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### **The Classification and the Computation of the Zeros of Quaternionic, Two-Sided Polynomials**

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# THE CLASSIFICATION AND THE COMPUTATION OF THE ZEROS OF QUATERNIONIC, TWO-SIDED POLYNOMIALS (2009/9/5)

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**Abstract.** Already for a long time it is known that quaternionic polynomials whose coefficients are located only at one side of the powers, may have two classes of zeros: isolated zeros and spherical zeros. Only recently a classification of the two types of zeros and a means to compute all zeros of such polynomials have been developed. In this investigation we consider quaternionic polynomials whose coefficients are located at both sides of the powers, and we show that there are three more classes of zeros defined by the rank of a certain real  $(4 \times 4)$  matrix. This information can be used to find all zeros in the same class if only one zero in that class is known. The essential tool is the description of the polynomial  $p$  by a matrix equation  $P(z) := \mathbf{A}(z)z + B(z)$ , where  $\mathbf{A}(z)$  is a real  $(4 \times 4)$  matrix determined by the coefficients of the given polynomial  $p$  and  $P, z, B$  are real column vectors with four rows. This representation allows also to include two-sided polynomials which contain several terms of the same degree. We applied Newton's method to  $P(z) = 0$ . This method turned out to be a very effective tool in finding the zeros. This method allowed also to prove, that the essential number of zeros of a quaternionic, two-sided polynomial  $p$  of degree  $n$  is, in general, not bounded by  $n$ . We conjecture that the bound is  $2n$ . There are various examples.

**Key words.** Classes of zeros; quaternionic, two-sided polynomials; structure of zeros of quaternionic, two-sided polynomials; number of zeros of quaternionic, two-sided polynomials; computation of the zeros of quaternionic, two-sided polynomials.

**AMS subject classifications.** 11R52, 12E15, 12Y05, 65H05

**1. Introduction.** We will treat quaternionic polynomials of the *two-sided type*

$$(1.1) \quad p(z) := \sum_{j=0}^n a_j z^j b_j, \quad z, a_j, b_j \in \mathbb{H}, a_0 b_0 \neq 0, a_n b_n \neq 0,$$

where  $\mathbb{H}$  is the skew field of quaternions. There will be a brief introduction shortly. These polynomials include also the *one-sided* polynomials, where all coefficients are located on the left side or the right side of the powers. A complete characterization of the types of zeros including an effective algorithm for finding all zeros of those polynomials was given by Janovská and Opfer (2009), [7]. For two-sided polynomials there is hardly any result in the literature, with the exception of an example by De Leo et alii, [11]. For one-sided polynomials there are several papers. The first is by Niven, 1941, [12], in which a general method for finding the zeros is described, which is still used more recently, e. g, [16], [3], [4]. More general information is contained in [15].

It should be noted, that the general form of a polynomial in quaternionic variables would consist of terms of the form  $t_j(z) := a_0 \cdot z \cdot a_1 \cdot z \cdot a_2 \cdots a_{j-1} \cdot z \cdot a_j$  such that the above  $p$  is only a very special type of quaternionic polynomial. See [13] for some statements on polynomials of general type and Section 2 of this paper on real zeros. The form (1.1) is not the most general form for a polynomial with coefficients on both sides of the powers. We will treat the more general case in Section 5.

We will give a short introduction in quaternionic algebra. By  $\mathbb{R}, \mathbb{C}$  we denote the fields of real and complex numbers, respectively, and by  $\mathbb{Z}$  the set of integers. The field of quaternions  $\mathbb{H}$  may be regarded as  $\mathbb{R}^4$  with a special multiplication rule. Let

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1,  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  denote the four standard units in  $\mathbb{H}$ . They obey the following multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1; \quad \mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}.$$

By writing  $a := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$ ,  $b := b_1 + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k}$ ,  $a_j, b_j \in \mathbb{R}, j = 1, 2, 3, 4$  we deduce the following general multiplication rule:

$$(1.2) \quad ab := (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4, a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3, \\ a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2, a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1),$$

where the notation  $a := (a_1, a_2, a_3, a_4)$  stands equivalently for  $a := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$ . The first component,  $a_1$ , of  $a$  will be called *real part* of  $a$  and denoted by  $\Re(a)$ . The second component,  $a_2$ , of  $a$  will be called *imaginary part* of  $a$  and denoted by  $\Im(a)$ . Note, that both numbers are real. The quantity  $|a|$  denotes the *absolute value* of  $a$ , defined by

$$|a| := \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}.$$

The *conjugate* of  $a$ , denoted by  $\bar{a}$ , is defined by

$$\bar{a} = (a_1, -a_2, -a_3, -a_4).$$

We will identify a real number  $a_1$  by the quaternion  $a := (a_1, 0, 0, 0)$  and a complex number  $a_1 + a_2\mathbf{i}$  by the quaternion  $a := (a_1, a_2, 0, 0)$ . By this identification the real part  $\Re$ , the imaginary part  $\Im$ , the absolute value  $|\cdot|$ , the conjugate  $\bar{\cdot}$  can be applied to real and complex numbers as well.

There are the following rules for  $a, b \in \mathbb{H}$ ,  $r \in \mathbb{R}$ :

$$(1.3) \quad \Re(ab) = \Re(ba), \quad ra = ar, \quad \text{and} \quad |ab| = |a||b|,$$

$$(1.4) \quad \overline{ab} = \bar{b}\bar{a},$$

$$(1.5) \quad a\bar{a} = \bar{a}a = |a|^2,$$

$$(1.6) \quad a^{-1} = \frac{\bar{a}}{|a|^2} \text{ for } a \neq 0.$$

For further use we will introduce *equivalence classes* of quaternions.<sup>1</sup>

DEFINITION 1.1. Two quaternions  $a, b \in \mathbb{H}$  are called *equivalent*, denoted by  $a \sim b$ , if

$$(1.7) \quad a \sim b \Leftrightarrow \exists h \in \mathbb{H} \setminus \{0\} \text{ such that } a = h^{-1}bh.$$

The set

$$(1.8) \quad [a] := \{u \in \mathbb{H} : u := h^{-1}ah \text{ for all } h \in \mathbb{H} \setminus \{0\}\}$$

will be called an *equivalence class* of  $a$  or, for short, *class* of  $a$ .

The fact that  $\sim$  indeed defines an equivalence relation is obvious. Equivalent quaternions  $a, b$  can easily be recognized by

$$(1.9) \quad a \sim b \Leftrightarrow \Re(a) = \Re(b) \text{ and } |a| = |b|, \text{ (cf. [9]).}$$

<sup>1</sup>Algebraists call these classes *conjugacy classes*, v. d. Waerden, p. 35, [17]. In matrix terms these classes are called *similarity classes*, Horn and Johnson, p. 44, [6].

Let  $a$  be real. Then  $[a] = \{a\}$ , which means, that in this case, the equivalence class consists only of one element,  $a$ . If  $a$  is not real, then  $[a]$  always contains infinitely many elements, and according to (1.7), (1.8), (1.9)  $[a]$  can be expressed by

$$(1.10) \quad [a] := \{u \in \mathbb{H} : \Re(u) = \Re(a), |u| = |a|\},$$

which may be regarded as the surface of a ball in  $\mathbb{R}^3$ . From (1.10) it follows, that

$$\bar{a} \in [a].$$

Let  $a \in \mathbb{H}$  be not real and  $a := (a_1, a_2, a_3, a_4)$ . Then  $[a]$  contains exactly two complex numbers,  $c \in \mathbb{C}$  and  $\bar{c} \in \mathbb{C}$  where  $c$  is determined by  $\Re(c) := a_1$  and  $\Im(c) := +\sqrt{a_2^2 + a_3^2 + a_4^2} > 0$ . The complex number  $c := a_1 + \sqrt{a_2^2 + a_3^2 + a_4^2} \mathbf{i} \in [a]$  will be called *complex representative* of  $[a]$ .

In a book by Lam, 2001, § 16, [10] we can find several interesting results on polynomials (of the one-sided type) over division rings. In particular, there is a theorem accredited to Gordon and Motzkin, saying that a polynomial of degree  $n$  has zeros in at most  $n$  equivalence classes.

**2. The real zeros of a general quaternionic polynomial.** The real zeros can always be found independently of the type of the polynomial, provided there are real zeros. Also the case where there are no real zeros can be detected. This will be the content of this section with the implication that the real zeros need not be treated in the remaining part of the paper. A general, quaternionic polynomial consists of a sum of terms of the type

$$t_j(z) := a_{0j} \cdot z \cdot a_{1j} \cdots a_{j-1,j} \cdot z \cdot a_{jj}, \quad z, a_{0j}, a_{1j}, \dots, a_{jj} \in \mathbb{H}, j \geq 0.$$

We call this term a *monomial of degree  $j$* . Since there may be several terms of the same degree we have to enumerate the terms. We do that in the form

$$(2.1) \quad t_{jk}(z) := a_{0j}^{(k)} \cdot z \cdot a_{1j}^{(k)} \cdots a_{j-1,j}^{(k)} \cdot z \cdot a_{jj}^{(k)}, \quad k = 1, 2, \dots, k_j, k_j \geq 0.$$

The case  $k_j = 0$  means that there is no monomial of degree  $j$ , or in other words,  $k = 1, 2, \dots, k_j$  defines the empty set of indices  $k$  in this case. A *general quaternionic polynomial of degree  $n$*  takes the form

$$(2.2) \quad p(z) := \sum_{j=0}^n \sum_{k=1}^{k_j} t_{jk}(z).$$

There are some recent results on these polynomials in a paper by Opfer, 2009, [13]. The essential result is by Eilenberg and Niven 1944, [2]. It says that such a polynomial has at least one zero, provided the number of monomials of degree  $n$  is only one. It is clear that also polynomials which contain terms like  $a \cdot z^2 \cdot b \cdot z^4 \cdot c$  are included in the form (2.1). One only needs to choose some of the coefficients to be real. Let  $z \in \mathbb{R}$  be a real zero of  $p$ , defined in (2.2). Since a real  $z$  commutes with all quaternions the polynomial can be written in the form

$$(2.3) \quad p(z) = \sum_{j=0}^n A_j z^j \quad \text{where} \quad A_j := \sum_{k=1}^{k_j} a_{0j}^{(k)} a_{1j}^{(k)} \cdots a_{jj}^{(k)}, \quad z \in \mathbb{R}.$$

In another paper, Janovská and Opfer [7] have given a complete algorithm for finding all zeros of quaternionic polynomials of the one-sided type mentioned in (2.3) including the determination of the type of zero. This includes finding the real zeros. For this purpose, one has to define a real *companion* polynomial  $q$  of degree  $2n$ , the zeros of which are pairwise complex conjugate or consists of pairs of the same real number. The real zeros of  $q$  are also real zeros of  $p$ . This is shown in [7]. Let us present a somewhat weird example.

EXAMPLE 2.1. Let

$$(2.4) \quad p(z) := z^2 + azbzc + dze + f.$$

The polynomial (2.3) reads in this case

$$p(z) = (1 + abc)z^2 + dez + f.$$

We choose

$$a := \mathbf{i}, b := \mathbf{j}, c := -\mathbf{k}, d := \mathbf{i} + \mathbf{j}, e := \mathbf{j} + \mathbf{k}, f := -1 - \mathbf{i} + \mathbf{j} - \mathbf{k},$$

such that  $p(z) = 2z^2 + (-1, 1, -1, 1)z + (-1, -1, 1, -1)$ . The companion polynomial  $q$  of degree four has one as a double zero and no other real zero. See [7]. Thus, the polynomial  $p$  in (2.4) has exactly one real zero, namely one. If in the general case the companion polynomial  $q$  has no real zero, then also the given polynomial  $p$  of (2.2) has no real zero. Because of these results, we will always disregard the discussion on real zeros in the sequel.

**3. Types of zeros of two-sided polynomials.** We will use as before in [7], [14] that all powers  $z^j, j \in \mathbb{Z}$  of a quaternion  $z$  have the form  $z^j = \alpha z + \beta$  with real  $\alpha, \beta$ . In order to determine the numbers  $\alpha, \beta$  we use the following lemma.

LEMMA 3.1. *Let  $z \in \mathbb{H}$ . Then we have*

$$(3.1) \quad z^j = \alpha_j z + \beta_j, \quad \alpha_j, \beta_j \in \mathbb{R}, \quad j = 0, 1, \dots, \text{ where}$$

$$(3.2) \quad \alpha_0 := 0, \quad \beta_0 := 1,$$

$$(3.3) \quad \alpha_{j+1} := 2\Re(z)\alpha_j + \beta_j,$$

$$(3.4) \quad \beta_{j+1} := -|z|^2\alpha_j, \quad j = 0, 1, \dots$$

For  $z \neq 0$  we may replace  $z$  with  $z^{-1}$  and we obtain a formula for negative  $j$ .

*Proof.* See [7], also for further comments on this recursion.  $\square$

By using the theory of difference equations, it is possible to find a closed form solution for  $\alpha_j$  (and thus, also for  $\beta_j$ ) for the case  $z \notin \mathbb{R}$ , namely

$$(3.5) \quad \alpha_j = \frac{\Im(u_1^j)}{\sqrt{|z|^2 - (\Re(z))^2}}, \quad j \geq 0,$$

where  $u_1$  is the complex solution of  $u^2 - 2\Re(z)u + |z|^2 = 0$  with positive imaginary part. Formula (3.5) for  $\alpha_j$  is of course easier to program than the iteration (3.2) to (3.4). However, since a power is involved, an economic use of (3.5) would also require an iteration.

COROLLARY 3.2. *Let  $z_1 \sim z_2$  be two different but equivalent elements in  $\mathbb{H}$ . Then, the two sequences  $\{\alpha_j\}, \{\beta_j\}$  generated according to Lemma 3.1 are the same for  $z_1$  and for  $z_2$ .*

*Proof.* The two sequences  $\{\alpha_j\}, \{\beta_j\}$  generated by formulas (3.2) to (3.4) depend only on  $\Re(z)$  and on  $|z|$ . However, these quantities are the same for  $z_1$  and  $z_2$  by (1.9).  $\square$

By means of (3.1) the polynomial  $p$  can be written as

$$(3.6) \quad p(z) := \sum_{j=0}^n a_j z^j b_j = \sum_{j=0}^n a_j (\alpha_j z + \beta_j) b_j$$

$$(3.7) \quad = \sum_{j=0}^n \alpha_j a_j z b_j + \sum_{j=0}^n \beta_j a_j b_j = C(z) + B(z), \text{ where}$$

$$(3.8) \quad C(z) := \sum_{j=0}^n \alpha_j a_j z b_j, \quad B(z) := \sum_{j=0}^n \beta_j a_j b_j.$$

LEMMA 3.3. *Let  $z_0$  be nonreal. Then,  $B(z)$ , defined in (3.8), is constant for  $z \in [z_0]$ . If  $p(z) = 0$  for some  $z \in \mathbb{H}$ , then  $C(z) = B(z) = 0$  or  $C(z) \neq 0$  and  $B(z) \neq 0$ .*

*Proof.* According to Corollary 3.2 the coefficients  $\alpha_j, \beta_j, j \geq 0$  are the same for all  $z \in [z_0]$ . Thus,  $B(z) = \text{const}$  for all  $z \in [z_0]$ . The last part is obvious.  $\square$

We apply a new idea, namely to use an isomorphic matrix representation for quaternions. For this purpose we introduce two mappings  $\mathfrak{1}_1, \mathfrak{1}_2 : \mathbb{H} \rightarrow \mathbb{R}^{4 \times 4}$  by

$$(3.9) \quad \mathfrak{1}_1(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4 \times 4},$$

$$(3.10) \quad \mathfrak{1}_2(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4 \times 4}.$$

The first mapping  $\mathfrak{1}_1$  represents the isomorphic image of a quaternion  $a = (a_1, a_2, a_3, a_4)$  in the matrix space  $\mathbb{R}^{4 \times 4}$ . Thus, we have  $\mathfrak{1}_1(ab) = \mathfrak{1}_1(a)\mathfrak{1}_1(b)$ . The two matrices  $\mathfrak{1}_1(a), \mathfrak{1}_2(b)$  coincide if and only if  $a = b \in \mathbb{R}$ . See [5, p. 5]. The second mapping  $\mathfrak{1}_2$ , introduced in 1995 by Aramanovitch, [1] has the remarkable property that it reverses the multiplication order

$$(3.11) \quad \mathfrak{1}_2(ab) = \mathfrak{1}_2(b)\mathfrak{1}_2(a).$$

From the definition (3.9), (3.10) it follows that

$$(3.12) \quad \mathfrak{1}_1(a)^T = \mathfrak{1}_1(\bar{a}), \quad \mathfrak{1}_2(b)^T = \mathfrak{1}_2(\bar{b}),$$

where the superscript  $T$  denotes transposition. It follows, that both matrices are orthogonal in the sense  $\mathfrak{1}_1(a)\mathfrak{1}_1(a)^T = \mathfrak{1}_1(a)\mathfrak{1}_1(\bar{a}) = |a|^2 \mathbf{I}$ ,  $\mathfrak{1}_2(b)\mathfrak{1}_2(b)^T = |b|^2 \mathbf{I}$ , where  $\mathbf{I}$  is the  $(4 \times 4)$  identity matrix.

Let  $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$ . We introduce a simple, but very useful *column* operator  $\text{col} : \mathbb{H} \rightarrow \mathbb{R}^4$  by

$$\text{col}(a) := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

The purpose of this column operator is to regard a quaternion as a matrix with one column and four rows.

LEMMA 3.4. *The column operator is linear over  $\mathbb{R}$ , i. e.*

$$\operatorname{col}(\alpha a + \beta b) = \alpha \operatorname{col}(a) + \beta \operatorname{col}(b), \quad a, b \in \mathbb{H}, \alpha, \beta \in \mathbb{R}.$$

*Proof.* Simple.  $\square$

The following lemma is very important in our context.

LEMMA 3.5. *For arbitrary quaternions  $a, b, c$  we have*

$$(3.13) \quad \operatorname{col}(ab) = {}_{1_1}(a)\operatorname{col}(b) = {}_{1_2}(b)\operatorname{col}(a),$$

$$(3.14) \quad \operatorname{col}(abc) = {}_{1_1}(a){}_{1_2}(c)\operatorname{col}(b).$$

*Proof.* See Aramanovitch [1995], [1], Gürlebeck and Sprössig, 1997, p. 6, [5].  $\square$

For more properties of these mappings see also [8]. Since the product  ${}_{1_1}(a){}_{1_2}(b)$  plays an important role for our investigations, we will use the abbreviation

$$(3.15) \quad {}_{1_3}(a, b) := {}_{1_1}(a){}_{1_2}(b) \in \mathbb{R}^{4 \times 4}, \quad a, b \in \mathbb{H}.$$

LEMMA 3.6. *The matrix  ${}_{1_3}(a, b)$  defined in (3.15) is normal and orthogonal in the sense*

$${}_{1_3}(a, b)^T {}_{1_3}(a, b) = {}_{1_3}(a, b){}_{1_3}(a, b)^T = |a|^2 |b|^2 \mathbf{I}.$$

*Thus, all eigenvalues of  ${}_{1_3}(a, b)$  have the same absolute value  $|a||b|$ .*

*Proof.* We have by definition and by (3.11), (3.12)

$$\begin{aligned} {}_{1_3}(a, b){}_{1_3}(a, b)^T &= {}_{1_1}(a){}_{1_2}(b)({}_{1_1}(a){}_{1_2}(b))^T \\ &= {}_{1_1}(a){}_{1_2}(b){}_{1_2}(\bar{b}){}_{1_1}(\bar{a}) = |a|^2 |b|^2 \mathbf{I}. \end{aligned}$$

And  ${}_{1_3}(a, b)^T {}_{1_3}(a, b) = |a|^2 |b|^2 \mathbf{I}$  follows from a very similar computation.  $\square$

THEOREM 3.7. *Let  $p(z) := C(z) + B(z)$  be defined as in (3.6) to (3.8). Then,*

$$(3.16) \quad \operatorname{col}(p(z)) = \left( \sum_{j=0}^n \alpha_j {}_{1_3}(a_j, b_j) \right) \operatorname{col}(z) + \sum_{j=0}^n \beta_j \operatorname{col}(a_j b_j)$$

$$(3.17) \quad =: \mathbf{A}(z)\operatorname{col}(z) + \operatorname{col}(B(z)), \quad \text{where}$$

$$(3.18) \quad \mathbf{A}(z) := \left( \sum_{j=0}^n \alpha_j {}_{1_3}(a_j, b_j) \right) \in \mathbb{R}^{4 \times 4}, \quad \operatorname{col}(B(z)) := \sum_{j=0}^n \beta_j \operatorname{col}(a_j b_j).$$

*Proof.* Apply the column operator and use Lemmata 3.4 and 3.5.  $\square$

LEMMA 3.8. *Let  $z_0$  be nonreal. Then, the matrix  $\mathbf{A}(z)$ , defined in Theorem 3.7, is constant for  $z \in [z_0]$ .*

*Proof.* Same as proof of Lemma 3.3 for  $B$ , since the  $\alpha_j$  depend only on  $\Re(z)$  and on  $|z|$ .  $\square$

Instead of considering the equation  $p(z) = 0$  we consider the equivalent equation

$$(3.19) \quad P(z) := \operatorname{col}(p(z)) = \mathbf{A}(z)\operatorname{col}(z) + \operatorname{col}(B(z)) = \operatorname{col}(0) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} =: 0.$$



From this formula we can gather important information.

**THEOREM 3.9.** *Let  $z$  be a nonreal zero of  $p$  such that equation (3.19) is valid. Then, this equation remains valid if in  $\mathbf{A}(z)$  and  $B(z)$  the zero  $z$  is replaced with the complex representative  $z_0$  of  $[z]$ .*

*Proof.* Follows from the fact that  $\mathbf{A}(z)$  and  $B(z)$  are constant on  $[z]$ , see Lemmata 3.3, and 3.8.  $\square$

**COROLLARY 3.10.** *In order to find the nonreal zeros  $z \in \mathbb{H}$  of  $p$ , defined in (1.1), it is sufficient to find the complex representatives  $z_0$  of  $[z]$ , where, in general,  $z_0$  is not a zero of  $p$ .*

*Proof.* If we know the complex representative  $z_0$  of  $z$ , where  $z$  is a zero of  $p$ , then, we can find all zeros  $z$  which are located in  $[z_0]$  by solving the linear equation (3.19), where  $\mathbf{A}(z), B(z)$  have to be replaced with the known quantities  $\mathbf{A}(z_0), B(z_0)$ , respectively.  $\square$

The matrix  $\mathbf{A}(z)$ , occurring in (3.19), may be singular or nonsingular.

**THEOREM 3.11.** *Let  $z_1, z_2$  be two different but equivalent zeros of a polynomial  $p$  defined in (1.1). Then,  $\mathbf{A} := \mathbf{A}(z_1) = \mathbf{A}(z_2)$ , and  $\mathbf{A}$  is singular, where  $\mathbf{A}(z)$  is defined in (3.18).*

*Proof.* Equation (3.19) reads  $\text{col}(p(z_j)) = \mathbf{A}(z_j)\text{col}(z_j) + \text{col}(B(z_j)) = 0$  for  $j = 1, 2$ . Since  $z_1 \sim z_2$  we have  $\mathbf{A}(z_1) = \mathbf{A}(z_2), B(z_1) = B(z_2)$  which follows from Lemmata 3.8 and 3.3. Taking differences, yields  $\text{col}(p(z_1) - p(z_2)) = \mathbf{A}\text{col}(z_1 - z_2) = 0$ . Since  $z_1 - z_2 \neq 0$ , the singularity of  $\mathbf{A}$  follows.  $\square$

From these results we obtain a classification of the zeros of  $p$  as follows:

**DEFINITION 3.12.** Let  $z$  be a zero of  $p$ , defined in (1.1), and let  $z_0 \in [z]$  be the complex representative of  $[z]$ . The zero  $z$  will be called *zero of type  $k$*  if  $\text{rank}(\mathbf{A}(z_0)) = 4 - k, 0 \leq k \leq 4$ . A zero of type 4 ( $\text{rank}(\mathbf{A}(z_0)) = 0$ ) will be called *spherical zero*. It has the property that all  $z \in [z_0]$  are zeros. A zero of type 0 will be called *isolated zero*. In this case  $z = -(\mathbf{A}(z_0))^{-1}\text{col}(B(z_0))$  is the only zero in  $[z_0]$ . We will also call a real zero an isolated zero.

The type is, roughly speaking, the dimension of the zero. Since the one-sided polynomials also belong to the class we are considering, zeros of types 0 and 4 will in fact occur. See [7]. From the study of the quadratic case in the next section, we shall see that zeros of type 2 will also exist. By some more tests with  $n = 4$  (deleted here), we found that all ranks (zero to four) are indeed possible for  $\mathbf{A}$ . Thus, the above classification mentions no superfluous case for  $n \geq 4$ . For  $n = 3$  we could not find a zero of type 3. In the next section we show that for  $n = 2$  the cases  $\text{rank}(\mathbf{A}) = 1$  (type 3) and  $\text{rank}(\mathbf{A}) = 3$  (type 1) are impossible.

**4. The quadratic case.** In this section we will study the quadratic case

$$(4.1) \quad p(z) := z^2 + azb + c, \quad a, b, c \in \mathbb{H}, a \notin \mathbb{R}, b \notin \mathbb{R}.$$

The cases  $a \in \mathbb{R}$  or  $b \in \mathbb{R}$  were already studied in [7]. We note, that it is not a restriction to assume that the highest coefficient (at  $z^2$ ) is one. Then, for the quadratic case we have (use Definition (3.18) for  $\mathbf{A}$  and (3.8) for  $B$  and  $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 2\Re(z), \beta_0 = 1, \beta_1 = 0, \beta_2 = -|z|^2$ )

$$(4.2) \quad \mathbf{A}(z) = 2\Re(z)\mathbf{I} + \mathbf{i}_3(a, b), \quad B(z) = c - |z|^2.$$

We note here that, by Lemma 3.6 and by Horn and Johnson, p. 110, #19, [6], the matrix  $\mathbf{A}(z)$  is normal.

**LEMMA 4.1.** *The rank of the matrix  $\mathbf{A}(z)$ , defined in (4.2) can only be even, i. e. the rank can be zero, two or four.*

*Proof.* Let  $\text{eig}(\mathbf{B})$  denote the column vector of all eigenvalues of a real square matrix  $\mathbf{B}$ . Then

$$(4.3) \quad \text{eig}(\mathbf{A}(z)) = \text{eig}({}_{13}(a, b)) + 2\Re(z) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The matrix  ${}_{13}(a, b)$  is orthogonal (cf. Lemma 3.6) such that all its eigenvalues have the same absolute value  $|a||b|$ . In particular, the eigenvalues of  ${}_{13}(a, b)$  are never zero. The four eigenvalues of  ${}_{13}(a, b)$  always come in two pairs, (a) either two pairs of complex conjugate numbers, or (b) one pair of complex conjugate numbers and one pair of the same real number or (c) two pairs of the same real number, where the two real pairs may only differ in sign. In case (a),  $\mathbf{A}(z)$  will be nonsingular, in case (b) it may be nonsingular or have rank two. In case (c),  $\mathbf{A}(z)$  may have rank zero, two or four.  $\square$

**THEOREM 4.2.** *For the zeros of a quadratic polynomial  $p$  defined in (4.1), there are the following possibilities:*

1. *All eigenvalues of  ${}_{13}(a, b)$  are nonreal. Then, only isolated zeros are possible.*
2. *There are real and complex eigenvalues. Then, isolated zeros or zeros of type 2 are possible.*
3. *All eigenvalues are real. Then, spherical zeros, zeros of type 2, and isolated zeros are possible.*

*Proof.* Follows from the foregoing lemma.  $\square$

We will show that spherical zeros are impossible if at least one of the coefficients  $a$  or  $b$  in (4.1) is nonreal.

**LEMMA 4.3.** *Let  $a, b \in \mathbb{H}$  and define  $\mathbf{J} := {}_{13}(a, b) := {}_{11}(a){}_{12}(b)$ , where  ${}_{11}, {}_{12}$  are defined in (3.9), (3.10). Then,  $\mathbf{J}$  has four identical real eigenvalues if and only if  $a, b \in \mathbb{R}$ .*

*Proof.* Let  $a, b \in \mathbb{R}$ . Then  $\mathbf{J} = ab\mathbf{I}$  and  $\mathbf{J}$  has four identical real eigenvalues  $ab$ . Now, assume that  $\mathbf{J}$  has four identical real eigenvalues  $c$ . Put  $a := (a_1, a_2, a_3, a_4)$ ,  $b := (b_1, b_2, b_3, b_4)$ . In this case the characteristic polynomial is  $\chi_{\mathbf{J}}(x) := \det(\mathbf{J} - x\mathbf{I}) = (x - c)^4$ . It follows that  $c^4 = \det({}_{13}(a, b)) = |a|^4|b|^4$ , and thus,  $c = \pm|a||b|$ . It also follows that the trace is  $\text{tr}(\mathbf{J}) = 4a_1b_1 = 4c$ . Therefore,  $a_1b_1 = \pm|a||b|$ . This implies  $a, b \in \mathbb{R}$ .  $\square$

Because of the eigenvalue formula (4.3), this lemma implies that the matrix  $\mathbf{A}$  can be the zero matrix (i. e.  $\text{rank}(\mathbf{A}) = 0$ ) only if  $a, b \in \mathbb{R}$ .

**EXAMPLE 4.4.** Let

$$p(z) := z^2 + \mathbf{i}z\mathbf{j} + \mathbf{k}.$$

Some tests show that  $z_0 := \frac{1}{2}(-1 + \sqrt{3}\mathbf{i})$  is a complex representative of a zero. In this case

$$\mathbf{A}(z_0) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad B(z_0) = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and  $\text{rank}(\mathbf{A}(z_0)) = 2$ . The zeros  $z := (z_1, z_2, z_3, z_4) \in [z_0]$  obey the following equations:

$$z_1 = \Re(z_0) = -0.5, \quad |z|^2 = |z_0|^2 = 1, \quad \mathbf{A}(z_0)\text{col}(z) + B(z_0) = 0.$$

There are two solutions, namely

$$u_1 := \frac{1}{2}(-1, -1, 1, 1), \quad u_2 := \frac{1}{2}(-1, 1, -1, 1).$$

EXAMPLE 4.5. Let

$$p(z) := z^2 + \mathbf{i}z\mathbf{j} + 1.$$

In this case we find that  $z_{\pm} := \frac{1}{2}(\pm 1 + \sqrt{3}\mathbf{i})$  are two, non equivalent, complex representatives of a zero and for  $z_- := \frac{1}{2}(-1 + \sqrt{3}\mathbf{i})$  we have

$$\mathbf{A}(z_-) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad B(z_-) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and  $\text{rank}(\mathbf{A}(z_-)) = 2$ . The zeros  $z := (z_1, z_2, z_3, z_4) \in [z_0]$  obey the equations already mentioned in Example 4.4 with the solutions

$$u_1 := \frac{1}{2}(-1, -1, 1, -1), \quad u_2 := \frac{1}{2}(-1, 1, -1, -1).$$

For the other representative,  $z_+ := \frac{1}{2}(1 + \sqrt{3}\mathbf{i})$ , we have

$$\mathbf{A}(z_+) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad B(z_+) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and  $\text{rank}(\mathbf{A}(z_+)) = 2$  and the corresponding zeros are

$$u_3 := \frac{1}{2}(1, -1, -1, -1), \quad u_4 := \frac{1}{2}(1, 1, 1, -1).$$

The quadratic polynomial presented here in this example has four nonspherical zeros.

EXAMPLE 4.6. Let

$$p(z) := z^2 + \mathbf{i}z\mathbf{j} + 1 + \mathbf{k}.$$

In this case we find that  $z_0 := 1 + \mathbf{i}$  is a complex representative of a zero and we have

$$\mathbf{A}(z_0) = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}, \quad B(z_0) = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and  $\text{rank}(\mathbf{A}(z_0)) = 4$ . The zero  $z := (z_1, z_2, z_3, z_4) \in [z_0]$  is the unique solution of  $\mathbf{A}(z_0)z + B(z_0) = 0$  which is

$$u_1 := (1, 0, 0, -1).$$

At this stage we do not claim that we have found all zeros of the foregoing examples. However, we see that the number of nonspherical zeros may exceed the degree, a phenomenon, which cannot be observed with one-sided polynomials. But if we only count the number of different equivalence classes which contain zeros, then, without looking for further zeros, this number does not exceed the polynomial degree. We will return to this topic in Section 8.

**5. Multiple terms of the same degree.** Since one cannot combine terms of the same degree, like  $azb + czd$ , the form (1.1) is not the most general form of a polynomial with coefficients at either side of the powers. Therefore, we specialize the most general form of a polynomial, already given in (2.1) and (2.2) in the following way:

$$(5.1) \quad t_{jk}(z) := a_j^{(k)} z^j b_j^{(k)}, \quad k = 1, 2, \dots, k_j, \quad k_j \geq 0, \quad k_n = k_0 = 1,$$

$$(5.2) \quad p(z) := \sum_{j=0}^n \sum_{k=1}^{k_j} t_{jk}(z), \quad t_{n1} \neq 0, \quad t_{01} \neq 0.$$

The condition  $k_n = 1$  together with  $t_{n1} \neq 0$  ensures that there is exactly one term with degree  $n$  which is not vanishing. This allows to normalize the highest term to  $z^n$ . According to Eilenberg and Niven, [2], this condition guarantees the existence of at least one zero. However, the following development will also work if we have several terms of the highest degree, thus, allowing  $k_n \geq 1$ . The condition  $k_0 = 1$  is not a restriction, since the constant terms could be combined to one term. The condition  $t_{01} \neq 0$  implies that the origin  $z = 0$  is never a zero.

The theory already developed in Section 3 will also work here. We apply the column operator to  $p$ , again using the representation  $z^j = \alpha_j z + \beta_j$ , developed in (3.1) to (3.4) and the matrix  ${}_3$  defined in (3.15) and obtain

$$(5.3) \quad \text{col}(p(z)) = \sum_{j=0}^n \sum_{k=1}^{k_j} \text{col}(t_{jk}(z)),$$

$$(5.4) \quad \text{col}(t_{jk}(z)) = \text{col}(a_j^{(k)} z^j b_j^{(k)}) = \text{col}(a_j^{(k)} (\alpha_j z + \beta_j) b_j^{(k)})$$

$$(5.5) \quad = \alpha_j \text{col}(a_j^{(k)} z b_j^{(k)}) + \beta_j \text{col}(a_j^{(k)} b_j^{(k)})$$

$$(5.6) \quad =: \mathbf{A}_{jk} \text{col}(z) + \text{col}(B_{jk}), \quad \text{where}$$

$$(5.7) \quad \mathbf{A}_{jk} := \alpha_j {}_3(a_j^{(k)}, b_j^{(k)}), \quad B_{jk} := \beta_j a_j^{(k)} b_j^{(k)}.$$

If we put

$$(5.8) \quad \mathbf{A} := \sum_{j=0}^n \sum_{k=1}^{k_j} \mathbf{A}_{jk}, \quad B := \sum_{j=0}^n \sum_{k=1}^{k_j} B_{jk},$$

we obtain exactly the representation (3.19). The classification of the zeros, given in Definition 3.12, will still be valid, as well as the further statements in Section 3. But Theorem 4.2 on quadratic polynomials will, in general, not be true. A quadratic polynomial will read

$$(5.9) \quad p(z) := z^2 + \sum_{k=1}^K a^{(k)} z b^{(k)} + c,$$

and the corresponding matrix representation is

$$(5.10) \quad \text{col}(p(z)) = (2\Re(z)\mathbf{I} + \sum_{k=1}^K {}_3(a^{(k)}, b^{(k)})) \text{col}(z) + \text{col}(c - |z|^2)$$

$$(5.11) \quad =: \mathbf{A} \text{col}(z) + \text{col}(B).$$

In this case, the matrix  $\mathbf{A}$  contains a sum of  ${}_3$  matrices and Lemma 4.1 will not be valid any more. We have to expect the possibilities of the general case.

We will treat three examples of the quadratic case

$$(5.12) \quad p(z) := z^2 + azb + czd + e.$$

If  $b = c = 1$ , then by putting  $z := u - d$  we obtain the one-sided case (see [11], p. 311)

$$(5.12') \quad \tilde{p}(u) := p(u - d) := u^2 + (a - d)u - ad + e, \quad z := u - d.$$

EXAMPLE 5.1. In  $p$  of (5.12) we choose

$$(a) \quad a := (0, -1, 0, 0), b := (0, 1, 1, 0), c := (0, 1, -1, 1), \\ d := (0, 1, 1, 0), e := (0, 3, -1, -3).$$

In this case  $z_0 := 1 + \sqrt{6}\mathbf{i}$  belongs to a class of zeros. And

$$\mathbf{A}(z_0) := \begin{pmatrix} 3 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \quad \text{col}(B(z_0)) := \begin{pmatrix} -7 \\ 3 \\ -1 \\ -3 \end{pmatrix}.$$

where  $\text{rank}(\mathbf{A}) = 2$ . There are two solutions

$$u_1 := (1, -2, 1, 1), \quad u_2 := (1, -1, 1, 2).$$

Let us now choose

$$(b) \quad a := (0, 1, 1, 0), b := (1, 0, -1, 0), c := (0, 0, 1, 1), \\ d := (0, 1, 1, 0), e := (16, 4, -16, 6).$$

Some experiments show that  $z_0 := 1 + \sqrt{29}\mathbf{i}$  belongs to a class of zeros. Then

$$\mathbf{A}(z_0) := \begin{pmatrix} 2 & -2 & 0 & -2 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 2 & -2 \\ -2 & -2 & 0 & 2 \end{pmatrix}, \quad \text{col}(B(z_0)) = \begin{pmatrix} -14 \\ 4 \\ -16 \\ 6 \end{pmatrix}.$$

and  $\text{rank}(\mathbf{A}) = 3$  and we have to solve  $\mathbf{A}\text{col}(z) + \text{col}(B) = 0$  for  $z \in [z_0]$ . The only solution is

$$u := (1, -2, 3, -4).$$

The third example, with  $\text{rank}(\mathbf{A}) = 4$ , is

$$(c) \quad a := (-4, -1, 4, 2), b := (3, -3, 3, -3), c := (0, -5, 0, -1), \\ d := (4, -3, -5, 1), e := (258, 208, 239, 220).$$

For  $z_0 = 2 + \sqrt{83}\mathbf{i}$  we obtain

$$A(z_0) := \begin{pmatrix} -31 & -12 & -1 & 20 \\ -34 & -21 & 0 & 29 \\ -1 & -20 & -9 & 36 \\ 48 & -19 & -34 & 29 \end{pmatrix}, \quad \text{col}B(z_0) := \begin{pmatrix} 171 \\ 208 \\ 239 \\ 220 \end{pmatrix}.$$

And the only solution of  $\mathbf{A}(z_0)\text{col}(z) + \text{col}(B(z_0)) = 0$  is

$$u := (2, -3, 5, -7).$$

As in the previous examples, we do not claim that we have found all zeros. However, there are some additional remarks in the end of Section 8, p. 14.

The investigation of this section makes also sense for the linear case,  $n = 1$ , if we would delete the condition  $K := k_1 = 1$ . In this case we would have  $p(z) := \sum_{k=1}^K a^{(k)} z b^{(k)} + c$  and  $\text{col}(p(z)) = \mathbf{A} \text{col}(z) + \text{col}(c)$ , where  $\mathbf{A} = \sum_{k=1}^K 1_3(a^{(k)}, b^{(k)})$ . Since  $\mathbf{A}, \text{col}(c)$  do not depend on  $z$ , the equivalence classes have to be replaced with the full space  $\mathbb{H}$ . A zero of type  $k, 0 \leq k \leq 4$ , is then a zero in a  $k$ -dimensional subspace of  $\mathbb{H}$ . Because of the loosening of the condition  $k_1 = 1$ , the equation  $p(z) = 0$  may have no solution, like  $p(z) := az - za + 1$ . The linear case, also for systems is treated in more detail in Janovská and Opfer, 2008, [8].

**6. Numerical considerations for finding the zeros of a two-sided polynomial.** The decisive equations (see (3.16) to (3.19)) read

$$(3.19) \quad P(z) := \text{col}(p(z)) = \mathbf{A}(z)\text{col}(z) + \text{col}(B(z)) = 0,$$

where the last zero stands for the zero column vector with four rows. A standard technique for solving such a system is Newton's method. In short, this technique results in solving the following linear equation for  $s$ , repeatedly:

$$(6.1) \quad P(z) + P'(z)s = 0; \quad z := z + s,$$

where in the beginning one needs an *initial guess*  $z$ . In order to compute the  $(4 \times 4)$  *Jacobi matrix*  $P'$  we use numerical differentiation. Let  $e_k, k = 1, 2, 3, 4$  be one of the four standard unit vectors in  $\mathbb{R}^4$  and  $z := (z_1, z_2, z_3, z_4)$ . Then,

$$(6.2) \quad \frac{\partial P}{\partial z_k}(z) \approx \frac{P(z + h e_k) - P(z)}{h}, \quad k = 1, 2, 3, 4, \quad h \approx 10^{-7},$$

$$(6.3) \quad P'(z) := \left( \frac{\partial P}{\partial z_1}(z), \frac{\partial P}{\partial z_2}(z), \frac{\partial P}{\partial z_3}(z), \frac{\partial P}{\partial z_4}(z) \right).$$

The choice  $h \approx 10^{-7}$  is the standard choice for computers with machine precision of  $\approx 10^{-15}$ . This choice implies a good balance between the round off and truncation errors. For completeness, we also give a formula for the exact Jacobian in the next section. See also the comments on a comparison of the numerical version with the exact version.

EXAMPLE 6.1. We choose the following not quite trivial polynomial of degree 4,

$$(6.4) \quad p(z) := \sum_{j=0}^4 a_j z^j b_j, \text{ where}$$

$$\begin{aligned} a_4 &:= (1, 1, 0, 0), & b_4 &:= (-1, -1, -1, 0), \\ a_3 &:= (-1, 0, 1, 1), & b_3 &:= (0, -1, 0, 1), \\ a_2 &:= (0, -1, 1, 1), & b_2 &:= (1, 0, 0, 0), \\ a_1 &:= (0, 1, 0, 0), & b_1 &:= (-1, 0, 1, -1), \\ a_0 &:= (1, 0, -1, 1), & b_0 &:= (-1, -1, 0, 0). \end{aligned}$$

We tried several initial guesses  $z$  where all four components of  $z$  varied randomly in  $[-2, 2]$ . We found that in all cases convergence took place, though in a variety of steps. Typically, towards the end of the iteration, convergence was quadratic. Choosing the four components of  $z$  randomly in  $[-2, 2]$  implies that for all guesses

we have  $|z| \leq 4$ . There are some statements on the location of the zeros of general, quaternionic polynomials in [13]. These results, applied to this example, say that all zeros  $z$  must be in the ball  $|z| \leq 2 + \sqrt{2} \approx 3.4$ . By applying the described Newton method, we found four zeros, all with type = 0, see Table 6.2.

TABLE 6.2. Four zeros  $u_1, u_2, u_3, u_4$  of  $p$ , defined in (6.4).

| $u_1$             | $u_2$             | $u_3$             | $u_4$             |
|-------------------|-------------------|-------------------|-------------------|
| 0.71351949935964  | -0.56474491922429 | -0.81751299009456 | 1.36509146887082  |
| 0.53959756736776  | 0.51869896708659  | 0.31625929302548  | 0.08286248144125  |
| -0.47236626339089 | -0.69234698020809 | 0.06765291838453  | 0.71279659128071  |
| -0.78476277296416 | -1.19972763447271 | 0.27386723230307  | -0.33902637867895 |

A comparison of the random selection of the initial guesses with a uniform selection of the initial guesses in the cube  $[-2, 2]^4$  reveals that the random selection finds all zeros by far much quicker than the uniform selection of the initial guesses.

**7. The exact form of the Jacobi matrix  $P'$ .** In view of the definition (6.3) we have to compute the four partial derivatives  $\frac{\partial P}{\partial z_k}$ ,  $k = 1, 2, 3, 4$ . Let the polynomial  $p$  be given in the form (1.1) with coefficients  $a_j, b_j \in \mathbb{H}$ ,  $j = 0, 1, \dots, n$  and let  $e_k$  be the  $k$ -th standard unit vector in  $\mathbb{R}^4$ . Using the definitions (3.19), (3.18), (3.2) to (3.4), (3.15) for  $P, \mathbf{A}, \text{col}(B), \alpha_j, \beta_j, j = 0, 1, \dots, 13$ , respectively, we obtain for  $1 \leq k \leq 4$

$$(7.1) \quad \frac{\partial P}{\partial z_k}(z) = \frac{\partial \mathbf{A}(z)}{\partial z_k} \text{col}(z) + \mathbf{A}e_k + \frac{\partial \text{col}(B(z))}{\partial z_k}$$

$$(7.2) \quad = \left( \sum_{j=0}^n \frac{\partial \alpha_j(z)}{\partial z_k} \mathbf{1}_3(a_j, b_j) \right) \text{col}(z) + \mathbf{A}e_k + \sum_{j=0}^n \frac{\partial \beta_j(z)}{\partial z_k} \text{col}(a_j b_j).$$

In order to complete the computation, we need the derivatives  $\frac{\partial \alpha_j(z)}{\partial z_k}, \frac{\partial \beta_j(z)}{\partial z_k}$ . We put

$$(7.3) \quad \alpha_{k,j}(z) := \frac{\partial \alpha_j(z)}{\partial z_k}, \quad \beta_{k,j}(z) := \frac{\partial \beta_j(z)}{\partial z_k} \quad \text{for } k = 1, 2, 3, 4, j = 0, 1, \dots$$

By differentiation of the recursion (3.2) to (3.4) we obtain for  $j = 0, 1, \dots$ ,

$$(7.4) \quad \alpha_{k,j+1} := \begin{cases} 2\alpha_j + 2z_1\alpha_{1,j} + \beta_{1,j} & \text{for } k = 1, \\ 2z_1\alpha_{k,j} + \beta_{k,j} & \text{for } 2 \leq k \leq 4, \end{cases}$$

$$(7.5) \quad \beta_{k,j+1} := -2z_k\alpha_j - |z|^2\alpha_{k,j} \text{ for } k = 1, 2, 3, 4.$$

Since  $\alpha_0 = \beta_1 = 0, \alpha_1 = \beta_0 = 1$ , all derivatives of these quantities vanish. Thus,  $\alpha_{k,0} = \alpha_{k,1} = \beta_{k,0} = \beta_{k,1} = 0, k = 1, 2, 3, 4$ . It should be noted, that the quantities  $\alpha_j$  are required for the computation of the derivatives  $\alpha_{k,j}, \beta_{k,j}$ .

Let  $P'$  be the exact Jacobi matrix and  $\tilde{P}' := P' + E$ , where  $E$  is a real  $(4 \times 4)$  matrix containing (elementwise) artificial random noise in the range  $[-0.01, 0.01]$ . Then we found, at least for polynomials with small degree, that the use of  $\tilde{P}'$  instead of the exact  $P'$  had almost no influence on the speed of the convergence. We also found, that the zeros presented for the examples treated do not change if instead of the numerical Jacobian we use the exact Jacobian. We also would like to mention, that the singularity of the matrix  $\mathbf{A}$  (Example 5.1, (a) and (b), p. 11) does not imply that the Jacobian is singular. Actually, we did not find one case where the Jacobian was singular.

**8. The number of zeros of two-sided polynomials.** Since the polynomial  $p(z) := z^2 + 1$  has already infinitely many zeros in  $\mathbb{H}$ , it makes no sense to count the individual zeros.

DEFINITION 8.1. Let  $p$  be any quaternionic polynomial of degree  $n \geq 2$ . By  $\#Z(p)$  we understand the number of equivalence classes in  $\mathbb{H}$  which contain zeros of  $p$ . We call this number *essential number of zeros of  $p$* .

By this definition,  $p(z) := z^2 + 1$  has one essential zero, since  $\mathbf{i}$  and  $-\mathbf{i}$  are located in the same equivalence class. All polynomials with real coefficients and degree  $n$  have at most  $n$  essential zeros in  $\mathbb{H}$ . All quaternionic, one-sided polynomials of degree  $n$  have at most  $n$  essential zeros. See Lam, §16, [10], Pogorui and Shapiro, [14], Janovská and Opfer, [7]. If we look at our examples in this paper, it could be guessed, that this is also true for two-sided polynomials. However, this is not the case.

THEOREM 8.2. *Let  $p$  be a quaternionic, two-sided polynomial of degree  $n$ . Then,  $\#Z(p)$ , the essential number of zeros of  $p$ , is, in general, not bounded by  $n$ .*

*Proof.* We will present an example. Let

$$(8.1) \quad p(z) := a_3 z^3 b_3 + a_2 z^2 b_2 + a_1 z b_1 + c_0, \text{ where}$$

$$\begin{aligned} a_3 &:= (1, 1, 0, 0), & b_3 &:= (-1, -1, -1, 0), \\ a_2 &:= (-1, 0, 1, 1), & b_2 &:= (0, -1, 0, 1), \\ a_1 &:= (0, -1, 1, 1), & b_1 &:= (1, 0, 0, 1), \\ c_0 &:= (2, 0, 0, 0). \end{aligned}$$

Observe that  $a_3 b_3 + a_2 b_2 + a_1 b_1 + c_0 = 0$ , which implies  $p(1) = 0$ . By the method described and also applied in Example 6.1, we found the additional, non equivalent four zeros with type = 0, listed in Table 8.3.

TABLE 8.3. Four additional zeros  $u_1, u_2, u_3, u_4$  of  $p$ , defined in (8.1).

| $u_1$             | $u_2$             | $u_3$             | $u_4$             |
|-------------------|-------------------|-------------------|-------------------|
| 0.12795969606090  | 0.56107593303891  | -0.90447809795613 | 0.89282516158148  |
| 0.95656086766094  | -0.86385162132303 | -0.29483216684617 | -0.10344502760252 |
| -0.44112817648356 | 0.10671040311206  | -0.45438193404159 | 0.47734930925792  |
| -1.25375585340058 | 0.17393396219483  | -0.18098296526254 | -0.23733895232048 |

Thus, the polynomial  $p$  is of degree three and the essential number of zeros of  $p$  exceeds the degree by at least two.  $\square$

If we apply the numerical techniques presented in Section 6 to the three quadratic polynomials of Section 4 we obtain: Example 4.4: There are two further non equivalent zeros both of type 0. The essential number of zeros is (at least) three. Example 4.5: There are two further non equivalent zeros both of type 0. The essential number of zeros is (at least) four. Example 4.6: There is one more zero of type 0, and there are two more equivalent zeros of type 2. The essential number of zeros is (at least) three. An application to Example 5.1, reveals in (a) two additional, non equivalent zeros, in (b) one additional non equivalent zero, in (c) one additional non equivalent zero. Thus, the essential number of zeros is in (a): (at least) three, in (b) (at least) two, in (c) (at least) two. Therefore, in several examples, the essential number of zeros exceeds the degree. Since the number of parameters of the polynomial  $p$ , defined in (1.1), is  $2n + 1$  we will make some guess on the maximal essential number of zeros.

CONJECTURE 8.4. *Let  $p$  be a polynomial of degree  $n$  of the form described in (1.1). Then, the essential number of zeros of  $p$  will not exceed  $2n$ .*

The conjectured bound is sharp, since the essential number of zeros of the quadratic polynomial of Example 4.5, p. 9 is four.



**9. Multiple zeros and the application of Newton's method.** We had many opportunities to employ Newton's method for finding the zeros of two-sided, quaternionic polynomials and in all cases the method worked very well in the sense that the precision of the zeros was close to machine precision and the convergence speed was locally quadratic. Multiple zeros would, however, reduce or destroy the precision of the zeros and also the convergence speed, and we would like to add that in the case of two-sided polynomials, multiple, quaternionic zeros cannot occur, with the only exception of  $p(z) := (z - a)^2$ .

**THEOREM 9.1.** *Let  $p$  be a polynomial of the form*

$$p(z) := q(z)(z - a)^k r(z), \text{ where } q(a) \neq 0, r(a) \neq 0, k \geq 1, \text{ and } a \in \mathbb{H} \setminus \mathbb{R}.$$

*If  $k = 2$  and at least one of the polynomials  $q, r$  is not constant or if  $k \geq 3$ , then, the polynomial  $p$  is not two-sided.*

*Proof.* Let  $k = 2$ ,  $a \in \mathbb{H} \setminus \mathbb{R}$  and let  $q$  have degree at least one. Then  $q(z)(z - a)^2 = q(z)(z^2 - az - za + a^2) = q(z)z^2 - q(z)az - q(z)za + q(z)a^2$ . The term  $q(z)az$  is not part of a two-sided polynomial. The remaining part is analogous.  $\square$

Thus, the two-sided polynomials different from  $p(z) := (z - a)^2$  cannot have multiple quaternionic zeros. Since  $p(z) := (z - a)^2$  is not a one-sided polynomial, we can also state, that one-sided polynomials do not have multiple, quaternionic zeros at all. [This section was added in proof.]

**10. Summary.** For quaternionic polynomials  $p$  of the two-sided type, allowing also several terms of the same degree (see (5.1), (5.2) for description) we have shown that the zeros  $z$  may fall into five different classes (Definition 3.12), where for each zero the class can be determined by looking at the rank of a  $(4 \times 4)$  matrix  $\mathbf{A}(z)$  which is defined for the general case in (5.8). If a zero in one class has been found, the described technique allows to find all zeros in the same class.

The representation of a given quaternionic, two-sided polynomial  $p$  in the form  $P(z) := \mathbf{A}(z)z + B(z)$  which was already used for the classification of the zeros can also be applied successfully to finding the zeros, by applying Newton's method to  $P(z) = 0$ . It shows the typical feature, that it may be slow in the beginning, but it will terminate then very quickly with quadratic rate. The possibility of finding the zeros quickly also was the source for finding out that the degree  $n$  of a polynomial  $p$  is, in general, not a bound for the essential number of zeros of  $p$ .

Finally, we would like to mention that without using two essential ideas of other authors, this paper could not have been written. One idea by Anatoliy Pogorui and Michael Shapiro, [14], was to write the powers  $z^j$  in the form  $\alpha z + \beta$  with real  $\alpha, \beta$ , which reduces the two-sided polynomials essentially (i. e. up to constants) to a sum of terms of the form  $azb$ . This idea also gave birth to the introduction of equivalence classes of zeros in  $\mathbb{H}$ . Another idea, by Ludmilla Aramanovitch, [1], was the introduction of the matrix  ${}_2$  (formula (3.10)) which permitted to pull out the variable  $z$  from  $azb$ , cf. formula (3.14). Both ingredients allowed the development of the important formula (3.19), p. 6.

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