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Nonsmooth Optimal Control Problems with Switching Functions of Order Zero

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The paper is concerned with general optimal control problems (OCP) which are characterized by a nonsmooth ordinary state differential equation. More precisely, we assume that the right-hand side of the state equation is piecewise smooth and that the switching points, which separate these pieces, are determined as roots of a state- and control dependent (smooth) switching function. For this kind of optimal control problems necessary conditions are developed. Special attention is payed to the situation that the switching function vanishes identically along a nontrivial subarc. Such subarcs, which are called singular state subarcs, are investigated with respect to the necessary conditions and to the junction conditions. In extension to earlier results cf. Ref.5, in this paper the case of a zero-order switching function is considered.

1. Nonsmooth Optimal Control Problems, Regular Case.

We consider a general OCP with a piecewise defined state differential equation. The problem has the following form.

Problem (P1). Determine a piecewise continuous control function $u : [a,b] \to \mathbb{R}$, such that the functional

$$I = g(x(b)) \tag{1}$$

is minimized subject to the following constraints (state equations, boundary conditions, and control constraints)

$$x'(t) = f(x(t), u(t)), \quad t \in [a,b] \quad \text{a.e.}, \tag{2a}$$

$$r(x(a), x(b)) = 0, \tag{2b}$$

$$u(t) \in U = [u_{\min}, u_{\max}] \subset \mathbb{R}. \tag{2c}$$

The right-hand side of the state equation (2a) may be of the special form

$$f(x,u) = \begin{cases} f_1(x,u), & \text{if } S(x,u) \le 0, \\ f_2(x,u), & \text{if } S(x,u) > 0, \end{cases} \tag{3}$$

where the functions $S : \mathbb{R}^{n+1} \to \mathbb{R}$, $f_k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ $(k = 1,2)$, and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$, $\ell \in \{0, \dots, 2n\}$, are assumed to be sufficiently smooth. S is called the *switching function* of Problem (P1).

Our aim is to derive necessary conditions for Problem (P1). To this end, let (x^0, u^0) denote a solution of the problem with a piecewise continuous optimal control function u^0.

Further, we assume that the problem is *regular* with respect to the minimum principle, that is: For each λ, $x \in \mathbb{R}^n$ both *Hamiltonians*

$$H_j(x, u, \lambda) := \lambda^{\mathrm{T}} f_j(x, u), \quad j = 1, 2, \tag{4}$$

possess a unique minimum u_j^0 with respect to the control $u \in U$.

Finally, for this Section, we assume that the following regularity assumption holds.

Regularity Condition (R). There exists a finite grid $a =: t_0 < t_1 < \ldots < t_s < t_{s+1} := b$ such that the optimal switching function $S[t] := S(x^0(t), u^0(t))$ is either positive or negative in each open subinterval $]t_{j-1}, t_j[$, $j = 1, \dots, s+1$.

Note, that the one-sided-limits $u(t_j^\pm)$ exist due to the assumption of the piecewise continuity of the optimal control. Now, we can summarize the necessary conditions for Problem (P1). Here, on each subintervall $[t_j, t_{j+1}]$, we denote $H(x, u, \lambda) := H_k(x, u, \lambda)$ where $k \in \{1, 2\}$ is chosen according to the sign of S in the corresponding subinterval.

Theorem 1.1.

With the assumptions above the following necessary conditions hold.

There exist an adjoint variable $\lambda : [a, b] \to \mathbb{R}^n$, which is a piecewise C^1–function, and Lagrange multipliers $\nu_0 \in \{0, 1\}$, $\nu \in \mathbb{R}^\ell$, such that (x^0, u^0) satisfies

$$\lambda'(t) = -H_x(x^0(t), u^0(t), \lambda(t)), \quad t \in [a, b] \text{ a.e.} \qquad \text{(adjoint equations)}, \tag{5a}$$

$$u^0(t) = \operatorname{argmin}\{H(x^0(t), u, \lambda(t)) : \ u \in U\} \qquad \text{(minimum principle)}, \tag{5b}$$

$$\lambda(a) = -\frac{\partial}{\partial x^0(a)} \left[\nu^{\mathrm{T}} r(x^0(a), x^0(b))\right] \qquad \text{(natural boundary conditions)}, \tag{5c}$$

$$\lambda(b) = \frac{\partial}{\partial x^0(b)} \left[\nu_0\, g(x^0(b)) + \nu^{\mathrm{T}} r(x^0(a), x^0(b))\right], \tag{5d}$$

$$\lambda(t_j^+) = \lambda(t_j^-), \quad j = 1, \dots, s, \qquad \text{(continuity condition)}, \tag{5e}$$

$$H[t_j^+] = H[t_j^-], \quad j = 1, \dots, s, \qquad \text{(continuity condition)}. \tag{5f}$$

Proof. Without loss of generality, we assume, that there is just *one* point $t_1 \in]a,b[$, where the switching function $S[\cdot]$ changes sign. Moreover, we assume that the following *switching structure* holds

$$S[t] \quad \begin{cases} < 0, & \text{if} \quad a \le t < t_1 \\ > 0, & \text{if} \quad t_1 < t \le b. \end{cases} \tag{6}$$

We compare the optimal solution (x^0, u^0) only with those admissible solutions (x, u) of the problem which have the same switching structure (6). Each candidate of this type can by associated with its separated parts $(\tau \in [0,1])$

$$\begin{aligned} x_1(\tau) &:= x(a + \tau(t_1 - a)), & x_2(\tau) &:= x(t_1 + \tau(b - t_1)), \\ u_1(\tau) &:= u(a + \tau(t_1 - a)), & u_2(\tau) &:= u(t_1 + \tau(b - t_1)). \end{aligned} \tag{7}$$

Now, $(x_1, x_2, t_1, u_1, u_2)$ performs an abmissible and $(x_1^0, x_2^0, t_1^0, u_1^0, u_2^0)$ an optimal solution of the following auxillary optimal control problem.

Problem (P1'). Determine a piecewise continuous control function $u = (u_1, u_2) : [0,1] \to \mathbb{R}^2$, such that the functional

$$I = g(x_2(1)) \tag{8}$$

is minimized subject to the constraints

$$x_1'(\tau) = (t_1 - a)\, f_1(x_1(\tau), u_1(\tau))\,, \quad \tau \in [0,1], \quad \text{a.e.,} \tag{9a}$$

$$x_2'(\tau) = (b - t_1)\, f_2(x_2(\tau), u_2(\tau)), \tag{9b}$$

$$t_1'(\tau) = 0, \tag{9c}$$

$$r(x_1(0), x_2(1)) = 0\,, \tag{9d}$$

$$x_2(0) - x_1(1) = 0, \tag{9e}$$

$$u_1(\tau),\ u_2(\tau) \in U \subset \mathbb{R}. \tag{9f}$$

Problem (P1') is a classical optimal control problem with a smooth right-hand side, and $(x_1^0, x_2^0, t_1^0, u_1^0, u_2^0)$ is a solution of this problem. Therefore, we can apply the well–known necessary conditions of optimal control theory, cf. References 2–4, i.e there exist continuous and piecewise continuously differentiable adjoint variables $\lambda_j : [0,1] \to \mathbb{R}^n$, $j = 1,2$, and Lagrange-multpliers $\nu_0 \in \{0,1\}$, $\nu \in \mathbb{R}^\ell$, and $\nu_1 \in \mathbb{R}^n$, such that with the Hamiltionian

$$\widetilde{H} := (t_1 - a)\, \lambda_1^{\mathrm{T}} f_1(x_1, u_1) + (b - t_1)\, \lambda_2^{\mathrm{T}} f_2(x_2, u_2), \tag{10}$$

and the augmented performance index

$$\Phi := \nu_0\, g(x_2(1)) + \nu^{\mathrm{T}} r(x_1(0), x_2(1)) + \nu_1^{\mathrm{T}} (x_2(0) - x_1(1))\,, \tag{11}$$

the following conditions hold

$$
\begin{aligned}
\lambda_1' &= -\widetilde{H}_{x_1} &=& -(t_1-a)\,\frac{\partial}{\partial x_1}\left(\lambda_1^{\mathrm{T}} f_1(x_1,u_1)\right), & \text{(12a)}\\
\lambda_2' &= -\widetilde{H}_{x_2} &=& -(b-t_1)\,\frac{\partial}{\partial x_2}\left(\lambda_2^{\mathrm{T}} f_2(x_2,u_2)\right) & \text{(12b)}\\
\lambda_3' &= -\widetilde{H}_{t_1} &=& -\lambda_1^{\mathrm{T}} f_1(x_1,u_1) \;+\; \lambda_2^{\mathrm{T}} f_2(x_2,u_2), & \text{(12c)}
\end{aligned}
$$

$$u_k(\tau) \;=\; \operatorname{argmin}\{\lambda_k(\tau)^{\mathrm{T}} f_k(x_k(\tau),u):\;\; u\in U\},\;\; k=1,2 \qquad \text{(12d)}$$

$$\lambda_1(0) \;=\; -\Phi_{x_1(0)} \;=\; -\frac{\partial}{\partial x_1(0)}(\nu^{\mathrm{T}} r), \quad \lambda_1(1) \;=\; \Phi_{x_1(1)} \;=\; -\nu_1, \qquad \text{(12e)}$$

$$\lambda_2(0) \;=\; -\Phi_{x_2(0)} \;=\; -\nu_1, \quad \lambda_2(1) \;=\; \Phi_{x_2(1)} \;=\; \frac{\partial}{\partial x_2(1)}(\nu_0\, g \;+\; \nu^{\mathrm{T}}\, r), \qquad \text{(12f)}$$

$$\lambda_3(0) \;=\; \lambda_3(1) \;=\; 0. \qquad \text{(12g)}$$

Now, due to the autonomy of the state equations and due to the regularity assumptions above, both parts $\lambda_1^{\mathrm{T}} f_1$ and $\lambda_2^{\mathrm{T}} f_2$ of the Hamiltonian are constant on $[0,1]$. Thus, λ_3 is a linear function which vanishes due to the boundary conditions (12g). Together with the relation (12c) one obtains the continuity of the Hamiltonian (5f).

If one recombines the adjoints

$$\lambda(t) \;:=\; \begin{cases} \lambda_1\!\left(\dfrac{t-a}{t_1-a}\right), & t \in [a,t_1[,\\[2ex] \lambda_2\!\left(\dfrac{t-t_1}{b-t_1}\right), & t \in [t_1,b], \end{cases} \qquad \text{(13)}$$

one obtains the adjoint equation (5a) from Eq. (12a-b), the minimum principle (5b) from Eq. (12d), and the natural boundary conditions and the continuity conditions (5c-e) from Eq. (12e-f).

□

It should be remarked that the results of Theorem 2.1. easily can be extended to nonautonomous optimal control problems with nonsmooth state equations. This holds too, if the performance index contains an additional integral term $I = g(x(t_b)) + \int_{t_a}^{t_b} f_0(t,x(t),u(t))dt$. Both extensions can be treated by standard transformation techniques which transform the problems into the form of Problem (P1). The result is, that for the extended problems, one simply has to redefine the Hamiltonian by

$$H(t,x,u,\lambda,\nu_0) \;:=\; \nu_0\, f_0(t,x,u) \;+\; \lambda^{\mathrm{T}} f(t,x,u). \qquad \text{(14)}$$

Example (1.1) The following example is taken from the well-known book of Clark. It describes the control of an electronic circuit which encludes a diode and a condensor. The control u is the initializing voltage, the state variable x denotes the voltage at the condensor. The resuling optimal control problem is given as follows.

Minimize the functional

$$I(u) \;=\; \frac{1}{2}\int_0^2 u(t)^2\, dt \qquad \text{(15)}$$

with respect to the state equation

$$x'(t) \;=\; \begin{cases} a\,(u-x), & \text{if} \quad S = x - u \le 0, \\ b\,(u-x), & \text{if} \quad S = x - u \;>\; 0, \end{cases} \tag{16}$$

and the boundary conditions

$$x(0) \;=\; 4, \qquad x(2) \;=\; 3. \tag{17}$$

First, we consider the smooth case, i.e. we choose $a = b = 2$. The solution easily can be found applying the classical optimal control theory. The Hamiltonian is given by

$$H \;=\; u^2/2 \;+\; a\,\lambda\,(u-x),$$

which yields the adjoint equation $\lambda' = a\,\lambda$, the optimal control $u = -a\,\lambda$. Thus, we obtain the linear *two-point boundary value problem*

$$\begin{aligned} x' &= -a^2\,\lambda - a\,x, \qquad && x(0) = 4, \\ \lambda' &= a\,\lambda, && x(2) = 3. \end{aligned} \tag{18}$$

The (unique) solution for the parameter $a = 2$ is given in Figure 1.

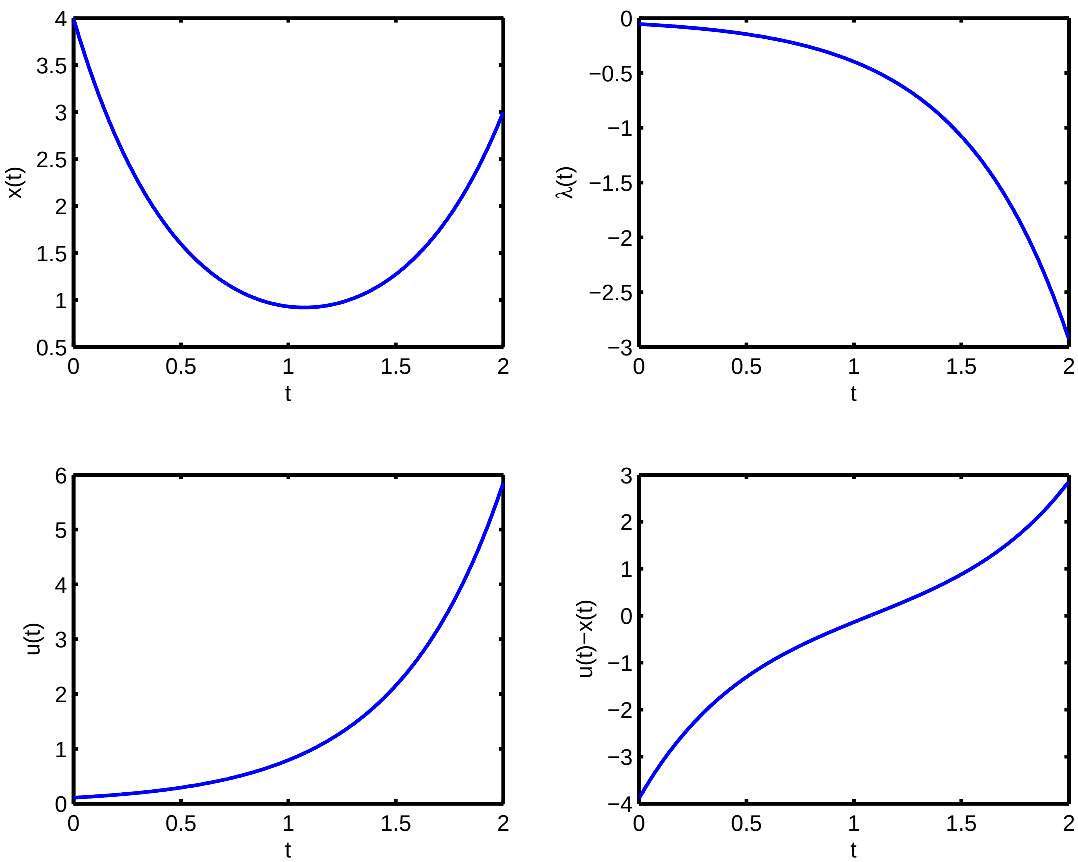

Fig. 1 Example 1.1: Smooth Case.

For the nonsmooth case, $a \neq b$, we assume that there is just one point $t_1 \in]0, 2[$ where the switching function changes sign. Further, due to the results for the smooth problem shown in Fig. 1, we assume the solution structure

$$S[t] \begin{cases} > 0, & \text{if} \quad 0 \le t < t_1, \\ < 0, & \text{if} \quad t_1 < t \le 2. \end{cases} \tag{19}$$

According to Theorem 1.1 we obtain the following necessary conditions for the solution (x^0, u^0):

(i) $t \in [0, t_1]$:

$$H = H_2 = \frac{1}{2} u^2 + b\,\lambda\,(u - x),$$
$$\lambda' = b\,\lambda, \quad u = -b\,\lambda.$$

(ii) $t \in [t_1, 2]$:

$$H = H_1 = \frac{1}{2} u^2 + a\,\lambda\,(u - x),$$
$$\lambda' = a\,\lambda, \quad u = -a\,\lambda.$$

The continuity condition (5f) yields

$$H[t_1^+] - H[t_1^-] = (b - a)\,\lambda(t_1) \left[\frac{a+b}{2}\,\lambda(t_1) + x(t_1) \right] = 0.$$

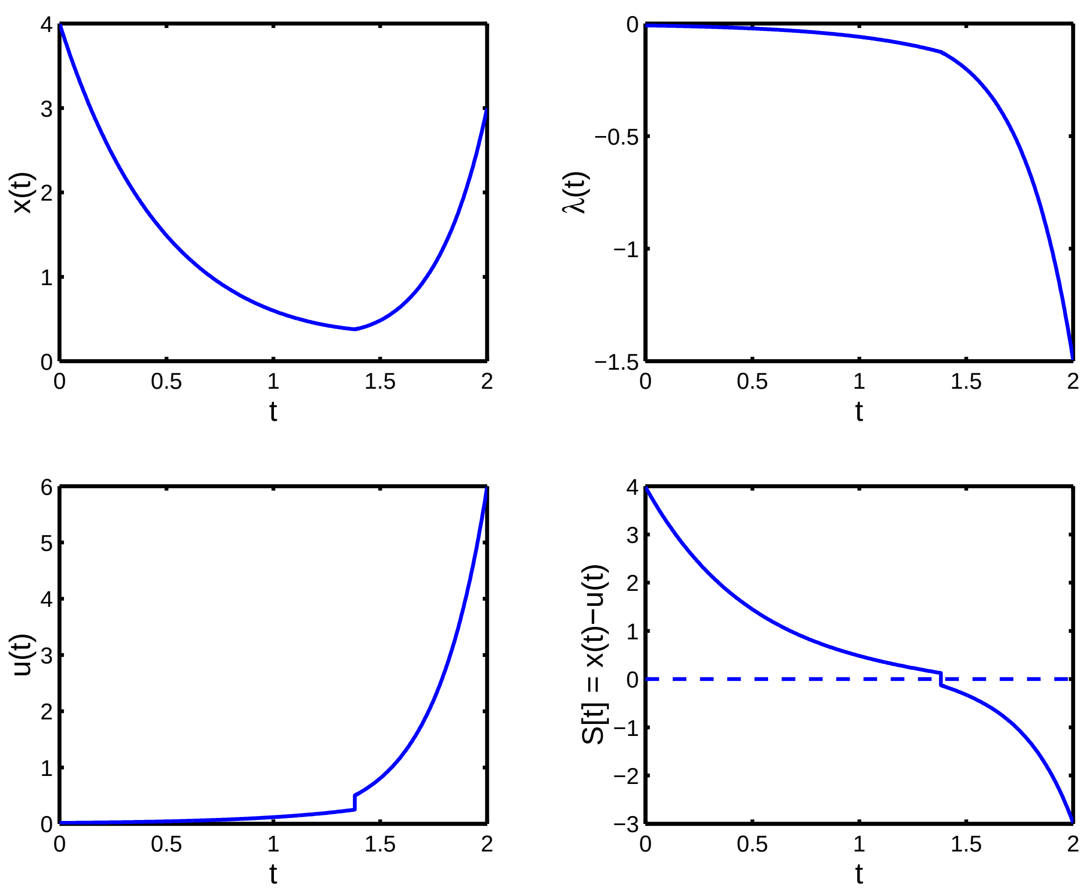

Fig. 2 Example 1.1: Nonsmooth and Regular Case, $a = 4$, $b = 2$.

So, we obtain the following *three-point boundary value problem*

$$
\begin{aligned}
x' &= \begin{cases} -b\,(b\,\lambda + x) & : \quad t \in [0, t_1], \\ -a\,(a\,\lambda + x) & : \quad t \in [t_1, 2], \end{cases} \\
\lambda' &= \begin{cases} b\,\lambda & : \quad t \in [0, t_1], \\ a\,\lambda & : \quad t \in [t_1, 2], \end{cases} \\
x(0) &= 4, \qquad x(2) \; = \; 3, \qquad \frac{a+b}{2}\,\lambda(t_1) + x(t_1) \; = \; 0.
\end{aligned}
\tag{20}
$$

In Figure 2 the numerical solution of this boundary value problem is shown for the parameters $a = 4$ and $b = 2$. The solution is obtained via the multiple shooting code BNDSCO, cf. References 6-7. One observes that the preassumed sign distribution of the switching function is satisfied. Further, the optimal control and the optimal switching function is discontinous at the switching point t_1.

For the parameters $a = 2$ and $b = 4$ the solution of the boundary value problem (20) is shown in Figure 3. Here, the preassumed sign distribution of the switching function is not satisfied. So, the estimated switching structure for these parameters is not correct and we have to consider the singular case, i.e. the switching function vanishes identically along a nontrivial subarc.

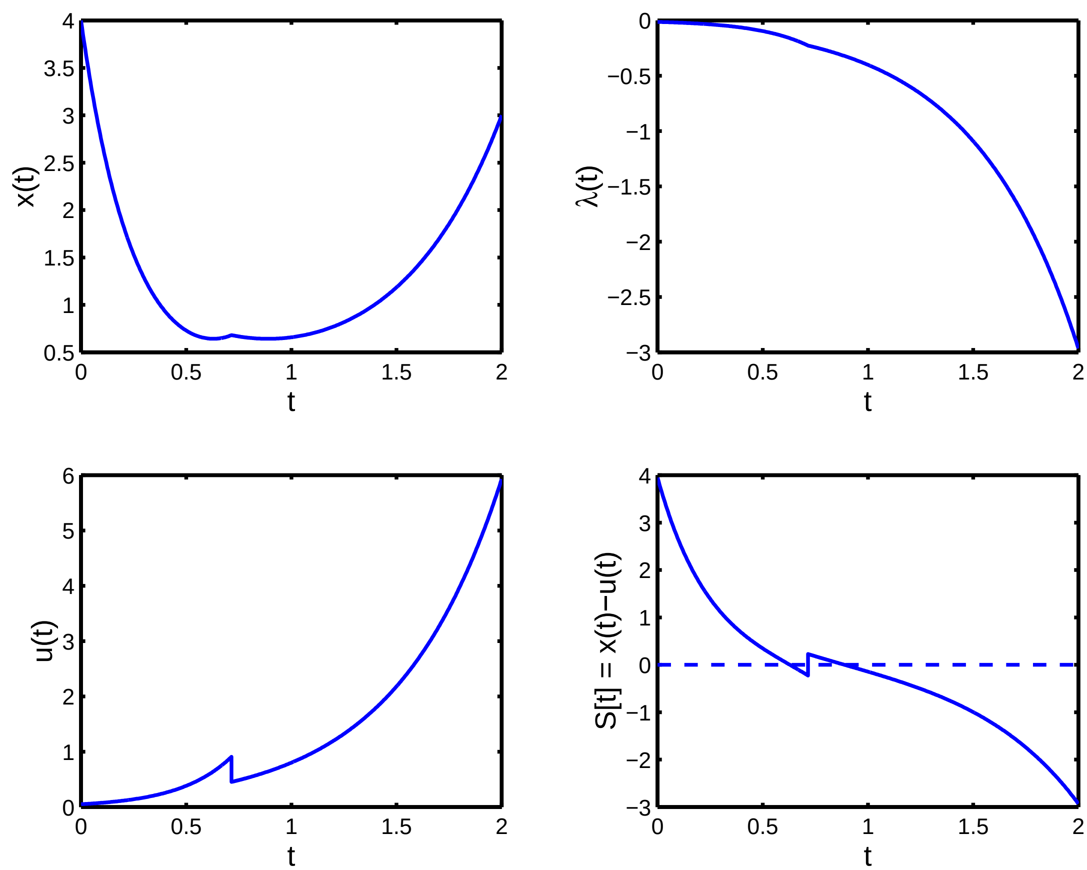

Fig. 3 Example 1.1: Nonsmooth and Regular Case, $a = 2$, $b = 4$.

2. Nonsmooth Optimal Control Problems, Singular Case.

In this section we continue the investigation of the general optimal control problem (P1). However, we drop the regularity condition (R). More precisely, we assume that a solution (x^0, u^0) of the optimal control problem contains a finite number of nontrivial subarcs, where the switching function vanishes identically. These subarcs are called *singular state subarcs*, cf. the analogous situation of singular control subarcs, cf. Ref. 1. In order to have a well-defined problem, we now have to consider the dynamics on the singular manifold $S(x,u) = 0$. Therefore, we generalize the problem formulation (P1) a bit, and allow the system to possess an independent dynamic on the singular subarcs.

Problem (P2). Determine a piecewise continuous control function $u : [a,b] \to \mathbb{R}$, such that the functional

$$I = g(x(b)) \tag{21}$$

is minimized subject to the following constraints (state equations, boundary conditions, and control constraints)

$$x'(t) = f(x(t), u(t)), \quad t \in [a,b] \quad \text{a.e.}, \tag{22a}$$

$$r(x(a), x(b)) = 0, \tag{22b}$$

$$u(t) \in U = [u_{\min}, u_{\max}] \subset \mathbb{R}, \tag{22c}$$

where the right-hand side f is of the special form

$$f(x,u) = \begin{cases} f_1(x,u), & \text{if } S(x,u) < 0, \\ f_2(x,u), & \text{if } S(x,u) = 0, \\ f_3(x,u), & \text{if } S(x,u) > 0, \end{cases} \tag{23}$$

with smooth functions $f_k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $k = 1,2,3$. All other assumptions with respect to Problem (P1) may be satisfied also for (P2).

Again, our aim is to derive necessary conditions for (P2). To this end, we assume that there exists a finite grid $a < t_1 < \ldots < t_s < b$ such that the t_j are either isolated points where the switching function $S[t] := S(x^0(t), u^0(t))$ changes sign or entry or exit points of a singular state subarc.

We assume, that the switching function is of order zero with respect to the control u, i.e.

$$S_u(x^0(t), u^0(t)) \neq 0 \tag{24}$$

holds along each singular state subarc. By the implicit function theorem, the equation $S(x,u) = 0$ can be solved (locally unique) for u. Thus, we assume that there exists a continuously differentiable function $u = V(x)$ which solves the equation above. With this, we define

$$\widehat{f}_2(x) := f_2(x, V(x)). \tag{25}$$

For the regular subarcs we introduce the Hamiltonian

$$H(x,u,\lambda) := H_j(x,u,\lambda) := \lambda^{\mathrm{T}} f_j(x,u), \tag{26}$$

where $j \in \{1,3\}$ is chosen in the corresponding regular subinterval $[t_k, t_{k+1}]$ according to the sign of S. For the singular subarcs we set

$$H(x,u,\lambda) \;:=\; H_2(x,u,\lambda) \;:=\; \lambda^{\mathrm{T}} \widehat{f}_2(x). \tag{27}$$

In the following, we summarize the necessary conditions for Problem (P2).

Theorem 2.1.

With the assumptions above the following necessary conditions hold.

There exist an adjoint variable $\lambda : [a,b] \to \mathbb{R}^n$, *which is a continuous and piecewise* C^1*–function, and Lagrange multipliers* $\nu_0 \in \{0,1\}$, $\nu \in \mathbb{R}^\ell$, *such that* (x^0, u^0) *satisfies the conditions*

$$\lambda'(t) \;=\; -H_x(x^0(t), u^0(t), \lambda(t)), \quad t \in [a,b], \;\text{ a.e.} \tag{28a}$$

$$u^0(t) \;=\; \begin{cases} \operatorname{argmin}\{H(x^0(t), u, \lambda(t)) : \; u \in U\}, \;\text{ on regular subarcs,} \\ V(x^0(t)), \;\text{ on singular subarcs,} \end{cases} \tag{28b}$$

$$\lambda(a) \;=\; -\frac{\partial}{\partial x^0(a)} \left[\nu^{\mathrm{T}} r(x^0(a), x^0(b))\right], \tag{28c}$$

$$\lambda(b) \;=\; \frac{\partial}{\partial x^0(b)} \left[\nu_0\, g(x^0(b)) \;+\; \nu^{\mathrm{T}} r(x^0(a), x^0(b))\right], \tag{28d}$$

$$\lambda(t_j^+) \;=\; \lambda(t_j^-), \quad j = 1, \ldots, s, \tag{28e}$$

$$H[t_j^+] \;=\; H[t_j^-], \quad j = 1, \ldots, s. \tag{28f}$$

Note, that on a singular subarc there holds no minimum principle for the control which is completely determined by the switching equation $S(x,u) = 0$.

Proof of Theorem 2.1. For simplicity, we assume, that the switching function $S[\cdot]$ along the optimal trajectory has just *one* singular subarc $[t_1, t_2] \subset]a,b[$, and that the following *switching structure* holds

$$S[t] \quad \begin{cases} < \; 0, & \text{if} \quad a \le t < t_1, \\ = \; 0, & \text{if} \quad t_1 \le t \le t_2, \\ > \; 0, & \text{if} \quad t_2 < t \le b. \end{cases} \tag{29}$$

Again, we compare the optimal solution (x^0, u^0) with those admissible solutions (x,u) of the problem which have the same switching structure. Each candidate is associated with its separated parts ($\tau \in [0,1]$, $t_0 := a$, $t_3 := b$)

$$\begin{aligned} x_j(\tau) &:= x(t_{j-1} + \tau(t_j - t_{j-1})), \quad j = 1,2,3, \\ u_j(\tau) &:= u(t_{j-1} + \tau(t_j - t_{j-1})), \quad j = 1,3. \end{aligned} \tag{30}$$

Now, $(x_1, x_2, x_3, t_1, t_2, u_1, u_3)$ performs an abmissible and $(x_1^0, x_2^0, x_3^0, t_1^0, t_2^0, u_1^0, u_3^0)$ an optimal solution of the following auxillary optimal control problem.

Problem (P2'). Determine a piecewise continuous control function $u = (u_1, u_3) : [0,1] \to \mathbb{R}^2$, such that the functional

$$I = g(x_3(1)) \tag{31}$$

is minimized subject to the constraints ($t_0 := a$, $t_3 := b$, $\tau \in [0,1]$)

$$x_j'(\tau) = \begin{cases} (t_j - t_{j-1})\, f_j(x_j(\tau), u_j(\tau))\,, & \text{a.e.}, \quad j = 1,3, \\ (t_2 - t_1)\ \widehat{f}_2(x_2(\tau))\,, & \text{a.e.}, \quad j = 2, \end{cases} \tag{32a}$$

$$t_k'(\tau) = 0, \quad k = 1,2, \tag{32b}$$

$$r(x_1(0), x_3(1)) = 0\,, \tag{32c}$$

$$x_2(0) - x_1(1) = x_3(0) - x_2(1) = 0, \tag{32d}$$

$$u_1(\tau),\ u_2(\tau),\ u_3(\tau) \in U \subset \mathbb{R}. \tag{32e}$$

Problem (P2') again is a classical optimal control problem with a smooth right-hand side. We can apply the classical necessary conditions of optimal control theory, cf. Hestenes, Ref. 10. If S satisfies the constraint qualification (36), there exist continuous and continuously differentiable adjoint variables λ_j , $j = 1,2,3$, and Lagrange-multpliers $\nu_0 \in \{0,1\}$, $\nu \in \mathbb{R}^\ell$, and $\nu_1,\ \nu_2 \in \mathbb{R}^n$, such that with the Hamiltonian

$$\widetilde{H} := (t_1 - a)\, \lambda_1^{\mathrm{T}} f_1(x_1, u_1) + (t_2 - t_1)\, \lambda_2^{\mathrm{T}} \widehat{f}_2(x_2) + (b - t_2)\, \lambda_3^{\mathrm{T}} f_3(x_3, u_3) \tag{33}$$

and the augmented performance index

$$\Phi := \nu_0\, g(x_3(1)) + \nu^{\mathrm{T}} r(x_1(0), x_3(1)) + \nu_1^{\mathrm{T}}(x_2(0) - x_1(1)) + \nu_2^{\mathrm{T}}(x_3(0) - x_2(1)), \tag{34}$$

the following conditions hold

$$\lambda_1' = -\widetilde{H}_{x_1} = -(t_1 - a)\ \left(\lambda_1^{\mathrm{T}} f_1\right)_{x_1}, \tag{35a}$$

$$\lambda_2' = -\widetilde{H}_{x_2} = -(t_2 - t_1)\ \left(\lambda_2^{\mathrm{T}} \widehat{f}_2\right)_{x_2} \tag{35b}$$

$$\lambda_3' = -\widetilde{H}_{x_3} = -(b - t_2)\, (\lambda_3^{\mathrm{T}} f_3)_{x_3} \tag{35c}$$

$$\lambda_4' = -\widetilde{H}_{t_1} = -\lambda_1^{\mathrm{T}} f_1 + \lambda_2^{\mathrm{T}} \widehat{f}_2, \tag{35d}$$

$$\lambda_5' = -\widetilde{H}_{t_2} = -\lambda_2^{\mathrm{T}} \widehat{f}_2 + \lambda_3^{\mathrm{T}} f_3, \tag{35e}$$

$$u_j(\tau) \;=\; \mathrm{argmin}\{\lambda_j(\tau)^{\mathrm{T}} f_j(x_j(\tau),u): \;\; u \in U\}, \;\; j=1,3, \tag{35f}$$

$$\lambda_1(0) \;=\; -\Phi_{x_1(0)} \;=\; -(\nu^{\mathrm{T}} r)_{x_1(0)}, \quad \lambda_1(1) \;=\; \Phi_{x_1(1)} \;=\; -\nu_1, \tag{35g}$$

$$\lambda_2(0) \;=\; -\Phi_{x_2(0)} \;=\; -\nu_1, \quad \lambda_2(1) \;=\; \Phi_{x_2(1)} \;=\; -\nu_2, \tag{35h}$$

$$\lambda_3(0) \;=\; -\Phi_{x_3(0)} \;=\; -\nu_2, \quad \lambda_3(1) \;=\; \Phi_{x_3(1)} \;=\; (\ell_0\, g + \nu^{\mathrm{T}} r)_{x_3(1)}, \tag{35i}$$

$$\lambda_4(0) \;=\; \lambda_4(1) \;=\; \lambda_5(0) \;=\; \lambda_5(1) \;=\; 0. \tag{35j}$$

Due to the autonomy of the optimal control problem, all three parts $\lambda_1^{\mathrm{T}} f_1$, $\lambda_2^{\mathrm{T}} \widehat{f_2}$, and $\lambda_3^{\mathrm{T}} f_3$ of the Hamiltonian are constant. Therefore, the adjoints λ_4 and λ_5 vanish and we obtain the global continuity of the augmented Hamiltonian (33).

If one recombines the adjoints

$$\lambda(t) \;:=\; \begin{cases} \lambda_1\big(\dfrac{t-a}{t_1-a}\big), & t \,\in\, [a,t_1[, \\ \lambda_2\big(\dfrac{t-t_1}{t_2-t_1}\big), & t \,\in\, [t_1,t_2], \\ \lambda_3\big(\dfrac{t-t_2}{b-t_2}\big), & t \,\in\,]t_2,b], \end{cases} \tag{36}$$

the state and control variables accordingly, one obtains all the necessary conditions of the Theorem.

□

Again, we mention that the results of Theorem 2.1. easily can be extended to nonautonomous nonsmooth optimal control problems and to optimal control problems with performance index of Bolza type, as well.

Example (2.1) Again, we consider the example of Clark, Ref. 3, but now we try to find solutions which contain a singular state subarc. The optimal control problem is given as follows.

Minimize the functional

$$I(u) \;=\; \frac{1}{2} \int_0^2 u(t)^2\, dt \tag{37}$$

with respect to the state equation

$$x'(t) \;=\; \begin{cases} a\,(u-x), & \text{if} \quad S = x-u \le 0, \\ b\,(u-x), & \text{if} \quad S = x-u \,>\, 0, \end{cases} \tag{38}$$

and the boundary conditions

$$x(0) \;=\; 4, \qquad x(2) \;=\; 3. \tag{39}$$

If we assume that there is exactly on singular state subarc, or more precisely

$$S[t] \begin{cases} > 0, & \text{if} \quad 0 \le t < t_1, \\ = 0, & \text{if} \quad t_1 \le t \le t_2, \\ < 0, & \text{if} \quad t_2 < t \le 2, \end{cases} \tag{40}$$

we obtain the following necessary conditions due to Theorem 2.1.

(i) $t \in [0, t_1]$:
$$H = H_3 = \frac{1}{2}u^2 + b\,\lambda\,(u - x),$$
$$\lambda' = b\,\lambda, \quad u = -b\,\lambda.$$

(ii) $t \in [t_1, t_2]$:
$$H = H_2 = \frac{1}{2}x^2,$$
$$\lambda' = -x, \quad u = V(x) = x.$$

(iii) $t \in [t_2, 2]$:
$$H = H_1 = \frac{1}{2}u^2 + a\,\lambda\,(u - x),$$
$$\lambda' = a\,\lambda, \quad u = -a\,\lambda.$$

The continuity of the Hamiltonian yields with

$$\begin{aligned} H[t_1^-] &= \frac{1}{2}b^2\,\lambda(t_1)^2 + b\,\lambda(t_1)\,(-b\,\lambda(t_1) - x(t_1)) \\ &= -\frac{1}{2}b\,\lambda(t_1)\,(b\,\lambda(t_1) + 2\,x(t_1)) \\ H[t_1^+] &= \frac{1}{2}x(t_1)^2. \end{aligned}$$

the interior boundary condition $x(t_1) + b\,\lambda(t_1) = 0$. The analogous condition holds at the second switching point t_2

Altogether we obtain the following *multipoint boundary value problem.*

$$\begin{aligned} x' &= \begin{cases} -b\,(b\,\lambda + x) & : \ t \in [0, t_1], \\ 0 & : \ t \in [t_1, t_2], \\ -a\,(a\,\lambda + x) & : \ t \in [t_2, 2], \end{cases} \\ \lambda' &= \begin{cases} b\,\lambda & : \ t \in [0, t_1], \\ -x & : \ t \in [t_1, t_2], \\ a\,\lambda & : \ t \in [t_2, 2], \end{cases} \\ x(0) &= 4, \qquad x(2) = 3, \\ x(t_1) &+ b\,\lambda(t_1) = 0, \quad x(t_2) + a\,\lambda(t_2) = 0. \end{aligned} \tag{41}$$

For the parameters $a = 2$, and $b = 4$ the numerical solution is shown in Figure 4. One observes the singular subarc with the switching points $t_1 \doteq 0.632117$, $t_2 \doteq 0.882117$.

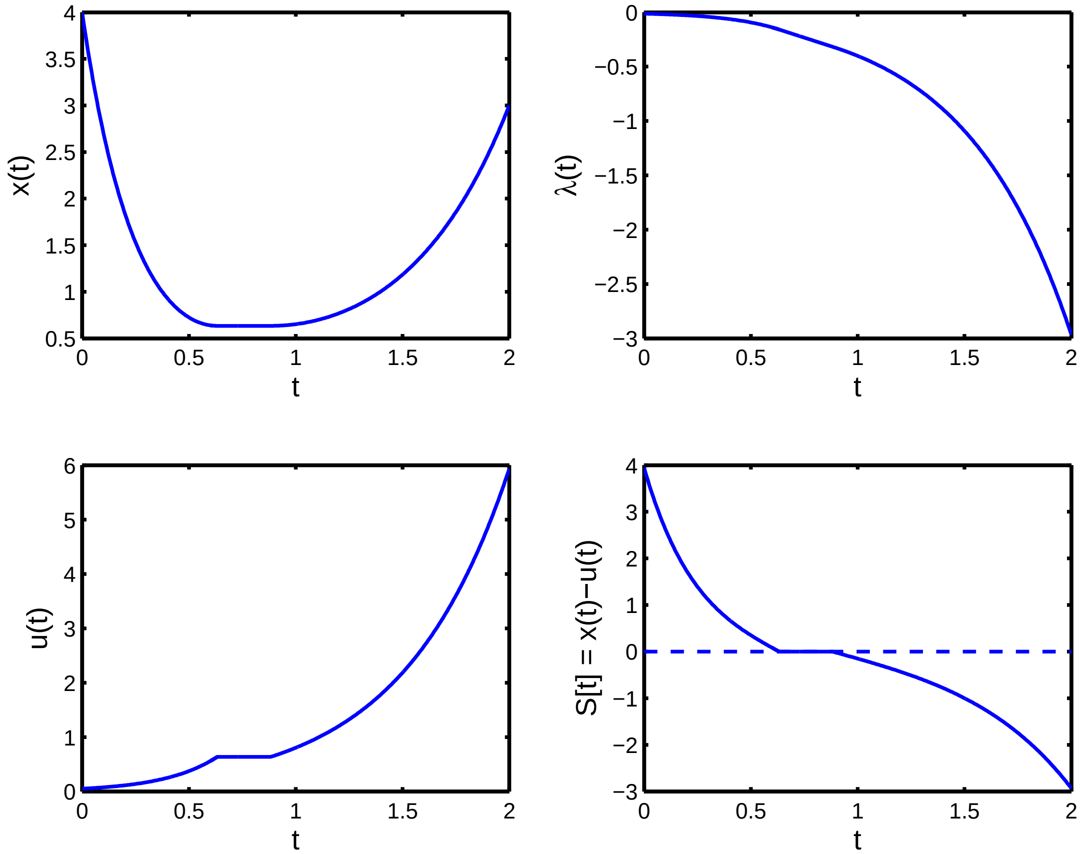

Fig. 4 Example 2.1: Nonsmooth and Singular Case, $a = 2$, $b = 4$.

3. Conclusions

In this paper optimal control problems with nonsmooth state differential equations are considered. Two solution typs are distinguished. In the first part of the paper regular solutions have been considered. The regularity is characterized by the assumption that the switching function changes sign only at isolated points. In the second part so called singular state subarcs are admitted. These are nontrivial subarcs, where the switching function vanishes identically. For both situations necessary conditions are derived from the classical (smooth) optimal control theory.

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