cdot ab = a^2 b$ and they satisfy the *left inverse property*, $\lambda_{a^{-1}} = \lambda_a^{-1}$ for all a, b. These well-known facts will be used freely and without explicit reference (see [**4**, (3.10)] for proofs and more).

A *semi-automorphism* of a groupoid L is a bijection $\sigma : L \to L$ with the properties $\sigma(1) = 1$ and $\sigma(a \cdot ba) = \sigma(a) \cdot \sigma(b)\sigma(a)$ for all $a, b \in L$. In Bol loops, the map $x \mapsto x^{-1}$ is a semi-automorphism [**8**, IV.6.12]

One usually denotes the loop multiplication by juxtaposition: xy. However, in this paper, we will define several loop operations on the same set L. They will be denoted by $x * y$, $h(x, y)$, etc. For the left translations of these operations we use

$$\lambda_x^*(y) = x * y, \quad \lambda_x^h(y) = h(x, y), \quad \text{etc.}$$

In a similar way we treat right translations $\varrho_x(y) := yx$.

In section 3 we'll use some isotopy theory, in particular for Bol loops. All that we need can be found in [**8**] or [**4**].

2. Extensions of loops

Definition *The algebra* $(L, \cdot, h, u, 1)$ *is called an E-triple if:*

(i) $(L, \cdot)$ *and* (L, h) *are Bol loops with the same unit element* 1*;*
(ii) $(L, \cdot)$ *is an involutorial Bol loop;*
(iii) $u : L \to L$ *is a semi-automorphism of* $(L, \cdot)$ *with* $u^2 = \mathrm{id}$*;*

(iv) The identities

$$(\lambda_x^h)^{-1}\lambda_y\lambda_x^h = \lambda_{h(\bar{x},yx)}, \tag{1}$$

$$(\lambda_x^h)^{-1}\lambda_y\lambda_x^h = \lambda_{u(h(x,u(y)\bar{x}))}, \tag{2}$$

$$\lambda_{u(x)}\lambda_y^h\lambda_x = \lambda^h_{u(x)\cdot h(y,x)} \tag{3}$$

hold for all $x, y \in L$, where $\bar{x}$ denotes the inverse of x with respect to h, i.e., $h(\bar{x},x) = h(x,\bar{x}) = 1$.

To the set L construct a set L' disjoint from L with $|L'| = |L|$ and a bijection $\alpha : L \leftrightarrow L'$ (e.g., $L' = L \times \{1\}, \alpha(x) = (x,1)$).

Theorem 1. *For an E-triple $(L, \cdot, h, u)$ define the following multiplication on the set $E = L \cup L'$:*

$$x \circ y = \begin{cases} x \cdot y & \text{if } x, y \in L; \\ \alpha(h(\alpha^{-1}(x), y)) & \text{if } x \in L', y \in L; \\ \alpha(g(x, \alpha^{-1}(y))) & \text{if } x \in L, y \in L'; \\ f(\alpha^{-1}(x), \alpha^{-1}(y)) & \text{if } x, y \in L'; \end{cases} \tag{4}$$

where the operations f, g on L are defined by $g(x,y) = u(x)y$ and $f(x,y) = (\lambda_x^h)^{-1}(y) = h(\bar{x}, y)$.

Then the following hold:

(i) $(E, \circ)$ is an involutorial Bol loop containing $(L, \cdot)$ as a subloop of index 2.

(ii) $(E, \circ)$ is uniquely defined by the E-triple $(L, \cdot, h, u)$ up to isomorphism, i.e., independent of the choice of α.

(iii) Every loop E, containing $(L, \cdot)$ as a subloop of index 2 occurs in this way.

Proof. To show (i), we need to prove the identities $1 \circ x = x \circ 1 = x$, $x \circ x = 1$ and $x \circ (y \circ (x \circ z)) = (x \circ (y \circ x)) \circ z$.

The first one is easy. The second is trivial for $x \in L$, so let $x \in L'$, then $x \circ x = f(\alpha^{-1}(x), \alpha^{-1}(x)) = \overline{\alpha^{-1}(x)}\alpha^{-1}(x) = 1$. There remains to show the Bol identity.

In order to simplify our notation, we abbreviate $\alpha(x) = x'$ for $x \in L$. For $x, y, z \in L$, we have to consider the following cases,

$a)\ x \circ (y \circ (x \circ z)) = (x \circ (y \circ x)) \circ z$	$x(y \cdot xz) = (x \cdot yx)z$
$b)\ x \circ (y \circ (x \circ z')) = (x \circ (y \circ x)) \circ z'$	$g(x, g(y, g(x,z))) = g(x \cdot yx, z)$
$c)\ x' \circ (y \circ (x' \circ z)) = (x' \circ (y \circ x')) \circ z$	$f(x, g(y, h(x,z))) = f(x, g(y,x)) \cdot z$
$d)\ x \circ (y' \circ (x \circ z)) = (x \circ (y' \circ x)) \circ z$	$g(x, h(y, x \cdot z)) = h(g(x, h(y,x)), z)$
$e)\ x \circ (y' \circ (x \circ z')) = (x \circ (y' \circ x)) \circ z'$	$x \cdot f(y, g(x,z)) = f(g(x, h(y,x)), z)$
$f)\ x' \circ (y \circ (x' \circ z')) = (x' \circ (y \circ x')) \circ z'$	$h(x, y \cdot f(x,z)) = g(f(x, g(y,x)), z)$
$g)\ x' \circ (y' \circ (x' \circ z)) = (x' \circ (y' \circ x')) \circ z$	$h(x, f(y, h(x,z))) = h(h(x, f(y,x)), z)$
$h)\ x' \circ (y' \circ (x' \circ z')) = (x' \circ (y' \circ x')) \circ z'$	$f(x, h(y, f(x,z))) = f(h(x, f(y,x)), z)$

a) holds by definition, b) follows from the Bol identity for "$\cdot$" and the fact that u is a semi-automorphism. Plugging in the definition g) and h) become

$$h(x, h(\bar{y}, h(x,z))) = h(h(x, h(\bar{y}, x)), z)$$

and

$$h(\bar{x}, h(y, h(\bar{x}, z))) = h(\overline{h(x, h(\bar{y}, x))}, z) = h(h(\bar{x}, h(y, \bar{x})), z)$$

respectively, which hold since h is a Bol operation, and therefore $x \mapsto \bar{x}$ is a semi-automorphism of (L, h).

Furthermore, we have the following equivalent expressions:

$$\begin{array}{rll}
c) & \iff & h(\bar{x}, u(y)h(x,z)) = h(\bar{x}, u(y)x) \cdot z \\
 & \iff & (\lambda_x^h)^{-1}\lambda_{u(y)}\lambda_x^h = \lambda_{h(\bar{x},u(y)x)} \qquad \iff \quad (1) \\
d) & \iff & u(x)h(y,xz) = h(u(x)\cdot h(y,x), z) \\
 & \iff & \lambda_{u(x)}\lambda_y^h\lambda_x = \lambda^h_{u(x)\cdot h(y,x)} \qquad \iff \quad (3) \\
e) & \iff & xh(\bar{y}, u(x)z) = h(\overline{u(x)h(y,x)}, z) \\
 & \iff & \lambda_x(\lambda_y^h)^{-1}\lambda_{u(x)} = (\lambda^h_{u(x)\cdot h(y,x)})^{-1} \qquad \iff \quad (3) \\
f) & \iff & h(x, yh(\bar{x},z)) = u(h(\bar{x}, u(y)x))z \\
 & \iff & \lambda_x^h\lambda_y(\lambda_x^h)^{-1} = (\lambda_{\bar{x}}^h)^{-1}\lambda_y\lambda_{\bar{x}}^h = \lambda_{u(h(\bar{x},u(y)x))} \qquad \iff \quad (2).
\end{array}$$

Hence $(E,\circ)$ is an involutorial Bol loop and (i) holds.

(ii) will be simple consequence of the general Theorem 2 on isomorphisms of extensions.

For (iii), let $E = (E,\cdot)$ be an involutorial Bol loop with subloop L of index 2. Let us take an arbitrary element $s \in E\backslash L$. We define $L' = sL$, $\alpha(x) = sx$ $(x \in L)$ and the operations

$$sh(x,y) = sx \cdot y, \quad \text{and} \quad u(x) = s \cdot xs$$

on L. By

$$h(x,y) = s(sx\cdot y) = s(s(x/s \cdot sy)) = x/s \cdot sy,$$

(L,h) is a subloop of a principal isotope of E, hence Bol. It is known (and directly verified) that u is a semi-automorphism of E, see [**8**, IV.6.11 p. 117]. The facts that 1 is a unit for h and $u(1) = 1$ are clear.

Now define the operations g and f as above. For g we have

$$sg(x,y) = s\cdot u(x)y = s((s\cdot xs)y) = s(s(x\cdot sy)) = x\cdot sy.$$

For f, the relation $sy = sh(x, f(x,y)) = sx \cdot f(x,y)$ implies

$$f(x,y) = sx \cdot sy.$$

Finally, the converse of the calculations in (i) implies the properties $(1)-(3)$. Thus, the triple $(L,\cdot,h,u)$ is an E-triple indeed and (iii) holds. □

Remark. One easily shows that for arbitrary loops $(L,\cdot)$, (L,h) with unit 1 and bijection $u : L \to L$ with $u(1) = 1$, (4) defines a loop $(E,\circ)$. Moreover, it is not difficult to construct examples showing that the three conditions $(1)-(3)$ in the definition of E-triples are independent.

3. Isomorphisms of loop extensions

We'll now answer the question "When do different E-triples give rise to isomorphic loop extensions?".

THEOREM 2. *Let $(L,\cdot,h,u)$, $(L,*,h^*,u^*)$ be E-triples with corresponding loop extensions $(E,\circ)$, $(E,\bullet)$, given by maps $\alpha,\alpha^* : L \leftrightarrow L'$, respectively. Assume that $\beta : (E,\circ) \to (E,\bullet)$ is a bijection with $\beta(L) = L$, and put $\varphi := \beta|_L : L \to L$, $\theta := (\alpha^*)^{-1}\beta\alpha$ and $k := \theta(1)$.*

(i) If β is an isomorphism, then we have for all $x,y \in L$

(5) $$(\theta,\varphi,\theta) : (L,h) \to (L,h^*) \quad \text{is an isotopism}$$

(6) $$u^*(x) = \theta\big(u\varphi^{-1}(x)\cdot\theta^{-1}(1)\big)$$

(ii) If $\varphi : (L, \cdot) \to (L, *)$ is an isomorphism, satisfying the conditions (5) and (6), then β is an isomorphism.

Proof. Notice that (5) implies by general isotopy theory [**8**, III.2.1, p. 60] or [**4**, (4.2.1)]

$$\theta = \lambda_k^{h^*} \varphi.$$

This will be used in the sequel without reference.

(i) Let $x, y \in L$, then we have

$$\begin{aligned} \beta(\alpha(x) \circ y) = \beta\alpha(x) \bullet \varphi(y) \quad &\iff \\ \beta\alpha(h(x,y)) = \alpha^* h^* \big((\alpha^*)^{-1} \beta\alpha(x), \varphi(y)\big) \quad &\iff \\ \theta h(x,y) = h^*(\theta(x), \varphi(y)) \quad &\iff \quad (5). \end{aligned}$$

Moreover,

$$\begin{aligned} \beta(x \circ \alpha(\theta^{-1}(1))) = \varphi(x) \bullet \beta\alpha\theta^{-1}(1) \quad &\iff \\ \beta\alpha\big(u(x) \cdot \theta^{-1}(1)\big) = \alpha^* \big(u^* \varphi(x) * (\alpha^*)^{-1} \beta\alpha\theta^{-1}(1)\big) \quad &\iff \\ \theta\big(u(x) \cdot \theta^{-1}(1)\big) = u^* \varphi(x). \end{aligned}$$

Replacing x with $\varphi^{-1}(x)$ gives (6).

(ii) To show $\beta(a \circ b) = \beta(a) \bullet \beta(b)$ for all $a, b \in E$, there are four cases to consider. The case $a, b \in L$ is trivial. The case $a \in L'$, $b \in L$ comes directly from the proof of (i) and (5).

Let ι, $\iota^* : L \to L$ denote the inverse maps of h, h^*, respectively.

In the case $a \in L$, $b \in L'$ put $a = x$, $b = \alpha(y)$ then $x, y \in L$. We have

$$(\alpha^*)^{-1} \beta(x \circ \alpha(y)) = \theta(u(x)y) = h^*(k, \varphi u(x) * \varphi(y)),$$

and $\theta^{-1}(1) = \varphi^{-1}(\lambda_k^{h^*})^{-1}(1) = \varphi^{-1}\iota^*(k)$. Using this, (6) and (1) we can compute

$$\begin{aligned} &(\alpha^*)^{-1} \big(\beta(x) \bullet \beta\alpha(y)\big) = u^* \varphi(x) * \theta(y) = \theta(u(x) \cdot \varphi^{-1} \iota^*(k)) * \theta(y) \\ = \quad & h^*(k, \varphi u(x) * \iota^*(k)) * \lambda_k^{h^*} \varphi(y) = \lambda_k^{h^*} \lambda^*_{\varphi u(x)} (\lambda_k^{h^*})^{-1} \lambda_k^{h^*} \varphi(y). \end{aligned}$$

Therefore $\beta(x \circ \alpha(y)) = \beta(x) \bullet \beta\alpha(y)$.

For the case $a, b \in L'$ put $a = \alpha(x)$, $b = \alpha(y)$, then

$$\begin{aligned} \beta(\alpha(x) \circ \alpha(y)) = \beta\alpha(x) \bullet \beta\alpha((y) \quad &\iff \\ \varphi h(\iota(x), y) = h^*(\iota^*(\theta(x)), \theta(y)) \quad &\iff \\ (\iota^* \theta \iota, \theta, \varphi) : (L, h) \to (L, h^*) \quad &\text{is an isotopism.} \end{aligned}$$

Now, as in any Bol loop $\big(\lambda_k^{h^*} \varrho_k^{h^*}, (\lambda_k^{h^*})^{-1}, \lambda_k^{h^*}\big)$ is an autotopism of (L, h^*). Thus, by (5) we find that

$$\left(\lambda_k^{h^*} \varrho_k^{h^*}, (\lambda_k^{h^*})^{-1}, \lambda_k^{h^*}\right)^{-1} (\theta, \varphi, \theta) = \left((\varrho_k^{h^*})^{-1} \varphi, \theta, \varphi\right) : (L, h) \to (L, h^*)$$

is an isotopism. So we are left with showing that $(\varrho_k^{h^*})^{-1} \varphi = \iota^* \theta \iota$. By (5) we have

$$\begin{aligned} k = \theta h(x, \iota(x)) = h^*(\theta(x), \varphi\iota(x)) \quad &\iff \quad h^*(\iota^*\theta(x), k) = \varphi\iota(x) \\ &\iff \quad \varrho_k^{h^*} \iota^* \theta = \varphi\iota. \end{aligned}$$

Therefore β is indeed an isomorphism. $\square$

As an immediate consequence we obtain the

Proof of Theorem 1, (ii). Let us take two maps $\alpha_1, \alpha_2 : L \leftrightarrow L'$ and consider the Bol loops $(E, \circ_1)$ and $(E, \circ_2)$, obtained from the E-triple $(L, \cdot, h, u)$ and α_1, α_2, respectively. With notation from Theorem 2 we have $\cdot = *$, $h = h^*$, $u = u^*$. Define

$$\beta : E \to E; \;\; x \mapsto \begin{cases} x, & \text{if } x \in L, \\ \alpha_2\alpha_1^{-1}(x), & \text{if } x \in L', \end{cases}$$

Then $\varphi = \mathrm{id}$, $\theta = \mathrm{id}$ and $k = 1$. Thus Theorem 2 gives the result. □

Lemma 3. *Let $(L, \cdot)$ and (L, h) be Bol loops with the same unit 1, satisfying (1). Denote the inverse of $\ell \in L$ with respect to h by $\bar{\ell}$. Then the map*

$$\sigma_\ell : L \to L; \;\; x \mapsto h(\ell, x\bar{\ell})$$

is a semi-automorphism for all $\ell \in L$. Moreover, $\sigma_\ell^{-1} = \sigma_{\bar{\ell}}$.

Proof. Let $x, y \in L$. Using Bol, (1), and Bol again, we can compute

$$\begin{aligned} & \lambda_{h(\ell,x\bar{\ell})\cdot h(\ell,y\bar{\ell})h(\ell,x\bar{\ell})} = \lambda_{h(\ell,x\bar{\ell})}\lambda_{h(\ell,y\bar{\ell})}\lambda_{h(\ell,x\bar{\ell})} \\ = \;& \lambda_\ell^h\lambda_x(\lambda_\ell^h)^{-1}\lambda_\ell^h\lambda_y(\lambda_\ell^h)^{-1}\lambda_\ell^h\lambda_x(\lambda_\ell^h)^{-1} = \lambda_\ell^h\lambda_x\lambda_y\lambda_x(\lambda_\ell^h)^{-1} \\ = \;& \lambda_\ell^h\lambda_{x\cdot yx}(\lambda_\ell^h)^{-1} = \lambda_{h(\ell,(x\cdot yx)\bar{\ell})}, \end{aligned}$$

showing the first assertion. Using (1) we obtain

$$h(\ell, h(\bar{\ell}, x\ell)\bar{\ell}) = \lambda_\ell^h\lambda_{h(\bar{\ell},x\ell)}(\bar{\ell}) = \lambda_\ell^h(\lambda_\ell^h)^{-1}\lambda_x\lambda_\ell^h(\bar{\ell}) = x,$$

the second assertion. □

For the rest of this paper, we fix the sets $L, L', E = L \cup L'$ and the bijection $\alpha : L \leftrightarrow L'$.

Proposition 4. *Let $(L, \cdot, h, u)$ be an E-triple, (L, h^*) a Bol loop and $(\psi, \theta, \varphi) : (L, h) \to (L, h^*)$ an isotopism. For $x, y \in L$ put*

$$x * y := \varphi(\varphi^{-1}(x) \cdot \varphi^{-1}(y)) \quad \textit{and} \quad u^*(x) := \theta\left(u\varphi^{-1}(x) \cdot \theta^{-1}(1)\right).$$

Then $(L, *, h^*, u^*)$ is an E-triple, and

$$\beta : (E, \circ) \to (E, \bullet); \;\; x \mapsto \begin{cases} \varphi(x) & \textit{for } x \in L \\ \alpha\theta\alpha^{-1}(x) & \textit{for } x \in L' \end{cases}$$

is an isomorphism, where $(E, \bullet)$ is the extension, belonging to $(L, *, h^*, u^*)$.

Proof. Let $a := \psi(1)$, $k := \theta(1)$, then

$$(\psi, \theta, \varphi) = \left((\varrho_k^{h^*})^{-1}\varphi, (\lambda_a^{h^*})^{-1}\varphi, \varphi\right).$$

The unit of (L, h) is $ak = 1$, therefore $a = k^{-1}$. Moreover, since (L, h^*) is a Bol loop, $\left(\lambda_k^{h^*}\varrho_k^{h^*}, (\lambda_k^{h^*})^{-1}, \lambda_k^{h^*}\right)$ is an autotopism of (L, h^*), and

$$\begin{aligned} \left(\lambda_k^{h^*}\varrho_k^{h^*}, (\lambda_k^{h^*})^{-1}, \lambda_k^{h^*}\right)(\psi, \theta, \varphi) \;&=\; \left(\lambda_k^{h^*}\varphi, \varphi, \lambda_k^{h^*}\varphi\right) \\ &=\; (\theta, \varphi, \theta) : (L, h) \to (L, h^*) \end{aligned}$$

is an isotopism, too.

As before, we denote the inverse maps of h, h^* by ι, ι^*, respectively, and we write $\bar{x} := \iota(x)$, $x \in L$, when it's convenient.

Since $(\theta^{-1}, \varphi^{-1}, \theta^{-1}) : (L, h^*) \to (L, h)$ is an isotopism, too:

$$\theta^{-1} = \lambda_t^h\varphi^{-1} \quad \text{for} \quad t = \theta^{-1}(1), \quad \text{thus} \quad \varphi^{-1}\iota^*(x) = h(\overline{\theta^{-1}(x)}, t).$$

For the last equality compute $\theta^{-1}h^*(x,\iota^*(x))$ two ways, and rearrange. We'll have use for the map σ_ℓ, $\ell \in L$, as introduced in Lemma 3. Analogously, we define $\sigma_\ell^*(x) := h^*(\ell, x * \iota^*(\ell))$. For later use we precompute

$$\varphi^{-1}\sigma_x^*\varphi = \sigma_t^{-1}\sigma_{\theta^{-1}(x)}. \tag{7}$$

Indeed, using (1) for the last but one step, we get

$$\begin{aligned}
\varphi^{-1}h^*(x,\varphi(y) * \iota^*(x)) &= (\lambda_t^h)^{-1} h\left(\theta^{-1}(x), y \cdot \varphi^{-1}\iota^*(x)\right) \\
&= (\lambda_t^h)^{-1} h\left(\theta^{-1}(x), y \cdot h(\overline{\theta^{-1}(x)}, t)\right) \\
&= h\left(\bar{t}, \sigma_{\theta^{-1}(x)}(y) \cdot t\right) = \sigma_{\bar{t}}\sigma_{\theta^{-1}(x)}(y),
\end{aligned}$$

showing the claim.

For $x, y \in L$, using (1) for the last step, we compute

$$\begin{aligned}
\lambda_x^{h^*}\lambda_y^*(\lambda_x^{h^*})^{-1} &= \theta\theta^{-1}\lambda_x^{h^*}\varphi\varphi^{-1}\lambda_y^*\varphi\varphi^{-1}(\lambda_x^{h^*})^{-1}\theta\theta^{-1} \\
&= \theta\lambda_{\theta^{-1}(x)}^h\lambda_{\varphi^{-1}(y)}(\lambda_{\theta^{-1}(x)}^h)^{-1}\theta^{-1} = \theta\lambda_{\sigma_{\theta^{-1}(x)}\varphi^{-1}(y)}\theta^{-1}.
\end{aligned}$$

Also, we obtain by (7) and (1)

$$\begin{aligned}
\lambda^*_{h^*(x,y*\iota^*(x))} &= \varphi\lambda_{\varphi^{-1}\sigma_x^*(y)}\varphi^{-1} = \varphi\lambda_{\sigma_t^{-1}\sigma_{\theta^{-1}(x)}\varphi^{-1}(y)}\varphi^{-1} \\
&= \varphi(\lambda_t^h)^{-1}\lambda_{\sigma_{\theta^{-1}(x)}\varphi^{-1}(y)}\lambda_t^h\varphi^{-1} = \theta\lambda_{\sigma_{\theta^{-1}(x)}\varphi^{-1}(y)}\theta^{-1}.
\end{aligned}$$

Therefore (1) holds for $(L, *, h^*, u^*, 1)$.

A tedious calculation gives the other conditions for an E-triple $(L, *, h^*, u^*)$. However, we don't really need this fact, for the verification that β is an isomorphism. Indeed, we know by the remark after Theorem 1, that $(E, \bullet)$ is a loop. In the proof of Theorem 2(ii), we only used (1) and that $(L, *)$ and (L, h^*) are Bol loops with the same unit element. The other properties of E-triples are not really needed. Hence here, β is an $(E, \circ) \to (E, \bullet)$ isomorphism. This means that $(E, \bullet)$ is an involutorial Bol loop. $\square$

4. Involutorial Bol loops of order 16

In the last two sections we give two applications of the theory of extensions. First, we give a complete classification of involutorial Bol loops of order 16.

In this chapter, the underlying set of loops of order n will always be $\{1, \dots, n\}$, the unit element 1. Furthermore, we will represent loops by their set $\{\lambda_1 = \mathrm{id}, \dots, \lambda_n\}$ of left translations. In this section, $L, L_1, \dots$ and $h, h_1, \dots$ will denote loop representations in this sense.

The set of left translations is a *transversal* in the left translation group $\mathcal{M}_\ell$ generated by the left translation. If T is the union of conjugacy classes of involutions of G, then it determines an involutorial Bol loop as we have seen in the beginning of the paper.

Representing loops with their set of left translation makes possible to use group theoretical tools for calculations as well. In particular, we can use the computer algebra program GAP4 [**3**].

Let now

$$h = \{\lambda_x | x = 1, \dots, n\}, \qquad h^* = \{\lambda_x^* | x = 1, \dots, n\}$$

be loop representations. The map $\varphi \in \mathrm{Sym}(\{1,\dots,n\})$ is a loop isomorphism if and only if $\varphi(h(x,y)) = h^*(\varphi(x),\varphi(y))$ holds, which is equivalent with $\lambda^*_{\varphi(x)} = \varphi\lambda_x\varphi^{-1}$, that is,

$$h^* = \varphi h \varphi^{-1}.$$

Conversely, if $\varphi(1) = 1$ and $h^* = \varphi h \varphi^{-1}$ hold, then φ is an $h \to h^*$ isomorphism.

Definition *Let the loop representation h and the permutation φ be given. Then the loop h^* defined by $\varphi(h(x,y)) = h^*(\varphi(x),\varphi(y))$ will be denoted by h^φ.*

In the case $h^* = h^\varphi$, it is clear that if u is a (semi-)automorphism of h, then $u^* = u^\varphi = \varphi u^* \varphi^{-1}$ is a (semi-)automorphism of h^*.

Lemma 5. *Let L be a loop representation and $G = \langle L\rangle$ its left translation group. Then, an element $\tilde\varphi \in \mathrm{Aut}(G)$ is induced by loop automorphism $\varphi \in \mathrm{Aut}(L)$ by conjugation if and only if $\tilde\varphi$ leaves L and the stabilizer G_1 of 1 invariant.*

Proof. That an element $\varphi \in \mathrm{Aut}(L)$ induces $\tilde\varphi \in \mathrm{Aut}(G)$ by conjugation with these properties follows from $\varphi L \varphi^{-1} = L$ and $\varphi(1) = 1$. Let us now suppose that $\tilde\varphi \in \mathrm{Aut}(G)$ satisfies $\tilde\varphi(G_1) = G_1$ and $\tilde\varphi(L) = L$. Obviously, there is a one-to-one correspondence between the underlying set $\{1,\dots,n\}$ and the cosets $G : G_1$. By $\tilde\varphi(xG_1) = \tilde\varphi(x)G_1$, the first property implies that $\tilde\varphi$ is induced by a permutation φ of $\{1,\dots,n\}$ with $\varphi(1) = 1$. As we have seen before, $\varphi L \varphi^{-1} = L$ means that $\varphi \in \mathrm{Aut}(L)$, which follows from $\tilde\varphi(L) = L$. □

Theorem 6. *There are precisely 34 non-isomorphic involutorial Bol loops of order 16, and a complete list of them exists.*

Proof. In this proof, we give the sketch of the program which can be used to make all the necessary calculation. See [**9**] for an implementation of the ideas below, using the computer algebra program GAP4 [**3**]. We also prove the correctness of the algorithm assuming no significant error in GAP4.

By the proof of Corollary 3.3 of [**7**] every Bol loop of order 2^n has a subloop of index 2. It therefore suffices to consider extensions of Bol loops of order 8.

The first step of the proof of the theorem is the definition of some global variables.

Preamble
$n :=$ positive integer;
$\mathcal{H} := \{h_1,\dots,h_s\} :=$ isotopy class representatives of
Bol loops on $\{1,\dots,n\}$ with unit 1;
$L_j =$ involutorial Bol loop h_{i_j}, $j = 1,\dots,t$;
$U_i := \{$semi-automorphisms u of L_i with $u(1) = 1$ and $u^2 = \mathrm{id}\}$,
$i = 1,\dots,t$;
$R_i :=$ system of left coset representatives of
$\mathrm{Sym}(\{2,\dots,n\}) : \mathrm{Aut}(L_i)$, $i = 1,\dots,t$;

LoopExt :=
function(L, h : *loop representations*, u : *involution*);
$g := (x,y) \mapsto u(x)\cdot y$;
$f := (x,y) \mapsto$ solution of $h(x,z) = y$ in z;

$J := x \mapsto$ inverse $\bar{x}$ of x w.r.t. h;
if L, h, u, g, f, J satisfies (1), (2), (3);
then construct E following (4); **return** E;
else **return** *fail*;
fi
end

By Theorem 1, the function `LoopExt` returns a loop extension. In the next step, we construct all involutorial Bol loops on the set $\{1, \ldots, 2n\}$ with possible multiplicities.

1 **Construction of loop extensions**
2 *extensions* $:= \emptyset$;
3 **for** $i \in \{1, \ldots, t\}$ **do**
4 **for** $\pi \in R_i$ **do**
5 **for** $u \in \pi U_i \pi^{-1}$ **do**
6 **for** $h \in \{h_1, \ldots, h_s\}$ **do**
7 $E := \mathit{LoopExt}(L_i^\pi, h, u)$;
8 **if** $E \neq \mathit{fail}$ **then** $\mathit{Add}(\mathit{extensions}, E)$; **fi**;
9 **od**;
10 **od**;
11 **od**;
12 **od**;

Let $L_i \in \mathcal{L}$ be an involutorial Bol loop. Clearly, the group $\mathrm{Sym}(\{2, \ldots, n\})$ acts transitively on the isomorphism class of L_i in the set of loops with unit 1. In this action, the stabilizer of L_i is the subgroup $\mathrm{Aut}(L_i) \leq \mathrm{Sym}(\{2, \ldots, n\})$. Hence, the set

$$\{L_i^\pi | \pi \in R_i\}$$

consists of the complete isomorphism class of L_i (with unit 1). Moreover, for an element $\pi \in R_i$, the set $\pi U_i \pi^{-1}$ contains all semi-automorphisms u of L_i^π with $u(1) = 1$ and $u^2 = \mathrm{id}$. Thus, by Proposition 4, the above procedure produces at least one element of the isomorphism classes of involutorial Bol loops of order $2n$.

Let us now consider the case $n = 8$. Then, by [**2**], up to isotopism there are 5 associative and 2 non-associative Bol loops of order 8 and up to isomorphism there is one associative and one non-associative involutorial Bol loop of order 8. Hence, $|\mathcal{H}| = 7$ and $\mathcal{L} = \{L_1, L_2\}$. Clearly, $|\mathrm{Aut}(L_1)| = 168$ and by [**6**], $|\mathrm{Aut}(L_2)| = 8$. Thus, $|R_1| = 30$ and $|R_2| = 630$. Moreover, the computer gives $|U_1| = 232$ and $|U_2| = 12$. This means that the line 7 of the above procedure is executed 101 640 times; this takes less than 3 minutes on an average home computer.[1] The number of resulting loop extensions is 1 068; which we store in the list `extensions` $= \{E_i | i = 1, \ldots, 1\,068\}$.

The last step is to show that any of the loops constructed above is isomorphic to precisely one element of a list `LoopIsomClassRepr` $= \{B_i | i = 1, \ldots, 34\}$, containing 34 involutorial Bol loops. Therefore, we have given in [**9**] a list

$$\texttt{LoopIsomData} = \{(n_i, \varphi_i) | n_i \in \{1, \ldots, 34\},\ \varphi_i \text{ permutation}, i = 1, \ldots, 1\,068\}$$

[1] Processor: AMD 400 MHz, RAM: 64 MB, operation system: Linux.

of pairs such that $\varphi_i E_i \varphi_i^{-1} = B_{n_i}$ holds for all $i = 1, \ldots, 1\,068$. This means that any involutorial Bol loop of order 16 is isomorphic to an element of `LoopIsomClassRepr`.

Finally, we have to show that the elements of `LoopIsomClassRepr` are not isomorphic. Let $G_i = \langle B_i \rangle$, $i = 1, \ldots, 34$, be the left translation group of B_i. The loops B_i are constructed in such a way that their groups are either non-isomorphic or equal as permutation groups on $\{1, \ldots, 16\}$. This can be checked by the GAP command `IsomorphismGroups`. With the GAP command `AutomorphismGroup`, we can calculate the automorphism groups $\mathrm{Aut}(G_i)$.

If L_1 and L_2 are loops and we know $\mathrm{Aut}(L_1)$ and $\mathrm{Aut}(L_2)$, then applying Lemma 5 and using the command `RepresentativeOperation` the following algorithm can calculate an isomorphism between L_1 and L_2 rather quickly.

```
Construction of loop isomorphisms
LoopIsomByLeftGroup :=
function(L1, L2 : loop representations);
    if ⟨L1⟩ ≠ ⟨L2⟩
      then return fail;
       else G := ⟨L1⟩
              G1 := Stabilizer(G, 1);
              A := Aut(G);
              β̃ := RepresentativeOperation(N_A(G1), L1, L2, OnSets);
              if β̃ ≠ fail
                then β := permutation inducing β̃ by Lemma 5;
                       return β;
                 else return fail;
              fi;
    fi;
end;
```

Using this algorithm, it turns out that the 34 elements of `LoopIsomClassRepr` are indeed non-isomorphic. The calculations take less than 7 minutes. ☐

Corollary 7 (Some statistics).

(i) *Up to equivalence of permutation groups, precisely* 10 *groups can occur as the left translation group of an involutorial Bol loop of order* 16. *Their sizes are* 16, 32, 64 *or* 128.

(ii) *Up to isomorphism, there are precisely* 3 *involutorial Bol loops of order* 16 *with trivial center. These loops are solvable but not nilpotent.*

(iii) *Up to isomorphism, there is a unique involutorial Bol loop of order* 16 *with a unique maximal subloop. This loop is not nilpotent, its right translation group* $\mathcal{M}_r$ *is an extension of* $A_7 \times A_7$ *by* $\mathbb{Z}_4$*, which is a subgroup of index* 2 *in its multiplication group* $\mathcal{M}$.

(iv) *There is an involutorial Bol loop* B *of order 16 such that* $Z(B) = 2$ *and* $\mathrm{Aut}(B)$ *acts transitively on* $B \setminus Z(B)$.

Proof. Easy calculation with the elements of `LoopIsomClassRepr` from [**9**]. ☐

5. Loops with large right nucleus

In this section, we classify all finite involutorial Bol loops B whose right nucleus

$$\mathcal{N}_r = \{n \in B \mid x(yn) = (xy)n \text{ holds for all } x, y \in B\}$$

is a subgroup of index 2 in B. We give the construction by its representation and then we deduce the loop operation explicitely, as well.

We define the following groups. Let $A = (A, +)$ be an elementary Abelian 2-group,

$$\begin{array}{ll} H = A \times A, & \\ \eta : H \to H;\ \ \eta(a,b) = (b,a) & \qquad \eta \in \mathrm{Aut}(H), \\ G = H \rtimes \langle \eta \rangle, & \\ \Omega = \{(0,a);\ a \in A\} \leq G. & \end{array}$$

Notice that we consider H as a subset of G and η as an element, so we can write (a,b), $(a,b)\eta$ for (a,b,id), (a,b,η), respectively. Let $f : A \to A$ be an arbitrary map with $f(0) = 0$ and let us define the subset

$$L = L_f = \{(a, f(a));\ a \in A\} \cup \{(b,b)\eta;\ b \in A\}$$

of G. Then, L is a transversal of G/Ω. Indeed, for all $a, b \in A$ we have

$$(a,b) = (a, f(a))(0, b + f(a)) \quad \text{and} \quad (a,b)\eta = (b,b)\eta(0, a+b),$$

both decompositions being clearly unique.

Therefore, L carries a natural multiplication "$*$" as introduced in the preliminaries. Also recall the definition of $D = D(L) = \langle d_{s,t} \mid s, t \in L \rangle$.

We first put down some properties of G. They are all easy consequences of the definition.

Lemma 8. *The conjugates of Ω in G are Ω and $\eta\Omega\eta^{-1} = \{(a,0);\ a \in A\}$, since Ω is normalized by $H = A \times A$. The center of G is $\mathcal{Z}(G) = \{(a,a);\ a \in A\}$.* □

It is known that left loops are equivalent with group transversals and a transversal of G/H corresponds to a loop if and only if it is a transversal of $G/(gHg^{-1})$ for all $g \in G$, see [**1**] and [**4**, (2.7.4)].

To avoid clumsy notation we abbreviate the elements of L:

$$[a,0] = (a, f(a)) \quad \text{and} \quad [a,1] = (a,a)\eta \quad \text{for all } a \in A.$$

Moreover, we write $\bar{a} = (0,a) \in \Omega$, the map $A \to \Omega;\ a \mapsto \bar{a}$ is clearly an isomorphism.

Lemma 9. *L is an alternative left loop of exponent 2. For all $a, b \in A$ we have*

$$\begin{array}{rcl} d_{[a,0],[b,0]} & = & \overline{f(a) + f(b) + f(a+b)} \\ d_{[b,1],[a,0]} = d_{[a,0],[b,1]} & = & \overline{a + f(a)} \\ d_{[a,1],[b,1]} & = & \overline{a + b + f(a+b)}\,. \end{array} \tag{8}$$

Therefore $D = \left\langle \overline{a + f(a)};\ \ a \in A \right\rangle$.

Proof. Since L is a transversal and each element of L has group order 2, L is alternative left loop of exponent 2. A simple calculation shows for all $a, b \in A$

$$\begin{aligned} [a,0]\,[b,0] &= [a+b,0]\,\overline{f(a)+f(b)+f(a+b)} \\ [a,0]\,[b,1] &= [f(a)+b,1]\,\overline{a+f(a)} \\ [b,1]\,[a,0] &= [a+b,1]\,\overline{a+f(a)} \\ [a,1]\,[b,1] &= [a+b,0]\,\overline{a+b+f(a+b)}\,. \end{aligned}$$

This gives the equalities. Moreover, the right hand sides of the equations in the lemma are all in the group generated by $\{\overline{a+f(a)};\ \ a \in A\}$ in Ω. Therefore D can be generated as stated. ☐

As a direct consequence of the displayed equations in the proof, we obtain the loop operation of L_f explicitely:

$$\begin{aligned} [a,0]*[b,0] &= [a+b,0] \\ [a,0]*[b,1] &= [f(a)+b,1] \\ [b,1]*[a,0] &= [a+b,1] \\ [a,1]*[b,1] &= [a+b,0]. \end{aligned} \tag{9}$$

Lemma 10. *For all $\omega \in \Omega$ we have $\omega L \omega^{-1} \subseteq L$. Let $\widehat{\omega}$ be the permutation of L induced by $\omega \in \Omega$. Then the map $\omega \mapsto \widehat{\omega}$ is an embedding of Ω in the automorphism group of the left loop L. In particular, $\mathcal{D}(L) = \widehat{D} \leq \mathrm{Aut}(L)$ and L is an A_ℓ-left-loop.*

Proof. Assume $\omega = (0,a)$. If $x = (b, f(b))$ then clearly $\omega x \omega^{-1} = x$. If $x = (b,b)\eta$ then $\omega x \omega^{-1} = (a+b, a+b)\eta \in L$. Thus the first statement holds and by Lemma 5 $\hat{\omega}$ is an automorphism of L. Since no ω centralizes L, $\Omega \to \widehat{\Omega} \leq \mathrm{Aut}(L)$ is an embedding. By [**4**, (2.8.6)] $\mathcal{D} = \widehat{D} \leq \mathrm{Aut}(L)$ and L is an A_ℓ-left-loop. ☐

Proposition 11.

(i) L is a loop if and only if f is bijective.
(ii) L is a Bol loop if and only if $f^2 = \mathrm{id}$.
(iii) L is a group if and only if $f = \mathrm{id}$.

Proof. (i) If f is bijective, then by Lemma 8 L is a transversal to all conjugates of Ω in G. Now [**1**] or [**4**, (2.7.4)] show that L is in fact a loop. Conversely, if L is a loop, then by [**4**, (2.7.4)] and Lemma 8 again, L must be a transversal of

$$\eta \Omega \eta^{-1} = \{(a,0);\ \ a \in A\}.$$

Hence for $b \in A$ there exist unique $u, v \in A$ such that $(0,b) = (u, f(u))(v,0)$. Therefore $f(u) = b$ can be solved in a unique way and f is invertible.

(ii) If L is a Bol loop, then $(b,b)\eta(a,f(a))(b,b)\eta = (f(a),a)$ must be in L for all $a, b \in A$. Therefore, $f^2(a) = a$.

For the converse, we have

$$\begin{aligned} (a,f(a))(b,f(b))(a,f(a)) &= (b,f(b)) \in L \\ (a,f(a))(b,b)\eta(a,f(a)) &= (a+f(a)+b, a+f(a)+b,\eta) \in L \\ (b,b)\eta(a,f(a))(b,b)\eta &= (f(a),a) = (f(a), f(f(a))) \in L \\ (b,b)\eta(a,a)\eta(b,b)\eta &= (a,a)\eta \in L \end{aligned}$$

for all $a, b \in A$, which shows that L is a Bol loop.

(iii) is a direct consequence of Lemma 9. ☐

Notice that $(A,+,+,f)$ is an E-triple, because any bijection $f: A \to A$ with $f(0)=0$ is a semi-automorphism of the elementary Abelian 2-group A. Moreover, conditions (1)–(3) are easily verified using associativity and commutativity of "+".

In the sequel, we assume that $f: A \to A$ is a bijection and $f \neq \mathrm{id}$, hence L is a non-associative loop. We determine the nuclei of L.

PROPOSITION 12.
(i) $\mathcal{N}_r(L) = \{[a,0];\ a \in A\}$, *thus* $|L:\mathcal{N}_r| = 2$.
(ii) $\mathcal{Z}(L) = \mathcal{N}_\ell = \mathcal{N}_m = \{[a,0];\ a \in A, f(a+b) = a + f(b)\, \forall b \in A\}$.
(iii) *If* $|A| \geq 8$ *then* $f: A \to A$ *can be chosen is such a way that* $f(0)=0$, $f^2 = \mathrm{id}$ *and* $\mathcal{Z}(L_f) = \{[0,0]\}$ *holds.*

Proof. (i) For any $1 \neq \omega \in \Omega$, the set of fixed points of $\hat{\omega}$ is $N = \{[a,0];\ a \in A\}$. By Lemma 10, we have $N \subseteq \mathcal{N}_r$. Since N has index 2 in L and L is not a group, the statement is proved Proposition 11(iii).

(ii) $\mathcal{Z}(L) = \mathcal{N}_\ell = \mathcal{N}_m \subseteq \mathcal{N}_r$ is true by [**5**, (4.4.1)]. By Lemma 9, $[b,1] \notin \mathcal{N}_\ell$ for all $b \in A$. The element $[a,0]$ is in $\mathcal{N}_\ell$ if and only if for all $b \in A$

$$\begin{aligned}(a,f(a))(b,b)\eta &= (a+b, f(a)+b)\eta \quad \text{and}\\ (a,f(a))(b,f(b)) &= (a+b, f(a)+f(b))\end{aligned}$$

are elements of L. Therefore $[a,0] \in \mathcal{N}_\ell$ precisely when $f(a) = a$ and $f(a+b) = a + f(b)$ hold.

(iii) If A is infinite then we simply choose f with only fixed point 0 . If $|A| \geq 8$ finite then we can choose the elements $a,b,c,d \in A$ such that $\{0,a,b,c,a+b,a+c,d\}$ be 7 distinct elements of A. Putting $f(0)=0$, $f(a)=a$, $f(b)=c$, $f(a+b)=d$ we can extend f to an involution with only fixed points 0 and a. Then (ii) implies (iii). □

Remark. The 3 non-nilpotent involutorial Bol loops of order 16 of the last section come form this construction, as well.

THEOREM 13. *Let* B *be a finite involutorial Bol loop such that its right nucleus has index 2. Then* B *is isomorphic to the loop* L_u, *where* $A = \mathcal{N}_r$ *is the elementary Abelian group of order* $|B|/2$ *and* $u: A \to A;\ x \mapsto s \cdot xs$, *for some fixed* $s \in B \setminus \mathcal{N}_r$.

Proof. Define the operation $h(x,y)$ on $\mathcal{N}_r$ by

$$sh(x,y) = (sx)y.$$

Since $x,y \in \mathcal{N}_r$, this implies $h(x,y) = x \cdot y$. Hence, B is isomorphic to an extension $(E,\circ)$, which belongs to the E-triple $(\mathcal{N}_r,\cdot,\cdot,u)$ (cf. the proof of Theorem 1 (iii)).

Now construct the loop L_u from the group $A = \mathcal{N}_r$. If $\alpha: \mathcal{N}_r \to \mathcal{N}_r'$ is used to obtain the loop $(E,\circ)$, then comparison of (4) and (9) shows that

$$\beta: E \to L_u;\ x \mapsto \begin{cases} [x,0], & \text{if } x \in A,\\ [\alpha^{-1}(x),1], & \text{if } x \in A' \end{cases}$$

is an isomorphism. □

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