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Numerical Computation of a Singular State Subarc in an Economic Optimal Control Model

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Abstract. In this paper optimal control problems are considered which are characterized by a nonsmooth state differential equation. More precisely, it is assumed that the right–hand side of the state equation is piecewise smooth and that the junction points between smooth subarcs are determined as roots of a state-dependent switching function. For this kind of optimal control problems necessary conditions are developed. Special attention is payed to the situation that the switching function vanishes identically along a nontrivial subarc. Such subarcs, which are called singular state subarcs, are investigated with respect to the necessary conditions and to the junction conditions. In this paper we assume that the switching function is of first order with respect to the control. It turns out that the necessary conditions are similar to those for first order state-constrained optimal control problems.

The theory is applied to an economic optimal control model due to Pohmer (1985), which describes the personal income distribution of a typical consumer, who wants to maximize the total utility of his lifetime by controling the consumption, the rate of the total time used for working, and the rate of working time used for education and extended professional training. The state variables are the human capital and the capital itself. The utility function contains different parts which represent the influence of consumption, time of recreation, and human capital. Into this problem a parameter enters which describes the interest rate of capital. It is obvious that this parameter in general will differ for positive and negative values of the capital. Thus, the resulting optimal control problem in a natural way becomes a nonsmooth one. For this problem, the necessary conditions are derived and numerical solutions are presented which are obtained by an indirect optimal control method. It turns out that for a certain distance of the positive and negative interest rate, the optimal solution contains a singular state subarc.

Key Words. Nonsmooth Optimal Control Problems, Singular State Subarcs, Personal Income Distribution,

AMS (MOS) subject classifications. 34B10, 34H05, 49K15, 49N60, 65L10.

1. Introduction

The paper is concerned with general optimal control problems (OCP) which are characterized by a nonsmooth ordinary state differential equation. More precisely, we assume that the right-hand side of the state equation is piecewise smooth and that the switching points, which separate these pieces, are determined as roots of a state-dependent (smooth) switching function.

Nonsmooth optimal control problems of this type rarely have been mentioned in the literatur, cf. for example Ref. 15. of course, they are special examples for the rather general theory of Clark, Ref. 9. Such problems sometimes occur in applications. So, for example, in aeronautical applications the trajectories may be divided into inner– and outer atmospheric parts. This may result in an discontinuous modelling of the atmospheric density depending on a certain value of the altitude, cf. Ref.1-2, 8. However, for a proper modelling of this problem the discontinuity of the atmospheric density will be kept small, such that its influence on the computed optimal control history may be neglected.

In other applications discontinuities of state equations may have more severe effects on the numerical solution. As an example we consider an economic model for the optimal personal income distribution. The model is due to Pohmer (Ref. 4, 12, 17) and it is given in form of a deterministic optimal control problem with two state variables (human capital H, and capital K) and three controls variables which describe the consumption and the total time allocation in time for working, education and recreation. Into this problem a parameter i enters which describes the interest of capital and which takes two different values depending on the sign of the capital. So, the optimal control problem in fact has a discontinuous state equation with state-dependent points of discontinuity. For a certain difference between this two parameter values the solution contains a subarc where the switching function - here this is simply the state K - vanishes identically. Such a subarc plays a similar role as a singular control subarc and it is therefore called a singular state subarc. We investigate the necessary conditions on this subarc and the junction conditions which have to be satisfied at the junction points between regular and singular subarcs. Further, numerical solutions obtained by an indirect multiple shooting method are presented.

The paper is organized as follows: In the first part we consider a general nonsmooth optimal control problem and derive corresponding necessary conditions in form of a multipoint boundary value problem. To this end, we assume that the switching function along the solution trajecory only has isolated roots (regularity assumption). We apply the these necessary conditions to the economic model of Pohmer. For small differences between the two values for the interest rate the regularity assumption holds and the theory leads to a well-defined multipoint boundary value problem which is solved numerically. For larger differences the regularity assumption fails. Therefore, we develop necessary conditions for solutions with singular state subarcs and apply these results to the economic model.

Finally, we interprete a certain limit situation (characterized by a maximal singular subarc) as the solution of the optimal control problem with an additional state variable inequality constraint.

2. Nonsmooth Optimal Control Problems, Regular Case.

We consider a general OCP with a piecewise defined state differential equation. The problem has the following form.

Problem (P1). Determine a piecewise continuous control function $u : [a,b] \to \mathbb{R}^m$, such that the functional

$$I = g(x(b)) \tag{1}$$

is minimized subject to the following constraints (state equations, boundary conditions, and control constraints)

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [a,b] \quad \text{a.e.}, \tag{2a}$$

$$r(x(a), x(b)) = 0, \tag{2b}$$

$$u(t) \in U \subset \mathbb{R}^m. \tag{2c}$$

The control region U is assumed to be a (compact and convex) cuboid of the form $U = \Pi_i [u_{i,\min}, u_{i,\max}]$. Further, the right-hand side of the state equation (2a) may be of the special form

$$f(x,u) = \begin{cases} f_1(x,u), & \text{if } S(x) \le 0, \\ f_2(x,u), & \text{if } S(x) > 0, \end{cases} \tag{3}$$

where the functions $S : \mathbb{R}^n \to \mathbb{R}$, $f_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ $(k = 1,2)$, and $r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^\ell$, $\ell \in \{0, \ldots, 2n\}$, are assumed to be sufficiently smooth. S is called the *switching function* of Problem (P1).

Our aim is to derive necessary conditions for Problem (P1). To this end, let (x^0, u^0) denote a solution of the problem with a piecewise continuous optimal control function u^0.

Further, we assume that the problem is *regular* with respect to the minimum principle, that is: For each $\lambda \in \mathbb{R}^n$ and $x = x^0(t)$ the *Hamiltonian*

$$H(x,u,\lambda) := H_j(x,u,\lambda) := \lambda^{\mathrm{T}} f_j(x,u) \tag{4}$$

has a unique minimum with respect to the control $u \in U$. Here the parameter $j \in \{1,2\}$ is determined depending on the state x such that $S(x) \ge 0$, or $S(x) < 0$, respectively. Note, that at a root of the switching function S both Hamilton-functions H_1, and H_2 are well-defined.

Finally, for this Section, we assume that the following regularity assumption holds.

Regularity Condition (R). The switching function along the optimal trajectory, $S[t] := S(x^0(t))$, vanishes only at a finite number of isolated roots t_j, where $a < t_1 < \ldots < t_s < b$.

Now, we can summarize the necessary conditions for Problem (P1).

Theorem 2.1.

With the assumptions above the following necessary conditions hold.

There exist an adjoint variable $\lambda : [a, b] \to \mathbb{R}^n$ *, which is a piecewise* C^1*–function, and Lagrange multipliers* $\nu_0 \in \{0, 1\}$ *,* $\nu \in \mathbb{R}^\ell$ *,* $\kappa \in \mathbb{R}^s$ *, such that* (x^0, u^0) *satisfies*

$$\dot{\lambda}(t) = -H_x(x^0(t), u^0(t), \lambda(t)), \quad t \in [a, b] \text{ a.e.} \qquad \text{(adjoint equations)}, \tag{5a}$$

$$u^0(t) = \operatorname{argmin}\{H(x^0(t), u, \lambda(t)) : \ u \in U\} \qquad \text{(minimum principle)}, \tag{5b}$$

$$\lambda(a) = -\frac{\partial}{\partial x^0(a)} [\nu^{\mathrm{T}} r(x^0(a), x^0(b))] \quad \text{(natural boundary conditions)}, \tag{5c}$$

$$\lambda(b) = \frac{\partial}{\partial x^0(b)} [\nu_0 \, g(x^0(b)) + \nu^{\mathrm{T}} r(x^0(a), x^0(b))], \tag{5d}$$

$$\lambda(t_j^+) = \lambda(t_j^-) + \kappa_j \, S_x(x^0(t_j)), \quad j = 1, \ldots, s, \qquad \text{(jump condition)}, \tag{5e}$$

$$H[t_j^+] = H[t_j^-], \quad j = 1, \ldots, s, \qquad \text{(continuity condition)}. \tag{5f}$$

Proof. Without loss of generality, we assume, that the switching function $S[\cdot]$ along the optimal trajectory has just *one* isolated root $t_1 \in]a, b[$, i.e. $s = 1$, and that the following *switching structure* holds

$$S[t] \begin{cases} < 0, & \text{if } \ a \le t < t_1 \\ > 0, & \text{if } \ t_1 < t \le b. \end{cases} \tag{6}$$

We compare the optimal solution (x^0, u^0) only with those admissible solutions (x, u) of the problem which have the same switching structure (6). Each candidate of this type can by associated with its separated parts ($\tau \in [0, 1]$)

$$\begin{aligned} x_1(\tau) &:= x(a + \tau(t_1 - a)), & x_2(\tau) &:= x(t_1 + \tau(b - t_1)), \\ u_1(\tau) &:= u(a + \tau(t_1 - a)), & u_2(\tau) &:= u(t_1 + \tau(b - t_1)). \end{aligned} \tag{7}$$

Now, $(x_1, x_2, t_1, u_1, u_2)$ performs an abmissible and $(x_1^0, x_2^0, t_1^0, u_1^0, u_2^0)$ an optimal solution of the following auxillary optimal control problem.

Problem (P1'). Determine a piecewise continuous control function $u = (u_1, u_2) : [0, 1] \to \mathbb{R}^{2m}$, such that the functional

$$I = g(x_2(1)) \tag{8}$$

is minimized subject to the constraints

$$x_1'(\tau) \;=\; (t_1 - a)\, f_1(x_1(\tau), u_1(\tau))\,, \quad \tau \,\in\, [0,1], \quad \text{a.e.}, \tag{9a}$$

$$x_2'(\tau) \;=\; (b - t_1)\, f_2(x_2(\tau), u_2(\tau)), \tag{9b}$$

$$t_1'(\tau) \;=\; 0, \tag{9c}$$

$$r(x_1(0), x_2(1)) \;=\; 0\,, \tag{9d}$$

$$x_2(0) \;-\; x_1(1) \;=\; 0, \tag{9e}$$

$$S(x_1(1)) \;=\; 0, \tag{9f}$$

$$u_1(\tau),\; u_2(\tau) \;\in\; U \;\;\subset \mathbb{R}^m. \tag{9g}$$

Problem (P1') is a classical optimal control problem with a smooth right-hand side, and $(x_1^0, x_2^0, t_1^0, u_1^0, u_2^0)$ is a solution of this problem. Therefore, we can apply the well–known necessary conditions of optimal control theory, i.e with the Hamiltionian

$$\widetilde{H} \;:=\; (t_1 - a)\, \lambda_1^{\mathrm{T}} f_1(x_1, u_1) \;+\; (b - t_1)\, \lambda_2^{\mathrm{T}} f_2(x_2, u_2), \tag{10}$$

the following conditions hold

$$\lambda_1' \;=\; -\widetilde{H}_{x_1} \;=\; -(t_1 - a)\, \frac{\partial}{\partial x_1} \left(\lambda_1^{\mathrm{T}} f_1(x_1, u_1)\right), \tag{11a}$$

$$\lambda_2' \;=\; -\widetilde{H}_{x_2} \;=\; -(b - t_1)\, \frac{\partial}{\partial x_2} \left(\lambda_2^{\mathrm{T}} f_2(x_2, u_2)\right) \tag{11b}$$

$$\lambda_3' \;=\; -\widetilde{H}_{t_1} \;=\; -\lambda_1^{\mathrm{T}} f_1(x_1, u_1) \;+\; \lambda_2^{\mathrm{T}} f_2(x_2, u_2), \tag{11c}$$

$$u_k(\tau) \;=\; \operatorname{argmin}\{\lambda_k(\tau)^{\mathrm{T}} f_k(x_k(\tau), u) : \;\; u \in U\}, \;\; k = 1, 2 \tag{11d}$$

$$\lambda_1(0) \;=\; -\frac{\partial}{\partial x_1(0)} (\nu^{\mathrm{T}} r), \quad \lambda_1(1) \;=\; -\nu_1 \;+\; \nu_2\, S_x(x_1(1)), \tag{11e}$$

$$\lambda_2(0) \;=\; -\nu_1, \quad \lambda_2(1) \;=\; \frac{\partial}{\partial x_2(1)} (\ell_0\, g \;+\; \nu^{\mathrm{T}}\, r), \tag{11f}$$

$$\lambda_3(0) \;=\; \lambda_3(1) \;=\; 0. \tag{11g}$$

Now, due to the autonomy of the state equations and due to the regularity assumptions above, both parts $\lambda_1^{\mathrm{T}} f_1$ and $\lambda_2^{\mathrm{T}} f_2$ of the Hamiltonian are constant on $[0,1]$. Thus, λ_3 is a linear function which vanishes due to the boundary conditions (11g). Together with the relation (11c) one obtains the continuity of the Hamiltonian (5f).

If one recombines the adjoints

$$\lambda(t) \;:=\; \begin{cases} \lambda_1(\dfrac{t - a}{t_1 - a}), & t \,\in\, [a, t_1[, \\[2ex] \lambda_2(\dfrac{t - t_1}{b - t_1}), & t \,\in\, [t_1, b], \end{cases} \tag{12}$$

one obtains the adjoint equation (5a) from Eq. (11a-b), the minimum principle (5b) from Eq. (11d), and the natural boundary conditions and the jump conditions (5c-e) from Eq. (11e-f). □

It should be remarked that the results of Theorem 2.1. easily can be extended to nonautonomous optimal control problems with nonsmooth state equations. This holds too, if the performance index contains an additional integral term $I = g(x(t_b)) + \int_{t_a}^{t_b} f_0(t, x(t), u(t))dt$. Both extensions can be treated by standard transformation techniques which transform the problems into the form of Problem (P1). The result is, that for the extended problems, one simply has to redefine the Hamiltonian by

$$H(t, x, u, \lambda, \nu_0) \;:=\; \nu_0\, f_0(t, x, u) \;+\; \lambda^{\mathrm{T}} f(t, x, u) \tag{13}$$

and one has to substitute the continuity condition (5f) by the jump condition

$$H[t_j^+] \;=\; H[t_j^-] \;-\; \kappa_j\, S_t(t_j, x^0(t_j)). \tag{14}$$

3. An Economic Model of Personal Income Distribution

We apply the necessary conditions of Theorem 2.1. to an economic optimal control model of personal income distribution. The model is due to Pohmer (Ref. 17). It describes the behaviour of a typical consumer who wants to maximize a total utility functional by controlling the *consumption* $c(t)$, the *rate of total time used for working* $\ell(t)$, and the *rate of working time used for education or extended professional training* $s(t)$. The state variables are the *human capital* $H(t)$ and the *capital* $K(t)$.

The utility function contains three parts corresponding to the influence of consumption, of the recreation time (leisure time), and of the human capital. Further, the bequest of the consumer is taken into account within the perfomance measure.

The optimal control probelm is formulated as follows

Problem (E). Minimize the functional

$$I(c, s, \ell) \;=\; -\int_0^T U(t, c, \ell, H)\, \mathrm{e}^{-\rho t}\, dt \;-\; \Gamma\, \frac{K(T)^{1-\kappa}}{1-\kappa} \tag{15}$$

with respect to the state equations

$$\dot{H}(t) \;=\; \sigma\, H(t)^{\varepsilon}\, s(t)\, \ell(t) \;-\; \delta\, H(t), \tag{16a}$$

$$\dot{K}(t) \;=\; i\, K(t) \;+\; r\, H(t)\, g(s(t))\, \ell(t) \;-\; c(t), \tag{16b}$$

the initial conditions

$$H(0) \;=\; H_0, \qquad K(0) \;=\; K_0, \tag{17}$$

and the control constraints

$$c(t) > 0, \qquad 0 \le s(t) \le 1, \qquad 0 \le \ell(t) < 1. \tag{18}$$

The utility function U is formulated as follows

$$U = \frac{c^{1-\alpha}}{1-\alpha} + \xi \frac{(1-\ell)^{1-\beta}}{1-\beta} + \nu\, t\, \frac{H^{1-\gamma}}{1-\gamma}. \tag{19}$$

The function $g(s)$ is modulated by the parabola

$$g(s) := 1 - (1-a)\,s - a\,s^2. \tag{20}$$

The parameters of the problem are summarized in the following table.

Table 1. Parameters of Problem (E)

α	=	2	β	=	1.5
γ	=	0.8	κ	=	0.8
ξ	=	0.4	ν	=	0.0015
Γ	=	0.2	ρ	=	0.01
σ	=	1	ε	=	0.35
δ	=	0.01	r	=	1
a	=	0.3	T	=	75
H_0	=	1	K_0	=	40

In addition to these parameters, the interest rate of capital i is fixed in dependence on the sign of the capital $K(t)$:

$$i := \begin{cases} i_1 = 0.04, & \text{if} \quad K(t) \ge 0, \\ i_2 \ge i_1, & \text{if} \quad K(t) < 0. \end{cases} \tag{21}$$

A. Numerical Solution of Problem (E) for $\mathbf{i_2 = i_1}$.

In the following, we summarize the necessary conditions for problem E. First, we consider the smooth case of the OCP, i.e. $i_2 = i_1$.

If the adjoint variables of the state (H, K) take the form $\lambda_H\, \mathrm{e}^{-\rho t}$ and $\lambda_K\, \mathrm{e}^{-\rho t}$, and if we assume regularity, i.e. $\nu_0 = 1$, the Hamiltonian is given by

$$\begin{aligned} \mathcal{H} \;=\; & -\left(\frac{c^{1-\alpha}}{1-\alpha} + \xi \frac{(1-\ell)^{1-\beta}}{1-\beta} + \nu\, t\, \frac{H^{1-\gamma}}{1-\gamma}\right) \mathrm{e}^{-\rho t} \\ & + (\lambda_H\, \mathrm{e}^{-\rho t})\,[\sigma\, H^{\varepsilon}\, s\, \ell \;-\; \delta\, H] \\ & + (\lambda_K\, \mathrm{e}^{-\rho t})\,[i\, K \;+\; r\, H\, g(s)\, \ell \;-\; c]\,. \end{aligned} \tag{22}$$

From this, we obtain the adjoint equations

$$\dot{\lambda}_H = \nu\, t\, H^{-\gamma} + \lambda_H \left(\rho + \delta - \varepsilon\, \sigma\, H^{\varepsilon-1}\, s\, \ell\right) - \lambda_K\, r\, g(s)\, \ell, \tag{23a}$$

$$\dot{\lambda}_K = (\rho - i)\, \lambda_K, \tag{23b}$$

and the natural boundary conditions

$$\lambda_H(T) = 0, \qquad \lambda_K(T) = -\Gamma K(T)^{-\kappa}\, e^{\rho T}. \tag{24}$$

For a solution of problem E it is reasonable to assume that $H(t) > 0$ and $K(T) > 0$ hold. Then, from the differential equation (23b) and the corresponding boundary condition (24) it follows, that $\lambda_K(t) < 0$ holds for all $t \in [0, T]$.

By a simple analysis of (22) we conclude that for fixed state and adjoint variables, the Hamiltonian $\mathcal{H}$ has a global minimium with respect to the control variables (c, s, ℓ) over the control region (18) which is given by the relations

$$c^* = (-\lambda_K)^{-1/\alpha}, \tag{25a}$$

$$s^* = \begin{cases} 0, & \text{if } s_0 < 0, \\ s_0, & \text{if } s_0 \in [0,1], \\ 1, & \text{if } s_0 > 1, \end{cases} \tag{25b}$$

$$\ell^* = \begin{cases} 0, & \text{if } \Phi(s^*) \geq -\xi, \\ \ell_0, & \text{if } \Phi(s^*) < -\xi, \end{cases} \tag{25c}$$

where

$$s_0 := \frac{a-1}{2\,a} + \frac{\lambda_H\, \sigma\, H^{\varepsilon-1}}{2\,a\,r\,\lambda_K}, \tag{26a}$$

$$\ell_0 := 1 - (-\xi/\Phi(s^*))^{1/\beta}, \tag{26b}$$

$$\Phi(s) := \lambda_H\, \sigma\, H^{\varepsilon}\, s + \lambda_K\, r\, H\, g(s). \tag{26c}$$

Further, the global minimum (c^*, s^*, ℓ^*) is a strict one, if $\ell^* \neq 0$ holds. In the case $\ell^* = 0$, precisely the control variables of the form $(c^*, s, 0)$, $0 \leq s \leq 1$ are the global minima of $\mathcal{H}$ on (18).

For the numerical solution of the resulting boundary value problem we assume the following reasonable **control structure** (with unknown switching points t_j, $j = 1, 2, 3$):

$$s(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq t_1, \\ s_0, & \text{if } t_1 \leq t \leq t_2, \\ 0, & \text{if } t_2 \leq t \leq T, \end{cases} \qquad \ell(t) = \begin{cases} \ell_0, & \text{if } 0 \leq t \leq t_3, \\ 0, & \text{if } t_3 \leq t \leq T. \end{cases} \tag{27}$$

With this assumption, the multipoint boundary value problem coming from the first-order necessary conditions and the minimum principle completely is derived. We have to integrate the differential equations (16), (23) using the control functions (25). The boundary conditions are given by (15), (22). The interior boundary conditions are just the continuity conditions of the control functions

$$s(t_j^+) \;=\; s(t_j^-),\;\; j=1,2, \qquad \ell(t_3^+) \;=\; \ell(t_3^-). \tag{28}$$

The numerical solution obtained by the multiple shooting code BNDSCO, cf. Ref. 7, 16, 18, are shown in the Figures 1 and 2.

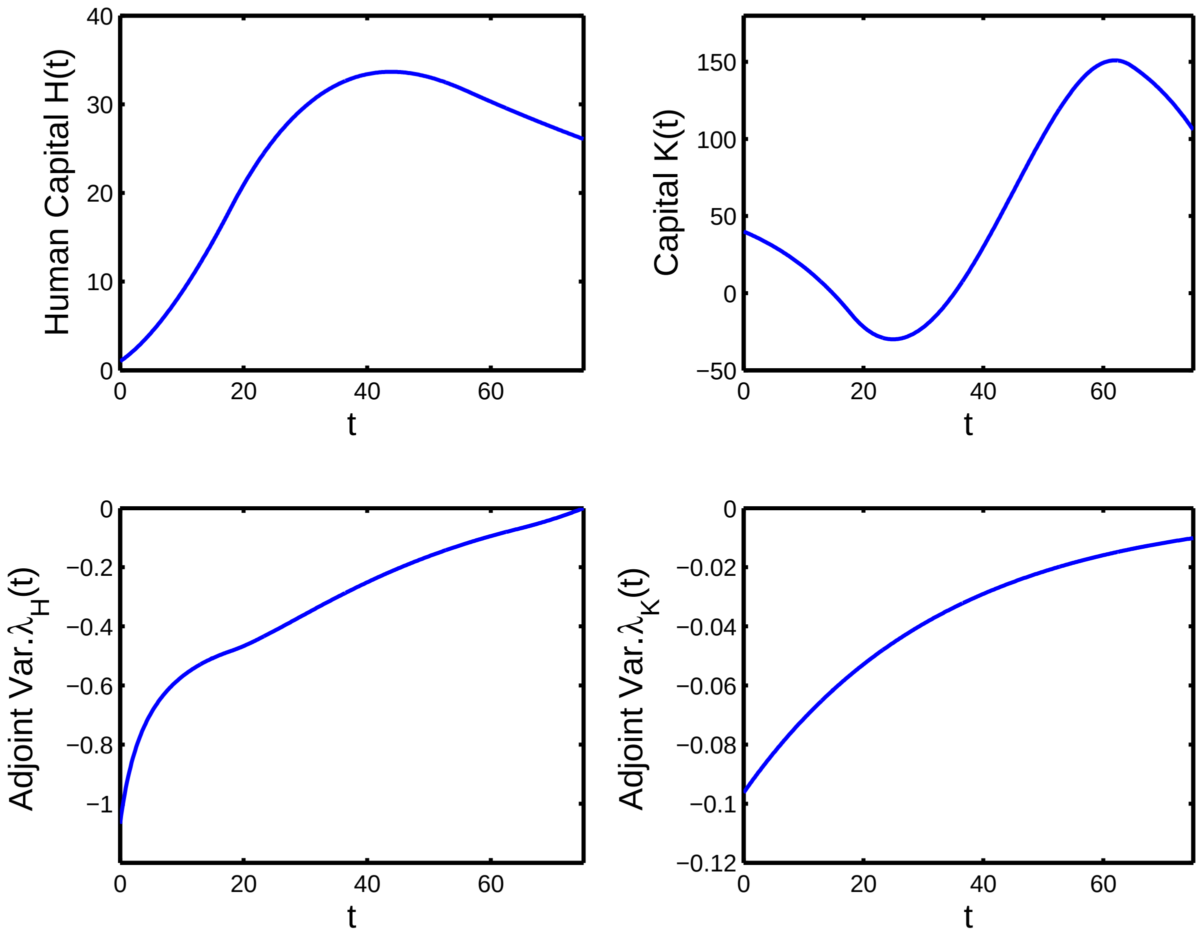

Fig. 1 Problem E: State and Adjoint Variables for Problem E, Smooth Case.

In the main, the numerical results correspond to the common expectations with respect to the economic behaviour of a real human being. In the first eighteen years of lifetime there is a phase of pure education ($s \equiv 1$). It follows a long period (38 years) of training on the job, where the part of the total working time used for extended professional training decreases. The next nine years, from the age of 56 till the age of 65 there is a phase of pure working, however also with a decreasing working time. Finally, it follows the retirement phase. With respect to the human capital one observes a broad maximum within the period from the age of 30 till the age of 60. Between the age of 15 and the age of 35 the model yields a period of negative capital (depts) which is due to the demand of a reliable consumption.

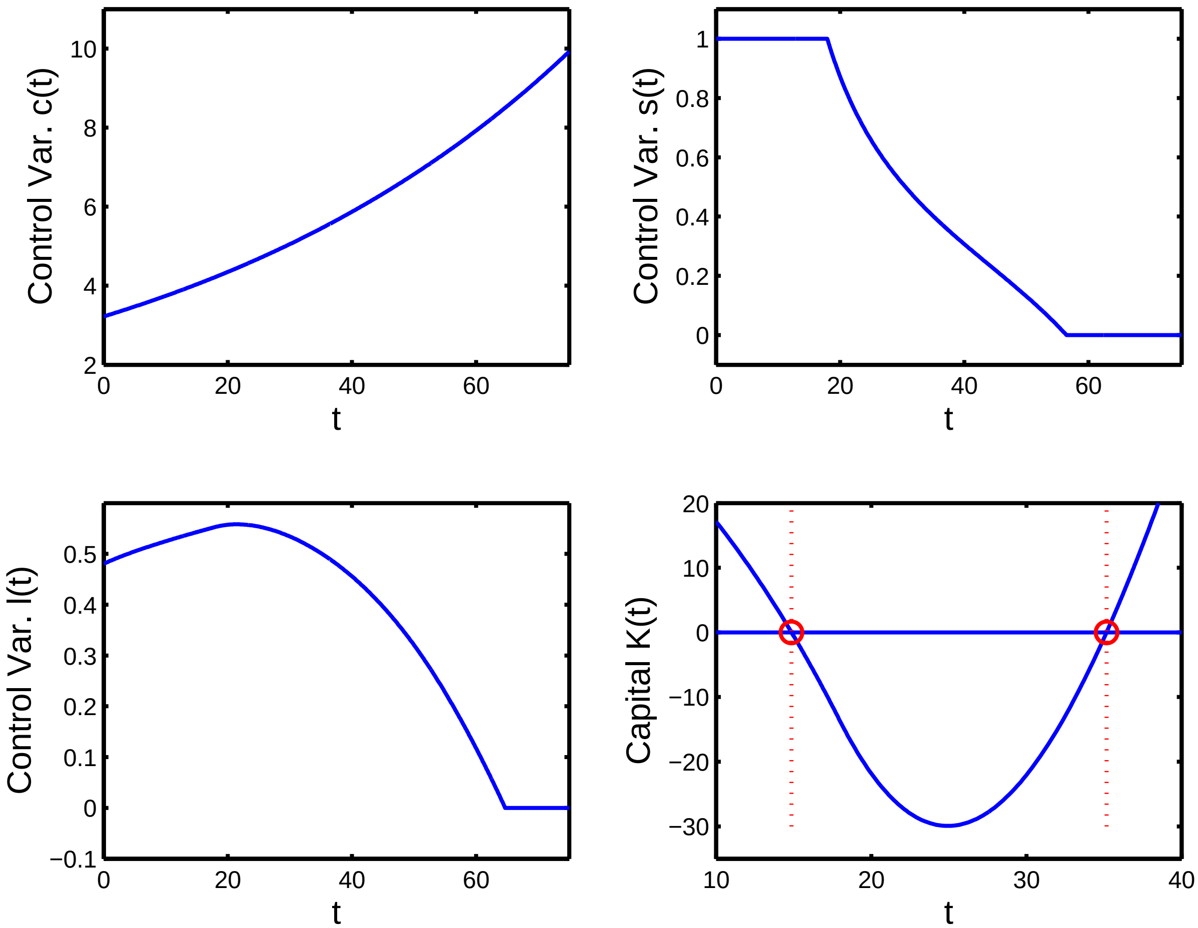

Fig. 2 Problem E: Control Variables for Problem E, Smooth Case.

B. Numerical Solution of Problem (E) for $\mathbf{i_2 > i_1}$.

For the nonsmooth case $i_2 > i_1$ we apply the necessary conditions of Theorem 2.1, i.e. we assume a certain regularity of the solution. Especially, we assume, that the switching function $S(H,K) := K$ has only isolated roots and, that the **solution structure** with respect to the switching function corresponds to the solution of the smooth case:

$$S[t] \;=\; K(t) \begin{cases} > \;0, & \text{if} \quad 0 \le t < t_4, \\ < \;0, & \text{if} \quad t_4 < t < t_5, \\ > \;0, & \text{if} \quad t_5 < t \le T. \end{cases} \tag{29}$$

Here, t_4 and t_5 denote the additional switching points, the two roots of capital K. These switching points are determined by the interior boundary conditions

$$K(t_4) \;=\; K(t_5) \;=\; 0. \tag{30}$$

The necessary conditions with respect to the adjoint equations (23), the natural boundary conditions (24) and the optimal control law (25-26) remain unchanged. This holds too for the optimal control structure (27) and for the interior boundary conditions (28).

The only difference to the smooth case is that the parameter i has to be changed within the interval $[t_4, t_5]$. Further, one has the jump condition (5e), which here takes the form $\lambda_K(t_j^+) = \lambda_K(t_j^-) + \kappa_j, \ j = 4, 5$, and the continuity condition (5f) with respect to the Hamiltonian. Due to the switching condition (30) this later condition is satisfied for $\kappa_j = 0$, cf. (22). In fact, this relation has been fulfilled for all numerical solutions we obtained for the resulting multipoint boundary value problem.

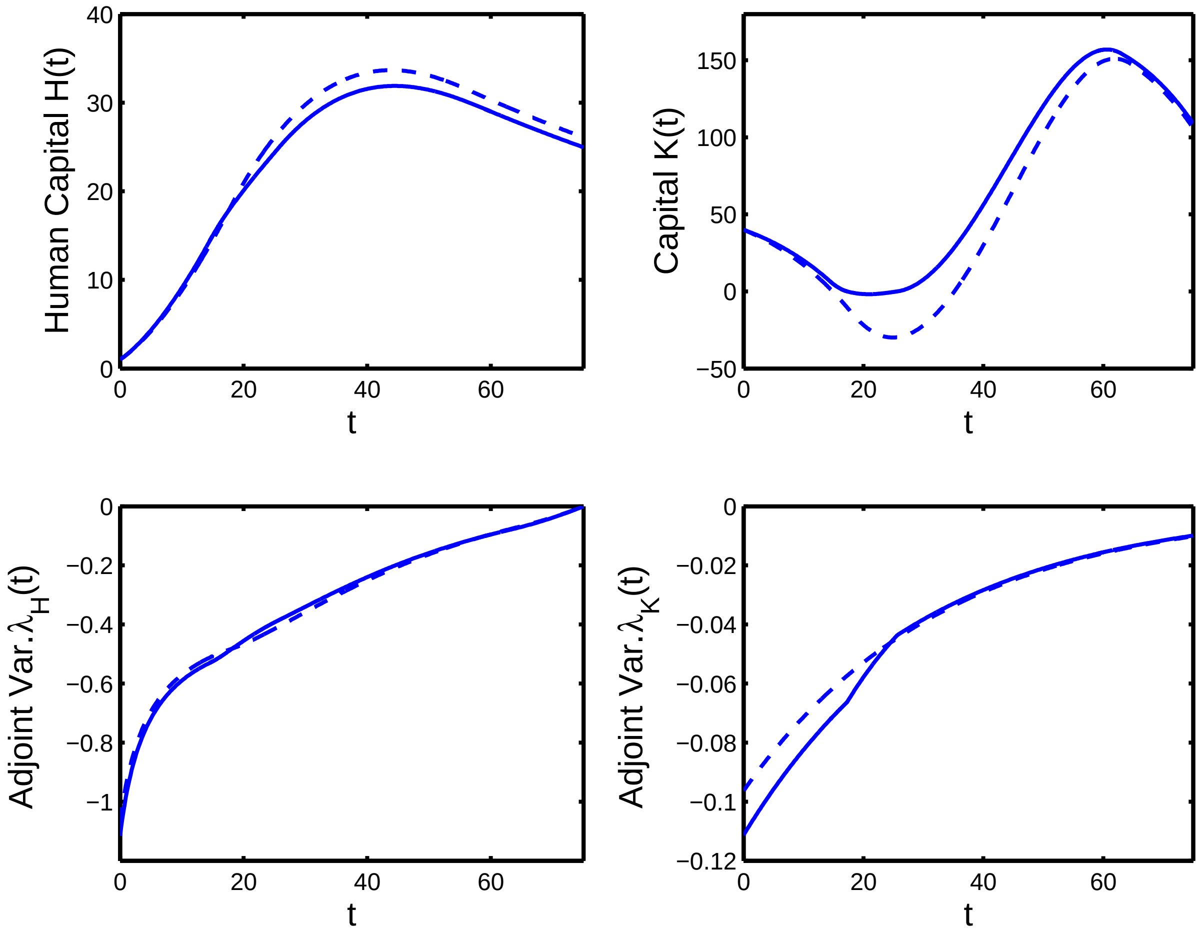

Fig. 3 Problem E: State and Adjoint Variables for Problem E, $i_2 = 0.06$

Numerically, we solve the multipoint boundary value problem using the multiple shooting routine BNDSCO by means of a homotopy chain with respect to the parameter i_2 ranging within the interval $0.04 \le i_2 \le i_2^*$, where $i_2^* \doteq 0.065193\,3305$.

In the Figures 3 and 4 the numerical solution for the parameter $i_2 = 0.06$ is shown. The dashed curves correspond to the smooth solution with $i_2 = i_1$.

In comparison of these two solutions one observes some typical results: For the solution with $i_2 > i_1$, the interval with a negative value of the capital becomes much smaller, already at the age of 14 the phase of on the job training starts, and the education part is considerably reduced during the first part of this period. Further, we observe a plain reduction of the human capital. Some concrete data are given in Table 2. Here, t_1 denotes the end of the education phase, t_2 the end of the on the job training phase, t_3 the age of retirement, and $[t_4, t_5]$ the interval with negative capital. I denotes the value of the functional, which has been minimzed, and $\|H\|_\infty$ the maximal value of human capital.

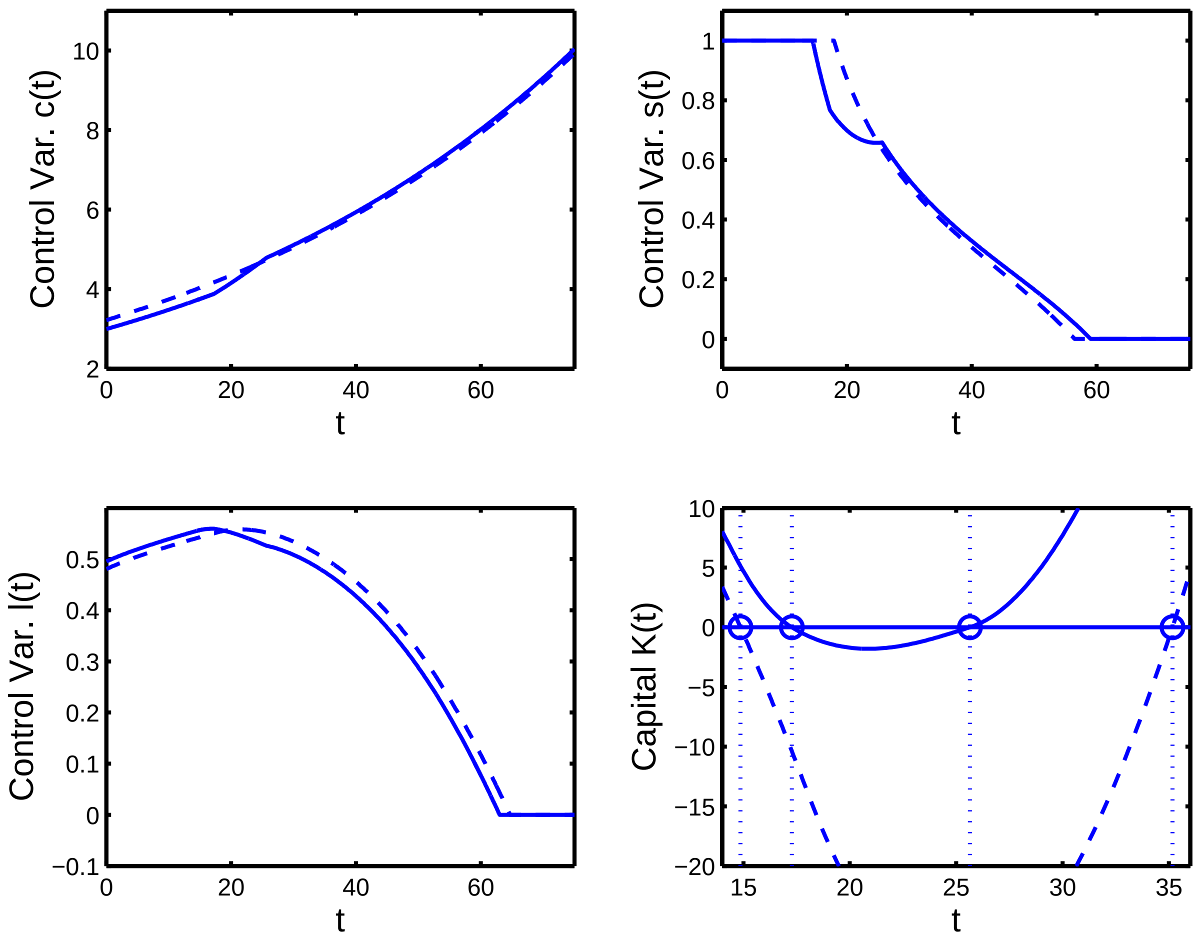

Fig. 4 Problem E: Control Variables for Problem E, $i_2 = 0.06$

Table 2. Problem (E), some characteristic quantities.

i_2	t_1	t_2	t_3	t_4	t_5	I	$\|H\|_\infty$
0.04	17.9	56.5	64.7	14.9	35.2	39.02	33.66
0.05	15.2	58.5	63.3	16.0	29.0	39.10	32.19
0.06	14.5	59.0	63.0	17.3	25.6	39.12	31.88
i_2^*	14.4	59.0	63.0	17.9	24.7	39.12	31.87

The family of regular solutions of the optimal control problem E depending on the parameter i_2 ends at the certain value $i_2^* \doteq 0.065193$ which is characterized by a horizontal slope of the function K at the exit point t_5 of the negative subarc. For parameters $i_2 > i_2^*$ one has to expect that the regularity assumption concerning the roots of K is not satisfied, and we have to apply new necessary conditions for this situation.

4. Nonsmooth Optimal Control Problems, Singular Case.

In this section we continue the investigation of the general optimal control problem (P1). However, we drop the regularity condition (R). More precisely, we assume that a solution (x^0, u^0) of the optimal control problem contains a finite number of nontrivial subarcs, where the switching function vanishes identically. These subarcs are called *singular state subarcs*, cf. the analogous situation of singular control subarcs, Ref. 3, 13. In order to have a well-defined problem, we now have to consider the dynamics on the singular manifold $S(x) = 0$. Therefore, we generalize the problem formulation (P1) a bit, and allow the system to possess an independent dynamic on the singular subarcs.

Problem (P2). Determine a piecewise continuous control function $u : [a, b] \to \mathbb{R}^m$, such that the functional

$$I = g(x(b)) \tag{31}$$

is minimized subject to the following constraints (state equations, boundary conditions, and control constraints)

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [a, b] \quad \text{a.e.}, \tag{32a}$$

$$r(x(a), x(b)) = 0, \tag{32b}$$

$$u(t) \in U \subset \mathbb{R}^m, \tag{32c}$$

where U is a (compact and convex) cuboid, and the right-hand side f is of the special form

$$f(x, u) = \begin{cases} f_1(x, u), & \text{if } S(x) < 0, \\ f_2(x, u), & \text{if } S(x) = 0, \\ f_3(x, u), & \text{if } S(x) > 0, \end{cases} \tag{33}$$

with smooth functions $f_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k = 1, 2, 3$. All other assumptions with respect to Problem (P1) may be satisfied also for (P2).

Again, our aim is to derive necessary conditions for (P2). To this end, we assume that there exists a finite grid $a < t_1 < \ldots < t_s < b$ such that the t_j are either isolated roots of the switching function $S[t] := S(x^0(t))$ or entry or exit points of a singular state subarc. By J_{reg} we denote the set of indices of the isolated roots t_j, by J_{entry} those of the entry points, and by J_{exit} those of the exit points of the singular state subarcs.

We introduce the extended Hamiltonian (here also denoted by H)

$$H(x, u, \lambda, \mu) := H_j(x, u, \lambda, \mu) := \lambda^{\mathrm{T}} f_j(x, u) + \mu\, S^{(1)}(x, u), \tag{34}$$

where $j \in \{1, 2, 3\}$ is chosen according to the sign of $S(x)$, and μ denotes a Lagrange multiplier. $S^{(1)}$ is the first total time derivative of the switching function

$$S^{(1)}(x, u) := S_x(x)^{\mathrm{T}} f_2(x, u). \tag{35}$$

We assume, that the switching function is of first order with respect to the control u, i.e.

$$S_u^{(1)}(x^0(t), u^0(t)) \neq 0 \tag{36}$$

holds along each singular state subarc. Again, we assume regularity with respect to the minimum principle.

In the following, we summarize the necessary conditions for Problem (P2).

Theorem 4.1.

With the assumptions above the following necessary conditions hold.

There exist an adjoint variable $\lambda : [a,b] \to \mathbb{R}^n$ *, which is a piesewise* C^1*–function, and Lagrange multipliers* $\nu_0 \in \{0,1\}$ *,* $\nu \in \mathbb{R}^\ell$ *,* $\kappa_j \in \mathbb{R}\ (j \in J_{reg} \cup J_{entry})$ *, and a piecewise continuous Lagrange multiplier* $\mu : [a,b] \to \mathbb{R}$ *such that* (x^0, u^0) *satisfies the conditions*

$$\dot\lambda(t) = -H_x(x^0(t), u^0(t), \lambda(t), \mu(t)), \quad t \in [a,b], \text{ a.e.} \quad \text{(adjoint equations)}, \tag{37a}$$

$$u^0(t) = \operatorname{argmin}\{H(x^0(t), u, \lambda(t), \mu(t)) : \ u \in U\}, \quad \text{(minimum principle)}, \tag{37b}$$

$$\mu(t)\, S(x^0(t)) = 0, \quad \text{(complementarity condition)}, \tag{37c}$$

$$\lambda(a) = -\frac{\partial}{\partial x^0(a)}\,[\nu^{\mathrm{T}} r(x^0(a), x^0(b))], \quad \text{(natural boundary conditions)}, \tag{37d}$$

$$\lambda(b) = \frac{\partial}{\partial x^0(b)}\,[\nu_0\, g(x^0(b)) + \nu^{\mathrm{T}} r(x^0(a), x^0(b))], \tag{37e}$$

$$\lambda(t_j^+) = \lambda(t_j^-) + \kappa_j\, S_x(x^0(t_j)), \quad j \in J_{\mathrm{reg}} \cup J_{\mathrm{entry}}\,, \quad \text{(jump condition)}, \tag{37f}$$

$$H[t_j^+] = H[t_j^-], \quad j = 1, \ldots, s, \quad \text{(continuity condition)}. \tag{37g}$$

Note, that at the entry point of a singular subarc, the adjoint variables may be discontinuous. Further, due to the (possibly discontinuous) Lagrange multplier μ, the optimal controls, in general, need not to be continuous at entry- or exit point. In analogy to the Bryson-Denham-Dreyfus conditions for state constrained optimal control problems, cf. Ref. 5, 6, 11, 14, the Lagrange multiplier μ can be expressed explicitly as a function of state and adjoint variables. This fact allows the numerical computation via the solution of a multipoint boundary value problem even for complicated switching functions.

Proof of Theorem 4.1. For simplicity, we assume, that the switching function $S[\cdot]$ along the optimal trajectory has just *one* singular subarc $[t_1, t_2] \subset]a, b[$, and that the following *switching structure* holds

$$S[t] \quad \begin{cases} < 0, & \text{if} \quad a \le t < t_1, \\ = 0, & \text{if} \quad t_1 \le t \le t_2, \\ > 0, & \text{if} \quad t_2 < t \le b. \end{cases} \tag{38}$$

We compare the optimal solution (x^0, u^0) with those admissible solutions (x, u) of the problem which have the same switching structure. Each candidate is associated with its separated parts $(\tau \in [0,1],\ t_0 := a,\ t_3 := b)$

$$\begin{aligned} x_j(\tau) &:= x(t_{j-1} + \tau(t_j - t_{j-1})), \quad j = 1,2,3, \\ u_j(\tau) &:= u(t_{j-1} + \tau(t_j - t_{j-1})), \quad j = 1,2,3. \end{aligned} \tag{39}$$

Now, $(x_1, x_2, x_3, t_1, t_2, u_1, u_2, u_3)$ performs an abmissible and $(x_1^0, x_2^0, x_3^0, t_1^0, t_2^0, u_1^0, u_2^0, u_3^0)$ an optimal solution of the following auxillary optimal control problem.

Problem (P2'). Determine a piecewise continuous control function $u = (u_1, u_2, u_3) : [0,1] \to \mathbb{R}^{3m}$, such that the functional

$$I = g(x_3(1)) \tag{40}$$

is minimized subject to the constraints $(t_0 := a,\ t_3 := b)$

$$x_j'(\tau) = (t_j - t_{j-1})\, f_j(x_j(\tau), u_j(\tau)), \quad \tau \in [0,1], \quad \text{a.e.}, \quad j = 1,2,3, \tag{41a}$$

$$t_k'(\tau) = 0, \quad k = 1,2, \tag{41b}$$

$$r(x_1(0), x_3(1)) = 0, \tag{41c}$$

$$x_2(0) - x_1(1) = x_3(0) - x_2(1) = 0, \tag{41d}$$

$$S(x_2(\tau)) = 0, \quad \forall\, \tau \in [0,1], \tag{41e}$$

$$u_1(\tau),\ u_2(\tau),\ u_3(\tau) \in U \subset \mathbb{R}^m. \tag{41f}$$

Problem (P2') again is a classical optimal control problem with a smooth right-hand side, and with a state variable equality constraint. We can apply the classical necessary conditions of optimal control theory, cf. Hestenes, Ref. 10. If S satisfies the constraint qualification (36), there exist a continuous Lagrange multiplier $\widetilde{\mu}$, and continuously differentiable adjoint variables λ_j, $j = 1,2,3$, such that with the Hamiltonian

$$\widetilde{H} := \sum_{j=1}^{3} (t_j - t_{j-1})\, \lambda_j^{\mathrm{T}} f_j(x_j, u_j) + \widetilde{\mu}\,(t_2 - t_1)\, S^{(1)}(x_2, u_2), \tag{42}$$

and the augmented performance index

$$\begin{aligned} \Phi &:= \ell_0\, g(x_3(1)) - \kappa\, S(x_2(0)) + \nu^{\mathrm{T}} r(x_1(0), x_3(1)) \\ &+ \nu_1^{\mathrm{T}}(x_2(0) - x_1(1)) + \nu_2^{\mathrm{T}}(x_3(0) - x_2(1)), \end{aligned} \tag{43}$$

the following conditions hold

$$\lambda_1' = -\widetilde{H}_{x_1} = -(t_1 - a)\, \left(\lambda_1^{\mathrm{T}} f_1\right)_{x_1}, \tag{44a}$$

$$\lambda_2' = -\widetilde{H}_{x_2} = -(t_2 - t_1)\, \left[(\lambda_2^{\mathrm{T}} f_2)_{x_2} + \widetilde{\mu}(\tau)\, S_{x_2}^{(1)}(x_2, u_2)\right] \tag{44b}$$

$$\lambda_3' = -\widetilde{H}_{x_3} = -(b - t_2)\, \left(\lambda_3^{\mathrm{T}} f_3\right)_{x3} \tag{44c}$$

$$\lambda_4' = -\widetilde{H}_{t_1} = -\lambda_1^{\mathrm{T}} f_1 + \lambda_2^{\mathrm{T}} f_2 + \widetilde{\mu}(\tau)\, S^{(1)}(x_2, u_2), \tag{44d}$$

$$\lambda_5' = -\widetilde{H}_{t_2} = -\lambda_2^{\mathrm{T}} f_2 + \lambda_3^{\mathrm{T}} f_3 - \widetilde{\mu}(\tau)\, S^{(1)}(x_2, u_2), \tag{44e}$$

$$u_j(\tau) \;=\; \operatorname{argmin}\{\lambda_j(\tau)^{\mathrm{T}} f_j(x_j(\tau),u): \;\; u \in U\}, \;\; j=1,3, \tag{44f}$$

$$u_2(\tau) \;=\; \operatorname{argmin}\{\lambda_2(\tau)^{\mathrm{T}} f_2(x_2(\tau),u) \;+\; \widetilde{\mu}(\tau)\, S^{(1)}(x_2(\tau),u): \;\; u \in U\}, \tag{44g}$$

$$\lambda_1(0) \;=\; -\Phi_{x_1(0)} \;=\; -(\nu^{\mathrm{T}} r)_{x_1(0)}, \quad \lambda_1(1) \;=\; \Phi_{x_1(1)} \;=\; -\nu_1, \tag{44h}$$

$$\lambda_2(0) \;=\; -\Phi_{x_2(0)} \;=\; -\nu_1 \;+\; \kappa\, S_x(x_2(0)), \tag{44i}$$

$$\lambda_2(1) \;=\; \Phi_{x_2(1)} \;=\; -\nu_2, (44j) \tag{44j}$$

$$\lambda_3(0) \;=\; -\Phi_{x_3(0)} \;=\; -\nu_2, \quad \lambda_3(1) \;=\; \Phi_{x_3(1)} \;=\; (\ell_0\, g + \nu^{\mathrm{T}} r)_{x_3(1)}, \tag{44k}$$

$$\lambda_4(0) \;=\; \lambda_4(1) \;=\; \lambda_5(0) \;=\; \lambda_5(1) \;=\; 0. \tag{44l}$$

Due to the autonomy of the optimal control problem, the parts $\lambda_j^{\mathrm{T}} f_j$, $j=1,2,3$, of the Hamiltonian are constant; further $S^{(1)}(x_2^0,u_2^0)$ vanishes. Therefore, the adjoints λ_4 and λ_5 vanish and we obtain the global continuity of the augmented Hamiltonian (34).

If one recombines the adjoints

$$\lambda(t) \;:=\; \begin{cases} \lambda_1\left(\dfrac{t-a}{t_1-a}\right), & t \in [a,t_1[, \\[2ex] \lambda_2\left(\dfrac{t-t_1}{t_2-t_1}\right), & t \in [t_1,t_2], \\[2ex] \lambda_3\left(\dfrac{t-t_2}{b-t_2}\right), & t \in]t_2,b], \end{cases} \tag{45}$$

and defines the Lagrange multiplier μ by

$$\mu(t) \;:=\; \begin{cases} \widetilde{\mu}\left(\dfrac{t-t_1}{t_2-t_1}\right), & t \in [t_1,t_2], \\[2ex] 0, & t \notin [t_1,t_2], \end{cases} \tag{46}$$

one obtains all the necessary conditions (37) by the relations (44). □

Again, we mention that the results of Theorem 4.1. easily can be extended to nonautonomous optimal control problems with nonsmooth state equations and to optimal control problems with performance index of Bolza type, as well.

5. Singular Solution of the Economic Model Problem

In Section 3 we considered a homotopy path for the economic optimal control model with respect to the parameter i_2 which describes the interest rate for negative capital (indeptedness). The path ends (with regular solutions) for a parameter value $i_2^* \doteq 0.065193\,3305$, which is characterized by the condition $\dot{K}(t_5) = 0$, cf. Figure 5. Here, $[t_4, t_5]$ denotes the subintervall with nonpositive values of the capital K, whereas t_1, t_2, and t_3 are switching points with respect to the control constraints. t_1 and t_2 are the switching points with respect to the control s, t_3 is concerned with the control ℓ, cf. Eq. (27), and Fig. 2.

In this section we are interested in the continuation of this homotopy path for parameters $i_2 > i_2^*$. It is reasonable to assume that solutions of the corresponding optimal control Problem (E) for those parameters may contain a singular state subarc $K = 0$, which is situated near the switching point t_5, i.e. we assume the switching structure

$$S[t] \;=\; K(t) \begin{cases} > \;0, & \text{if} \quad 0 \le t < t_4, \\ < \;0, & \text{if} \quad t_4 < t < t_5, \\ = \;0, & \text{if} \quad t_5 \le t \le t_6, \\ > \;0, & \text{if} \quad t_6 < t \le T. \end{cases} \tag{47}$$

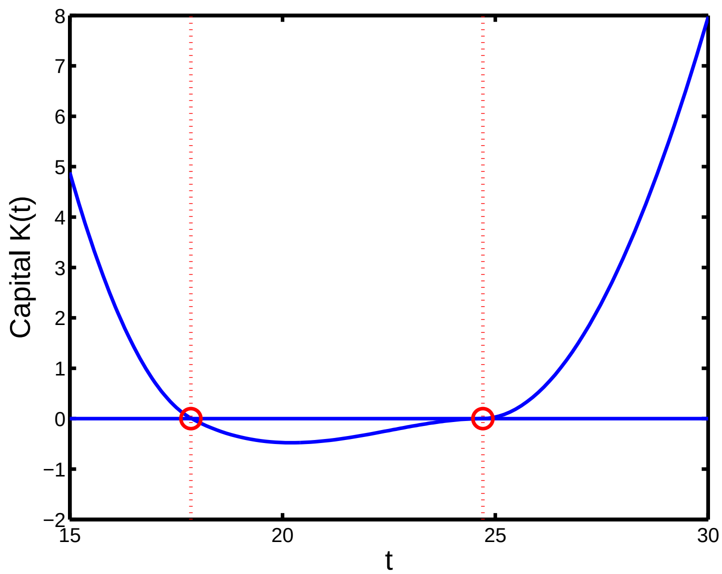

Fig. 5 Problem E: State Variable K for Problem E, $i_2 = i_2^*$

In order to derive necessary conditions we apply Theorem 4.1, or its straight forward modification for nonautonomous optimal control problems, respectively. The dynamic along a singular state subarc is described by the original state equations (16), where into the right-hand side $i = i_1$ is substituted.

First, we observe

$$S(H,K) \;:=\; K \;=\; 0 \;\Rightarrow\; S^{(1)}(H,K,c,s,\ell) \;=\; i\,K \;+\; r\,H\,g(s)\,\ell \;-\; c,$$

such that the constraint qualification (order one condition) $S_u^{(1)} \neq 0$ along a (nontrivial) singular state subarc is satisfied.

The augmented Hamiltonian is given by

$$\widetilde{\mathcal{H}} \;=\; \mathcal{H} \;+\; \left(\mu(t)\,\mathrm{e}^{-\rho t}\right)\left[i\,K \;+\; r\,H\,g(s)\,\ell \;-\; c\right], \tag{48}$$

where $\left(\mu(t)\,\mathrm{e}^{-\rho t}\right)$ denotes the Lagrange multiplier with respect to the singular state subarc, $\mu(t) = 0$ for $t \notin [t_5, t_6]$.

Along the singular arc the state and adjoint differential equations are given by

$$\dot{H}(t) \;=\; \sigma\,H^{\varepsilon}\,s\,\ell \;-\; \delta\,H, \tag{49a}$$

$$\dot{K}(t) \;=\; r\,H\,g(s)\,\ell \;-\; c, \tag{49b}$$

$$\dot{\lambda}_H(t) \;=\; \nu\,t\,H^{-\gamma} \;+\; \lambda_H\,\left(\rho + \delta - \varepsilon\,\sigma\,H^{\varepsilon-1}\,s\,\ell\right) \;-\; \lambda_K^*\;r\,g(s)\,\ell, \tag{49c}$$

$$\dot{\lambda}_K(t) \;=\; (\rho \;-\; i)\,\lambda_K, \tag{49d}$$

where $\lambda_K^*(t) \;:=\; \lambda_K(t) + \mu(t)$. Here, the problem is that the Lagrange multiplier μ is not known. So we are not able to use Eq. (49) directly for the numerical integration.

We apply the minimum principle, and use the side constraint

$$S^{(1)} \;=\; r\,H\,g(s)\,\ell \,-\, c \;=\; 0. \tag{50}$$

Due to the special form of the augmented Hamiltonian (48) the equations (25) and (26) remain valid along the singular subarc; one only has to substitute the adjoint λ_K by the augmented adjoint λ_K^*.

Thus, if we assume that the optimal control lies within the interior of the control region (the validity of this assumption can be proved a posteriori), we obtain the relations

$$\widetilde{\mathcal{H}}_c \,=\, 0 \quad\Rightarrow\quad c^* \,=\, (-\lambda_K^*)^{-1/\alpha} \tag{51a}$$

$$\widetilde{\mathcal{H}}_s \,=\, 0 \quad\Rightarrow\quad \Phi'(s^*) \,=\, 0, \quad \text{where} \quad \Phi(s) \,:=\, \lambda_H\,\sigma\,H^{\varepsilon}\,s \,+\, \lambda_K^*\,r\,H\,g(s), \tag{51b}$$

$$\widetilde{\mathcal{H}}_\ell \,=\, 0 \quad\Rightarrow\quad \Phi(s^*) \,+\, \xi\,(1-\ell^*)^{-\beta} \;=\; 0. \tag{51c}$$

Now, the relations (50), (51) can be rewritten as follows: From (51b), (51a), and (50) we get

$$\lambda_K^* \;=\; -\,\frac{\lambda_H\,\sigma\,H^{\varepsilon-1}}{r g'(s)}, \qquad c^* \;=\; (-\lambda_K^*)^{-1/\alpha}, \qquad \ell^* \;=\; \ell^*(s) \;=\; \frac{c^*}{r\,H\,g(s)}, \tag{52}$$

so that the quantities λ_K^*, c^*, and ℓ^* can be evaluated if the optimal control $s \;=\; s^*$ is known.

Now, we apply the necessary condition (51c). We find the control s^* as the (uniquely determined) root of the function

$$\Psi(s) \;:=\; \lambda_H\,\sigma\,H^{\varepsilon}\,\left(s - g(s)/g'(s)\right) \,+\, \xi\,\left(1 - \ell^*(s)\right)^{-\beta}, \tag{53}$$

where ℓ^* is evaluated by Eq.(52).

The numerical solution along the singular state subarc is obtained as follows. We use the equations (49a-d) for the numerical integration. In each evaluation step of the right-hand side, the control s^* is computed numerically as root of Ψ using $\ell^*(s)$ from Eq. (52) by bisection method. Finally, the Lagrange multiplier $\mu(t)$ is evaluated by means of $\mu = \lambda_K^* - \lambda_K$.

The boundary points of the singular subarc $[t_5, t_6]$ are determined by the corresponding interior boundary conditions. First, we have a jump condition concerning λ_K, cf. Eq. (37f):

$$\lambda_K(t_5^+) \;=\; \lambda_K(t_5^-) \;+\; \kappa, \quad \lambda_H(t_5^+) \;=\; \lambda_H(t_5^-). \tag{54}$$

The continuity of the Hamiltonian at t_5, cf. Eq. (37g), is satisfied due to the interior boundary conditions

$$K(t_5) \;=\; 0, \qquad \dot{K}(t_5^-) \;=\; 0, \tag{55}$$

the continuity of $\mathcal{H}$ at t_6 is obtained by the boundary condition

$$\mu(t_6^-) \;=\; \mu(t_6^+) \;=\; 0. \tag{56}$$

With these conditions we get a well-defined multipoint boundary value problem which is solved numerically by the multiple shooting code BNDSCO.

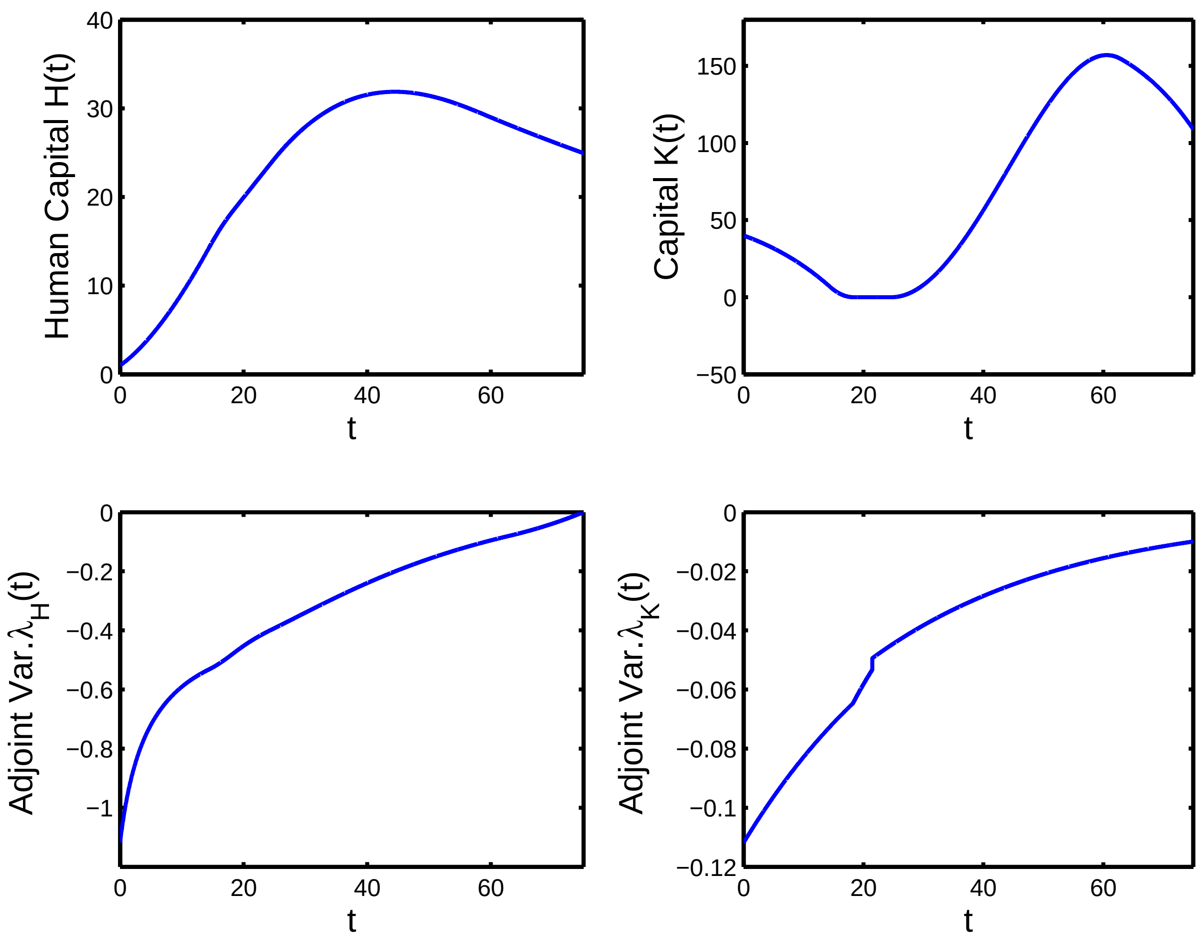

Fig. 6 Problem E: State and Adjoint Variables for Problem E, $i_2 = 0.07$

For the parameter $i_2 = 0.07$ the state and adjoint time histories are shown in Figure 6. One observes the discontinuity of λ_K at t_5 and its continuity (cf. Eq. (56)) at t_6 .

The corresponding control variables and the Lagrange multiplier μ are shown in Figure 7. Finally, a zoom on the state $K(t)$ along the singular arc is shown in the Figure 8.

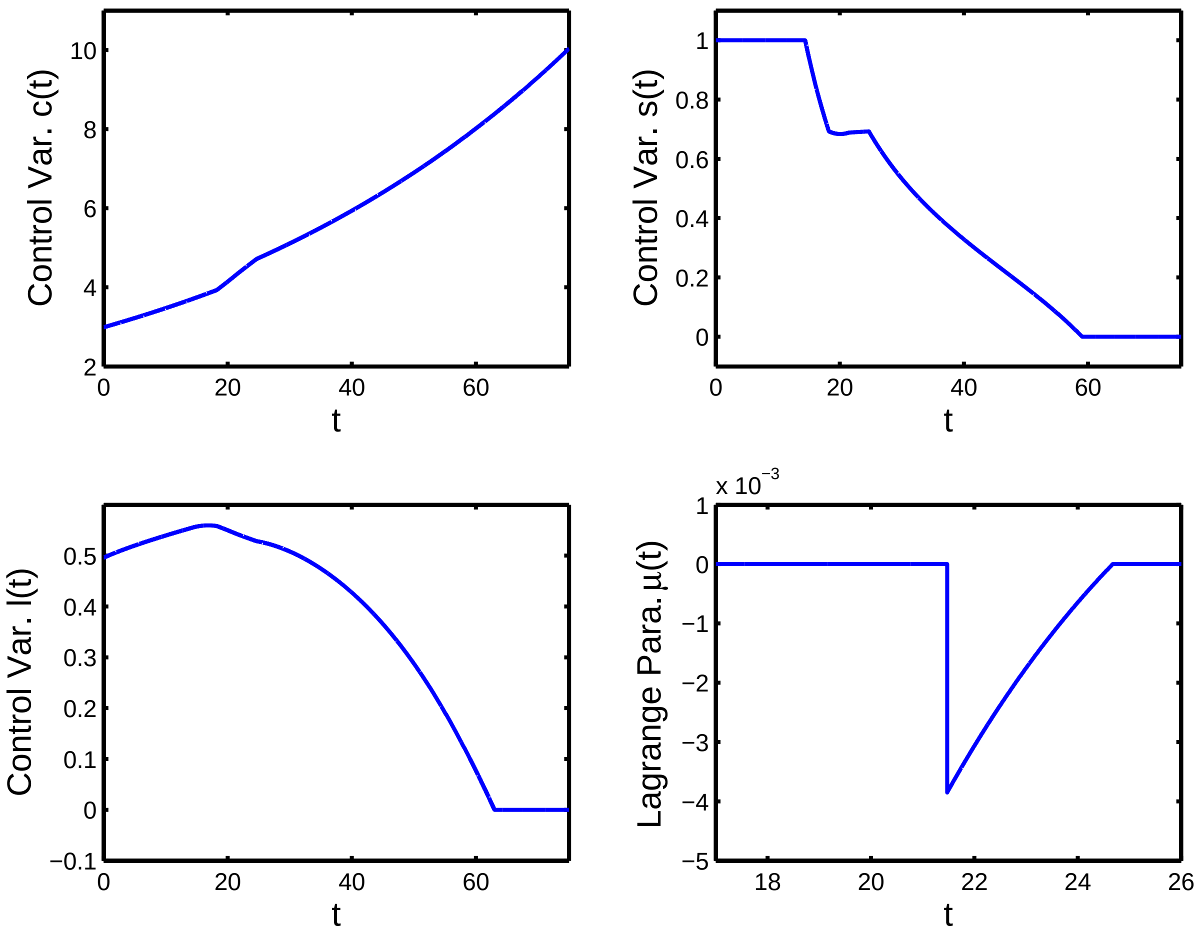

Fig. 7 Problem E: Control Variables and Lagrange Multiplier for Problem E, $i_2 = 0.07$

Some characteristic quantities of these solutions are given in Table 3.

On this part of the homotopy path the changes are more or less restricted to the separation of the $K \leq 0$–subarc into its regular and its singular part. The regular part very soon (parameter values $i_2 \approx 0.075$, becomes small. For $i_2^\infty \doteq 0.075925$ this is just the intervall $[18.365, 18.369]$.

Table 3. Problem (E), some characteristic quantities.

i_2	t_4	t_5	t_6	κ
0.067	18.035	23.387	24.678	0.001303
0.070	18.230	21.475	24.679	0.003852
0.075	18.361	18.804	24.679	0.009101
i_2^∞	18.365	18.369	24.679	0.010203

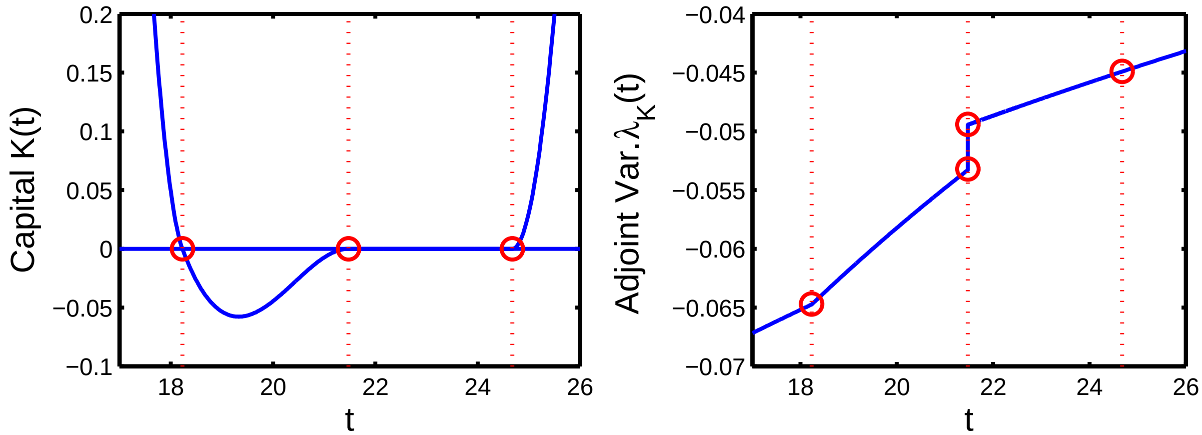

Fig. 8 Problem E: Singular State Subarc for Problem E, $i_2 = 0.07$

6. The State Constraint Economic Model Problem

It may be worth to mention that the solution data for the last parameter value i_2^∞ correspond with good accuracy to the solution of the optimal control problem (E) with the additional state variable inequality constraint $K \geq 0$.

The corresponding optimal control problem is given as follows.

Problem (E'). Minimize the functional

$$I(c,s,\ell) \;=\; -\int_0^T U(t,c,\ell,H)\,\mathrm{e}^{-\rho t}\,dt \;-\; \Gamma\,\frac{K(T)^{1-\kappa}}{1-\kappa} \tag{57}$$

with respect to the state equations

$$\dot{H}(t) \;=\; \sigma\,H(t)^{\varepsilon}\,s(t)\,\ell(t) \;-\; \delta\,H(t), \tag{58a}$$

$$\dot{K}(t) \;=\; i\,K(t) \;+\; r\,H(t)\,g(s(t))\,\ell(t) \;-\; c(t), \tag{58b}$$

the initial conditions

$$H(0) \;=\; H_0, \qquad K(0) \;=\; K_0, \tag{59}$$

the control constraints

$$c(t) \;>\; 0, \qquad 0 \;\leq\; s(t) \;\leq\; 1, \qquad 0 \;\leq\; \ell(t) \;<\; 1, \tag{60}$$

and the state variable inequalitiy constraint

$$\widetilde{S}(H,K) \;:=\; -K \;\leq\; 0. \tag{61}$$

We observe that the state constraint $\widetilde{S}$ is of order one. In order to clarify the relation to the singular state solution, we apply the necessary conditions of Bryson et al. (Ref. 5). Here, the augmented Hamiltonian is given by

$$\widetilde{\mathcal{H}} \;=\; \mathcal{H} \;-\; \left(\widetilde{\mu}(t)\,\mathrm{e}^{-\rho\,t}\right)\,\left[i\,K \;+\; r\,H\,g(s)\,\ell \;-\; c\right], \tag{62}$$

Note, that K vanishes along a boundary subarc. So, if we identify $\widetilde{\mu} := -\mu$, the augmented Hamiltonian corresponds completely to Eq. (48). This correspondence holds too for the optimal control laws given in Eq. (51), where we have $\lambda_K^* := \lambda_K - \widetilde{\mu}$.

Further, along a boundary subarc the state and adjoint differential equations correspond to Eq. (49) for the singular subarc, and even the jump conditions (54) and the interior boundary conditions (55) and (56) remain valid for the end points of the boundary subarc.

The numerical solution of the correponding multipoint boundary value problem reveals the boundary subarc $[18.3648, 24.6787]$, and the Lagrange parameter $\kappa \doteq 0.010213$, cf. the results in Table 3.

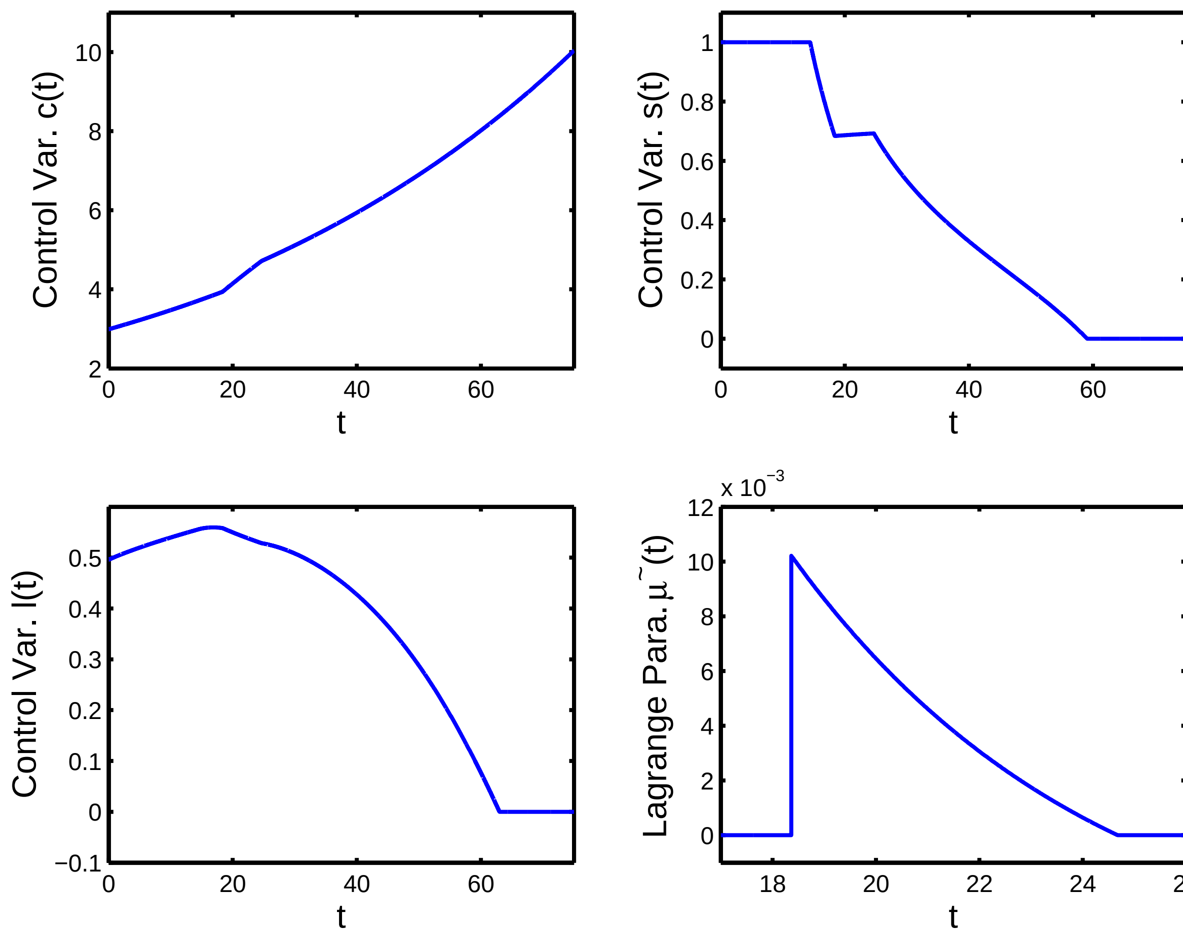

Fig. 9 Problem E': Control Variables and Lagrange Multiplier

In Figure 9 the resulting optimal control histories for the state constrained optimal control problem (E') are shown as well as the Lagrange multiplier $\widetilde{\mu}$. One observes that the necessary conditions $\widetilde{\mu} \geq 0$, and $\widetilde{\mu}' \leq 0$ are satisfied.

In Figure 10 the state variable K and the corresponding adjoint variable λ_K are shown in a neighborhood of the boundary subarc.

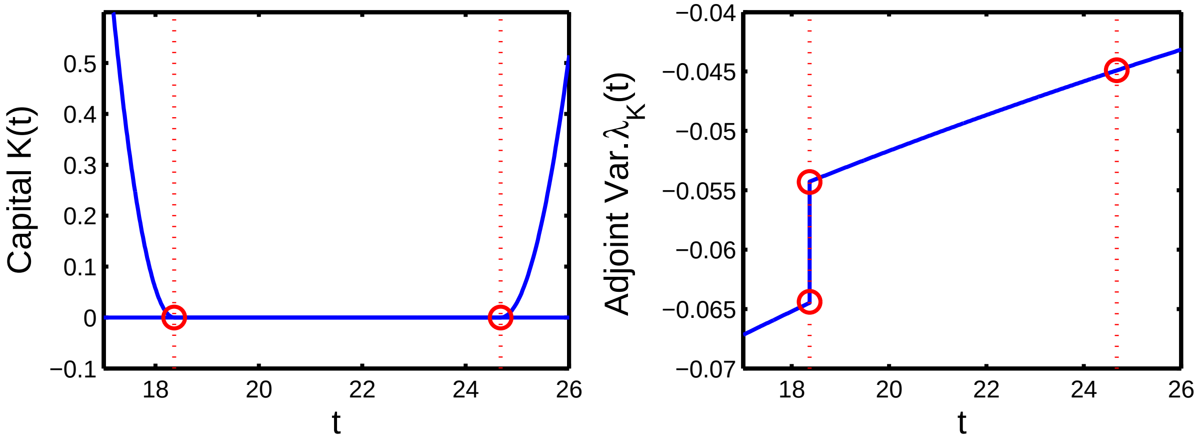

Fig. 10 Boundary Subarc for Problem E'

7. Conclusions

In this paper optimal control problems with nonsmooth state differential equations are considered. Two solution typs are distinguished. In the first part of the paper regular solutions have been considered. The regularity is characterized by the assumption that the switching function only has isolated roots. In the second part so called singular state subarcs are admitted. This are nontrivial subarcs, where the switching function vanishes identically. For both situations necessary conditions are derived from the classical (smooth) optimal control theory.

The results are applied to an economic optimal control model, which describes the personal income distribution of a typical consumer. This is an optimal control problem with two state variables (capital K and human capital H) and three control variables. Into the problem a parameter enters which descibed the interest rate of capital and which taked two different values depending on the sign of the state K. For small differences between these parameter values the solution turns out to be regular, for a larger difference the solution contains a singular state subarc. Numerical solutions of this problem obtained by the indirect multiple shooting method have been presented.

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