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Dedicated to the memory of Walter Hengartner

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SHIFT GENERATED HAAR SPACES ON TRACK FIELDS

GERHARD OPFER*

Dedicated to the memory of Walter Hengartner[1]

Abstract. The general aim is to show that $G(z) := 1/z^2$ is never a universal Haar space generator for all compact sets K in $\mathbb{C}$. For many cases that was already shown in papers by HENGARTNER & OPFER [5, 2002], [6, 2005]. The remaining cases are those for which K is convex (different from ellipses) and $K = \overline{K^\circ}$, where K° is the interior of K and $\overline{K^\circ}$ is the closure of K° and where the boundary of K is smooth. We show for several cases of compact, convex sets that G is not a 2-dimensional Haar space generator for K implying that it is not a universal Haar space generator for K. We will be guided by a model of a track field: a rectangle with two half disks attached on two opposite sides of the rectangle. We also show, that the above G is not a 3-dimensional Haar space generator for all regular polygons (with smoothed vertices). The definition of Haar spaces and Haar space generators will be given in the main text. The paper contains as a byproduct an overview of the joint work of Walter Hengartner and the present author.

Ամփոփում: Հոդվածի ընդհանուր նպատակն է ապացուցել, որ $G(z) := 1/z^2$ ֆունկցիան չի կարող հանդիսանալ ունիվերսալ Հաարի փարածության ծնիչ բոլոր կոմպակտ K բազմությունների համար $\mathbb{C}$-ում: Շատ դեպքերում դա արդեն ցույց էր տրված Հենգարտների և Օփֆերի [5, 2002], [6, 2005] հոդվածներում: Բաց են մնացել հետևյալ դեպքերը, երբ K-ն էլիպսից տարբեր ուռուցիկ բազմություն է, կամ էլ, երբ K-ն ողորկ եզրով, $K = \overline{K^\circ}$ պայմանին բավարարող բազմություն է, որտեղ $\overline{K^\circ}$-ն K-ի ներքին կետերի բազմության փակումն է: ՈՒռուցիկ կոմպակտ բազմությունների մի քանի դեպքերի համար ցույց ենք տալիս, որ G-ն K-ի Հաարի երկու չափանի, և հետևաբար նաև, ունիվերսալ տարածության ծնիչ չէ: Առաջնորդվել ենք մարզադաշտի օրինակով՝ ուղղանկյուն, որի երկու հանդիպակաց կողմերին կցված են կիսաշրջաններ: Մենք ցույց ենք տալիս նաև, որ վերը նշված G-ն Հաարի երեք չափանի տարածության ծնիչ չէ բոլոր կանոնավոր բազմանիստերի համար (ողորկացված անկյուններով): Հաարի տարածությունների և նրանց ծնիչների սահմանումները տրված են հիմնական տեքստում: Հոդվածը պարունակում է հեղինակի և Վալթեր Հենգարտների համատեղ աշխատանքի հակիրճ շարադրանքը:

Key words. Complex Haar spaces, complex approximation, shift generated Haar spaces on convex sets, universal Haar space generators, admissible convex sets, Haar spaces on polygons

AMS subject classifications. 30C15, 30E10, 41A50, 41A52

1. Introduction. We will start by explaining what a Haar space is and shall mention some of its properties. Let $\mathbb{C}$ denote the field of all complex numbers and let $K \subset \mathbb{C}$ be a non empty, compact subset of $\mathbb{C}$ and $X :=$

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[1] An obituary including a photograph has been published by Bshouty & Fournier [1].

C(K) the space of all continuous, complex valued functions equipped with the uniform norm $||f|| := \max_{z\in K} |f(z)|$.

DEFINITION 1.1. With the above terminology let $n \in \mathbb{N} := \{1, 2, \ldots\}$ be fixed and

$$t_j, t_k \in K,\ t_j \neq t_k,\ j \neq k \text{ and } \eta_j \in \mathbb{C},\ j, k = 1, 2, \ldots, n. \tag{1.1}$$

Any n-dimensional linear subspace V of C(K) will be called a *Haar*[2] *space for* K if the interpolation problem

$$h(t_j) = \eta_j, \quad j = 1, 2, \ldots, n$$

has a unique solution $h \in V$.

Let $W \subset X$ be a non empty but otherwise arbitrary set and $f \in X$. An *approximation problem* consists in finding all $\hat{w} \in W$ with $\hat{w} \in P_W(f)$ where

$$P_W(f) := \{\hat{w} : ||f - \hat{w}|| = \inf_{w\in W} ||f - w||\}. \tag{1.2}$$

The set $P_W(f)$ may be empty or contain several elements. All elements $\hat{w} \in P_W(f)$ will be called *best (uniform) approximations of* f *with respect to* W. We are mainly interested in the case where $P_W(f)$ contains exactly one element for all f, which means that the approximation problem is uniquely solvable for all $f \in X$. In this case $P_W : X \to W$ is a mapping, called a *projection* or a *projection map*. The approximation problem will be called *linear* if W is a linear subspace of X with finite dimension. The importance of Haar spaces is expressed in the following theorem.

THEOREM 1.2. *Let $V \subset X :=$ C(K) be an n-dimensional linear subspace of X. Then the following statements are equivalent.*

1. *V is a Haar space for K.*
2. *Let $h_1, h_2, \ldots, h_n$ be a basis for V. Then the matrix*

$$\mathbf{M} := (h_j(t_k)), \quad j, k = 1, 2, \ldots, n$$

   is non singular for all choices of $t_k \in K$, $k = 1, 2, \ldots, n$ which obey (1.1).
3. *All elements $v \in V\backslash\{0\}$ have at most $n - 1$ zeros in K.*
4. *Each element $f \in X$ has a unique best (uniform) approximation in V.*

[2] Alfred Haar, Hungarian mathematician, 1885–1933.

Proof: MEINARDUS[7, 1967, p. 1–19], HAAR[4, 1918]. $\square$

EXAMPLE 1.3. (a) Let $K \subset \mathbb{C}$ and let Π_{n-1} be the n-dimensional linear space of all polynomials of degree at most $n-1$ with complex coefficients. Then, Π_{n-1} is a Haar space of dimension n for all K with sufficiently many points.
(b) Let $K \subset \mathbb{C}$ and $0 \in K$. Then

$$V := \langle z, z^3, \ldots, z^{2n-1} \rangle$$

is an n-dimensional linear space which is not a Haar space for K. By $\langle \ldots \rangle$ we understand the linear hull (also called *the span*) of the elements ... between the brackets.

2. Shift generated Haar spaces. Let $K \subset \mathbb{C}$ be non empty and compact and $t_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$ be mutually distinct.

EXAMPLE 2.1. In this example we define two shift generated linear spaces of dimension n:

$$V := \langle \frac{1}{z-t_1}, \frac{1}{z-t_2}, \ldots, \frac{1}{z-t_n} \rangle, \quad z \in K,$$
$$W := \langle \exp(-(z-t_1)^2), \exp(-(z-t_2)^2), \ldots, \exp(-(z-t_n)^2) \rangle, \quad z \in K.$$

It is easy to see that V is a Haar space if we choose all $t_j \notin K$, regardless of the definition of K. The only restriction on K is that it contains sufficiently many points. The second space W is a Haar space in case $K \subset \mathbb{R}$ and it is in general not a Haar space if $K \subset \mathbb{C}$. This is implied by the periodicity $\exp(z) = \exp(z + 2k\pi\,\mathrm{i}\,)$ for all $z \in \mathbb{C}$ and all $k \in \mathbb{Z}$. Both spaces, V, W are shift generated, V by $G(z) := \frac{1}{z}$ and W by $G(z) := \exp(-z^2)$ in the sense that they coincide with

$$V_n := \langle G(z-t_1), G(z-t_2), \ldots, G(z-t_n) \rangle, \quad z \in K, \tag{2.1}$$

where in the case of W the multipliers of the span have to be restricted to $\mathbb{R}$.

The question which functions G generate (real) Haar spaces by applying (2.1) was posed by CHENEY & LIGHT[2, 2000, p. 76].

DEFINITION 2.2. Let $K \subset \mathbb{C}$ be non empty and compact and let $G : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be a function defined on $\mathbb{C}\backslash\{0\}$ with values in $\mathbb{C}$.

1. Let $n \in \mathbb{N}$ be fixed. The function G will be called an *n-dimensional Haar space generator for K* if for each set of n pairwise distinct points

$t_1, t_2, \ldots, t_n \in \mathbb{C}\backslash K$ (i. e. outside of K) the functions h_j defined by $h_j(z) := G(z - t_j)$, $j = 1, 2, \ldots, n$ span an n-dimensional Haar space for K.

2. The function G is called a *universal Haar space generator for* K if G is an n-dimensional Haar space generator for K for all $n \in \mathbb{N}$.

The set of universal Haar space generators is not empty. Take $G(z) := \frac{1}{z}$ and refer to V of Example 2.1. Slightly more general is the following example of a universal Haar space generator G:

$$G(z) := \frac{\exp(az + b)}{z}, \quad a, b \in \mathbb{C}. \tag{2.2}$$

It is easy to show, that the space defined in (2.1) for this G is a Haar space for all n and all K (non empty, compact, sufficiently many points).

3. Shift generated Haar spaces on disks. In our first paper Hengartner and the present author[5, 2002] investigated the case where $K := \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disk and $G \in H(\mathbb{C}\backslash\{0\})$, which means that G is holomorphic on $\mathbb{C}$ with the possible exception of the origin. We also say that G is an *analytic Haar space generator*, tacitly assuming that G is defined on $\mathbb{C}$ with the exception of the origin. We obtained the following main result.

Theorem 3.1. *Let K be the unit disk and $G \in H(\mathbb{C}\backslash\{0\})$. Then, G is a universal Haar space generator if and only if G is of the form (2.2).*

For the proof we proceeded stepwise. First we assumed that G is a one dimensional Haar space generator which is equivalent to the fact that $G(z-t)$ has no zeros in K for all $t \notin K$. Then we assumed that G is a one and two dimensional Haar space generator, etc. In this way we found the following surprising result.

Theorem 3.2. *Let $G \in H(\mathbb{C}\backslash\{0\})$ be an n-dimensional Haar space generator for the unit disk for $n = 1, 2, 3, 4$. Then G is a universal Haar space generator. This result is best possible in the sense that 4 cannot be replaced by a smaller number.*

We have always assumed that G is holomorphic on $\mathbb{C}\backslash\{0\}$. If we would admit entire functions for G (i. e. holomorphic on the whole of $\mathbb{C}$), then our proofs show that we would not find any universal Haar space generator. Thus, the hole in the domain of definition for G is essential.

4. Shift generated Haar spaces on general compact sets. In a second paper HENGARTNER and the present author [6, 2005] investigated the case of a general compact set $K \in \mathbb{C}$. We found the following main result.

THEOREM 4.1. *An analytic universal Haar space generator G for an arbitrary, non empty, compact set K (with sufficiently many points) must be necessarily of one of the two forms:*

$$G(z) := \frac{\exp(az+b)}{z} \quad \textit{or} \tag{4.1}$$

$$G(z) := \frac{\exp(az+b)}{z^2}, \ \textit{where } a, b \in \mathbb{C}. \tag{4.2}$$

By K° we denote the interior of K, by $\overline{K^\circ}$ we denote the closure of K°. In order to prove Theorem 4.1 we had to distinguish between the following two cases:

(i) $K \backslash \overline{K^\circ} \neq \emptyset$,
(ii) $K = \overline{K^\circ}$.

The first case would apply if K° is empty. An example is a *segment* S in $\mathbb{C}$:

$$S := [z_1, z_2] := \{z : z = (1-\lambda)z_1 + \lambda z_2,\ \lambda \in [0,1]\}.$$

We have found an important additional information for the case (4.2) of Theorem 4.1.

THEOREM 4.2. *An analytic universal Haar space generator G of the form (4.2) in Theorem 4.1 is possible only under the following additional conditions for K:*

(i) $K = \overline{K^\circ}$,
(ii) *K is convex,*
(iii) *the boundary ∂K of K has no corner,*
(iv) *K is not an ellipse (including disks).*

Actually, the authors conjectured that case (4.2) will never happen. It was already shown by HENGARTNER and the present author [6, 2005, Lemma 1.6, part 1.] that case (4.2) can be reduced to the simpler case

$$G(z) := \frac{1}{z^2}, \quad z \neq 0. \tag{4.3}$$

CONJECTURE 4.3. *Let K have the properties mentioned in Theorem 4.2. Then G defined by $G(z) := \frac{1}{z^2}$ is never a universal Haar space generator.*

It should be repeated that the conjecture is true in case K does not have the properties mentioned in Theorem 4.2. In order to prove the conjecture it is sufficient to prove, that G is not an n-dimensional Haar space generator for one specific $n > 1$ since G is always a one dimensional Haar space generator (G is non vanishing for all $z \neq 0$). So it might be of interest to study some special cases, e. g. a track field.

5. Track fields. A track field is a sort of oval which in our model will consist of a rectangle adjoined by two halfdisks. For two given positive reals $c, d \in \mathbb{R}$ let

$$R(c,d) := \{z \in \mathbb{C} : |\Re z| \leq c, |\Im z| \leq d\}$$

be a rectangle, where $\Re, \Im$ stand for real part, imaginary part, respectively. Now define two halfdisks

$$D_l := \{z \in \mathbb{C} : |z + c| \leq d,\ \Re z \leq -c\}, \quad D_r := \{z \in \mathbb{C} : |z - c| \leq d,\ \Re z \geq c\}.$$

Then, a *track field* is defined by

$$T(c,d) := R(c,d) \cup D_l \cup D_r, \quad c > 0,\ d > 0. \tag{5.1}$$

Apparently, $T(c,d)$ is the closure of its interior points, is convex, is not an ellipse and the boundary has no corner. All conditions of Theorem 4.2 are satisfied. One example of a track field is shown in Figure 5.1.

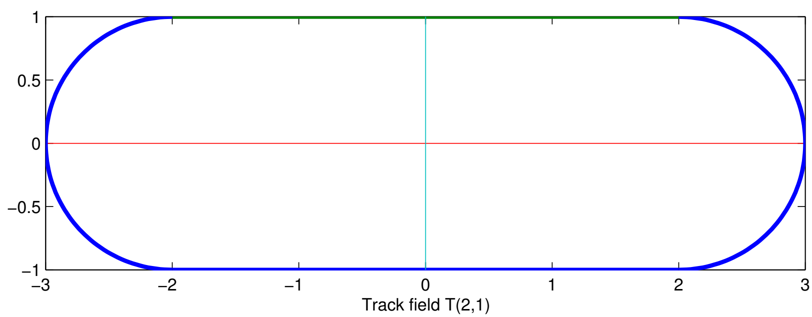

FIGURE 5.1. Example of track field for $c = 2,\ d = 1$

Instead of putting the half disks on the left and right side of the rectangle $R(c,d)$ we could have put half disks on top and on the bottom of the rectangle. However, n-dimensional Haar space generators G of the form (4.3) are invariant under transformations of K of the type $\alpha K + \beta$ where $\alpha \in \mathbb{C}\backslash\{0\}$ and $\beta \in \mathbb{C}$, [6, Lemma 1.6, part 2]. We want to show that G, defined in (4.3) is not a universal Haar space generator for $T := T(c, d)$. For this it is sufficient to show that G is not a 2-dimensional Haar space generator. We repeat three lemmata (Lemma 2.3, Lemma 2.4, Lemma 4.10) from [6] which are valid for all compact sets K.

LEMMA 5.2. *Let G defined by $G(z) := \dfrac{1}{z^2}$, $s, t \notin K, s \neq t$ and $v_1(z) := G(z-s), v_2(z) := G(z-t)$. Then $V := \langle v_1, v_2 \rangle$ is a Haar space of dimension two if and only if*

$$\mu(z; s, t) := \frac{G(z-s)}{G(z-t)} = \Big(\frac{z-t}{z-s}\Big)^2, \quad z \in K, \quad s, t \in \mathbb{C}\backslash K, s \neq t \tag{5.2}$$

is injective on K which means $\mu(u; s, t) = \mu(v; s, t)$ implies $u = v$.

LEMMA 5.3. *Let $G(z) := \dfrac{1}{z^2}$ be a two dimensional Haar space generator for any compact K. Then the function*

$$F(t; u, v) := \Big(\frac{t-u}{t-v}\Big)^2, \quad t \in \mathbb{C}\backslash K \quad u, v \in K, u \neq v \tag{5.3}$$

is injective in $\mathbb{C}\backslash K$.

LEMMA 5.4. *Let $K \subset \mathbb{C}$ be a compact set containing the two distinct points $z_1, z_2 \in K$. If both points $u_{1,2} := 0.5(z_1 + z_2 \pm \mathrm{i}\,(z_1 - z_2))$ do not belong to K, then $G(z) = 1/z^2$ is not a two dimensional Haar space generator for K.*

This lemma is good enough to solve the track field problem.

THEOREM 5.5. *Let $K := T(c, d)$ be a given track field, defined in(5.1). Then, $G(z) := 1/z^2$ is not a 2-dimensional Haar space generator for $T(c, d)$.*

Proof: Define $z_{1,2} := \pm(c + d)$. Then $u_{1,2} := 0.5(z_1 + z_2 \pm \mathrm{i}\,(z_1 - z_2)) = \pm\,\mathrm{i}\,(c + d)$ and $z_{1,2} \in K$, $u_{1,2} \notin K$. Lemma 5.4 proves the theorem. □

We see that in the limit case $c = 0$ the track field $T(c, d)$ degenerates to a disk with radius d and the above proof would not work. In this case, $G(z) := 1/z^2$ is indeed a 2-dimensional Haar space generator for the disk. See [6, Lemma 4.12]. This is another proof for the fact that the dimension n is not continuous with respect to the monotone convergence of compact sets ([6, Lemma 1.6, part 5]). But we have also shown in [5, proof of Lemma 19], that $G(z) := 1/z^2$ is not a 3-dimensional Haar space generator for any disk with positive radius.

6. Admissible convex sets. The proof for the above case (Theorem 5.5) can be transferred to all convex sets for which Lemma 5.4 is applicable. In that lemma two points $u_{1,2} := 0.5(z_1+z_2 \pm \mathrm{i}\,(z_1-z_2))$ are computed from two given, distinct points $z_1, z_2 \in K$. Define the two segments $S_1 := [z_1, z_2]$, $S_2 := [u_1, u_2]$. It is easy to see that $u_1 - u_2 = \mathrm{i}\,(z_1 - z_2)$ and $(u_1+u_2)/2 = (z_1+z_2)/2$. For the segments that means that they are diagonals of a square. Let us denote this square by $Q(z_1, z_2)$. It is that square whose one diagonal is the segment $S_1 := [z_1, z_2]$. Lemma 5.4 now says that $G(z) = 1/z^2$ is not a 2-dimensional Haar space generator for a compact, convex set K if there are two distinct points $z_1, z_2 \in K$ such that the other two corners of the square $Q(z_1, z_2)$ are outside of K.

Definition 6.1. Let $K \subset \mathbb{C}$ be a non empty convex set. We shall call K *admissible* if there are two points $z_1, z_2 \in K$ such that the square $Q(z_1, z_2)$ defined above has the property that the two other corners of $Q(z_1, z_2)$ are outside of K.

An example of an admissible set (an ellipse) is sketched in Figure 6.4. Let Δ be the regular (equilateral) triangle with the corners $(-1, 0), (1, 0), (0, \sqrt{3})$. Then, Δ is admissible. To see this, choose $z_1 := 0 \in \Delta$, $z_2 = \sqrt{3}\,\mathrm{i} \in \Delta$. Then, $u_1 = 0.5\sqrt{3}(-1 + \mathrm{i}\,)$, $u_2 = 0.5\sqrt{3}(1 + \mathrm{i}\,)$ are both outside of Δ.

Theorem 6.2. *Let P_n be a regular polygon with $n \geq 3$ vertices. (i) If n is odd, then P_n is admissible. (ii) If n is even, then (a) P_n is admissible if n is not divisible by four. (b) Otherwise (i. e. n is divisible by four) P_n is not admissible.*

Proof: Assume that the vertices of P_n are represented by $v_j := \exp(\frac{2j\pi\,\mathrm{i}}{n})$, $j = 0, 1, \ldots, n-1$. Let n be odd and let $n_1 := (n-1)/2$, $n_2 := (n+1)/2$. Then $n_2 - n_1 = 1$. Let $z_1 := v_0$, and $z_2 := 0.5(v_{n_1} + v_{n_2})$. Clearly, $z_1 \in P_n$ and the convexity of P_n implies $z_2 \in P_n$. The two other corners of $Q(z_1, z_2)$ are $u_{1,2} := 0.5(z_1 + z_2 \pm \mathrm{i}\,(z_1 - z_2))$ and they are outside of P_n. Let n be even. We use mainly the same construction. Let $z_1 := v_0, z_2 := v_{n/2}$. Since z_1, z_2 are both vertices located on the x-axis, we have $u_{1,2} = \pm\mathrm{i}$. In case n is divisible by four, u_1 coincides with vertex $v_{n/4}$ and u_2 coincides with vertex $v_{n/4+n/2}$. In case n is not divisible by four, the two points u_1, u_2 are different from vertices and are therefore necessarily outside of P_n. $\square$

The general theory says that convex sets K with corners never have $G(z) := 1/z^2$ as universal Haar space generators. So polygons are actually not of interest. However, we may think of slightly perturbed polygons where the corners have been smoothed.

The problem could be solved if one could prove the following conjecture.

CONJECTURE 6.3. *Let $K \subset \mathbb{C}$ be a compact, convex set with the following properties:*

(i) $K = \overline{K^\circ}$,

(ii) *the boundary ∂K of K is smooth,*

(iii) *K is not admissible.*

Then, $G(z) := 1/z^2$ is not a 3-dimensional Haar space generator for K and thus, not a universal Haar space generator for K.

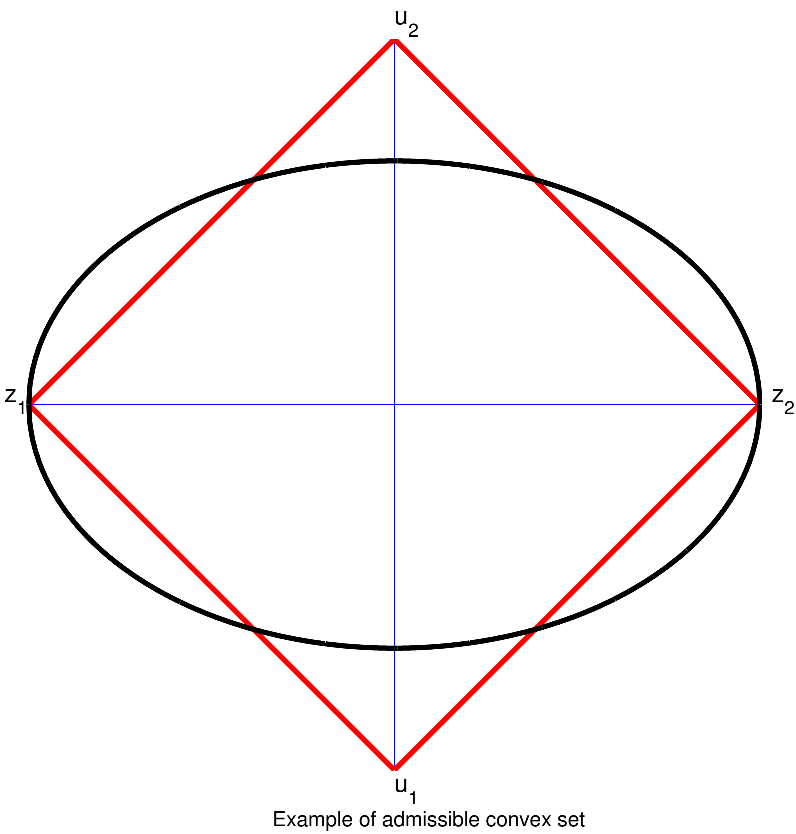

FIGURE 6.4. Example of an admissible set (ellipse)

Examples different from a disk, satisfying (i) to (iii) are according to our

Theorem 6.2 regular polygons with $n = 4k, k = 1, 2, \ldots$ vertices with (slightly) rounded corners. We will show that $G(z) := 1/z^2$ is not a 3-dimensional Haar space generator for all such regular polygons. We start with the following theorem.

THEOREM 6.5. *Let $Q = \{z : |\Re z| \le 1, |\Im z| \le 1\}$ be a square and $\tilde{Q}$ the same square with rounded corners (e. g. by using small circular arcs near the corners). Then $G(z) := 1/z^2$ is not a 3-dimensional Haar space generator for $\tilde{Q}$ and thus, not a universal Haar space generator for $\tilde{Q}$.*

Proof: We show that $V_3 := \langle G(z - t_1), G(z - t_2), G(z - t_3)\rangle$ is not a Haar space for suitable shifts t_1, t_2, t_3. We take $t_2 := \exp(-2\pi\,\mathrm{i}/3)t_1 =: at_1, t_3 := \exp(2\pi\,\mathrm{i}/3)t_1 =: bt_1$ and leave t_1 as a real parameter, to be suitably adjusted. One element in V_3 is

$$v(z) := \frac{1}{(z-t_1)^2} + \frac{a^2}{(z-t_2)^2} + \frac{b^2}{(z-t_3)^2} = \frac{1}{(z-t_1)^2} + \frac{a^2}{(z-at_1)^2} + \frac{b^2}{(z-bt_1)^2}.$$

Since $ab = 1$ we have

$$v(z) = \frac{1}{(z-t_1)^2} + \frac{1}{(bz-t_1)^2} + \frac{1}{(az-t_1)^2}.$$

Now, $v(z) = 0$ if and only if $z \in \{z_1, z_2, z_3\}$ where $z_j^3 = -1/2t_1^3$, $j = 1, 2, 3$. If we choose $t_1 = 1.2$, then $t_{2,3} = -0.6 \pm 1.0392\,\mathrm{i}$. Thus, $t_{1,2,3} \notin \tilde{Q}$. For the zeros we obtain $z_1 := -0.9524, z_{2,3} = 0.4762 \pm 0.8248\,\mathrm{i}$ which are all in $\tilde{Q}$. $\square$

It should be observed that the above proof will work also for values of t_1 which are slightly different from the given value 1.2. For $t_1 \in [1.16, 1.26]$ the proof still works, but for $t_1 \le 1.15$ and $t \ge 1.27$ the proof does not work. Nevertheless, the idea of the proof is good enough to settle the problem for all regular polygons. It should be noted that this proof is adapted from [5, Proof of Lemma 19].

THEOREM 6.6. *Let P_n be a regular polygon with $n \ge 3$ vertices and $\tilde{P}_n$ the same polygon with slightly rounded vertices. Then, G defined by $G(z) = 1/z^2$ is not a universal Haar space generator for $\tilde{P}_n$.*

Proof: We only need to show, that $G(z) = 1/z^2$ is not a 3-dimensional Haar space generator for $\tilde{P}_{4k}$, $k \ge 2$. The case $\tilde{P}_4$ was already settled in Theorem 6.5. All other cases are settled by Theorem 6.2. We use the same proof as for Theorem 6.5 and assume that the vertices have the form $v_j := \sqrt{2}\exp\left(\frac{2j\pi\,\mathrm{i}}{4k}\right)\exp\left(\frac{2\pi\,\mathrm{i}}{8}\right)$, $j = 0, 1, \ldots, 4k-1$. This form guarantees that the vertices of the square Q defined in Theorem 6.5 are included in that definition of the vertices. Also note that all polygons are included in the centered disk of radius $\sqrt{2}$. Now we refer to the proof of Theorem 6.5 and put $t_1 := 3/2$. Then $t_{2,3} = 3/4(-1 \pm \sqrt{3}\,\mathrm{i}\,)$ and $|t_{2,3}| = 3/2$. Hence, all shifts $t_{1,2,3}$ are outside the disk of radius $\sqrt{2}$ and thus, outside of all $\tilde{P}_n$. The zeros $z_1 = -3 \cdot 2^{-4/3} = -1.1906$, $z_{2,3} = 0.5953 \pm 1.0310\,\mathrm{i}$ are inside $\tilde{P}_8$ and inside $\tilde{P}_{12}$. We have $P_4 \subset P_{12} \subset P_{20} \cdots$ and $P_8 \subset P_{16} \subset P_{24} \cdots$ and therefore, the zeros are all inside of P_{4k}, for all $k \geq 2$. □

7. Extension to unbounded sets. It is interesting that even for non compact sets in $\mathbb{C}$ some analogue results can be derived. However, the class of continuous functions has to be restricted such that the uniform norm is still (finitely) defined. The results of this section are by Maude Giasson, Walter Hengartner and the present author, [3, 2003]. Let $F \subset \mathbb{C}$ be non empty and closed. We will restrict the continuous functions to the cases where

$$||f||_F := \sup_{z \in F} |f(z)|$$

is still finite. For this purpose it is sufficient to require that

$$\lim_{z \in F, z \to \infty} f(z) = 0. \tag{7.1}$$

We will denote the class of continuous functions for which (7.1) is valid by $\mathrm{C}^0(F)$. There is the following basic theorem.

Theorem 7.1. *Let $F \subset \mathbb{C}$ be non empty and closed and V an n-dimensional linear subspace of $\mathrm{C}^0(F)$. The approximation problem has a unique solution in V for all $f \in \mathrm{C}^0(F)$ if and only if V is a Haar space for F.*

Proof: [3, Theorem 1.2]. □

It should be noted that the above theorem is not valid for the larger class of continuous and bounded functions. There is a counterexample in [3, Example 1.3].

Theorem 7.2. *Let $F \subset \mathbb{C}$ be non empty and closed and*

$$G(z) := \frac{\mathrm{e}^{az+b}}{z} \text{ and } ||\mathrm{e}^{az}||_F < \infty \quad a, b \in \mathbb{C}, z \neq 0.$$

Then $G \in H(\mathbb{C}\backslash\{0\})$ and G is a universal Haar space generator for F.

Proof: [3, Example 1.5]. □

The proof is actually straightforward, and depending on F there is a table [3, Table 1] showing the actual restriction of a induced by $||\mathrm{e}^{az}||_F < \infty$. To mention two examples, let $F := \mathbb{R}_+$. Then $\Re(a) \le 0$. In case $F := \{z : |z| \ge R\}$ for a positive R, we have $a = 0$.

There are several theorems in which universal Haar space generators are characterized for closed, but unbounded sets. Let e. g. F contain $\{z : |z| \ge R\}$, then under some additional conditions on F a universal analytic Haar space generator must necessarily have the form $G(z) = 1/z$ ([3, Theorem 4.5]).

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REFERENCES

[1] D. Bshouty & R. Fournier: Editorial note (includes photograph), Computational Methods and Function Theory **4** (2004), pp. i–ix.

[2] E. W. Cheney & W. Light: A course in approximation theory, Brooks/Cole, Pacific Grove, 2000, 359 p.

[3] M. Giasson, W. Hengartner & G. Opfer: *Shift generated Haar spaces on unbounded closed domains in the complex plane* . Computational Methods and Function Theory **3** (2003), pp. 151–164. Hamburger Beiträge zur Angewandten Mathematik, Serie A, Preprint 175, August 2003, 14 p.

[4] A. Haar: *Die Minkowskische Geometrie und die Annäherung an stetige Funktionen.* Math. Ann. **78** (1918), pp. 294–311.

[5] W. Hengartner & G. Opfer: *Complex Haar Spaces Generated by Shifts Applied to a Single Function.* In: C. K. Chui, L. L. Schumaker, J. Stöckler (eds), Approximation Theory X: Abstract and Classical Analysis, Vanderbilt University Press, Nashville, TN, 2002, pp. 223–238. Hamburger Beiträge zur Angewandten Mathematik, Serie A, Preprint 167, November 2001, 14 p.

[6] W. Hengartner & G. Opfer: *Shift generated Haar spaces on compact domains in the complex plane*. Constr. Approx. **22** (2005), pp. 113–132. Hamburger Beiträge zur Angewandten Mathematik, Serie A, Preprint 174, July 2003, 18 p.

[7] G. Meinardus: Approximation of Functions: Theory and Numerical Methods. Springer, Berlin, Heidelberg, New York, 1967, 198 p.