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Higher Order Acoustic Perturbations for a Loudspeaker with Axisymmetric Enclosure I : The Mathematical Model

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Abstract

We derive higher order acoustic perturbations for the sound field generated by a bass–reflex loudspeaker with an axisymmetric enclosure. The model is based on an asymptotic expansion of the field quantities described by the isentropic compressible Euler's equation in cylindrical coordinates assuming that the maximal displacement of the voice coil is small compared to the dimension of the enclosure. Moreover, assuming a simplified model geometry we derive an explicit time–periodic solution for the asymptotic system, which should serve as a reference solution for numerical investigations of the system.

1 Introduction

The present work is motivated by the following industrial application: the optimal shape design of the enclosure of modern bass–reflex loudspeakers relies more and more on simulation tools, which should be able to predict the influence of small changes in the geometry of the enclosure on the sound field without performing time–consuming and costly experiments.

Commercial software tools to predict sound generation are based on the theory of linear acoustics, i.e. one describes the sound field using the wave equation or, when considering time–periodic problems, the Helmholtz's equation. The most widely spread code for linear acoustics is *SYSNOISE* which discretizes the Helmholtz's equation either using the boundary element method (BEM) or the finite element approach (FEM) [1].

Modeling sound generation in a loudspeaker using linear acoustic theory is valid as long as the change of the enclosure volume in the loudspeaker due to the moving membrane is negligible. A problem arises when the dimension of the loudspeaker becomes small and one should take into account the dynamic change of the enclosure volume. In particular, nonlinear phenomena will influence the sound generation, indicated by the presence of higher order harmonics in the sound field, which reduces the desired quality of the

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loudspeaker, generally expressed in terms of the so–called distortion factor. This problem appears more strong in the case of a bass–reflex loudspeaker, because they are operated at a higher power and the deflection of the membrane is more large than at lower frequencies.

To describe nonlinear phenomena in a loudspeaker, which is the basis to understand how to optimize the shape in order to reduce the presence of higher order harmonics, one should go back to the fundamental conservation principles from fluid dynamics, e.g. the compressible Euler's or Navier–Stokes equations. Nevertheless, the main transport phenomena is the propagation of sound, i.e. one should treat the low–Mach number limit of the Euler's or Navier–Stokes equations. Moreover, due to the moving membrane one should perform in principle even numerical simulations on a time–dependent domain, which is far away from being trivial, because the deflection of the membrane still remains small compared to the overall dimension of the loudspeaker.

A possible way out to overcome the above mentioned difficulties is to use a mathematical model which is based on formal asymptotic expansions of the sound field generated by the loudspeaker. In [2] the author derived an asymptotic approximation of Euler's equation assuming that a single air particle oscillates around its equilibrium position by a small displacement proportional to the maximal displacement of the membrane. An expansion up to second order yields an inhomogeneous Helmholtz's equation where the inhomogeneity couples the perturbation with the first order pressure obtained from standard linear acoustic theory.

A different asymptotic approach was recently used by the authors to derive higher order acoustic perturbations for a simplified one–dimensional piston problem, where the piston on the right end of the slab undergoes a time–periodic movement with small maximal deflection [3]. The asymptotic model is derived from a formal asymptotic expansion of the one–dimensional isentropic Euler's equation by transforming the time–dependent slab geometry onto a fixed interval.

In the following present work we consider the isentropic Euler's equations formulated on a time–dependent two–dimensional domain representing the enclosure of an axisymmetric loudspeaker including the moving membrane. The shape of the enclosure corresponds to an experimental set–up which will be used by the Fraunhofer–Institute for Industrial Mathematics (ITWM), Kaiserslautern, Germany, in order to validate the asymptotic approach given in [2] and to compare with what is presented in the following. Assuming that the maximal deflection ε of the membrane is small, we apply an asymptotic expansion of the Euler's equation in cylindrical coordinates. Moreover, to get rid of the time–dependent domain, we transform the given boundary conditions on the moving membrane to conditions on a fixed boundary using the concept of Taylor expansions [4].

In the present work we restrict ourselves to the formal derivation of our asymptotic acoustic model and derive under certain simplification explicit time–periodic solutions. Numerical simulations of the model and – if available – a comparison with experimental data will be presented in a separated paper [5].

The remainder of the paper is organized as follows. In Section 2 we introduce in more detail the enclosure geometry and indicate briefly which asymptotic model we will use to describe the sound generation. In Section 3 we formally derive our asymptotic model starting with the axisymmetric Euler's equations in cylindrical coordinates. Section 4 deals with the corresponding model when looking for time–periodic solutions and we derive an explicit time–periodic solution for a simplified geometry, namely a planar and infinitely large membrane. This explicit solution should serve as a reference solution when performing numerical experiments. Finally we give some conclusions in Section 5.

2 The Problem Setting

For a general description of the functionality of loudspeakers we refer the interested readers to the HyperPhysics–homepage of C. R. Nave at the Georgia State University [6]. The loudspeaker which will be used in the experimental set–up at the Fraunhofer–ITWM has an axisymmetric symmetry as shown in Figure 1.

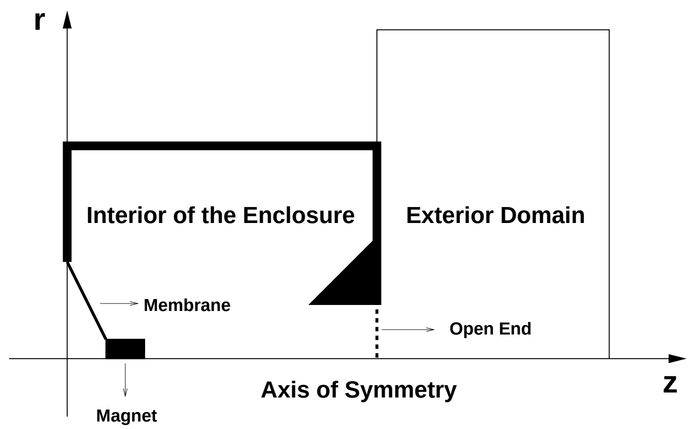

Figure 1: Schematic illustration of the loudspeaker with axisymmetric enclosure.

Fixed at the axis of symmetry we have the magnet which includes the voice coil (not shown in the figure). Attached to the voice coil is the membrane or speaker cone which is further attached to the outer ring of the speaker support. The movable parts of the loudspeaker are the voice coil and attached to the coil the membrane. The enclosure volume has an open end opposite to the loudspeaker and we are mainly interested on the sound generation in the interior of the enclosure as well as the exterior domain shown in Figure 1.

To model the sound generation we start with the isentropic Euler's equation in cylindrical coordinates formulated on the time–dependent domain shown in Figure 1. The idea now is the following: because the coil movement is small compared to the dimension of the speaker enclosure, we apply an asymptotic expansion of the field quantities, i.e. the density, the pressure and the velocity field, in terms of the maximal displacement of the voice coil or equivalently the moving membrane. In particular, we assume that the left end of the domain is described by a function $Z(t,r) = \varepsilon h(t,r)$, where ε is the maximal deflection of the membrane and $h(t,r)$ models the dynamic shape of the speaker cone. Hence, the field quantities will be expanded in terms of $\varepsilon \ll 1$.

Next one should define appropriate boundary conditions at the fixed walls of the domain, the moving membrane as well as, if necessary, to prescribe the behavior at $z \to \infty$. At the fixed wall one has the standard boundary conditions for Euler's equation, namely the velocity in normal direction should vanish [7]. At the moving membrane the velocity v in z–direction should be equal to the time derivative of $Z(t,r)$, i.e.

$$v(t,r,Z(t,r)) = \varepsilon \frac{\partial h(t,r)}{\partial t} \qquad \forall\, t, r, > 0$$

and the velocity in r–direction is equal to zero. Boundary conditions for the pressure are more difficult to derive and we leave this problem open, because it is even strongly connected with the need to define numerical boundary conditions when applying numerical simulations on a bounded computational domain. In the limit $z \to \infty$ one may assume that the system behaves like a point source such that the sound field is far away from the loudspeaker a spherical wave.

We still have the problem that the mathematical model is formulated on a time–dependent domain. Because the maximal displacement of the coil is of order $O(\epsilon)$ one may use the technique to transform the given boundary condition on $Z(t,r) = \varepsilon h(t,r)$ back to the r–axis using the concept of Taylor's expansion, see [4]. This will be shown in detail in the next section, where we derive our asymptotic based on the preliminary explanations given above.

3 The Model Equations

We start with the axisymmetric Euler's equations in cylindrical coordinates (r, θ, z), where the z co–ordinate is aligned with the axis of symmetry of the loudspeaker. Assuming the angular velocity to be zero, Euler's equations are given by the system [7]

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho u)}{\partial r} + \frac{\partial (\rho v)}{\partial z} = 0 \tag{3.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0 \tag{3.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + v\frac{\partial v}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0 \tag{3.3}$$

where ρ, p, u and v denote the density, the pressure and the velocities in r and z direction, respectively. The system (3.1)–(3.3) is closed by the equation of state

$$\left(\frac{p}{p_0}\right) = \left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{3.4}$$

where γ defines the ratio of specific heat, ρ_0 and p_0 are the reference values for the density and the pressure, respectively.

In order to derive higher order perturbations we formally expand the field quantities ρ, p, u and v with respect to $\varepsilon \ll 1$ as

$$\rho = \rho^{(0)} + \varepsilon \rho^{(1)} + \varepsilon^2 \rho^{(2)} + o(\varepsilon^3) \tag{3.5}$$

$$u = \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + o(\varepsilon^3) \tag{3.6}$$

$$v = \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + o(\varepsilon^3) \tag{3.7}$$

$$p = p^{(0)} + \varepsilon p^{(1)} + \varepsilon^2 p^{(2)} + o(\varepsilon^3) \tag{3.8}$$

From (3.4) we obtain by Taylor expansion the expression

$$\begin{aligned} p &= p_0 \rho_0^{-\gamma} \Big[\rho^{(0)\gamma} + \varepsilon \Big(\gamma \rho^{(0)\gamma - 1} \rho^{(1)} \Big) \\ &\quad + \varepsilon^2 \Big(\gamma \rho^{(0)\gamma-1} \rho^{(2)} + \frac{\gamma(\gamma-1)}{2} \rho^{(0)\gamma-2} \rho^{(1)2} \Big) + O(\varepsilon^3) \Big] \end{aligned}$$

and this yields immediately the following relations between the density and pressure expansions

$$p^{(0)} = p_0 \rho_0^{-\gamma} {\rho^{(0)}}^{\gamma} \tag{3.9}$$

$$p^{(1)} = \gamma p_0 \rho_0^{-\gamma} {\rho^{(0)}}^{\gamma-1} \rho^{(1)} \tag{3.10}$$

$$p^{(2)} = p_0 \rho_0^{-\gamma} \left(\gamma {\rho^{(0)}}^{\gamma-1} \rho^{(2)} + \frac{\gamma(\gamma-1)}{2} {\rho^{(0)}}^{\gamma-2} {\rho^{(1)}}^{2} \right) \tag{3.11}$$

Substituting (3.5)–(3.8) into Euler's equations (3.1)–(3.3) and collecting terms of the order ε^0 yields the zeroth order approximation

$$\begin{aligned} \frac{\partial}{\partial t} \rho^{(0)} &= 0 \\ \frac{\partial}{\partial r} p^{(0)} &= 0 \\ \frac{\partial}{\partial z} p^{(0)} &= 0 \end{aligned}$$

i.e. the zeroth order density is independent of t, $\rho^{(0)} = \rho^{(0)}(r, z)$, and the pressure $p^{(0)}$ only depends on t, $p^{(0)} = p^{(0)}(t)$. Because zeroth order density and pressure are related by (3.9) we conclude that the zeroth order density and pressure are both constant in space and time and we use in the following $\rho^{(0)} = \rho_0$ and $p^{(0)} = p_0$.

Denoting by c the speed of sound,

$$c = \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2},$$

the expansion of the state equation becomes

$$p^{(1)} = c^2 \rho^{(1)} \tag{3.12}$$

$$p^{(2)} = c^2 \rho^{(2)} + \frac{c^2(\gamma - 1)}{2\rho_0} (\rho^{(1)})^2 \tag{3.13}$$

Using the result of the zeroth order approximation the first order system reads

$$\frac{\partial \rho^{(1)}}{\partial t} + \frac{\rho_0}{r} \frac{\partial (r u^{(1)})}{\partial r} + \rho_0 \frac{\partial v^{(1)}}{\partial z} = 0 \tag{3.14}$$

$$\frac{\partial u^{(1)}}{\partial t} + \frac{1}{\rho_0} \frac{\partial p^{(1)}}{\partial r} = 0 \tag{3.15}$$

$$\frac{\partial v^{(1)}}{\partial t} + \frac{1}{\rho_0} \frac{\partial p^{(1)}}{\partial z} = 0 \tag{3.16}$$

With $p^{(1)} = c^2 \rho^{(1)}$ we obtain

$$\frac{\partial p^{(1)}}{\partial t} + c^2 \rho_0 \left(\frac{\partial u^{(1)}}{\partial r} + \frac{\partial v^{(1)}}{\partial z} \right) = -\frac{c^2 \rho_0}{r} u^{(1)} \tag{3.17}$$

$$\frac{\partial u^{(1)}}{\partial t} + \frac{1}{\rho_0} \frac{\partial p^{(1)}}{\partial r} = 0 \tag{3.18}$$

$$\frac{\partial v^{(1)}}{\partial t} + \frac{1}{\rho_0} \frac{\partial p^{(1)}}{\partial z} = 0 \tag{3.19}$$

It is straightforward to derive from the system above a wave equation, e.g., for the first order pressure $p^{(1)}$, namely

$$\frac{\partial^2 p^{(1)}}{\partial t^2} - c^2 \left(\frac{\partial^2 p^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial p^{(1)}}{\partial r} + \frac{\partial^2 p^{(1)}}{\partial z^2} \right) = 0,$$

i.e. (3.17)–(3.19) are compatible with linear acoustics and describe the ground mode of the loudspeaker.

Collecting all terms of order ε^2 we obtain the first order perturbation with respect to linear acoustics. In particular, the second order continuity equation reads

$$\frac{\partial \rho^{(2)}}{\partial t} + \frac{\rho_0}{r}\frac{\partial}{\partial r}(r u^{(2)}) + \frac{1}{r}\frac{\partial}{\partial r}(r\rho^{(1)}u^{(1)}) + \rho_0 \frac{\partial v^{(2)}}{\partial z} + \frac{\partial}{\partial z}(\rho^{(1)}v^{(1)}) = 0$$

Rewriting the equation as

$$\frac{\partial \rho^{(2)}}{\partial t} + \frac{\rho_0}{r}\frac{\partial}{\partial r}(r u^{(2)}) + \rho_0 \frac{\partial v^{(2)}}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r\rho^{(1)}u^{(1)}) - \frac{\partial}{\partial z}(\rho^{(1)}v^{(1)})$$

and using (3.13) yields

$$\begin{aligned}\frac{\partial}{\partial t}\left(p^{(2)} - \frac{c^2(\gamma-1)}{2\rho_0}(\rho^{(1)})^2 \right) + \frac{c^2\rho_0}{r}\frac{\partial}{\partial r}(r u^{(2)}) + c^2\rho_0 \frac{\partial v^{(2)}}{\partial z} \\ = -\frac{c^2}{r}\frac{\partial}{\partial r}(r\rho^{(1)}u^{(1)}) - c^2\frac{\partial}{\partial z}(\rho^{(1)}v^{(1)})\end{aligned}$$

Hence, the second order continuity equation may be written in the form

$$\frac{\partial p^{(2)}}{\partial t} + c^2\rho_0 \left(\frac{1}{r}\frac{\partial}{\partial r}(r u^{(2)}) + \frac{\partial v^{(2)}}{\partial z} \right) = \text{R.H.S.}$$

with

$$\text{R.H.S.} = \frac{\gamma - 1}{c^2\rho_0} p^{(1)} \frac{\partial p^{(1)}}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}(r p^{(1)}u^{(1)}) - \frac{\partial}{\partial z}(p^{(1)}v^{(1)})$$

Reformulating the right hand side using

$$\frac{\partial p^{(1)}}{\partial t} + \frac{c^2\rho_0}{r}\frac{\partial}{\partial r}(r u^{(1)}) + c^2\rho_0\frac{\partial v^{(1)}}{\partial z} = 0$$

yields

$$\begin{aligned}\frac{\gamma}{c^2\rho_0} p^{(1)}\frac{\partial p^{(1)}}{\partial t} \underbrace{- \frac{1}{c^2\rho_0} p^{(1)}\frac{\partial p^{(1)}}{\partial t} - \frac{p^{(1)}}{r}\frac{\partial}{\partial r}(r u^{(1)}) - p^{(1)}\frac{\partial v^{(1)}}{\partial z}}_{=0} \\ -u^{(1)}\frac{\partial p^{(1)}}{\partial r} - v^{(1)}\frac{\partial p^{(1)}}{\partial z} = \text{R.H.S.}\end{aligned}$$

and we end up with the formula

$$\text{R.H.S.} = \frac{\gamma}{c^2\rho_0} p^{(1)}\frac{\partial p^{(1)}}{\partial t} - u^{(1)}\frac{\partial p^{(1)}}{\partial r} - v^{(1)}\frac{\partial p^{(1)}}{\partial z}$$

such that the second order continuity equation now reads

$$\frac{\partial p^{(2)}}{\partial t} + c^2\rho_0\left(\frac{1}{r}\frac{\partial}{\partial r}(ru^{(2)}) + \frac{\partial v^{(2)}}{\partial z}\right) = \frac{\gamma}{c^2\rho_0}p^{(1)}\frac{\partial p^{(1)}}{\partial t} - u^{(1)}\frac{\partial p^{(1)}}{\partial r} - v^{(1)}\frac{\partial p^{(1)}}{\partial z}$$

The second order moment equations are given by

$$\begin{aligned}
\rho_0\left(\frac{\partial u^{(2)}}{\partial t} + u^{(1)}\frac{\partial u^{(1)}}{\partial r} + v^{(1)}\frac{\partial u^{(1)}}{\partial z}\right) + \frac{\partial p^{(2)}}{\partial r} + \rho^{(1)}\frac{\partial u^{(1)}}{\partial t} &= 0\\
\rho_0\left(\frac{\partial v^{(2)}}{\partial t} + u^{(1)}\frac{\partial v^{(1)}}{\partial r} + v^{(1)}\frac{\partial v^{(1)}}{\partial z}\right) + \frac{\partial p^{(2)}}{\partial z} + \rho^{(1)}\frac{\partial v^{(1)}}{\partial t} &= 0
\end{aligned}$$

which we express in the more convenient form given by

$$\begin{aligned}
\frac{\partial u^{(2)}}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial r} &= -u^{(1)}\frac{\partial u^{(1)}}{\partial r} - v^{(1)}\frac{\partial u^{(1)}}{\partial z} - \frac{\rho^{(1)}}{\rho_0}\frac{\partial u^{(1)}}{\partial t}\\
\frac{\partial v^{(2)}}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial z} &= -u^{(1)}\frac{\partial v^{(1)}}{\partial r} - v^{(1)}\frac{\partial v^{(1)}}{\partial z} - \frac{\rho^{(1)}}{\rho_0}\frac{\partial v^{(1)}}{\partial t}
\end{aligned}$$

Using $\rho^{(1)} = p^{(1)}/c^2$ and the notation $U^{(1)} = (u^{(1)}, v^{(1)})^T$ the moment equations become

$$\begin{aligned}
\frac{\partial u^{(2)}}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial r} &= -U^{(1)}\cdot\nabla u^{(1)} - \frac{p^{(1)}}{c^2\rho_0}\frac{\partial u^{(1)}}{\partial t}\\
\frac{\partial v^{(2)}}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial z} &= -U^{(1)}\cdot\nabla v^{(1)} - \frac{p^{(1)}}{c^2\rho_0}\frac{\partial v^{(1)}}{\partial t}
\end{aligned}$$

where $\nabla := \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial z}\right)^T$.

Summarizing the results, the second order system reads

$$\frac{\partial p^{(2)}}{\partial t} + c^2\rho_0\left(\frac{1}{r}\frac{\partial}{\partial r}(ru^{(2)}) + \frac{\partial v^{(2)}}{\partial z}\right) = -U^{(1)}\cdot\nabla p^{(1)} + \gamma\frac{p^{(1)}}{c^2\rho_0}\frac{\partial p^{(1)}}{\partial t} \tag{3.20}$$

$$\frac{\partial u^{(2)}}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial r} = -U^{(1)}\cdot\nabla u^{(1)} - \frac{p^{(1)}}{c^2\rho_0}\frac{\partial u^{(1)}}{\partial t} \tag{3.21}$$

$$\frac{\partial v^{(2)}}{\partial t} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial z} = -U^{(1)}\cdot\nabla v^{(1)} - \frac{p^{(1)}}{c^2\rho_0}\frac{\partial v^{(1)}}{\partial t} \tag{3.22}$$

As mentioned previously, we like to get rid of the time–dependent domain by reformulating the velocity boundary conditions on the moving membrane to the fixed point $z = 0$ using Taylor's expansions. The velocity boundary conditions on moving membrane are given by

$$u(t, r, \varepsilon h) = 0, \quad v(t, r, \varepsilon h) = \varepsilon\frac{\partial h(t, r)}{\partial t} \qquad \forall\, t, r, > 0$$

Taylor's expansions around $z = 0$ yield

$$\begin{aligned}
u(t, r, \varepsilon h) &= u(t, r, 0) + \varepsilon h(t, r)\frac{\partial u}{\partial z}(t, r, 0) + O(\varepsilon^2)\\
v(t, r, \varepsilon h) &= v(t, r, 0) + \varepsilon h(t, r)\frac{\partial v}{\partial z}(t, r, 0) + O(\varepsilon^2)
\end{aligned}$$

and using the expansions for u and v we obtain the following velocity boundary conditions for the first and second order system:

$$\begin{aligned} u^{(1)}(t,r,0) &= 0 \\ u^{(2)}(t,r,0) &= -h(t,r)\frac{\partial u^{(1)}}{\partial z}(t,r,0) \\ v^{(1)}(t,r,0) &= \frac{\partial h}{\partial t}(t,r) \\ v^{(2)}(t,r,0) &= -h(t,r)\frac{\partial v^{(1)}}{\partial z}(t,r,0) \end{aligned}$$

4 Periodic Solutions

In this section we look for time–periodic solutions of (3.17)–(3.19) and (3.20)–(3.22), which may occur if the voice coil moves periodically in time. Moreover, we derive explicit solutions for some special cases assuming that the problem is considered on the unbounded domain $r > 0$, $z > 0$.

There are two reasons to derive explicit solutions even for such an – in some sense – academic model problem. First of all we show that the formal asymptotic expansion yields at least in this case a well–posed problem, which is far away from being trivial. Secondly, the explicit solutions may serve as reference solutions when performing numerical simulations. In particular, as already mentioned in Section 2, introducing a bounded computational domain naturally leads to the problem of numerical boundary conditions, which should not change the general behavior of the solution. Because our systems are hyperbolic we have to attack the problem to define appropriate non–reflecting boundary conditions.

Assuming a time–periodic movement of the membrane, i.e.

$$h(t,r) = e^{i\omega t}\tilde{h}(r)$$

leads to the Ansatz $p^{(1)}(t,r,z) = e^{i\omega t}\tilde{p}^{(1)}(r,z)$ and equivalent for $u^{(1)}$ and $v^{(1)}$. Then the first order system may be written in the form

$$i\omega p^{(1)} + c^2\rho_0\left(\frac{1}{r}\frac{\partial}{\partial r}(ru^{(1)}) + \frac{\partial v^{(1)}}{\partial z}\right) = 0 \tag{4.23}$$

$$i\omega u^{(1)} + \frac{1}{\rho_0}\frac{\partial p^{(1)}}{\partial r} = 0 \tag{4.24}$$

$$i\omega v^{(1)} + \frac{1}{\rho_0}\frac{\partial p^{(1)}}{\partial z} = 0 \tag{4.25}$$

where for simplicity we use the notation $p^{(1)}$ instead of $\tilde{p}^{(1)}$ and the same for the velocity field as well as for the function $\tilde{h}(r)$. The corresponding boundary conditions at $z = 0$ are given by

$$u^{(1)}(r,0) = 0, \qquad v^{(1)}(r,0) = i\omega h(r)$$

First we notice that from (4.24), (4.25) we directly obtain the additional constraint

$$\frac{\partial u^{(1)}}{\partial z} = \frac{\partial v^{(1)}}{\partial r}$$

Now there are two possible ways to proceed: one may decouple the whole system into three independent Helmholtz's equations, namely

$$\begin{aligned}\left(\frac{\partial^2 p^{(1)}}{\partial r^2}+\frac{1}{r}\frac{\partial p^{(1)}}{\partial r}+\frac{\partial^2 p^{(1)}}{\partial z^2}\right)+\frac{\omega^2}{c^2}p^{(1)} &= 0\\ \left(\frac{\partial^2 u^{(1)}}{\partial r^2}+\frac{1}{r}\frac{\partial u^{(1)}}{\partial r}+\frac{\partial^2 u^{(1)}}{\partial z^2}\right)+\left(\frac{\omega^2}{c^2}-\frac{1}{r^2}\right)u^{(1)} &= 0\\ \left(\frac{\partial^2 v^{(1)}}{\partial r^2}+\frac{1}{r}\frac{\partial v^{(1)}}{\partial r}+\frac{\partial^2 v^{(1)}}{\partial z^2}\right)+\frac{\omega^2}{c^2}v^{(1)} &= 0\end{aligned}$$

A second possibility is to rewrite the system above in the form

$$\frac{\partial u^{(1)}}{\partial r}+\frac{\partial v^{(1)}}{\partial z} = -\frac{i\omega}{c^2\rho_0}p^{(1)}-\frac{1}{r}u^{(1)} \tag{4.26}$$

$$\frac{\partial p^{(1)}}{\partial r} = -i\omega\rho_0 u^{(1)} \tag{4.27}$$

$$\frac{\partial p^{(1)}}{\partial z} = -i\omega\rho_0 v^{(1)} \tag{4.28}$$

with corresponding matrix notation for the vector $\Phi^{(1)} := (p^{(1)}, u^{(1)}, v^{(1)})^T$ given by

$$A_1\frac{\partial \Phi^{(1)}}{\partial r}+A_2\frac{\partial \Phi^{(1)}}{\partial z} = B^{(1)}(\Phi^{(1)})$$

The matrices A_1 and A_2 read

$$A_1 := \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad A_2 := \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

and right hand side $B^{(1)}(\Phi^{(1)})$ is

$$B^{(1)}(\Phi^{(1)}) := -i\omega\rho_0\left(\frac{1}{c^2\rho_0^2}p^{(1)}+\frac{1}{i\omega\rho_0 r}u^{(1)}, u^{(1)}, v^{(1)}\right)^T$$

The system (4.26)–(4.28) is strictly hyperbolic, because with $(\xi_1, \xi_2) \neq (0,0)$ the matrix $\xi_1 A_1 + \xi_2 A_2$ has the pairwise distinct real eigenvalues $0, \pm\sqrt{\xi_1^2+\xi_2^2}$ with corresponding linear independent eigenvectors

$$v_1 = \begin{pmatrix} 0\\ \xi_1/\xi_2\\ 1\end{pmatrix} \qquad v_{2/3} = \begin{pmatrix} \pm\sqrt{\xi_1^2+\xi_2^2}\\ 1\\ \xi_1/\xi_2\end{pmatrix}$$

Now we go to the second order system: first of all, the right hand side of the continuity equation is given by

$$\text{R.H.S.} = e^{2i\omega t}\left(-U^{(1)}\cdot\nabla p^{(1)}+\frac{i\omega\gamma}{c^2\rho_0}(p^{(1)})^2\right)$$

and the right hand sides of the momentum equation have a similar structure.

The factor $e^{2i\omega t}$ shows, that the second order system will describe the first harmonic of the sound field and we therefore make the Ansatz $p^{(2)}(t,r,z) = e^{2i\omega t}\tilde{p}^{(2)}(r,z)$ and corresponding expressions for the second order velocity field.

Substituting this Ansatz into (3.20)–(3.22) yields – again omitting the tilde – the system

$$\begin{aligned} 2i\omega p^{(2)} + c^2\rho_0\left(\frac{1}{r}\frac{\partial}{\partial r}(ru^{(2)}) + \frac{\partial v^{(2)}}{\partial z}\right) &= -U^{(1)}\cdot\nabla p^{(1)} + \frac{i\omega\gamma}{c^2\rho_0}(p^{(1)})^2 \\ 2i\omega u^{(2)} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial r} &= -U^{(1)}\cdot\nabla u^{(1)} - \frac{i\omega}{c^2\rho_0}p^{(1)}u^{(1)} \\ 2i\omega v^{(2)} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial z} &= -U^{(1)}\cdot\nabla v^{(1)} - \frac{i\omega}{c^2\rho_0}p^{(1)}v^{(1)} \end{aligned}$$

The corresponding matrix formulation is similar to the one of the first order system, namely

$$A_1\frac{\partial\Phi^{(2)}}{\partial r} + A_2\frac{\partial\Phi^{(2)}}{\partial z} = B^{(2)}(\Phi^{(1)},\Phi^{(2)})$$

In particular, the left hand side is identical, i.e. the second order system is again strictly hyperbolic, and the right hand side contains additionally the coupling with the first order vector $\Phi^{(1)}$. The corresponding boundary conditions now read

$$u^{(2)}(r,0) = -h(r)\frac{\partial u^{(1)}}{\partial z}(r,0) \qquad v^{(2)}(r,0) = -h(r)\frac{\partial v^{(1)}}{\partial z}(r,0)$$

4.1 Explicit solution for $h = \text{const.}$

It is straightforward to compute an explicit solution for the case $h = \text{const}$. We can assume that $u^{(1)}(r,z) = 0$ and the first order system reads

$$\frac{\partial v^{(1)}}{\partial z} = -\frac{i\omega}{c^2\rho_0}p^{(1)} \tag{4.29}$$

$$\frac{\partial p^{(1)}}{\partial r} = 0 \tag{4.30}$$

$$\frac{\partial p^{(1)}}{\partial z} = -i\omega\rho_0 v^{(1)} \tag{4.31}$$

From (4.30) we obtain $p^{(1)} = p^{(1)}(z)$ and (4.31) yields that even $v^{(1)}$ only depends on z. Hence the first order system reduces to the linear ODE system

$$\frac{d}{dz}\begin{pmatrix} p^{(1)} \\ v^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & -i\omega\rho_0 \\ -\frac{i\omega}{c^2\rho_0} & 0 \end{pmatrix}\cdot\begin{pmatrix} p^{(1)} \\ v^{(1)} \end{pmatrix}$$

with general solution

$$\begin{aligned} p^{(1)}(z) &= \alpha^{(1)}\sin\left(\frac{\omega z}{c}\right) + \beta^{(1)}\cos\left(\frac{\omega z}{c}\right) \\ v^{(1)}(z) &= \frac{i}{c\rho_0}\left(\alpha^{(1)}\cos\left(\frac{\omega z}{c}\right) - \beta^{(1)}\sin\left(\frac{\omega z}{c}\right)\right) \end{aligned}$$

where $\alpha^{(1)}$ and $\beta^{(1)}$ are two constants.
The boundary condition $v^{(1)}(0) = i\omega h$ gives $\alpha^{(1)} = c\rho_0\omega h$ and putting $\beta^{(1)} = 0$, which corresponds to the (unphysical) boundary condition $p^{(1)} = 0$, we obtain for the time dependent solutions the expressions

$$\begin{aligned} p^{(1)}(t,z) &= c\rho_0\omega h \cdot e^{i\omega t} \sin\left(\frac{\omega z}{c}\right) \\ v^{(1)}(t,z) &= i\omega h \cdot e^{i\omega t} \cos\left(\frac{\omega z}{c}\right) \end{aligned}$$

Assuming $u^{(2)}(r,z) = 0$ the second order system is given by

$$\begin{aligned} 2i\omega p^{(2)} + c^2\rho_0 \frac{\partial v^{(2)}}{\partial z} &= -v^{(1)}\frac{\partial p^{(1)}}{\partial z} + \frac{i\omega\gamma}{c^2\rho_0}(p^{(1)})^2 \\ \frac{\partial p^{(2)}}{\partial r} &= 0 \\ 2i\omega v^{(2)} + \frac{1}{\rho_0}\frac{\partial p^{(2)}}{\partial z} &= -v^{(1)}\frac{\partial v^{(1)}}{\partial z} - \frac{i\omega}{c^2\rho_0} p^{(1)} v^{(1)} \end{aligned}$$

Due to (4.32) we again get $p^{(2)} = p^{(2)}(z)$ and with (4.32) we can conclude $v^{(2)} = v^{(2)}(z)$. Moreover the right hand side of (4.32) vanishes identically because of (4.29). Hence we obtain the following linear and inhomogeneous ODE system

$$\frac{d}{dz}\begin{pmatrix} p^{(2)} \\ v^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & -2i\omega\rho_0 \\ -\dfrac{2i\omega}{c^2\rho_0} & 0 \end{pmatrix} \cdot \begin{pmatrix} p^{(2)} \\ v^{(2)} \end{pmatrix} + \frac{1}{c^2\rho_0}\begin{pmatrix} 0 \\ \dfrac{i\omega\gamma}{c^2\rho_0}(p^{(1)})^2 - v^{(1)}\dfrac{dp^{(1)}}{dz} \end{pmatrix}$$

Substituting the solution of the first order system gives

$$\frac{i\omega\gamma}{c^2\rho_0}(p^{(1)})^2 - v^{(1)}\frac{dp^{(1)}}{dz} = i\rho_0\omega^3 h^2\left(\gamma\sin^2\left(\frac{\omega z}{c}\right) - \cos^2\left(\frac{\omega z}{c}\right)\right)$$

and the general solution becomes

$$\begin{aligned} p^{(2)} &= \left(\alpha^{(2)} - (\gamma+1)\frac{\rho_0\omega^3h^2z}{4c}\right)\sin\left(\frac{2\omega z}{c}\right) + \left(\beta^{(2)} - \frac{1}{16}(\gamma+1)\rho_0\omega^2h^2\right)\cos\left(\frac{2\omega z}{c}\right) \\ &\quad + \frac{1}{4}(\gamma-1)\rho_0\omega^2h^2 \\ v^{(2)} &= \frac{i}{c\rho_0}\left[\left(\alpha^{(2)} - \frac{1}{4}(\gamma+1)\frac{\rho_0\omega^3h^2z}{c}\right)\cos\left(\frac{2\omega z}{c}\right)\right. \\ &\quad \left. - \left(\beta^{(2)} + \frac{1}{16}(\gamma+1)\rho_0\omega^2h^2\right)\sin\left(\frac{2\omega z}{c}\right)\right] \end{aligned}$$

From the boundary condition $v^{(2)}(0) = -h\dfrac{\partial v^{(1)}}{\partial z}(0) = 0$ we get $\alpha^{(2)} = 0$ and taking again $p^{(2)}(0) = 0$ yields

$$\beta^{(2)} = \frac{1}{16}\rho_0\omega^2h^2(5-3\gamma)$$

The solution now reads

$$\begin{aligned} p^{(2)} &= -\frac{1}{4}(\gamma+1)\frac{\rho_0\omega^3h^2z}{c}\sin\left(\frac{2\omega z}{c}\right) + \frac{1}{4}(\gamma-1)\rho_0\omega^2h^2\left(1-\cos\left(\frac{2\omega z}{c}\right)\right) \\ v^{(2)} &= -\frac{i}{4}(\gamma+1)\frac{\omega^3h^2z}{c^2}\cos\left(\frac{2\omega z}{c}\right) - \frac{i}{8}(3-\gamma)\frac{\omega^2h^2}{c}\sin\left(\frac{2\omega z}{c}\right) \end{aligned}$$

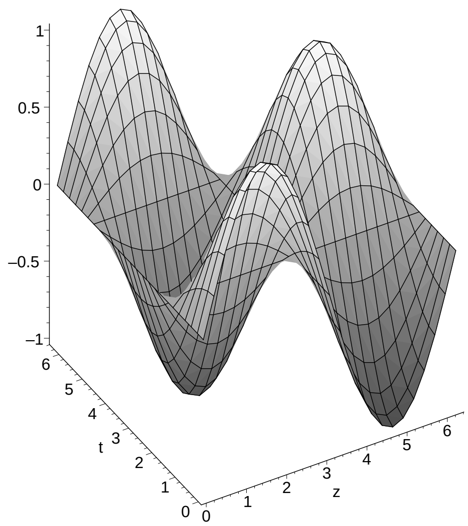

Figure 2: 1st order pressure field.

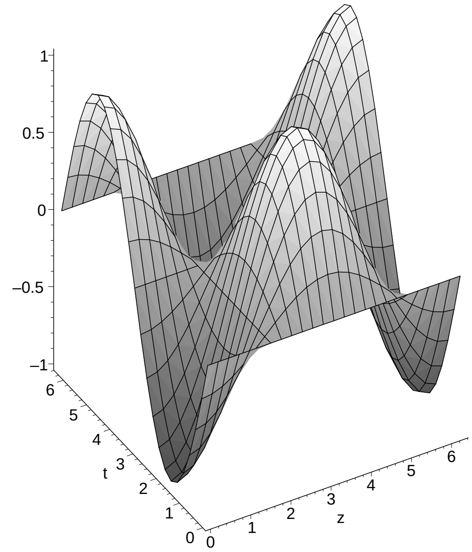

Figure 3: 1st order velocity field.

and the corresponding time–periodic solution is given by

$$p^{(2)} = e^{2i\omega t}\cdot\left[-\frac{1}{4}(\gamma+1)\frac{\rho_0\omega^3h^2z}{c}\sin\left(\frac{2\omega z}{c}\right)+\frac{1}{4}(\gamma-1)\rho_0\omega^2h^2\left(1-\cos\left(\frac{2\omega z}{c}\right)\right)\right]$$

$$v^{(2)} = -e^{2i\omega t}\cdot\left[\frac{i}{4}(\gamma+1)\frac{\omega^3h^2z}{c^2}\cos\left(\frac{2\omega z}{c}\right)+\frac{i}{8}(3-\gamma)\frac{\omega^2h^2}{c}\sin\left(\frac{2\omega z}{c}\right)\right]$$

A graphical representation of the first and second order pressure and velocity field is given in Figures 2–5, respectively. One clearly observe that the second order fields exactly describes the first harmonic in the sound field, i.e. the increasing amplitude in the second order quantities leads to an increase of the distortion factor.

5 Conclusion

In the present paper we derived an asymptotic model to describe higher order acoustic perturbations for a loudspeaker with an axisymmetric enclosure. The geometry of the enclosure was chosen in order to fit to an experimental set–up, which will be used later to validate the asymptotic model by experimental data.

The model studied in the present work is based on an expansion of the isentropic Euler's equation in cylindrical coordinates and leads to inhomogeneous linear hyperbolic systems for the pressure and the velocity field. In particular, the first order system is equivalent to the theory of linear acoustics, i.e. the first order pressure satisfies the standard wave equation or, in the time–periodic case, the Helmholtz's equation. The second order system has the same structure as the first order system, but contains an inhomogeneous right hand side, which couples the second order expansions with the first order ones.

Assuming a time–periodic movement of the membrane we derived the corresponding time–independent systems and show that the second order expansion exactly describe the first harmonic in the sound field. For a simplified model geometry we further obtained an explicit time–periodic solution which should serve as a reference solution when applying numerical simulation techniques.

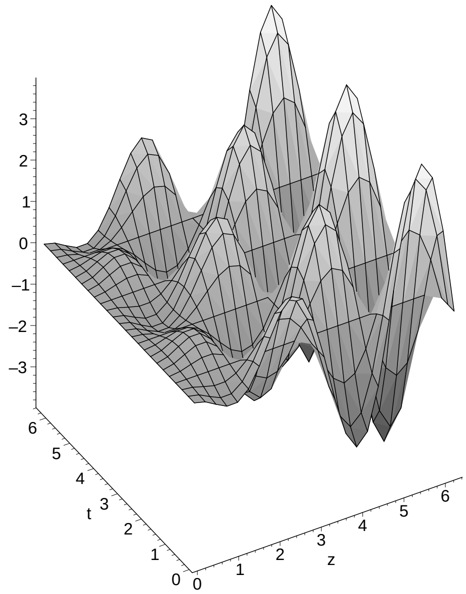

Figure 4: 2nd order pressure field.

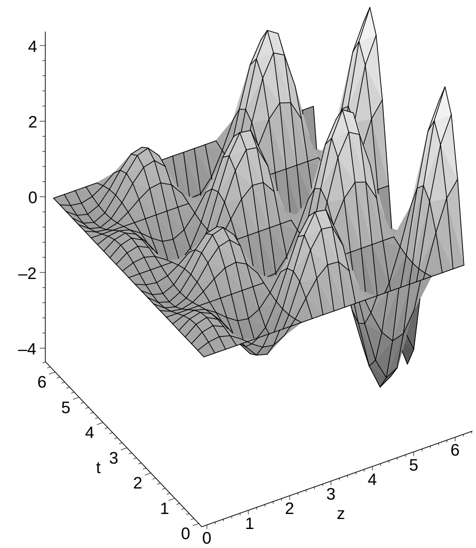

Figure 5: 2nd order velocity field.

The derivation of a numerical scheme seems to be straightforward, because the systems are linear and hyperbolic. Nevertheless, the inhomogeneous right hand side in the second order equation, which contains time– and space–derivatives of the first order solution, maybe strongly influence the quality of standard numerical schemes. The work on this topic is under progress.

Connected with the implementation of a numerical scheme there are several other problems which remain open. As mentioned in Section 2 we did not specify physical reasonable boundary conditions for the pressure field. This is attached to the problem how to define numerical boundary when applying numerical simulation on a bounded computational domain. Another important problem concerns the validation of the asymptotic approach presented above. We only gave a formal derivation without to touch the question whether the asymptotic expansions are valid or under what conditions they will be valid. Hence, remaining on this formal level it will be certainly necessary to validate the model by experimental data.

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