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Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

Convergence of discretization procedures for problems whose entropy solutions are uniquely characterized by additional relations

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Abstract
Weak solutions of given problems are sometimes not necessarily unique. Relevant solutions are then picked out of the set of weak solutions by so-called entropy conditions. Connections between the original and the numerical entropy condition were often discussed in the particular case of scalar conservation laws (e.g. [1], [2]), and also a general theory was presented for general scalar problems ([3], [4]). The entropy conditions were realized by certain inequalities not generalizable to systems of equations in a trivial way. It is a concern of this article to extend the theory in such a way that inequalities can be replaced by general relations, and this not only in an abstract way but also realized by examples.

1 Introduction

Often, (weak) solutions of problems like a system of conservation laws are not necessarily unique. In order to pick out of the set of solutions the particular one relevant from the point of view of the applications under consideration, additional relations to be fulfilled by the seeked solution are added to the original problem. We call a solution that fulfills these additional relations an *entropy solution*, and the set of these additional relations itself is called an *entropy condition*.
In order to ensure that numerical procedures leading to approximate solutions do really approximate the entropy solution instead of another (weak) solution, also the numerical solutions have to fulfill certain additional relations. The set of these conditions will be called a *numerical entropy condition*.

2 The conception

As in [4], let $X, Y, X_n \subset X$ be topological -normally metric- spaces ($n = 1, 2, \cdots$).

Let the originally given problem be written as

$$\tilde{A}\, u \;=\; w \tag{1}$$

with $\tilde{A} \,:\, \tilde{X} \to Y \,,\, \tilde{X} \subset X \,.$

If necessary, we replace problem (1) by a weak representation:

Find $u \in X$ so that

$$A(\hat{\Phi})\, u \;=\; a(\hat{\Phi}) \,,\, \forall\, \hat{\Phi} \in \hat{J} \tag{2}$$

where $\hat{J}$ is an index set, where $\left\{a(\hat{\Phi}) \,|\, \hat{\Phi} \in \hat{J}\right\} \subset Y$ is a given set, and where $\left\{A(\hat{\Phi}) \,|\, \hat{\Phi} \in \hat{J}\right\}$ is a set of operators with joint domain $D \subset X$ and with

$$A(\hat{\Phi}) \;:\; D \to Y \,,\, \forall\, \hat{\Phi} \in \hat{J} \,.$$

The elements $u \in S$ with

$$S \;=\; \{u \in X \,|\, u \text{ solves (2)}\}$$

are called *weak solutions* of (1) or simply *solutions* if the following implications hold:

a) $\tilde{X} \subset D \,\wedge\, u$ solves (1) $\Rightarrow\; u \in S$
b) $u \in S \,\cap\, \tilde{X} \;\Rightarrow\; u$ solves (1) .

We are now going to be only concerned with problem (2). The elements $\hat{\Phi} \in \hat{J}$ are called *test elements* [1].

Assume Z to be a topological space, too, let $\left\{\hat{A}_n \,:\, X_n \to Z \,;\, n = 1, 2, \cdots\right\}$ be a sequence of operators, and let $\{\hat{a}_n\} \subset Z$ be a sequence compact in Z .

For each fixed $n \in N$, we ask for an element $u_n \in X_n$ with

$$\hat{A}_n\, u_n \;=\; \hat{a}_n \quad (n = 1, 2, \cdots) \,. \tag{3}$$

(3) is looked upon as a numerical discretization procedure constructed in order to solve problem (2) approximately. Let this method be suitable, i.e.

$$S_n \;:=\; \{u_n \in X_n \,|\, u_n \,\text{solves (3)}\} \neq \emptyset \quad (n = 1, 2, \cdots) \,, \tag{4}$$

but each S_n is allowed to contain more than one element [2].

The elements $u_n \in S_n \quad (n = 1, 2, \cdots)$ are called *approximate solutions* or *numerical solutions* of problem (2) where suitable connections between the problems (2) and (3) have still to be formulated.

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[1] Here, $\hat{J} \subset X$ can occur.

[2] as it sometimes happens if implicit finite-difference methods are used in order to solve certain differential equations

Problem (3) was expected to be independent of test elements [3]. Nevertheless, we assume that (3) can also be formulated in a weak sense, namely that there is for every $\hat{\Phi} \in \hat{J}$ a sequence of operators

$$\left\{ A_n(\hat{\Phi}) \,|\, \hat{\Phi} \in \hat{J} \,,\, n \in \mathcal{N} \right\}$$

with

$$A_n(\hat{\Phi}) \,:\, X_n \to Y \,,\, \forall \,\hat{\Phi} \in \hat{J}$$

as well as a sequence $\left\{ a_n(\hat{\Phi}) \right\} \subset Y$ with

$$\lim_{n\to\infty} a_n(\hat{\Phi}) \;=\; a(\hat{\Phi}) \tag{5}$$

so that

$$A_n(\hat{\Phi})\, u_n \;=\; a_n(\hat{\Phi}) \,,\, \forall \,\hat{\Phi} \in \hat{J} \,,\, \forall \, u_n \in S_n \quad (n = 1, 2, \cdots) \,. \tag{6}$$

We call formula (6) a *weak formulation of the numerical procedure.*

3 A convergence theorem

Definition 3.1 A pair $[\{C_n\}\,,\, C]$ consisting of an operator sequence $\{C_n\}$ and of an operator C is called *asymptotically closed* if the implication

$$v_n \to v \,\wedge\, C_n\, v_n \to z \quad \Rightarrow \quad C\, v \;=\; z \tag{7}$$

holds.

Definition 3.2 An operator sequence $\{C_n\}$ is called *asymptotically regular* if the implication

$$\{C_n\, v_n\} \quad \text{compact in} \quad Y \quad \Rightarrow \quad \{v_n\} \quad \text{compact in} \quad X \tag{8}$$

holds.

Definition 3.3 The numerical procedure (3) is called *convergent* if set convergence

$$S_n \;\to\; S \tag{9}$$

is ensured in the following sense:

$\{S_n\}$ is *discretely compact*, i.e. each sequence $\{u_n \,|\, u_n \in S_n\,;\, n = 1, 2, \cdots\}$ is compact in X, and if u is the limit of a convergent subsequence, $u \in S$ follows.

Using these definitions, the following theorem can be stated:

Convergence Theorem:

(i) Let $\left[\left\{ A_n(\hat{\Phi}) \right\} \,,\, A(\hat{\Phi}) \right]$ be asymptotically closed for every fixed $\hat{\Phi} \in \hat{J}$;

(ii) let $\left\{ \hat{A}_n \right\}$ be asymptotically regular.

[3] because computers do not understand what test elements are

Then:

$$S_n \to S \tag{10}$$

holds.

For the proof, cf [4].

Remark 3.1: It should be mentioned that the assumption (ii) can be replaced by the weaker assumption

$$\{S_n\} \text{ discretely compact}$$

as far as the right sides $\{\hat{a}_n \mid n = 1, 2, \cdots\}$ are constant:

$$\hat{a}_1 = \hat{a}_2 = \hat{a}_3 = \cdots .$$

In this case, only

$$u \in S \quad \text{for every limit of a subset}$$

has to be shown.

Remark 3.2: If the result (10) is guaranteed, $S \neq \emptyset$ follows so that even the existence of (weak) solutions of the given problem is stated.

4 Existence and uniqueness of entropy solutions

Let R be a relation between X and the set $\hat{J}$. This means that there is a statement concerning ordered pairs $(u\,,\,\hat{\Phi})$ of elements $u \in X\,,\,\hat{\Phi} \in \hat{J}$ so that it can be decided whether or not this statement is true for the given pair.

If it is true, we write

$$uR\hat{\Phi}\,.$$

Assume that there is a uniqueness theorem available of the following type:

There is **at most** one element $u \in S$ with

$$uR\hat{\Phi}\,,\,\forall\,\hat{\Phi} \in \hat{J}\,. \tag{11}$$

Moreover, we assume that there is for each $n \in \mathcal{N}$ a relation R_n between X_n and $\hat{J}$, and **at least** one element $u_n \in S_n$ with

$$u_n R_n \hat{\Phi} \quad , \quad \forall\,\hat{\Phi} \in \hat{J}\,. \tag{12}$$

Finally, let the relations R_n $(n = 1, 2, \cdots)$ be *continuously convergent* to R in the following sense:

For every fixed $\hat{\Phi} \in \hat{J}$, the implication

$$\left\{ u_n \,|\, u_n \in X_n \,,\, u_n R_n \hat{\Phi} \right\} \to u \quad \Rightarrow \quad u R \hat{\Phi} \tag{13}$$

holds.

Theorem:

Under the assumptions of the convergence theorem and of this section, the entropy solution u_E exists uniquely, and each of the sequences $\{u_n \,|\, u_n \in S_n \,;\, n = 1, 2, \cdots\}$ converges to u_E.

Proof: The Convergence Theorem leads to the validity of property (10), i.e. each sequence $\{u_n \,|\, u_n \in S_n \,;\, n = 1, 2, \cdots\}$ contains a convergent subsequence $\{u_{n'} \,|\, n' \in \mathcal{N}' \subset \mathcal{N}\}$ with a certain limit $u \in S$.

Consider now especially a sequence

$$\left\{ u_n \,|\, u_n \in S_n \,;\, u_n R_n \hat{\Phi} \,,\, \forall \hat{\Phi} \in \hat{J} \,;\, n \in \mathcal{N} \right\} \;. \tag{14}$$

Sequences of this type exist because of the assumptions made before, and each of these sequences contains a convergent subsequence.

Take one of these sequences and then one of its convergent subsequences. Denote its limit by u_E . Hence, $u_E \in S$.

From (13),

$$u_E R \hat{\Phi} \,,\, \forall\, \hat{\Phi} \in \hat{J} \tag{15}$$

follows, and because of the uniqueness theorem, the whole sequence (14) converges to this limit u_E, and all sequences of type (14) behave so. Thus, u_E is the unique entropy solution. ■

5 Examples

5.1 One-dimensional scalar conservation law

Let

$$\Omega = \{(x,t) \,|\, x \in R \,,\, t \in [0,T]\} \;,\; X = L_1^{\text{loc}}(\Omega) \;,\; \tilde{X} = C^1(\Omega) \,,\, Y = C(\Omega) \;,$$

$$\tilde{A}\, u = \begin{cases} \partial_t u \,+\, \partial_x f(u) = 0 \,, & f \in C^1(R) \quad \text{strictly convex,} \quad f \geq 0 \,,\, f(0) = 0 \\ u(x,0) = u_0(x) & . \end{cases} \tag{16}$$

Let

$$\hat{J} = \left\{ \hat{\Phi} = (\Phi, c) \,|\, \Phi \in C_0^1(\Omega) := J,\, c \in \mathcal{R} \right\}$$

where J is the space of functions continuous on Ω and with compact support.

Formula (2) will then be realized by

$$\begin{aligned}\left[A(\hat{\Phi})\,u\right](x,t) &= -\int\limits_{\Omega} \left[\partial_t\,\Phi(x,t)\,u(x,t) \,+\, \partial_x\,\Phi(x,t)\,f(u(x,t))\right]\,d\Omega \\ &\quad -\int\limits_{R} \Phi(x,0)\,u_0(x)\,dx \\ &= 0\,.\end{aligned} \tag{17}$$

In order to concretize the numerical procedure (3), we are going to use an explicit one-step three-point FDM in conservation form:
For each fixed $n \in \mathcal{N}$, let $\Delta t = \frac{T}{n}$ the time step size and $\Delta x > 0$ the spatial step size where

$$\sigma = \frac{\Delta t}{\Delta x}$$

is assumed to be a prescribed constant.

With $u^m(x_i)$ expected to become an approximation to the solution $u(i\,\Delta x,\, m\,\Delta t)$ $(i = 0, \pm 1, \pm 2, \cdots;\ m = 0, 1, 2, \cdots)$, the general 3-point scheme is described as

$$u^{m+1}(x_i) = u^m(x_i) - \sigma\left\{g(u^m(x_{i+1}), u^m(x_i)) - g(u^m(x_i), u^m(x_{i-1}))\right\}\,. \tag{18}$$

Here, the *numerical flux* g is assumed to fulfill the *consistency condition*

$$g(v,v) = f(v) \quad \forall\, v \in \mathcal{R}\,, \tag{19}$$

and the numerical initial values are constructed by

$$u^0(x_i) = \frac{1}{\Delta x}\int\limits_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} u_0(\xi)\,d\xi\,. \tag{20}$$

We assume that the *CFL-condition*

$$\sigma\,|f'|^*_\infty \,<\, 1 \tag{21}$$

with $|f'|^*_\infty := \max\left\{|f'(u)|\,,\ |u| \le \|u_0\|_{L_\infty}\right\}$ is fulfilled, too.

$u_n \,\in\, L_1^{\mathrm{loc}}(\Omega) = X$ will then be defined as a piecewise constant function by

$$u_n(x,t) := u^m(x_i) \quad \text{for} \quad \left\{\begin{array}{rlll} x_i \le x < & x_{i+1} & (i & = \;0, \pm 1, \pm 2, \cdots) \\ m\,\Delta t \le t < & (m+1)\,\Delta t & (m & = \;0, 1, \cdots) \end{array}\right. \tag{22}$$

$(n = 0, 1, 2, \cdots)$.

The weak formulation (6) of our particular method (18) can then be read as

$$\int_{\Omega} \Phi(x,t) \left\{ \frac{1}{\Delta t} [u_n(x,t+\Delta t) \quad - \quad u_n(x,t)] + \frac{1}{\Delta x} [g(u_n(x+\Delta x,t), u_n(x,t)) \right.$$
$$\left. - \quad g(u_n(x,t), u_n(x-\Delta x,t))] \right\} d\Omega = 0 \tag{23}$$

so that the requirement (5) is fulfilled naturally.

In order to show the validity of the Convergence Theorem, it suffices to prove that $[\{A_n(\Phi)\}, A(\Phi)]$ is at least asymptotically closed with respect to sequences $\{v_n\}$ with

$$v_n = u_n \in S_n \ , \ (n = 0, 1, 2, \cdots) \ . \tag{24}$$

For these sequences, (7) holds with $z = 0$ so that

$$A(\Phi)\, u \ = \ 0 \tag{25}$$

follows for convergent sequences $u_n \ \to \ u$ by means of the Lax-Wendroff theorem [5].

By the way, (16) - (25) can also be considered as a description of the situation defined by systems of conservation laws provided that the CFL-condition (21) is formulated in a suitable way, and also the Lax-Wendroff theorem holds in this case. We are going to take advantage of that later.

But let now restrict ourselves to the scalar case where we will realize the numerical procedure (3) by means of the monotone Engquist-Osher scheme [2]. The numerical solutions $u_n \in S_n$ are then bounded with respect to the L_1-norm on each compact subset of Ω so that they form an L_1-contraction (cf. [1]) which makes these sequences convergent ones with respect to the L_1^{loc}-topology. (cf. [6]).

In order to ensure that the limit function coincides with the entropy solution, we consider the particular realization of the relation R by

$$uR\Phi \iff$$
$$- \int_{\Omega} \{\partial_t \Phi(x.t)\, V(u(x,t),c) \ + \ \partial_x \Phi(x,t)\, F(u(x,t),c)\}\, d\Omega \tag{26}$$
$$- \int_{R} \Phi(x,0)\, V(u_0(x),c)\, dx \leq 0\,, \qquad\qquad \forall \hat{\Phi} \in \hat{J}\,,$$

where $\{V(\cdot\,,\, c)\,|\, c \in R\}$ is a one-parameter family of real functions which are continuous, convex and piecewise differentiable with respect to x for every fixed $c \in R$.

We choose especially

$$V(u,c) = |u - c|\,, \tag{27}$$

call it the entropy functional and determine the entropy flux $F : \mathcal{R} \times \mathcal{R} \ \to \ \mathcal{R}$ by the reqirement to fulfill weakly

$$\partial_t\, V(u(x,t),c) \ + \ \partial_x\, F(u(x,t),c) = 0 \quad , \forall\, c \in \mathcal{R} \text{ and for every } \textbf{smooth} \text{ solution}\, u\,. \tag{28}$$

In the scalar case, there is at most one (weak) solution u fulfilling the inequality (26) for all $c \in \mathcal{R}$, indeed (cf. [7]) .

Let us now introduce a numerical flux function G by

$$G(\alpha, \beta; c) := F_+(\alpha, c) + F_-(\beta, c)$$

with

$$F_+(\alpha, c) = \begin{cases} F(\alpha, c) & , \quad \alpha \geq c \\ 0 & , \quad \alpha < c \end{cases} \quad , \quad F_-(\beta, c) = \begin{cases} 0 & , \quad \beta \geq c \\ F(\beta, c) & , \quad \beta < c \end{cases} \quad . \tag{29}$$

Because of formulas (27), (28), formula (29) together with (16) leads to

$$F_+(\alpha, c) = \begin{cases} f(\alpha) & , \quad \alpha \geq c \\ 0 & , \quad \alpha < c \end{cases} \quad , \quad F_-(\beta, c) = \begin{cases} 0 & , \quad \beta \geq c \\ -f(\beta) & , \quad \beta < c \end{cases} \quad . \tag{30}$$

If the relations $R_n \;\; (n = 1, 2, \cdots)$ will then be realized as

$$\begin{aligned} & u_n R_n \hat{\Phi} \quad \Longleftrightarrow \\ & \int_\Omega \Phi(x,t) \quad \Big\{ \tfrac{V(u_n(x,t+\Delta t),c) - V(u_n(x.t),c)}{\Delta t} \\ & \qquad + \tfrac{G(u_n(x,t), u_n(x+\Delta x,t); c) - G(u_n(x-\Delta x,t), u_n(x,t); c)}{\Delta x} \Big\} \, d\Omega \leq 0 \,, \\ & \qquad\qquad\qquad\qquad \forall c \in \mathcal{R}, \end{aligned} \tag{31}$$

the convergence property (13) will hold as it was shown in [3], [4].

6 One-dimensional systems of conservation laws

Let us now look at systems of conservation law problems of the type (16) and of two or more equations, say r equations $(2 \leq r \in \mathcal{N})$.
And these systems are assumed to be strictly hyperbolic and genuinely nonlinear where smooth solutions fulfill the equation (28) automaticly with a strictly convex function V.

Let the numerical solution be computed by means of the Glimm-finite-difference scheme (cf. [8]). And it follows from [8] that these numerical solutions converge for increasing n, i.e. for decreasing step sizes, to a (weak) solution u^E of the original problem. Moreover, u^E fulfills the Lax entropy conditions

$$\begin{aligned} \lambda_{k-1}(u_l^E) \;\; &< s < \;\; \lambda_k(u_l^E) \\ \lambda_k(u_r^E) \;\; &< s < \;\; \lambda_{k+1}(u_r^E) \end{aligned} \tag{32}$$

for an integer $k \in \{2, \cdots, r-1\}$.

Here, the values $\lambda_1, \cdots, \lambda_r$ are the eigenvalues of the Jacobian of the flux f, s is the velocity of a k-shock, and u_l^E , u_r^E are the values of u^E at the left or at the right side of this shock, respectively. We suppose that the inequalities (32) guarantee the uniqueness of u^E as it is suggested by arguments of information theory.

But (32) only holds if and only if the inequality

$$(V_l - V_r)s \leq F(V_l) - F(V_r) \tag{33}$$

holds along the shock, and this property is equivalent to the inequality (26), i.e. to

$$uR\hat{\Phi} \; .$$

Moreover, the Glimm scheme fulfills for positive step sizes also the relations

$$u_n R_n \hat{\Phi} \quad (n = 1, 2, \cdots)$$

because of (31), so that the considerations concerning the scalar case can immediately be transfered to the situation studied here (cf. [9], p. 337).

Here, in the case of the Glimm scheme, the numerical flux G reads as

$$\begin{aligned} \hat{J} &= J \\ G(u_i^n \, , \, u_{i+1}^n) &:= \frac{1}{\Delta t} \int\limits_{t_n}^{t_{n+1}} F\left(v^n(x_i + \frac{\Delta x}{2} \, , \, t) \right) \, dt \end{aligned} \tag{34}$$

where $v^n(x,t)$ solves the local Riemann problem

$$\begin{aligned} &\partial_t v^n + \partial_x f(v^n) = 0 \quad \text{on} \quad [x_i, x_{i+1}] \times [t_n, t_{n+1}] \\ &\text{where} \\ &v^n(x, t_n) = \begin{cases} u_i^n & \text{for} \quad x < x_{i+\frac{1}{2}} \\ u_i^{n+1} & \text{for} \quad x > x_{i+\frac{1}{2}} \end{cases} \quad . \end{aligned} \tag{35}$$

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