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Polygonal Representations of Digital Sets

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Abstract

In the context of discrete curve evolution the following problem is of relevance: Decompose the boundary of a plane digital object into convex and concave parts. Such a decomposition is very useful for describing the form of an object e.g. for shape databases. Although the problem is relatively trivial in ordinary plane geometry, in digital geometry its statement becomes a very difficult task due to the fact that in digital geometry there is no simple set–complement duality.

The paper is based on results given by Hübler, Klette and Voss [11, 11]. The main new contribution of the paper is the generalization of the concepts introduced by these authors to nonconvex sets.

The digital geometric 'low level' segmentation of the boundary of a digital object can be used as a starting basis for further reduction of the boundary by means of discrete evolution.

Keywords: digital geometry, digital convexity, polygonal representation, shape simplification

1 Introduction

The shape of a two–dimensional object can be represented by its boundary contour. For recognition of objects, their shapes have to be compared with given shapes e.g. from a database. Such investigations became relevant in connection with the MPEG–7 standardization [15, 16, 19]. In order to keep the computational amount for comparing shapes tractable, shapes are simplified either by continuous [29]) or by discrete evolution [15]. The latter approach has many advantages, there is no type change from the original object to the evolved objects, the evolved objects are trivially localized with respect to the original object in an image or a scene and properties of the original object are inherited by the evolved objects. One such property is the convexity or concavity of boundary parts. In an earlier paper [17] the decomposition of a boundary curve into convex and concave parts was performed by segmenting the boundary contour into digital line segments. Lakämper [14] was able to show that — under certain conditions — the digital evolution process has the property to identify digital lines before eliminating their mutual intersection points. So it seemed not to be necessary to deal with the detection of digital lines separately, they are detected anyway by discrete evolution 'automatically'.

The aim of this paper is to investigate the discrete evolution process and its properties specifically with respect to convex and concave parts of the boundary from a low–level point of view. Assume that we are given a shape S_Δ, which is (without loss of generality) a simply connected set in the digital plane $\mathbb{Z}^2$ (throughout this paper S_Δ denotes a subset of the digital plane whereas S is a set in $\mathbb{R}^2$). The question is raised: Can the digital set S_Δ be represented by a polygon in the plane $\mathbb{R}^2$ whose vertices are vertices of the (digital) boundary contour of the given set S_Δ such that the representing polygon is a Jordan contour in $\mathbb{R}^2$ which contains exactly the points of S_Δ in its interior? Furthermore, one wants the polygonal representing contour to be "minimal" in a certain sense. When such a set can be found easily, then one has the advantage of a reduction of data for representing the shape. If the vertices of the polygonal set are taken from the boundary of S_Δ, then the polygon can be represented by an ordered subset of the original representation of the boundary of S_Δ (e.g. the chain code of the boundary). Finally, the vertices of the representing polygon can be classified in an obvious way into convex and concave vertices.

In digital geometry [13] it is not a simple task to decide (digital) convexity of a set. More precisely, in 1928 Tietze proved a remarkable Theorem [24] stating that convexity of a set in $\mathbb{R}^2$ can be decided locally in a time which is proportional to the length of the boundary of the set. In $\mathbb{Z}^2$ one can easily see that convexity of a set cannot be decided locally [6]. So, it becomes an interesting question, how far one can decide whether a part of the boundary of a set is convex or not by a method which is "as local as possible".

There is a different aspect which plays a role in the subsequent discussion. In 1987 Scherl proposed a method in the context of document analysis [23] which was based on sets of descriptors which were obtained as points of local support with respect to a certain finite number of directions. This approach has practical advantages as well as theoretically appealing properties. In a certain sense, Scherl's descriptors segment the boundary of a set into parts which are "suspicious candidates" for being convex or concave parts.

Mokhtarian [19] (and others) describe a shape in terms of the "Curvature Scale Space". In the context of the present investigation we are dealing exclusively with polygons. For polygons the concept of curvature is rather trivial and one can ask how to relate the discrete evolution to its continuous counterpart. One step in this direction is the subdivision of the boundary of a set into convex and concave parts [17].

2 Definitions

2.1 The Digital Plane

The *digital plane* $\mathbb{Z}^2$ is the set of all points in the plane $\mathbb{R}^2$ having integer coordinates.

Given a point $P = (m,n)^\top \in \mathbb{Z}^2$. The 8–*neighbors* of P are all points with integer coordinates $(k,\ell)^\top$ such that

$$\max(|m-k|, |n-\ell|) \leq 1.$$

We number the 8–neighbors of P in the following way:

	$m-1$	m	$m+1$
$n+1$	$N_3(P)$	$N_2(P)$	$N_1(P)$
n	$N_4(P)$	P	$N_0(P)$
$n-1$	$N_5(P)$	$N_6(P)$	$N_7(P)$

Neighbors with even number are the *direct* or 4–*neighbors* of P, those with odd numbers are the *indirect neighbors*. The 8–*neighborhood* $\mathcal{N}_8(P)$ of P is the set of all 8–neighbors of P (excluding P), the 4–*neighborhood* $\mathcal{N}_4(P)$ of P is the set of all 4–neighbors of P.

We can introduce into the digital plane a "topology" [21]. A subset $S_\Delta \subseteq \mathbb{Z}^2$ is termed ρ–*connected* ($\rho \in \{4,8\}$) if for each two points $P, Q \in S_\Delta$ there is a sequence of points $P_0 = P, P_1, \cdots, P_n = Q$ with $P_j \in S_\Delta$ for $j = 0,1,\cdots,n$ and if $P_{j+1} \in \mathcal{N}_\rho(P_j)$ for $j = 0,1,\cdots,n-1$.

We assume here that the sets under consideration are equipped with the 8–topology and the complements of them with the 4–topology. Let S_Δ be an 8–connected subset of the digital plane and $\mathbb{C}S_\Delta$ its complement. The *boundary* of S_Δ is the set

$$\text{bd } S_\Delta := \{P \in S_\Delta \text{ and } P \text{ is a 4–neighbor of } \mathbb{C}S_\Delta\}.$$

On the set bd S_Δ we define a binary *successor relation*: For points $P, Q \in$ bd S_Δ holds suc (P, Q) (Q is a successor of P) if and only if

1. Q is an 8–neighbor of P, $Q = N_i(P)$, and
2. for all points $R \in S_\Delta$ which are 8–neighbors of both P and Q (i.e. $R \in S_\Delta \cap \mathcal{N}(P) \cap \mathcal{N}(Q)$) is $R = N_j(P)$ with $j = i + 1 \pmod 8$ or $j = i + 2 \pmod 8$.

The *border* or *oriented boundary* of a digital set S_Δ is the boundary of S_Δ equipped with the successor relation suc. When moving along the border, the points of S_Δ are always on the left–hand side of the oriented boundary.

Rosenfeld ([21], see also [7]) was able to show that this definition leads to a Jordan Curve Theorem in the digital plane. If S_Δ is a digital set whose boundary is a closed simple (not self–intersecting) and 8–connected set, then the boundary separates the digital plane into two components, the 8–connected points of S_Δ and the 4–connected points of the complement of S_Δ. Moreover, Rosenfeld showed that the (8–) connected components of the boundary of a set can be oriented in such a way that each boundary point has exactly one successor and exactly one predecessor on the oriented boundary (see [21, 8]).

In the sequel we concentrate exclusively on one single connected component of the border of a set. We may therefore assume without loss of generality that S_Δ is a bounded simply connected set. Moreover we assume that the border of S_Δ is oriented in such a way that S_Δ is always on the left hand side of the oriented border.

In continuous geometry and topology of the plane there exists a useful duality relation: If we replace a set S_Δ by its complement and if the boundary of S_Δ is a (finite or infinite) Jordan curve, then the orientiation of the boundary curve is simply inverted. Duality arguments are very convenient for defining convex and concave parts of the boundary of a set. Since it is very simple to agree on a notion of convexity of a boundary part, concave parts are simply defined as convex parts of the complement. The situation is not so simple and clear in discrete topology. The set and its complement have different topologies, to begin with. Furthermore, unlike in ordinary topology, the border of (the interior of) the set and the border of the (interior of the) complement may be essentially different.

In Figure 1 a part of the border of a set is shown. The point P_1 has two 8–successors in the border, P_2 and P_3. The border following algorithm (see [21, 8]) yields uniquely the succession $P_1 \to P_2 \to P_3$. If, however, the orientation of the border is reversed, P_2 is no longer a legal boundary point (of the complement) and we get the succession $P_3 \to P_1$. Consequently, if we apply the duality relation again, the information about P_2 is lost. In order to maintain perfect duality, the information that P_2 belongs to the border of the original set must be maintained explicitly by the duality transform. These pecularities of digital geometry are related to the fact that there are — in a certain sense — only “few” digital 8–curves possible in the digital plane [8].

A digital border can be described by means of a simple compact cyclically ordered data structure containing the coordinates of one of its points and a sequence of code numbers in {0, 1, 2, 3, 4, 5, 6, 7} indicating for each point on the contour which of its neighbors will be the next point on the contour. This data structure was proposed by Freeman [10] and is known as the *chain code*.

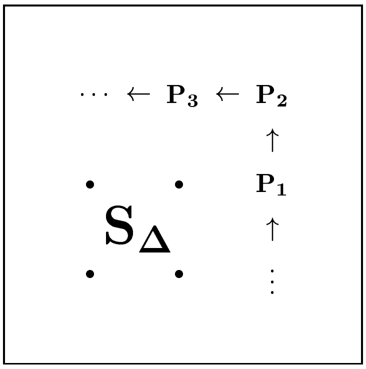


Figure 1: Part of the border of a set.

Arrows indicate the orientation of the border. If the orientation of the border is reversed, point P_2 is no longer a boundary point.

2.2 Polygonal Representation of Digitally Convex Sets

Digital sets in $\mathbb{Z}^2$ can be described by means of polygonal sets in $\mathbb{R}^2$. We therefore state some important properties of polygonal sets in the plane.

Definition 2.1 *A* polygonal curve $\Pi = (V, E)$ *in* $\mathbb{R}^2$ *consists of a cyclically ordered set of vertices* $V = (v_0, v_1, \cdots v_n) \subseteq \mathbb{R}^2$ *and a set of edges* $E \subseteq V \times V$. *E is the set of all line segments joining* v_i *and* v_{i+1}, $i = 0, 1, \cdots, n-1$.

Usually it is assumed that there are finitely many vertices. Sometimes also infinite edges are allowed.

A polygonal curve $\Pi = (V, E)$ *is*

bounded *if* V *is a finite set and if there are no infinite edges.*

closed *if each vertex point belongs to exactly two edges.*

simple *if two edges are either disjoint or meet in a vertex point, and if* $v_i \neq v_j$ *for* $i \neq j$.

Remark 2.1 *The famous* Jordan Curve Theorem *states that any simple closed (bounded) polygonal curve* Π *separates the plane into the interior and the exterior with respect to the curve.*

More formally: The set $\mathbb{R}^2 \setminus \Pi$ *consists of exactly two disjoint connected components. One of them is declared as the interior and the other as the exterior with respect to* Π *(see e.g. [1, §8.9]).*

There is no difficulty to treat infinite polygonal curves. In order to do this we have to allow for infinite edges.

Definition 2.2 *A* polygonal set Π *is a finite set* $\Pi_1, \Pi_2, \cdots, \Pi_n$ *of simple closed curves which are mutually disjoint.*

Remark 2.2 *If we assume that the polygonal curves Π_i defining a polygonal set Π are bounded, then there is a unique unbounded connected component. We may define a subset $S \subseteq \mathbb{R}^2$ by requiring that the unbounded connected component does not belong to S. Given any point $x \in \mathbb{R}^2$ then it is defined to belong to S if and only if any curve joining it to the unbounded connected component meets an odd number of polygonal curves. Thus, a polygonal set as defined above specifies uniquely a bounded subset S of $\mathbb{R}^2$.*

We assume that the polygons under consideration are oriented in such a way that the set which is defined by them is always on the left hand side of the oriented polygons. So, given a polygonal set Π, we can speak about interior points with respect to Π. We do not distinguish in notation between the polygonal boundaries and the sets defined by them. Furthermore, we assume that the boundaries belong to the set they represent (that means we consider topologically closed polygonal sets). In this context, the symbol "$x \in \Pi$" is well–defined.

For a polygonal set we can define in an obvious way such topological concepts like connected components, simply connected *polygonal sets or* holes *of a set.*

Definition 2.3 *Let Π be a polygonal curve and let $P_1 = (p_x^{(1)}, p_y^{(1)})$, $P_2 = (p_x^{(2)}, p_y^{(2)})$ and $P_3 = (p_x^{(3)}, q_x^{(3)})$ be three successive vertices on Π (in the order given by the orientation of the curve). Then Π is said to perform a* left turn *at point P_2 (or P_2 is a* convex vertex *of Π or P_2 is on the* right side *of the oriented line through P_1 and P_2) if the determinant*

$$\begin{vmatrix} p_x^{(2)} - p_x^{(1)} & p_x^{(3)} - p_x^{(2)} \\ p_y^{(2)} - p_y^{(1)} & p_y^{(3)} - p_y^{(2)} \end{vmatrix}$$

is positive. If the determinant is negative, Π performs a right turn *(or P_2 is a* concave vertex *of Π or P_2 is on the* left side *of the oriented line through P_1 and P_2). If the determinant vanishes, then points P_1, P_2 and P_3 are collinear.*

Definition 2.4 *Let Π be a polygonal curve. A part of the curve is said to be a* convex part *if it consists of a set $P_1, P_2, \cdots, P_k$ of successive vertices of Π together with the lines joining them such that P_1 and P_k are concave vertices of Π and $P_2, P_3, \cdots, P_{k-1}$ are convex vertices of Π. If Π has only convex vertices, the only convex part of Π is Π itself.*

A concave part *of Π is defined in the same manner by replacing in the above definition the terms 'convex' and 'concave'.*

Remark 2.3 *We here have a perfect duality: If we replace a polygonal set by its complement, the orientation of the boundary is reversed and convex parts become concave parts and vice versa.*

Definition 2.5 *Given a (without loss of generality) finite digital set $S_\Delta \subseteq \mathbb{Z}^2$. A* polygonal representation *of S_Δ is a polygonal set $\Pi = (V, E)$ such that*

$$x \in S_\Delta \qquad \Longleftrightarrow \qquad x \in \Pi \cap \mathbb{Z}^2.$$

A polygonal set representing a digital set is not unique. We are free to require a number of additional properties. The polygonal representation Π of a digital set S_Δ is said to be

discrete if all vertices of Π are in $\mathbb{Z}^2$,

topological if any two points $P, Q \in S_\Delta$ which can be joined by a 8–path $P = P_0, P_1, \cdots, P_m = Q$ such that $P_i \in S_\Delta$ for $i = 0, 1, \cdots, m$ and P_{i-1} and P_i are 8–neighbors for $i = 1, 2, \cdots, m$ belong to the same connected component of the polygonal set Π.

faithful if the convex parts of the boundary of S_Δ correspond to convex parts of the boundary of Π and if the same is true for the concave parts (here the terms "convex part" and "concave part" of a digital set are used intuitively, they will be defined later),

The chain code representation of a digital set is a discrete topological polygonal representation having a maximal number of vertices. It is, however, not faithful (see Figure 7).

Now we introduce a concept which is fundamental for this paper.

Definition 2.6 *A set $S_\Delta \subseteq \mathbb{Z}^d$ is termed digitally convex (or convex for short), whenever*

$$S_\Delta = \mathbb{Z}^d \cap \operatorname{conv} S_\Delta.$$

Where the symbol conv S *denotes the* convex hull *of a set S which is the smallest convex set containing S.*

We need a concept from ordinary convexity theory in the plane:

Definition 2.7 *Given a convex set $S \subseteq \mathbb{R}^d$ (i.e. $x \in S$, $y \in S$ $0 \leq \lambda \leq 1 \Longrightarrow \lambda x + (1-\lambda) y \in S$). A point $x_0 \in S$ is an* exposed point *of S if there exists a nonzero vector x^* such that $\langle x_0, x^* \rangle > \langle x, x^* \rangle$ for all $x \in S \setminus \{x_0\}$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$.*

If S_Δ is digitally convex then P is an exposed point of S_Δ if it is an exposed point of conv S_Δ.

The set $[x^* : \langle x_0, x^* \rangle] := \{x \in \mathbb{R}^d \mid \langle x_0, x^* \rangle = \langle x, x^* \rangle\}$ is termed a supporting hyperplane *at S in x.*

It is well–known from convexity theory that the convex hull of a polygonal set is the convex hull of its exposed points [26].

If a set S_Δ is digitally convex, there exists a uniquely determined polygonal representation which is discrete and faithful. This representation is the convex hull of S_Δ.

For digitally convex sets there exists a simple characterization of exposed points:

Lemma 2.1 *Let S_Δ be a digitally convex set. A point $P \in S_\Delta$ is an exposed point of S_Δ if and only if P is a vertex of each discrete polygonal representation of S_Δ.*

Proof
1. First assume that S_Δ is a digitally convex set and that $P \in S_\Delta$ is a vertex of each discrete polygonal representation of S_Δ. Since conv S_Δ is a polygonal representation of S_Δ, P is also a vertex of conv S_Δ. The exposed points of a convex set are exactly its vertices, hence P is an exposed point of conv S_Δ and also of S_Δ.
2. If P is an exposed point of S_Δ then there exists a nonzero vector x^* such that $\langle P, x^* \rangle > \langle Q, x^* \rangle$ for all $Q \in S_\Delta \setminus \{P\}$. Let Π be any discrete polygonal representation of S_Δ. Then $P \in \Pi$ and all vertices of Π are points of S_Δ and thus belong to the set (half space)

$$H := \left\{x \in \mathbb{R}^d \mid \langle P, x^* \rangle \geq \langle x, x^* \rangle\right\}.$$

This in turn implies $\Pi \subseteq H$. The point P is a (topological) boundary point of H and consequently it is also a boundary point of Π. In $\mathbb{R}^2$ this implies that P belongs to a edge joining two vertices of Π. If P were an interior point of such an edge, both vertices must belong to the boundary of H. However, the only vertex of Π which belongs to the boundary of H is P itself by definition of an exposed point. Hence, P is a vertex of Π $\square$

A problem which is closely related to the concept of digital convexity is the adequate definition of digital line segments. These are parts of the boundary which are models of real line segments. Rosenfeld [22] gave a characterization of such digital line segments (see also [20]). Hübler, Klette and Voss [12] characterized digtal line segments algorithmically. Debled–Renesson and Reveillès [5] presented a different characterization.

3 Scherl's Descriptors

Scherl [23] proposed in 1987 a method for representing digital sets. This method was developed for document processing applications. An "object" in this context is a connected component S_Δ in a binary document image. Scherl introduced so–called *shape descriptors* which are border points belonging to local extrema of linear functionals corresponding to the main directions in the digital plane (0^o, 45^o, 90^o, 135^o, 180^o, 225^o, 270^o and 315^o).

Definition 3.1 *Let $P_0, P_1, P_2, \cdots, P_{\nu-1}, P_\nu, P_{\nu+1}$ be successive points on the oriented border. For $k = 0, 1, 2, \cdots, 7$ define linear functionals*

$$\ell_k(x, y) = -x \cdot \sin k \cdot \frac{\pi}{4} + y \cdot \cos k \cdot \frac{\pi}{4}.$$

The border points P_1 and P_ν are termed descriptor points of type k *whenever the following conditions are true:*

1. $\ell_k(P_1) = \ell_k(P_2) = \cdots = \ell_k(P_\nu)$,
2. $\ell_k(P_0) < \ell_k(P_1)$
   or $\ell_k(P_0) = \ell_k(P_1)$ *and* $P_0 = N_{k+4 \pmod 8}(P_1)$
3. $\ell_k(P_{\nu+1}) < \ell_k(P_\nu)$
   or $\ell_k(P_{\nu+1}) = \ell_k(P_\nu)$ *and* $P_{\nu+1} = N_k(P_\nu)$.

*The border points P_1 and P_ν are termed T–*descriptor points of type k *(*Top–*descriptor points) or T_k–points if in addition $N_{k+2 \pmod 8}(P_1) \notin S_\Delta$ and $N_{k+2 \pmod 8}(P_\nu) \notin S_\Delta$. They are termed S–*descriptor points of type k *(*Saddle–*descriptor points) or S_k–points if $N_{k-2 \pmod 8}(P_1) \notin S_\Delta$ and $N_{k-2 \pmod 8}(P_\nu) \notin S_\Delta$.*

If P is a descriptor point of type k then the descriptor tangent *belonging to P is the set*

$$\{(x, y) \mid \ell_k(x, y) = \ell_k(P)\}.$$

We note that by definition of the border each descriptor point is a T–descriptor point or an S–descriptor point (or both). Each descriptor tangent meets S_Δ in one or more points. The first and the last points on the intersection of S_Δ and a descriptor tangent are the descriptor points belonging to the descriptor tangent. Whenever S_Δ is a convex set it has only T–descriptor points and these are exactly the exposed points of S_Δ. It is possible to give a characterization of sets without S–descriptor points. In order to do this we need two more Definitions.

Definition 3.2 *A pair of descriptor points P_1 and P_ν is* nondegenerate *if in Definition 3.1 in both cases the strict inequality sign is true, i.e. if $\ell_k(P_0) < \ell_k(P_1)$ and $\ell_k(P_{\nu+1}) < \ell_k(P_\nu)$.*

Definition 3.3 *A digital set S_Δ is* d–convex *(*directional convex*) if it is 8–connected and if all intersections of S_Δ with horizontal, vertical or diagonal grid lines are 8–connected.*

Lemma 3.1 *Let S_Δ be an 8–connected digital set which has only nondegenerate descriptor points.*

Then S_Δ is d–convex if and only if all its descriptor points are T–descriptor points.

Proof 1. Assume that S_Δ is not d–convex. Then there exists a grid line which intersects S_Δ in two disjoint 8–components. We assume without loss of generality that this grid line is vertical. The functional $\ell_2(x, y) = -x$ is constant on each vertical grid line.

Let P_1 be the lowest point of the upper component and P_2 be the highest point of the lower component on the vertical line. We assume for sake of simplicty that all points on the vertical line between P_1 and P_2 are not in S_Δ. Since S_Δ was assumed to be 8–connected, there exists an 8–path in S_Δ joining P_1 and P_2. Let P be a grid point which is between P_1 and P_2 on the vertical grid line. Then the horizontal grid line through P meets the 8–path. There exists between P and the intersection point of the horizontal line with the path a boundary point of S_Δ such that all points between this boundary point and P are not in S_Δ. Among these boundary points we select one which renders the functional ℓ_2 extremal. The extremal value is different from $\ell_(P_1)$. If the extremum is not a maximum we take the functional ℓ_6 instead. However, to a maximum of a functional ℓ_k there corresponds a pair of descriptor points. These descriptor points are by construction S–descriptor points.

2. Assume that there exist nondegenerate S–descriptor points P_1 and P_ν. Without loss of generality we can assume that the descriptor pair is of type $k = 0$. Since the descriptor pair is a S–descriptor pair, the points $N_6(P_1)$ and $N_6(P_\nu)$ are not in S_Δ. Since the descriptor points are nondegenerate, the predecessor of P_1 is $P_0 = N_5(P_1)$ and the successor of P_ν is $P_{\nu+1} = N_7(P_\nu)$. Then the horizontal line through $N_6(P_1)$ and $N_6(P_\nu)$ contains P_0 and $P_{\nu+1}$ which are in S_Δ and between these points there are $N_6(P_1)$ and $N_6(P_\nu)$ which are not in S_Δ, consequently S_Δ is not d–convex. □

Remark 3.1 *If the assumption of nondegeneracy is not fulfilled, the second part of the proof of the Lemma is not necessarily valid as the example in Figure 2 demonstrates.*

The ordered sequence of descriptors on the boundary carries information about the shape of the object and allows a rough reconstruction of it. Moreover, the descriptors have a very interesting interpretation in the image space of the Hough transform [9]. The descriptors can be computed in a time proportional to the length of the boundary. The succession of descriptors on the oriented boundary is not arbitrary (see [23, Section 5.2.1]) as is stated in the next Lemma.

Lemma 3.2 *Let P be a descriptor point on the border and let Q be the descriptor point following P on the oriented border. Then the following situations are possible:*

- *P and Q are both T_k (S_k–) points of the same type k. Then the border segment between P and Q has only chain code directions $k + 4 \pmod 8$ ($k \pmod 8$).*

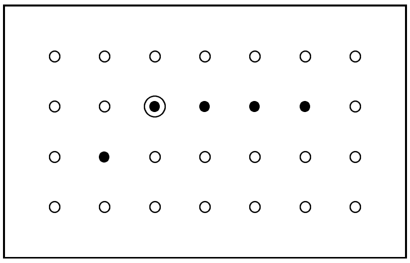


Figure 2: Example of a d–convex set.

The set S_Δ (points $\bullet$) is d–convex. The point marked $\odot$ is an S–descriptor point (as well as a T–descriptor point).

- *P is a T_k (S_k–) point and Q is a $T_{k+1 \pmod 8}$– ($S_{k-1 \pmod 8}$–) point. Then the border segment between P and Q has only chain code directions $k+4 \pmod 8$ and $k+5 \pmod 8$ (k and $k+1 \pmod 8$).*
- *P is a T_k– (S_k–) point and Q is an $S_{k+4 \pmod 8}$– ($T_{k+4 \pmod 8}$–) point. Then the border segment between P and Q has only chain code directions $k+4 \pmod 8$ and $k+5 \pmod 8$ (k and $k+1 \pmod 8$).*

Remark 3.2 *There exists the possibility that two (or more) descriptor points coincide. When we assume that the descriptors belonging to one point are sorted according to the cyclic order ($k_1 \preceq k_2 \iff k_2 - k_1 \in \{0,1,2,3,4\} \bmod 8$), then the assertions of the Lemma remain trivially true.*

Proof Let $i \in \{0, 1, \cdots, 7\}$ be the chain code direction leading from a point P to its successor on the oriented border. When moving from P to its successor, the functional ℓ_k is changed by the value given in Table 1.

We start the proof by considering the case that P is a descriptor point of type $k = 0$. By definition of a descriptor point, the functional ℓ_0 decreases when we move from P to its successor on the border and it does not decrease by moving from the prececessor of P to P. In Table 2 all 15 possible combinations of nondecreasing moves to P and decreasing steps to the successor of P are shown. We see that 4 of these combinations are not legal moves on the border (these are the moves numbered 1, 2, 5 and 9). From the remaining combinations 5 (moves 8, 11, 12, 14 and 15) yield the sitation that functional ℓ_1 strictly increases in the first step and strictly decreases in the following step, so that P is not only a T_0–point by assumption but also a T_1–point (note that in all these cases the 3–neighbor of the middle point of the configuration is not in S_Δ). In these cases the second assertion of the Lemma is true with $P = Q$.

Configurations 7, 10 and 13 have the property that ℓ_1 increases in the first step and remains constant in the second step. In all three cases the the second step has chain code direction 5. We see from Table 1 that only chain code directions 4 and 5 have the property that ℓ_0 is not increased and ℓ_1 is not decreased. Whenever a direction from the set $\{6, 7, 0\}$ occurs when moving from the successor of P along the boundary, then ℓ_1 is strictly decreased and we get a local maximum of ℓ_1, which means that we have a descriptor point of type 1. By inspecting all three cases we see that always the 3–neighbor of the maximizing point for ℓ_1 is not in S_Δ, hence this point is a T_1–point.

i	direction		type k							
			0	1	2	3	4	5	6	7
0	1	0	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$
1	1	1	1	0	-1	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$
2	0	1	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$
3	-1	1	1	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$	-1	0
4	-1	0	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$
5	-1	-1	-1	0	1	$\sqrt{2}$	1	0	-1	$-\sqrt{2}$
6	0	-1	-1	$-\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$
7	1	-1	-1	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	1	0

Table 1: Change of ℓ_k by a chain code step in direction i.

If a direction from the set $\{1, 2, 3\}$ occurs when moving from the successor of P along the boundary, then ℓ_0 is strictly increased which means that we have a strict local minimum of ℓ_0. This means that ℓ_4 has a strict local maximum. Detailed discussion of this case implies that this maximum point is a S_4–point.

There remain the configurations 3, 4 and 6. Configurations 3 and 6 are configurations of S_0–points. As above we first note that functionals ℓ_0 and ℓ_3 both do not increase simultaneously only for directions 0 and 7. Since direction 7 is diametral to direction 3, functional ℓ_7 does not decrease for directions 0 and 7. As above we argue that the S_0–point is followed by an S_7–point or a T_4–point and that between these two descriptor points only directions 0 and 7 occur.

In the case of configuration 4 the middle point is a T_0–, T_6– and T_7–point. There can be a number of subsequent directions 5 and then any other direction. If direction 1 follows, we have then the middle point is also a T_1– and T_5–point. If other directions follow, the point following the last direction 5 is either a T_1–point (directions 6, 7, 0) or a S_4–point (directions 2, 3, 4).

If we have a descriptor point of type 2, 4 or 6, we get the assertion of the Lemma by applying a trivial rectangular transformation of the digital plane into itself.

For a descriptor point of odd type $k = 1, 3, 5, 7$ we observe that the transformation with matrix

$$D_6 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

is a bijection from $\mathbb{Z}^2$ onto itself which maps chain code the direction 0 on direction 1. $\square$

In Figure 3 a digital set together with its descriptor tangents is shown.

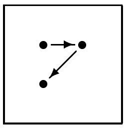
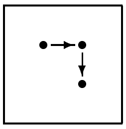
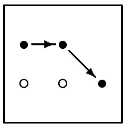
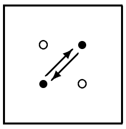
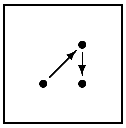
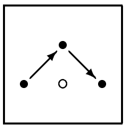
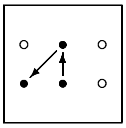
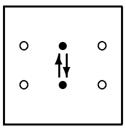
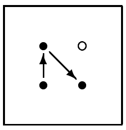
	Transition	Situation		ℓ_0		ℓ_1		Remarks
1	$0 \longrightarrow 5$		not possible					
2	$0 \longrightarrow 6$		not possible					
3	$0 \longrightarrow 7$		**S**	0	$-$	$-$	$-$	
4	$1 \longrightarrow 5$		**T**	$+$	$-$	0	0	
5	$1 \longrightarrow 6$		not possible					
6	$1 \longrightarrow 7$		**S**	$+$	$-$	0	$-$	
7	$2 \longrightarrow 5$		**T**	$+$	$-$	$+$	0	
8	$2 \longrightarrow 6$		**T**	$+$	$-$	$+$	$-$	double descriptor
9	$2 \longrightarrow 7$		not possible					

Table 2: List of possible configurations for a descriptor of type 0.

The points marked $\circ$ are not in S_Δ by definition of the border.
In the fifth and sixth column the changes of functionals ℓ_0 and ℓ_1 during the transition are indicated. $+$ denotes increase, $-$ decrease and 0 means no change.

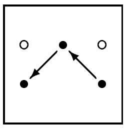
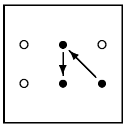
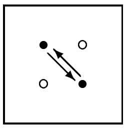
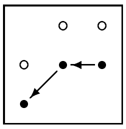
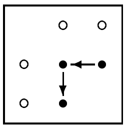
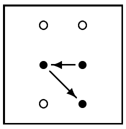
	Transition	Situation		ℓ_0		ℓ_1		Remarks
10	$3 \longrightarrow 5$		**T**	$+$	$-$	$+$	0	
11	$3 \longrightarrow 6$		**T**	$+$	$-$	$+$	$-$	double descriptor
12	$3 \longrightarrow 7$		**T**	$+$	$-$	$+$	$-$	double descriptor
13	$4 \longrightarrow 5$		**T**	0	$-$	$+$	0	
14	$4 \longrightarrow 6$		**T**	0	$-$	$+$	$-$	double descriptor
15	$4 \longrightarrow 7$		**T**	0	$-$	$+$	$-$	double descriptor

Table 2 (continued).

4 The Convex Case

The situation is very simple in the case of digitally convex sets. The following theory is due to Hübler, Klette and Voss ([12], see also [11], a brief account is given in [6]).

Situation Given a (digitally) convex set S_Δ with (oriented 8–) border Γ. We assume that Γ is given by its chain code representation. When we concentrate ourselves on the boundary segment which is delimited by two adjacent descriptor tangents, then we may assume without loss of generality that only two chain code directions, say 0 and 1, appear.

Definition 4.1 *For a code number $\nu \in \{0,1,2,3,4,5,6,7\}$ a ν–*piece *of the border is a maximal subset of Γ whose chain code representation consists only of the code number ν. The number of successive code numbers of a piece is called its* length. *A piece of length* 1 *is called a* singular piece.

Lemma 4.1 *If two successive pieces of the border of a convex set perform a right turn, then one of them is singular.*

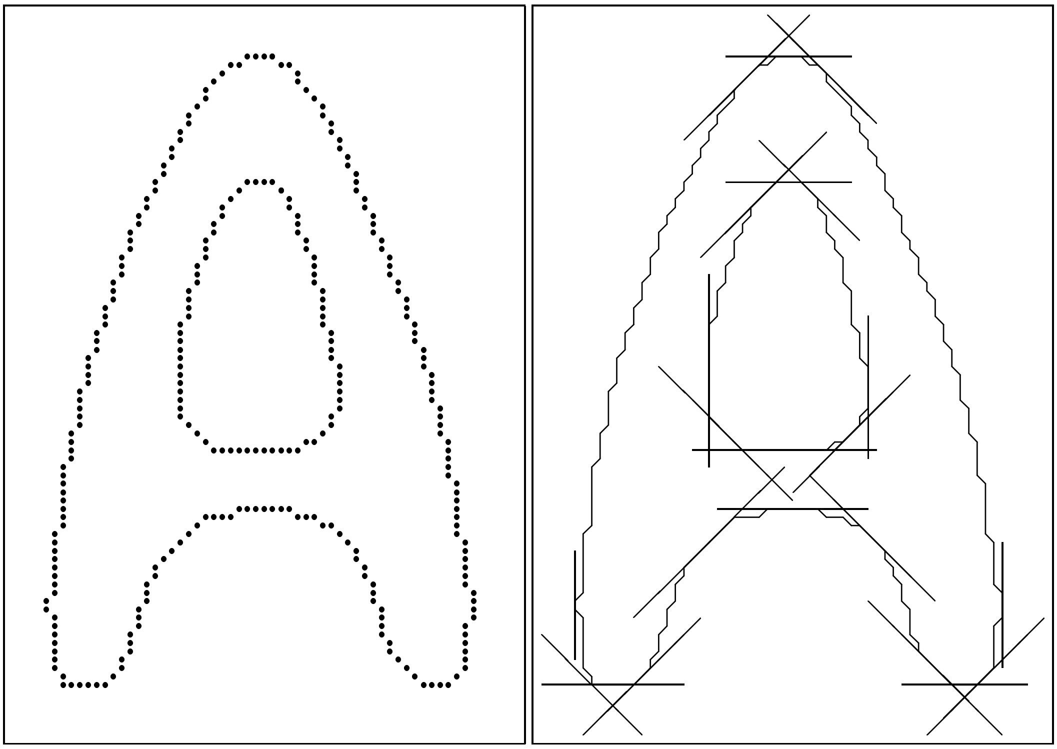
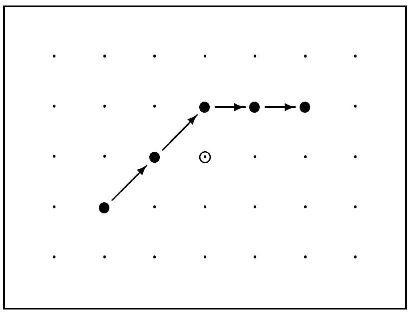


Figure 3: Digital set (Letter 'A').

In the left picture a digital set is given. Only boundary points are marked.
In the right picture the descriptor tangents are indicated.

Proof Assume that there are two boundary pieces having lengths at least two and forming a right turn. If the first of them has code number 1, then the situation is like this:



The point $\odot$ is on the line segment joining the first and the last point of the configuration, but it does not belong to the (convex) set S_Δ. $\square$

Lemma 4.2 *If two successive nonsingular pieces perform a left turn, then their intersection point is an exposed point of* S_Δ.

Proof If we assume (without loss of generality) that only code numbers 0 and 1 are present, then the situation of the Lemma can only occur if at least two code numbers 0 are followed by at least two code numbers 1. Assume that the 0–piece starts in $(0,0)$. Since S_Δ is convex, on the left–hand side of the origin there are no further points on the horizontal axis (note that the origin is the leftmost point on the 0–piece). Similarly, at the right–hand side of the 0–piece there are on the horizontal axis no further points of S_Δ. Consequently, the horizontal axis is tangent line for S_Δ. In a similar way one shows that the 45^o–line through the right end point of the 0–piece is also a tangent line for S_Δ. As a consequence, each line with slope strictly between 0^o and 45^o which passes through the right end point of the 0–piece is a supporting line meeting S_Δ only in this single point. □

We now concentrate on the case that one of the both directions is singular according to Lemma 4.1. The situation is then completely characterized by the lengths of the nonsingular pieces. There are two possibilities: Either the nonsingular pieces are horizontal or vertical (chain code directions 0, 2, 4 or 6) or either they are diagonal (chain code directions 1, 3, 5 or 7). We begin with the first case.

4.1 Nonsingular Horizontal or Vertical Pieces

In this section we will assume that all nonsingular pieces are horizontal or vertical, without loss of generality we may assume that all nonsingular pieces are 0–pieces which are separated by singular 1–pieces.

Lemma 4.3 *Given a sequence of successive (0–) pieces as follows: A piece of length m is followed by $\kappa \geq 0$ pieces of length $m+1$. Then the following piece has at most length $m+1$.*

Proof We start in point $(0,0)$ with a (singular) 1–piece. The end point of the first 0–piece of length m is the point $(m+1,1)$. The next 0–piece of length $m+1$ starts in $(m+2,2)$ and ends in $(2m+3,2)$. We get:

0–piece	Length	Start	End
	m	$(1,1)$	$(m+1,1)$
first	$m+1$	$(m+2,2)$	$(2m+3,2)$
second	$m+1$	$(2m+4,3)$	$(3m+5,3)$
…	…	…	…
κ–th	$m+1$	$(\kappa(m+2),\kappa+1)$	$((\kappa+1)m+2\kappa+1,\kappa+1)$

Continuing with a 0–piece of length $m+2$, we start in $((\kappa+1)m+2\kappa+2,\kappa+2)$ and end in $((\kappa+2)m+2\kappa+4,\kappa+2) = (\kappa+2)\cdot(m+2,1)$. The point $(m+2,1)$ does not belong to S_Δ.

If $\kappa = 0$ then exactly the same argumentation applies. □

Lemma 4.4 *Assume that a 0–piece of length $m > 2$ is followed by a 0–piece of length $\leq m-2$. Then the last point of the piece of length m is an exposed point of S_Δ.*

Proof As in the foregoing proof we start in $(0,0)$ with a singular 1–piece which is followed by a 0–piece of length m. This latter piece extends from point $(1,1)$ to point $(m+1,1)$ The following piece of length $\leq m-2$ starts in $(m+2,2)$ and ends at $(m+n+2,2)$ where $n \leq m-2$. The line

$y = \frac{x-1}{m}$ passes through the point $(m+1, 1)$ and runs below the first piece and the end point of the second piece. The configuration can be followed by Lemma 4.3 by pieces of length at most $m-1$ and thus they all are above the line.

At the left hand side of $(0,0)$ there are pieces whose lengths are not smaller than $m-1$ by Lemma 4.3. If they all have length $m-1$ they lie exactly on the line. In this case the line is slightly rotated around the point $(m+1, 1)$. if the first piece at the left side of $(0,0)$ has length smaller than $m-1$ then all points are below the line. □

Lemma 4.5 *If a 0–piece of length $m > 1$ is followed by a 0–piece of length $m-1$ then the last point of the first piece is an exposed point of S_Δ unless the first piece is preceded by a 0–piece of length $m-1$.*

Proof Again we assume that the first 0–piece is preceded by a singular 1–piece starting in $(0,0)$. Then we have

Piece	Length	Start	End
first	m	$(1,1)$	$(m+1,1)$
second	$m-1$	$(m+2,2)$	$(2m+1,2)$

For any real number b the line

$$y = \frac{1-b}{m+1} \cdot x + b$$

passes the point $(m+1, 1)$. If in addition $b < 0$ then the point $(0,0)$ is above the line and for $-\frac{1}{m} < b$ also the point $(2m+1, 2)$ is above the line. All following 0–pieces have at most length m by Lemma 4.3.

We now assume that a sequence of 0–pieces of length m follows the last piece of the configuration just considered. The ν–th piece of length m starts in point $((\nu+1)m + \nu + 1, \nu + 2)$ and ends in $((\nu+2)m + \nu + 1, \nu + 2)$. For $x = (\nu+2)m + \nu + 1$ we have

$$y = \frac{1-b}{m+1} \cdot ((\nu+2)m + \nu + 1) + b = \nu + 2 + \frac{1}{m+1}\left(-1 + b(-\nu m - m - \nu)\right)$$

and the latter is $< \nu + 2$ whenever $b > -\frac{1}{\nu m + m + \nu}$. As a consequence, for $-\frac{1}{\nu m + m + \nu} < b < 0$ the whole configuration just considered lies above the line.

If the first piece of the configuration is preceded by a piece of length n which in turn is preceded by a singular 1–piece, then the start point of this singular piece is $(-n-1, -1)$. For $x = -n-1$ we get

$$y = \frac{1-b}{m+1}(-n-1) + b = -1 + \frac{1}{m+1} \cdot (m - n + b(n+m+2))\,.$$

For $n \geq m$ obviously $y \leq -1$ and also the extended configuration is above the line.

The piece preceding the configuration considered cannot have length $n < m-1$ (Lemma 4.3, $\kappa = 0$). Consequently, there remains only the case $n = m-1$. $y < -1$ and $n = m-1$ means that $1 + b(2m+1) < 0$. We know from above that we need $-\frac{1}{\nu m + m + \nu} < b$ for some $\nu > 1$ in order that the last point of the piece of length m is an exposed point. Both inequalities imply

$$-\frac{1}{\nu m + m + \nu} < b < -\frac{1}{2m+1}.$$

However, for all $\nu \geq 1$ the leftmost term of this inequality is not smaller than the rightmost one. $\square$

Lemma 4.6 *The nonsingular pieces between two successive exposed points have at most two different lengths differing by one.*

Proof Without loss of generality we may assume that all nonsingular pieces are 0–pieces. Assume further that the first nonsingular piece has length m. By Lemma 4.3 ($\kappa = 0$) the following piece has at most length $m+1$. If the second piece has length $m+1$ then the following piece has at most length $m+1$, again by Lemma 4.3 ($\kappa = 1$). If the following pieces have lengths m or $m+1$ then the same Lemma tells us that they can be followed by a piece whose length is at most $m+1$. So we conclude that all pieces of the part of the border between two successive exposed points have at most lengths $\leq m+1$.

Whenever a piece of length m is followed by a piece of length $m-2$, we have an exposed point by Lemma 4.4. We now assume that there is a first piece of length $m-1$ among the pieces on the border part under consideration. The piece immediately preceding the piece of length $m-1$ cannot have length $m+1$ by Lemma 4.4 unless there is an exposed point. Therefore the piece of length $m-1$ is preceded by a piece of length m. By Lemma 4.5, however, also in this situation the last point of the piece of length m is an exposed point. $\square$

We collect the relevant results obtained so far in a Theorem:

Theorem 4.1 *Given a connected border part of a convex set which consists of a sequence of* 0*–pieces separated by singular* 1*–pieces. Assume that the first* 0*–piece has length* m*, then the lengths of the following* 0*–pieces are restricted to have the following values:*

$\geq m+2$	*not possible*	*Lemma 4.3*
$m+1$	*all following pieces have lengths* $\leq m+1$	*Lemma 4.3*
m	—	—
$m-1$	*all following pieces have lengths* $\leq m$ *or exposed point*	*Lemma 4.3* *Lemma 4.5*
$\leq m-2$	*exposed point*	*Lemma 4.4*

4.2 Nonsingular Diagonal Pieces

For handling the case of a sequence of diagonal pieces separated by singular horizontal or vertical pieces we can apply the following nonsingular linear transformation

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x-y \end{pmatrix}.$$

Since $T\begin{pmatrix} \kappa \\ \kappa \end{pmatrix} = \begin{pmatrix} \kappa \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} \kappa \\ 0 \end{pmatrix} = \begin{pmatrix} \kappa \\ \kappa \end{pmatrix}$, the transformation maps a sequence of nonsingular 0–pieces which are separated by singular 1–pieces into a sequence of nonsingular diagonal pieces separated by singular horizontal pieces. Note, however, that the direction of the border is inverted by the transformation ($\det T = -1$). Thus, the 0– and 1–pieces are mapped on 5– and 4–pieces, respectively. By a trivial rotation we arrive at a similar situation as in the case treated above.

Since the Lemmas 4.1 and 4.2 do not involve singular and nonsingular pieces, we only transform Lemma 4.3 and Lemmas 4.4 and 4.5 which deal with exposed points. We number the transformed Lemmas by the numbers used for the original Lemmas with a raised circle ($^\circ$).

Lemma 4.3° *Given a sequence of successive (1–) pieces as follows: $\kappa \geq 0$ pieces of length $m+1$ are followed by a piece of length m. Then the piece preceding this configuration has at most length $m+1$.*

Proof We assume that the last point of the configuration is in followed by a singular 0–piece which ends in $(0,0)$. Then we have the following situation:

1–piece	Length	Start	End
last	m	$(1,0)$	$(m+1,m)$
κ–th	$m+1$	$(m+2,m)$	$(2m+3,2m+1)$
$\kappa-1$–th	$m+1$	$(2m+4,2m+1)$	$(3m+5,3m+2)$
$\dots$	$\dots$	$\dots$	$\dots$
first	$m+1$	$(\kappa m+2\kappa,\kappa m+\kappa-1)$	$((\kappa+1)m+2\kappa+1,((\kappa+1)m+\kappa)$

If the first piece of length $m+1$ were preceded by a piece of lenght $m+2$ then the latter would start in $((\kappa+1)m+2\kappa+2,(\kappa+1)m+\kappa)$ and end in $((\kappa+2)m+2\kappa+4,(\kappa+2)m+\kappa+2) = (\kappa+2)(m+2,m+1)$. The point $(m+2,m+1)$, however, does not belong to S_Δ. Actually, it is above the point $(m+2,m)$ which is the last point of the last piece of length $m+1$. Since the border part under consideration runs from right to left, this point is outside the border. $\square$

Lemma 4.4° *For $m>2$ assume that a 1–piece of length $\leq m-2$ is followed by a 1–piece of length m. Then the first point of the piece of length m is an exposed point of S_Δ.*

Proof We transform the proof of Lemma 4.4. We have the following correspondences:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} m+1 \\ 1 \end{pmatrix} \longleftrightarrow \begin{pmatrix} m+1 \\ m \end{pmatrix},$$
$$\begin{pmatrix} m+2 \\ 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} m+2 \\ m \end{pmatrix},$$
$$\begin{pmatrix} m+n+2 \\ 2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} m+n+2 \\ m+n \end{pmatrix}$$

with $n \leq m-2$. The equation of the line $y = \frac{x-1}{m}$ is mapped onto the equation $y' = \frac{(m-1)x'+1}{m}$.

We get

x'	y'
0	$\frac{1}{m} > 0$
1	$1 > 0$
$m+1$	m
$m+2$	$m+1-\frac{1}{m} > m$
$m+n+2$	$m+n+1-\frac{n+1}{m} > m+n$

The line meets the boundary point $(m+1, m)$ and is above all the points of the configuration.

The remainder of the proof follows from Lemma 4.4. $\square$

Lemma 4.5° *If a 1–piece of length m is preceded by a 1–piece of length $m-1$ then the first point of the second piece (of length m) is an exposed point of S_Δ unless the last piece is followed by a 1–piece of length $m-1$.*

Proof We transform the proof of Lemma 4.5. We have the following correspondences:

$$\begin{array}{ccc}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\begin{pmatrix} 1 \\ 1 \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\
\begin{pmatrix} m+1 \\ 1 \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} m+1 \\ m \end{pmatrix}, \\
\begin{pmatrix} m+2 \\ 2 \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} m+2 \\ m \end{pmatrix}, \\
\begin{pmatrix} 2m+1 \\ 2 \end{pmatrix} & \longleftrightarrow & \begin{pmatrix} 2m+1 \\ 2m-1 \end{pmatrix}.
\end{array}$$

The equation of the line $y = \frac{1-b}{m+1} \cdot x + b$ is mapped onto the equation $y' = \frac{m+b}{m+1} \cdot x' - b$, where b fulfills the same conditions as in Lemma 4.5. We get

x'	y'
0	$-b > 0$
1	$m \cdot \frac{1-b}{m+1} > 0$
$m+1$	m
$m+2$	$m + \frac{m+b-1}{m+1} > m$
$2m+1$	$2m-1+\frac{1+mb}{m+1} > 2m-1$

The line meets the boundary point $(m+1, m)$ and is above all the points of the configuration.

The remainder of the proof follows from Lemma 4.5. $\square$

We collect the results in a Theorem:

Theorem 4.1° *Given a connected border part of a convex set which consists of a sequence of* 1–*pieces separated by singular* 0–*pieces. Assume that the first* 1–*piece has length* m*, then the lengths of the following* 1–*pieces are restricted to have the following values:*

$\leq m-2$	*not possible*
$m-1$	*all following pieces have lengths* $\geq m-1$
m	—
$m+1$	*all following pieces have lengths* $\geq m$ *or exposed point*
$\geq m+2$	*exposed point*

The general case of a sequence of ν–pieces separated by singular $\nu+1 \pmod 8$–pieces is obtained by a trivial rotation of the digital plane.

When we scan the boundary of an object in order to check whether the conditions of the Lemmas in this section are fulfilled, we can first determine all those exposed points which are characterized by Lemma 4.2, Lemma 4.4 and by Lemma 4.5. There may be of course further exposed points on the boundary of S_Δ. From Lemmas 4.1 and 4.2 we conclude that in a part of the boundary between two successive exposed points one direction occurs singularly. Lemmas 4.3 and 4.4 together with Lemma 4.5 imply that for the nonsingular pieces of such a part of the boundary the lengths of successive pieces can at most change by 1. Moreover, in such a part of the boundary the nonsingular pieces can at most have two different lengths. This result (as well as the methods used in this section) is very closely related to the characterization of digital lines by Hübler, Klette and Voss [12] (see also [11]).

Between two successive exposed points there are nonsingular pieces of one direction whose lengths can at most have two different values m and $m+1$. This situation can be mapped to the situation at the begin of the section by means of the so–called Hübler–transformation [11].

4.3 Polygonal Representations and Hübler Transforms

A linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is given by a 2×2 matrix (we use the same symbol for the matrix and the transformation defined by it)

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If the elements of the matrix T are integers, then the transformation T maps $\mathbb{Z}^2$ into $\mathbb{Z}^2$. If in addition, the determinant of T is ± 1, then T is invertible and the inverse matrix has integer elements, hence in this case T is a bijective mapping from $\mathbb{Z}^2$ into $\mathbb{Z}^2$. A matrix T having integer elements and determinant ± 1 is called a *unimodular matrix*. We denote the set of all unimodular integer 2–matrices by $\mathcal{M}_2$. It is immediately clear that $\mathcal{M}_2$ is a group with respect to matrix multiplication.

Bitter [2] has shown that all unimodular matrices can be obtained as finite products of three matrices, namely

$$D_6 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad D_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

D_4 is a rotation by 90^o with $D_4^4 = I$ (I denotes the identity matrix) and S_2 is a reflection at the diagonal line $y = x$ with the property $S_2^2 = I$.

The interpretation of the matrix D_6 is more complicated. First, it has the property $D_6^6 = I$. D_6 can be considered as an approximation of a 60^o rotation of the plane. The matrix

$$T_\triangle = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

maps the digital plane $\mathbb{Z}^2$ onto the plane

$$\mathbb{Z}^2_\triangle = \left\{ n_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + n_2 \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \;\middle|\; n_1, n_2 \in \mathbb{Z} \right\}$$

which is a digital model of the plane covered by a trianglular mesh. The rotation with matrix

$$R_{60} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

maps $\mathbb{Z}^2_\triangle$ onto itself, it yields a 60^o rotation of the discrete space $\mathbb{Z}^2_\triangle$. One easily calculates

$$D_6 = T_\triangle^{-1} R_{60} T_\triangle.$$

Trott [25] showed that the group of all unitary d–matrices can be generated by four elements (see also [4, 18]). In the two–dimensional case this reduces to three matrices, however, together with their inverses which amounts in four different generators in Bitter's sense.

A unitary matrix is a square matrix A with the property $A^\top A = AA^\top = I$ ($^\top$ denotes transposition). Unitary matrices leave the scalar product of two vectors (or the angle between two vectors) and the Euclidean length invariant. They describe motions of the plane. The unitary inimodal integermatrices are a subgroup $\mathcal{U}_2$ of the group $\mathcal{M}_2$ of all unimodal integer 2–matrices. This subgroup consists of 8 elements

$$\pm I, \pm D_4, \pm S_2, \pm D_4 S_2$$

which model rotations of the digital plane by 90^o, 180^O and 270^o (D_4, $-I$ and $-D_4$) and reflections at the tweo main diagonal and the horizontal and vertical through the origin ($\pm S_2$, $-D_4S_2$, D_4S_2).

Up to the sign there are four different elements in $\mathcal{U}_2$. Since $D_6^2 = -S_2 D_6 S_2$ and since $\mathcal{U}_2$ is a group, each matrix in $\mathcal{M}_2$ can be represented in the form

$$U_0 \prod_{i=1}^{n} D_6 U_i$$

with $U_i \in \mathcal{U}_2$ for $i = 0, 1, \cdots, n$. That means that up to the sign and a unimodular factor U_0 each matrix in $\mathcal{M}_2$ can be written as a finite product of three matices

$$\begin{array}{rcrcl} V_1 & := & -D_6 D_4 & = & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\ V_2 & := & D_6 S_2 & = & \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \\ V_3 & := & -D_6 D_4 S_2 & = & \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \end{array}$$

Let S_Δ be a digital set and Π a polygonal representation of S_Δ. Furthermore, let $T \in \mathcal{M}_2$ be a unimodal transformation of $\mathbb{Z}^2$. Then $T\Pi$ is a polygonal representation of TS_Δ. We note that the orientation of the border of a set is reverted when it is mapped by a unimodal transformation with determinant -1. The image of a border point is not always a border point. Exposed points, however, are mapped on border points by any unimodal transformation. Moreover, any discrete (faithful) polygonal representation of a digital set is mapped into a discrete (faithful) polygonal representation by a unimodular transformation. The image of a topological representation, however, is not necessarily topological.

These considerations lead to a first idea of an algorithm for determining exposed points of a digital set S_Δ:

Algorithm $\mathcal{E}_0(S_\Delta)$

Start Given a digital set $S_\Delta \in \mathbb{Z}^2$. Define a subset $S_\Delta^{(e)} \subseteq S_\Delta$ containing *potentially exposed points* of S_Δ. Initially $S_\Delta^{(e)}$ contains all boundary points of S_Δ.

Iteration Choose $\nu \in \{1, 2, 3\}$ and apply the unimodal transformation V_ν to S_Δ. All points in $S_\Delta^{(e)}$ which are not boundary points of $V_\nu S_\Delta$ are removed from the set $S_\Delta^{(e)}$.

Result The more different unimodal transformations were applied, the greater is the probability that the set $S_\Delta^{(e)}$ contains only exposed points.

In order to get a valid algorithm from $\mathcal{E}_0$, we must apply the unimodal transformations in the iteration step in a more systematic manner.

A special class of unimodal matrices are the matrices of the *Hübler transformation* [11]. For an integer ν we define a matrix by

$$H_\nu^\pm = \begin{pmatrix} \nu & \pm 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$H_\nu^\pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \nu \\ 1 \end{pmatrix}, \qquad H_\nu^\pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \qquad H_\nu^\pm \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \nu \pm 1 \\ 1 \end{pmatrix},$$

and

$$H_\nu^{\pm^{-1}} = \begin{pmatrix} 0 & 1 \\ \pm 1 & \mp\nu \end{pmatrix}.$$

The Hübler–transformation has the following properties:

- $H_\nu^\pm$ is a bijection of $\mathbb{Z}^2$ in $\mathbb{Z}^2$.
- $\det H_\nu^\pm = \mp 1$.
- Since the transformation $H_\nu^\pm : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is linear and bijective, images of convex sets and lines are convex sets and lines, respectively. Images of exposed points are exposed points. Since the transformation is nonsingular, the same assertion holds for the inverse transformation. Exposed points in the image space of the Hübler transformation which are in $\mathbb{Z}^2$ correspond to exposed points in the original space which are in $\mathbb{Z}^2$.

In Figure 4 a digital set and its Hübler–transform are given. Note that the transformed set is 8–connected but the images of border points of the original do not entirely cover the border of the image and not all images of border points are border points of the image.

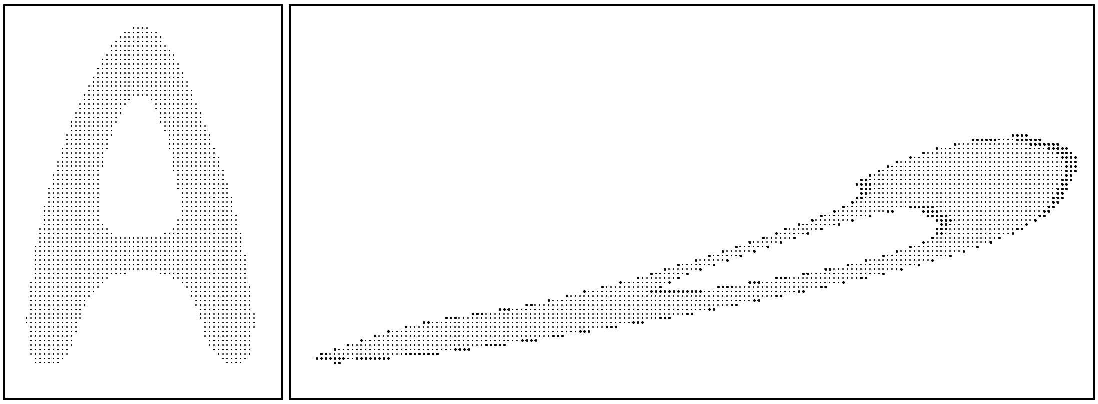


Figure 4: Hübler Transform of a digital set (Letter 'A').

In the left picture a digital set S_Δ is given.
The right picture shows the set $H_3^+ S_\Delta$. Images of border points are emphasized.

The border part Γ between two successive exposed points is characterized by the following properties:

- It is described by a chain code sequence consisting of two code numbers (without loss of generality we assume that these code numbers are 0 and 1).
- One of these code occurs singularly (without loss of generality it is code number 1).
- There exists a finite set of Hübler–transforms H_{m_1}, H_{m_2}, $\cdots$, H_{m_q} (the superscript '+' is omitted here) such that

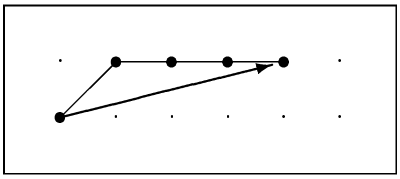
– Each transformed set $\Gamma_i = H_{m_i}^{-1} H_{m_{i-1}}^{-1} \cdots H_{m_1}^{-1} \Gamma$, $i = 1, 2, \cdots, q-1$, $\Gamma_0 = \Gamma$ has the same properties as Γ, i.e its border can be described by code numbers 0 and 1, and code number 1 occurs singularly.

– Γ_q is a horizontal line which is given by a finite number of code numbers 0.

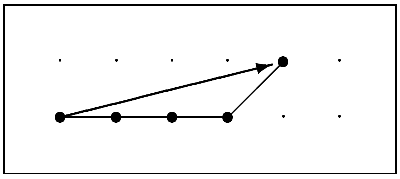
In Figure 5 it is shown how the process of successively applying inverse Hübler transforms yields a horizontal digital line Γ_q.

We now assume that a border part between two successive exposed points is transformed by a sequence of inverse Hübler transforms into a horizontal line segment, The border part under consideration is a horizontal, vertical or diagonal line segment if and only if $q = 0$. If this is not the case then H_q transforms the horizontal line Γ_q into a sequence of collinear points. These points are joined by legitimate chain code sequences consisting either of a singular code 1 followed by m_q nonsingular directions 0 (if S_Δ is on the left side of the border part) or else by m_q directions 0 followed by a singular direction 1 (if S_Δ is on the right side of the border part).

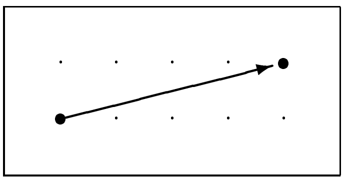
For a horizontal line segment on the border of Γ_q which joins two grid points there corresponds on Γ_{q-1} to a triangle of the form



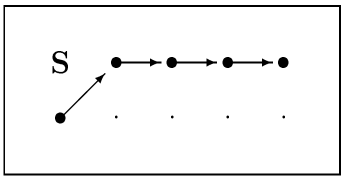
or



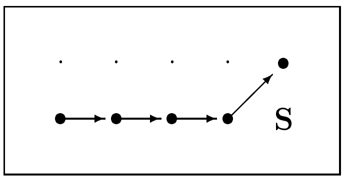
These triangles contain no grid points in the interior. There are two equivalent polygonal representations of the border part defined by these triangles, depending whether S_Δ is on the right or on the left side of the border:



and



or



The foregoing discussion immediately leads to the following algorithm for determining all exposed points of a convex set S_Δ:

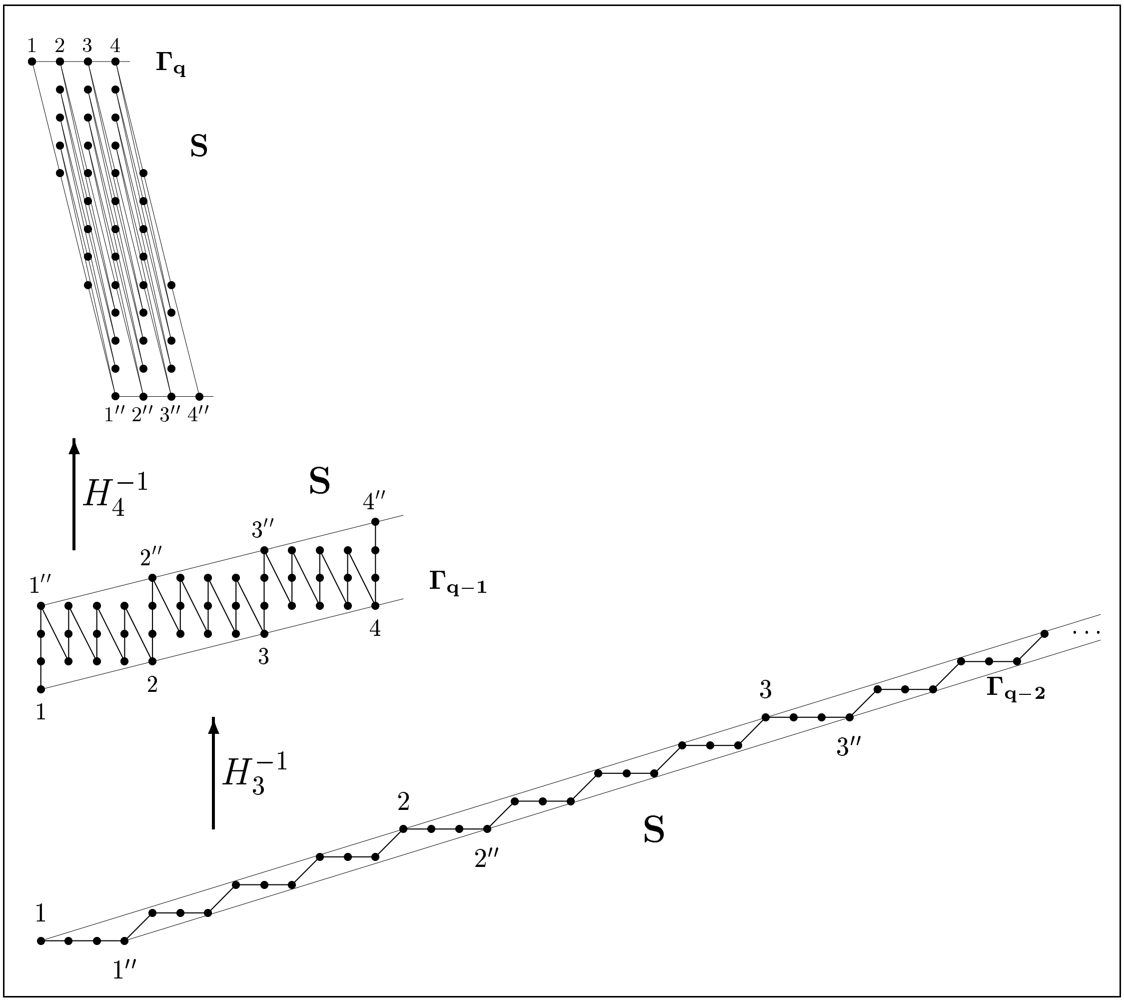


Figure 5: Example of the Inverse Hübler–Transformation.

In the lower part of the picture a part Γ_{q-1} of the border of a set is given. The set is assumed to be below the border. By means of the transformation H_3^{-1} this border part is transformed into the set in the middle. Now the set S_Δ is above the border curve Γ_{q-1} which is a digital curve running through the points 1 to 4 which are images of the corresponding points in the lower curve. The transform H_4^{-1} yields the upper set whose border Γ_q is a horizontal digital line consisting of the points 1 to 4. Here S_Δ is again below the border curve.

Algorithm $\mathcal{E}_1(S_\Delta)$

Start Given a digital set $S_\Delta \in \mathbb{Z}^2$. Let $S_\Delta^{(e)} \subseteq S_\Delta$ be the oriented boundary of S_Δ.

Iteration For any three successive points P, Q and R on $S_\Delta^{(e)}$ let $\Delta(P,Q,R)$ be the triangle spanned by them. If there is no grid point contained in the interior of $\Delta(P,Q,R)$ then Q is removed from $S_\Delta^{(e)}$ and R becomes the successor of P.

Result The iteration step is repeated until each such triangle contains grid points in its interior.

Remark 4.1 *The test whether a triangle whose vertices are grid points contains a grid point in its interior or not can be easily carried out by making use of Pick's Theorem (see [27, Section 4.2.3]*

Lemma 4.7 *For a convex set S_Δ algorithm $\mathcal{E}_1(S_\Delta)$ always terminates. After termination, $S_\Delta^{(e)}$ contains exactly all exposed points of S_Δ.*

The proof of this Lemma is an immedate consequence of the discussion above (see [12]).

In our context, Γ_q is a horizontal grid line segment. The horizontal line ℓ_q containing Γ_q has then the equation $y = 0$. By the transformation H_{m_q} we get the line $\ell_{q-1} = H_{m_q}\ell_q$ which has the equation

$$y = \frac{x}{m_q + 1}.$$

The transformation $H_{m_{q-1}}$ yields the line $\ell_{q-2} = H_{m_{q-1}}H_{m_q}\ell_q$ with equation

$$y = \frac{x}{m_{q-1} + 1 + \frac{1}{m_q+1}}.$$

$H_{m_{q-2}}$ yields the line $\ell_{q-3} = H_{m_{q-2}}H_{m_{q-1}}H_{m_q}\ell_q$ with equation

$$y = \frac{x}{m_{q-2} + 1 + \frac{1}{m_{q-1}+1+\frac{1}{m_q+1}}}. \tag{4.1}$$

Continuing inductively in this manner we get the continued–fraction characterization of a digital line by Bruckstein [3] and Voss [28].

Given a border segment Γ between two successive exposed points. This border segment can be described by a sequence of two code numbers differing by $1 \pmod 8$. We assume that these are code numbers 0 and 1. Furthermore we assume that q is such that Γ_q is a horizontal grid line segment. Then

- The line $H_{m_1}H_{m_2}\cdots H_{m_{q-1}}H_{m_q}\Gamma_q$ is a line in the original image which joins both exposed points bounding the border part under consideration.

- The line $\ell = H_{m_q} H_{m_{q-1}} \cdots H_{m_2} H_{m_1} \Gamma_q$, if it is written in the slope–intercept representation $\ell = \{(\xi, \eta) \mid \eta = m\xi - c\}$, has slope m between 0 and 1.
- The discrete border curve between the two exposed points can be constructed in the following way: For each vertical grid line of the form $\{(i, \eta) \mid i \in \mathbb{Z}, \eta \in \mathbb{R}\}$, the grid point which is nearest to ℓ on the same side as S_Δ is a border point.
  The set of all these points is an 8–curve.
- The border curve between the both exposed points has the chord property (see [22]).
- The line has rational slope m. Let $m = \frac{a}{b}$, $a, b \in \mathbb{Z}$, be an irreducible representation. Then we may represent the line ℓ by
  $$ax - by = \mu,$$
  where $\mu = -bc$.
- Γ is contained in the strip
  $$\{(\xi, \eta) \mid \mu \leq ax - by < \mu + b\}$$
  if S_Δ above the line ℓ or in the strip
  $$\{(\xi, \eta) \mid \mu - b < ax - by \leq \mu + b\},$$
  if S_Δ is below the line.

In case $q > 0$ the set $H_{m_q} \Gamma_q$ is a digital curve consisting of congruent triangles. This set is contained in the parallel strip bounded by the line $\ell = H_{m_q} H_{m_{q-1}} \cdots H_{m_2} H_{m_1} \Gamma_q$ and a line parallel to it going though the vertices of the triangles which are not on ℓ (see Figure 6). All border parts $\Gamma_i = H_{m_i}^{-1} H_{m_{i-1}}^{-1} \cdots H_{m_1}^{-1} \Gamma$ lie between the images of these two parallel lines. Therefore the original border part Γ is a digital line segment in the sense of Hübler, Klette and Voss [12] and of Debled–Rennesson [5].

In Figure 6 the application of the Hübler transforms for reconstructing the original border part is illustrated.

When we apply the Hübler transformation $(H^+_{m+1})^{-1}$ to the situation described at the end of Section 4.1 or Section 4.2, respectively (Theorem 4.1 or Theorem 4.1°, respectively) then the boundary part between two successive exposed points is mapped into a sequence of numbers 0 and 1. One of these numbers occurs singularly. We can interpret this sequence as a chain code of a digital curve. This curve can again be analyzed by the Lemmas of the preceding Sections. By virtue of the properties of the Hübler transformation, this latter curve reflects the convexity behavior of the original boundary part. Specifically we may detect new exposed points. Exactly as in the paper of Hübler, Klette and Voss [12] we can show that this process will terminate in a time proportional to the length of the boundary with a horizontal line. It is easily seen that the procedure will detect all exposed points of S_Δ and thus all vertices of the convex hull of S_Δ. Moreover, the parts between two successive exposed points will be digital lines. This follows immediately from the construction since the chain code sequences of these parts fulfill the conditions characterizing digital line segments according to Hübler, Klette and Voss [12] (see also [6]).

To state it differently, one can find in time depending linearly on the length of the boundary a uniquely determined polygonal representation Π of a digitally convex set S_Δ. This representation has the properties

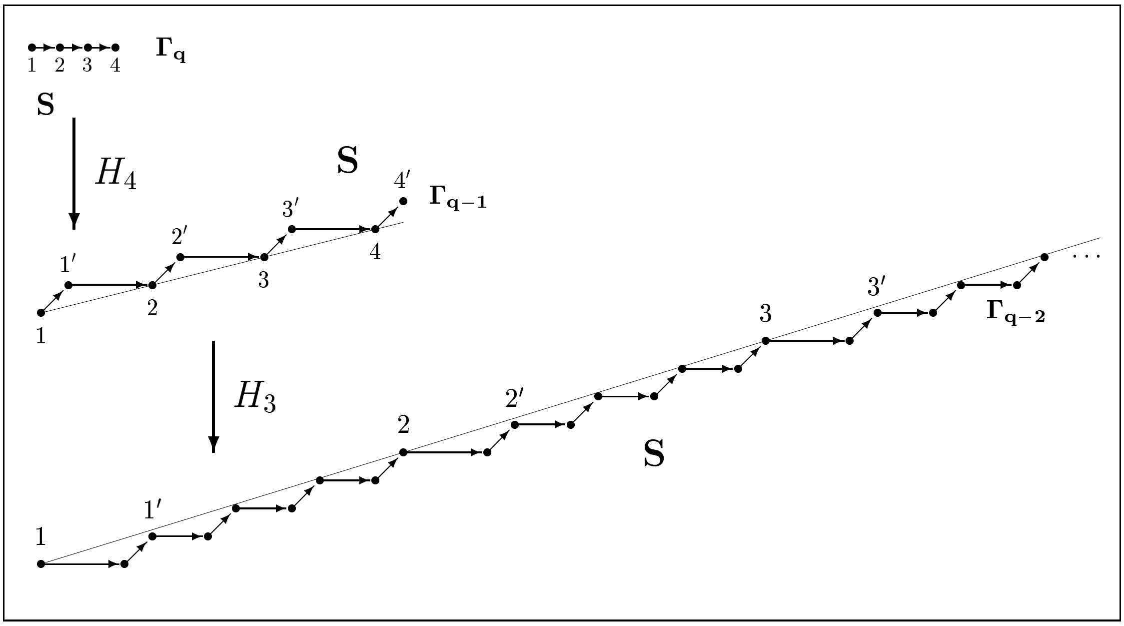


Figure 6: Example of the Hübler–Transformation.

It is here assumed that the transformation process ends with a configuration Γ_q consisting of four points on a horizontal line (direction 0). This configuration is shown in the upper part. The set S_Δ is assumed to lie below the curve.
By means of transformation H_4 (i.e. $m_q = 4$) the preceding configuration Γ_{q-1} is obtained. Note that here the set S_Δ is above the border curve. The thin line is the image under H_4 of the horizontal line in the upper curve.
The transformation H_3 ($m_{q-1} = 3$) yields the lower border curve Γ_{q-2} The set S_Δ is now again below the border curve. The thin line meets the curve in the points 1, 2, $\cdots$ which are the images of the corresponding points of the upper curve. The whole part of the border lies below the thin line. The points $1'$, $2'$, $\cdots$ are the images of the corresponding points of the middle curve. Triangle 1 $1'$ 2 of the middle curve is transformed into the flat triangle 1 $1'$ 2 of the lower curve.

1. All vertices of Π are points of S_Δ.
2. Π contains exactly the points of S_Delta.
3. Π is convex (actually it is the convex hull of S_Δ).
4. The vertices of Π are the exposed points of S_Δ.
5. Π is the polygon of minimal circumference containing S_Δ.

Specifically, for a convex digital set the ordinary convex hull is a polygonal representation which is as well discrete as faithful. It is clear that Π is characterized by the last two minimality conditions.

5 General Sets

Whereas the situation is very nice in the convex case, this is no longer true for general sets. Our goal will now be to find a polygonal representation of a (connected) digital set which will have as much of the properties of the convex case as possible.

The problems encountered in the nonconvex case can be illustrated by the example given in Figure 7. As indicated by the arrows, the set S_Δ is above the curve. The curve belongs to the nonconvex part of the border. The main trouble is caused by the both marked points of the configuration. There is no possibility to find in a canonic way — for example by an invariant approach — a polygonal representation which is both digital and faithful. We are therefore forced to abandon one of these two requirements. Here we concentrate on a digital polygonal representation. This is in accordance with the philosophy of discrete curve evolution where the the boundary of a digital set is evolved by eliminating successively boundary points [17].

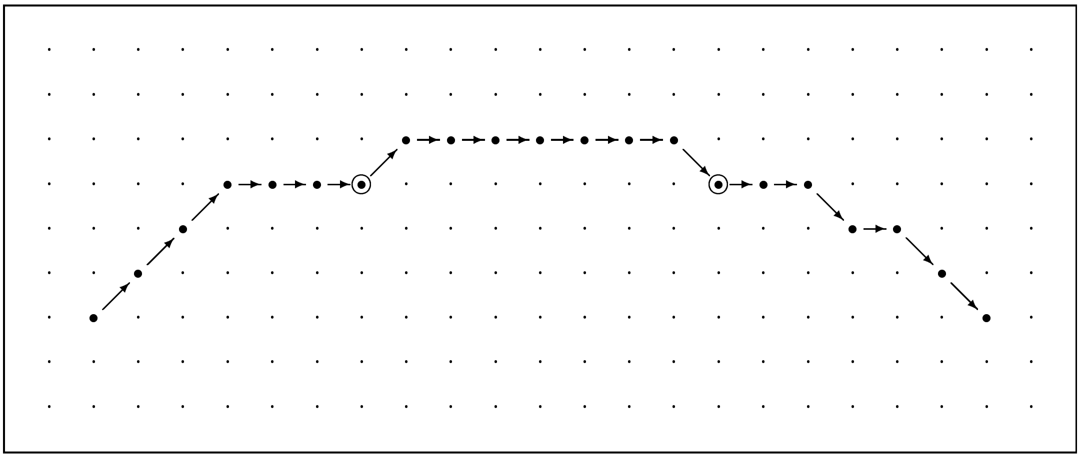


Figure 7: Detail of the nonconvex part of the border of Letter 'A'.

5.1 Duality

As already mentioned, there is no perfect set–complement duality in digital geometry. Specifically, the boundary of a set is not the boundary of the complement of the set. Whereas boundary points of S_Δ are all points in S_Δ having a direct neighbor not in S_Δ, a boundary point of the complement of S_Δ is a point not in S_Δ having an 8–neighbor in S_Δ. The boundary of S_Δ is an oriented 8–connected set (however, in general not an 8–curve) and the boundary of the complement is an oriented 4–curve. Indeed the border–following algorithm [21] yields not only the 8–border of a set but also the 4–border of its complement. For introducing duality, we should take into account these differences. However, dealing with two distinct borders would be too much of inconvenience. We therefore adopt an "assymetric" duality dealing only and exclusively with the 8–border of the set under consideration.

First we state a simple Lemma:

Lemma 5.1 *Let S_Δ be a digital set and denote by* bd S_Δ *its (8–) boundary.*

If S_Δ is convex then also $S_\Delta \setminus \text{bd } S_\Delta$ is convex.

Remark 5.1 *As can be seen by counterexamples, the converse of the Lemma is not generally true. There do exist digital sets S_Δ such that $S_\Delta \setminus \text{bd } S_\Delta$ is convex but S_Δ fails to be convex.*

If we consider a convex digital set S_Δ, then its boundary has the properties derived in Section 4. If we reverse the orientation of the border of S_Δ, we get a set which may not be a legitimate border of a set but which contains the border of a set as was indicated in Figure 1. We have in general ($\mathbb{C}S$ denoting the set theoretic complement of S)

$$\text{bd } (\mathbb{C}S \cup \text{bd } S) \subseteq \text{bd } S$$

with strict inclusion sign possible. It should be noted, however, that the exceptional situation of Figure 1 can only occur on the convex part of the border. By the Lemma, the reversed boundary is the boundary of a set which is the complement of the convex set $S_\Delta \setminus \text{bd } S_\Delta$, which means that the assertions of Section 4 can also be used for sets being the complement of a convex set.

The situation is illustrated by Figure 8 which shows the border of the hole of the Letter 'A' (see Figure 3). The convex hull of this part of the border is contained in the set and therefore it is not useful for representing the latter. On the other hand, the convex hull of the 4–border of the complement is a faithful representation of the complement component, however, this polygonal representation has complement points as vertices and is therefore also not very useful. The polygonal representation of the set which is based on 'exposed' points (to be defined) is not faithful. From the convex hull of the border we can conclude that this part of the border is indeed a 'concave part' since the convex hull contains all border points. Lemma 5.1 yields the assertion that also the set of all complement points defined by the border is a digitally convex set.

5.2 The Anticonvex Case

Our goal is to define 'exposed' points of the nonconvex part of a set. First we investigate 'anticonvex' sets which are complements of convex sets and proceed as in 4.

Situation Given a digital set S_Δ with (oriented 8–) border Γ. We assume that S_Δ is *anticonvex*, i.e. the complement of S_Δ is a digitally convex set.

Lemma 2.1 does not provide a way out of the dilemma since in the anticonvex case it can happen that there exists a pair of boundary points such that one of them necessarily belongs a vertex of any polygonal representation which does not contain the other point as a vertex (see Figure 7).

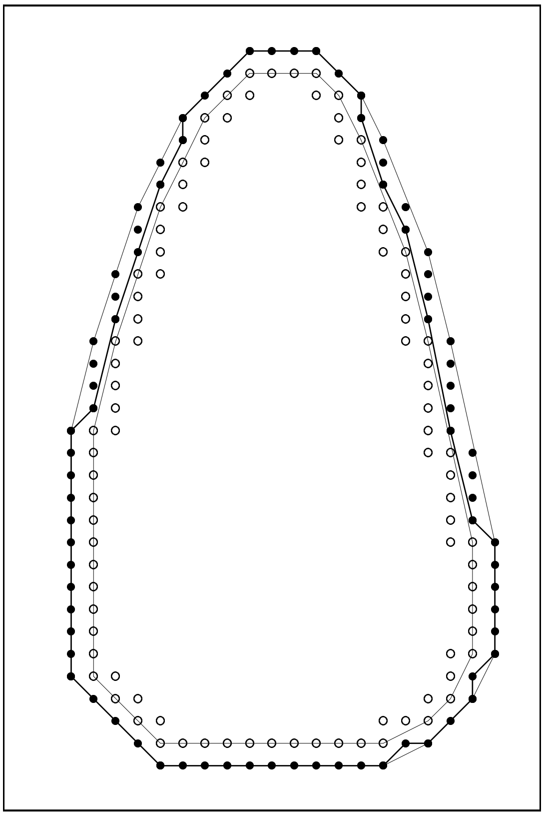


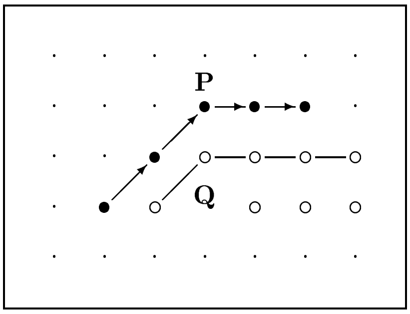
Figure 8: Nonconvex border of the hole of Letter ‘A’.

Points of the border are indicated by $\bullet$, points of the 4–border of the complement by $\circ$. The polygonal representation which is digital but not faithful is marked by thick lines. The polygonal representation of the border and the polygonal representation of the border of the complement are given by thin lines.

Definition 5.1 *Given an anticonvex digital set S_Δ. The point $P \in S_\Delta$ is a $*$–*exposed point *if*

- *P is an S–descriptor point of S_Δ, or*
- *P is the intersection point of two adjacent nonsingular border pieces performing a right turn, or*
- *P is a vertex of each discrete polygonal representation of S_Δ.*

We discuss the second case of the Definition. Obviously it is modelled according to Lemma 4.2. If we assume (without loss of generality) that only code numbers 0 and 1 are present, then this situation can only occur if at least two code numbers 1 are followed by at least two code numbers 0 (or vice versa). Since S_Δ is anticonvex, all points in the following picture marked $\circ$ belong to the complement of S_Delta. The convex hull of the border of the complement is indicated by lines.



By the same argument as in Lemma 4.2 we conclude that the upper horizontal line and the diagonal line of the convex hull are tangent lines of the convex complement. Again, each line with slope between 0^o and 45^o which passes through the exposed point of the complement Q is a supporting line meeting the complement only in this single point. Moreover, the point P is the only point of the boundary of S_Δ which is strictly separated from Q by all these lines.

We note that in the situation just described the characterization of exposed points from Lemma 2.1 does not carry over. Indeed, as in Figure 7 there are polygonal representations which do not contain the $*$–exposed point P as a vertex.

The following Lemma is just the anticonvex counterpart of Lemma 4.1:

Lemma 5.2 *If two successive pieces of the border of an anticonvex set perform a left turn, then one of them is singular.*

We now can translate all the Lemmas of Section 4 into the context of anticonvex sets. Again we assume that all 0–pieces mentioned are separated by singular 1–pieces.

Lemma 5.3 *Given a sequence of successive (0–) pieces as follows: κ pieces of length $m+1$ are followed by a piece of length m then the piece preceding the configuration has at most length $m+1$.*

Proof Assume that the piece preceding the configuration has length $m+2$ and is preceded by a singular 1–piece which starts in $(0,0)$. We get:

0–piece	Length	Start	End
	$m+2$	$(1,1)$	$(m+3,1)$
first	$m+1$	$(m+4,2)$	$(2m+5,2)$
second	$m+1$	$(2m+6,3)$	$(3m+7,3)$
...	...	...	...
κ–th	$m+1$	$(\kappa m+2\kappa+2,\kappa+1)$	$((\kappa+1)m+2\kappa+3,\kappa+1)$

The following 0–piece of length m starts in $((\kappa+1)m+2\kappa+4,\kappa+2)$ and ends in $((\kappa+2)m+2\kappa+4,\kappa+2)=(\kappa+2)\cdot(m+2,1)$. In other words: the line segment joining points $(0,0)$ and $((\kappa+2)m+2\kappa+4,\kappa+2)$ meets the point $(m+2,1)$ which belongs to S_Δ. It is the point of the boundary immediately preceding the end point of the piece of length $m+2$.

When we translate this line segment by the vector $(1,0)$ it starts in $(1,0)$ and ends in $((\kappa+2)m+2\kappa+5,\kappa+2)$. The point $(1,0)$ is a common direct neighbor of the boundary points $(0,0)$ and

$(0,1)$. By definition of the directed border it does not belong to S_Δ. The same is true for the point $((\kappa+2)m+2\kappa+5, \kappa+2) = (\kappa+2)\cdot(m+2,1)+(1,0)$ which is the 0–neighbor of the last point of the configuration under consideration. The line joining these two points meets the point $(m+3,1)$ which is a border point. Hence the complement of S_Δ is not convex, contrary to the assumption.

If $\kappa = 0$ then exactly the same argumentation applies. □

Lemma 5.4 *Assume that a 0–piece of length m is followed by a 0–piece of length $\geq m+2$. Then the last point of the first piece (of length m) is a $*$–exposed point of S_Δ.*

Proof We proceed as in the proof of Lemma 4.4 (note, however, that the border orientation is reverted here). If we assume that the singular 1–piece preceding the first 0–piece of length m starts in point $(0,0)$ we get the following table

Piece	Length	Start	End
first	m	$(1,1)$	$(m+1,1)$
second	$n \geq m+2$	$(m+2,2)$	$(n+m+2,2)$

The line $y = \frac{x}{m+2}+1$ corresponds to the supporting line constructed in Lemma 4.4. It goes through the first point of the piece of length n and runs above all remaining points of the configuration (solid line in Figure 9). Translating it by $(0,-1)$ we get a line running through points $(0,0)$ and $(2m+4,2)$ which are both border points and in addition it meets point $(m+2,2)$ which is a point not in S_Δ. The translated line is a supporting line of the convex complement of S_Δ (dashed line in Figure 9). Obviously, any discrete polygonal representation of S_Δ necessarily contains the point marked $\odot$ in Figure 9. Hence this point is a $*$–exposed point of S_Δ. □

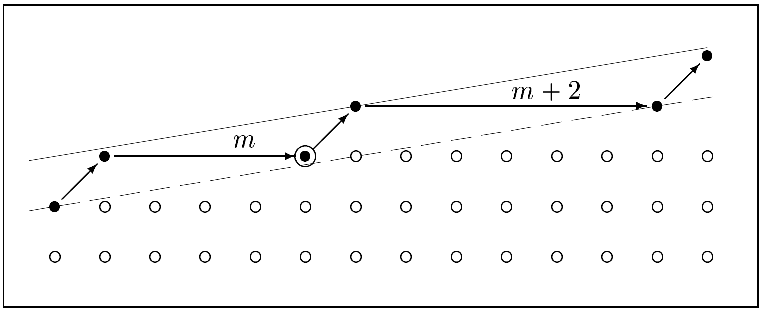


Figure 9: Illustration of a $*$–exposed point.

Border points of S_Δ are marked $\bullet$, points of the complement $\circ$. The tangent line separating an exposed point on the reverted boundary is indicated by a line. Translating the tangent line by $(0,-1)$ a line is obtained (dashed line) which is a supporting line for the complement. This dashed line contains the first point of the configuration and the last point of the 0–piece of length $m+2$. The point marked $\odot$ is a $*$–exposed point.

Lemma 5.5 *If a 0–piece of length m is followed by a 0–piece of length $m+1$ then the last point of the first piece is a $*$–exposed point of S_Δ unless the last piece is succeded by a 0–piece of length m.*

Proof Analogously as in Lemma 4.5 we construct a supporting line for the reverted boundary.

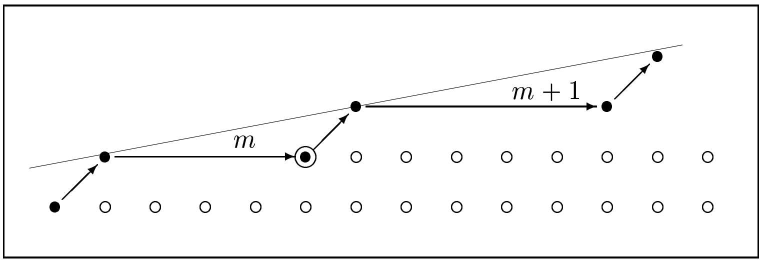


Figure 10: Illustration of a $*$–exposed point, second case

If we again start with the first singular piece in $(0,0)$ this line has the equation (solid line in Figure 10)

$$y = \frac{2-b}{m+2} \cdot x + b, \qquad \frac{m}{m+1} < b < 1.$$

This line meets the first point of the piece of length $m+1$ and runs above the remaining configuration. If the piece following the configuration has not length m we see as in Lemma 5.4 that the line translated by $(0,-1)$ is a supporting line for the complement and that the point marked $\odot$ in Figure 10 is a $*$–exposed point.

For the case that the piece following the configuration has length m we proceed as in Lemma 4.5. $\square$

We state some properties of $*$–exposed points:

- If P is a $*$–convex point then there exists a direct neighbor Q of P wich is an exposed point of the complement.
- If P is a $*$–convex point then either P is an exposed point of the reverted border or else there exists an 8–neighbor P' of P wich is an exposed point of the reverted border.

Again we collect the relevant results in a Theorem:

Theorem 5.1 *Given a connected border part of an anticonvex set which consists of a sequence of 0–pieces separated by singular 1–pieces. Assume that a 0–piece has length m, then the lengths of the foregoing 0–pieces are restricted to have the following values:*

$\geq m+2$	*not possible* *(exposed point)*	*Lemma 5.3* *(Lemma 4.4)*
$m+1$	*all foregoing pieces have lengths* $\geq m+1$ *or* $*$*–exposed point*	*Lemma 5.3* *Lemma 5.5*
m	—	—
$m-1$	*all foregoing pieces have lengths* $\geq m$	*Lemma 5.3*
$\leq m-2$	$*$*–exposed point*	*Lemma 5.4*

Theorem 5.2 *Given a connected border part of an anticonvex set which consists of a sequence of* 1*–pieces separated by singular* 0*–pieces. Assume that a* 1*–piece has length* m*, then the lengths of the foregoing* 1*–pieces are restricted to have the following values:*

$\leq m-2$	*not possible*
$m-1$	*all foregoing pieces have lengths* $\leq m-1$
m	—
$m+1$	*all foregoing pieces have lengths* $\leq m$ *or* $*$*–exposed point*
$\geq m+2$	$*$*–exposed point*

5.3 The General Case

We now collect all relevant Definitions and simultaneously formally define the concept of a $*$–exposed point.

Definition 5.2 *Given a digital set* S_Δ*. A point* $P_0 \in \text{bd } S_\Delta$ *is a* convex (locally) exposed point *(*concave (locally) exposed point *or* $*$–exposed point*) if*

- P_0 *is the intersection point of two nonsingular pieces which form a left (right) turn of the border (Lemma 4.2; Definition 5.1), or*
- P_0 *is the last (first) point of a* 0*–piece of length* m *which is followed by a* 0*–piece of length* $\leq m-2$ $(\geq m+2)$ *(Lemma 4.4; Definition 5.1), or*
- P_0 *is the first point of a* 1*–piece of length* m *which is preceded (succeded) by a* 1*–piece of length* $\leq m-2$ $(\geq m+2)$ *(Lemma* 4.4°*), or*
- P_0 *is the last point of a* 0*–piece of length* m *which is followed by a* 0*–piece of length* $m-1$ $(m+1)$ *(unless the first (last) piece is preceded (succeded) by a* 0*–piece of length* $m-1$ (m)*) (Lemma 4.5; Lemma 5.5), or*
- P_0 *is the first point of a* 1*–piece of length* m $(m-1)$ *which is preceded by a* 1*–piece of length* $m-1$ (m) *(unless the last piece is followed by a* 1*–piece of length* $m-1$*) (Lemma* 4.5°*), or*
- P_0 *belongs to a part of the boundary such that a Hübler–transform generates one of the last four situations.*

A point $P_0 \in \text{bd } S_\Delta$ *is a* locally exposed point *if it is a convex exposed point or a concave exposed point.*

Definition 5.3 *A border segment which is bounded by locally exposed points and contains no further locally exposed points is termed a* digital line segment.

This definition is just a reformulation of the definition of Hübler, Klette and Voss [12]. Specifically, it also coincides with Rosenfelds definition [22] and also with the definition of Debled–Rennesson and Reveillès [5].

We now are in a state to introduce the concept of convex and concave parts of the boundary of a digital set:

Definition 5.4 *Given a digital set S_Δ with boundary Γ. A subset Γ' of Γ is termed a* convex boundary part *if*

- *Γ' is (8–) connected,*
- *All exposed points of Γ which are on Γ' are convex exposed points unless they are end points of Γ',*
- *Γ' is maximal, i.e. there is no subset of Γ which has the first two properties and strictly contains Γ'.*

A subset Γ' of Γ is termed a concave boundary part *if*

- *Γ' is (8–) connected,*
- *All exposed points of Γ which are on Γ' are concave exposed points unless they are end points of Γ',*
- *Γ' is maximal, i.e. there is no subset of Γ which has the first two properties and strictly contains Γ'.*

We state the main result of this paper as a Theorem:

Theorem 5.3 *Let S_Δ be a digital set and let Γ be a connected component of the border of S_Δ. Let furthermore $\Gamma^{(e)} \subseteq \Gamma$ be the set of all locally exposed points of S_Δ which belong to Γ We assume that $\Gamma^{(e)}$ is equipped with an orientiation which is induced by the orientation of Γ (i.e. $P \in \Gamma^{(e)}$ is the successor of $Q \in \Gamma^{(e)}$ whenever there exists a sequence $P_0 = Q$, P_1, $\cdots$, P_{n-1}, $P_n = P$ of points in Γ such that P_i is the successor of P_{i-1} on Γ for $i = 1, 2, \cdots, n$, and if $P_i \notin \Gamma^{(e)}$ for $i = 1, 2, \cdots, n-1$).*

Then the polygonal set which is given by the cyclically ordered sequence of points in $\Gamma^{(e)}$ is a polygonal representation for the boundary component Γ. A complete polygonal representation of S_Δ is obtained by considering all boundary components of S_Δ. The parts of Γ which are bounded by two successive points in $\Gamma^{(e)}$ are digital lines.

The proof of this Theorem follows from the Lemmas and Theorems stated in this paper.

In Figure 11 an algorithm for determining all locally exposed points of a digital set is sketched.

Algorithms $\mathcal{E}_1(S_\Delta)$ and $\mathcal{E}(S_\Delta)$ yield the same result. Algorithm $\mathcal{E}_0(S_\Delta)$ generates a set of boundary points containing all exposed points. In order to guarantee termination of it, the unimodal

Algorithm $\mathcal{E}(S_\Delta)$

Start Given a digital set $S_\Delta \in \mathbb{Z}^2$. Define a subset $S_\Delta^{(e)} \subseteq S_\Delta$ containing all Scherl's descriptor points oriented by the orientation induced by the border of S_Δ.

Iteration Take any two successive exposed points obtained so far. Then the conditions of Theorems 4.1, 4.1°, 5.1 and 5.2 are fulfilled (up to 90^o–rotations). By means of these Theorems (after applying an appropriate Hübler transform, if necessary) it can be decided whether there are additional locally exposed points on the border segment bounded by the both exposed points under consideration.

If this is the case, these new locally exposed points are sorted into the ordered list $S_\Delta^{(e)}$ of exposed points obtained so far.

If not, the border segment is a digital line.

Result If no new exposed points are found, $S_\Delta^{(e)}$ contains all exposed points of S_Δ.

Figure 11: Algorithm for determining all locally exposed points.

transformations used have to be specified which is not done here. Algorithm $\mathcal{E}_1(S_\Delta)$ is very easy to implement and can be used on sets of moderate size. Algorithm $\mathcal{E}(S_\Delta)$ is a linear–time algorithm and therefore adviceable if larger sets should be processed.

The ordered sequences of locally exposed points generated by algorithms $\mathcal{E}_1(S_\Delta)$ or $\mathcal{E}(S_\Delta)$ yields a polygonal representation and in addition a decomposition into convex and concave parts of the border of a digital set. This representation, however, is not always faithful (see Figure 8).

6 Conclusions

For any digital set S_Δ which is given by its border Γ we can find a polygonal representation which is discrete but not necessarily faithful. It is, however, possible to segment the border into a convex and a nonconvex part (which overlap in line segements). This segmentation can be performed in time proportional to the length of the border by using the method proposed here which is related to the approach of Hübler, Klette and Voss [12] or the characterization of digital lines by Debled–Rennesson and Reveillès [5]. A large number of experiments was performed in our group to test this segmentation [17]. The results of these tests are documented in [15].

The polygonal representation of a set obtained by Algorithm $\mathcal{E}(S_\Delta)$ can be used as a basis for further simplification of the representing polygonal set by eliminating vertices (discrete evolution [17]).

For illustration of the approach the polygonal representation of Letter 'A' is shown in Figure 12.

An alternative possibility would be the polygonal representation of digital sets which are no longer discrete but faithful.

The concept of Scherl's descriptors can be generalized in an obvious way by considering more general tangent directions. [9]. In the context of digital borders one can make use of the fact that there is only a finite number of possible tangent directions. In a more general setting one can even define descriptors for boundary curves in $\mathbb{R}^2$ [9].

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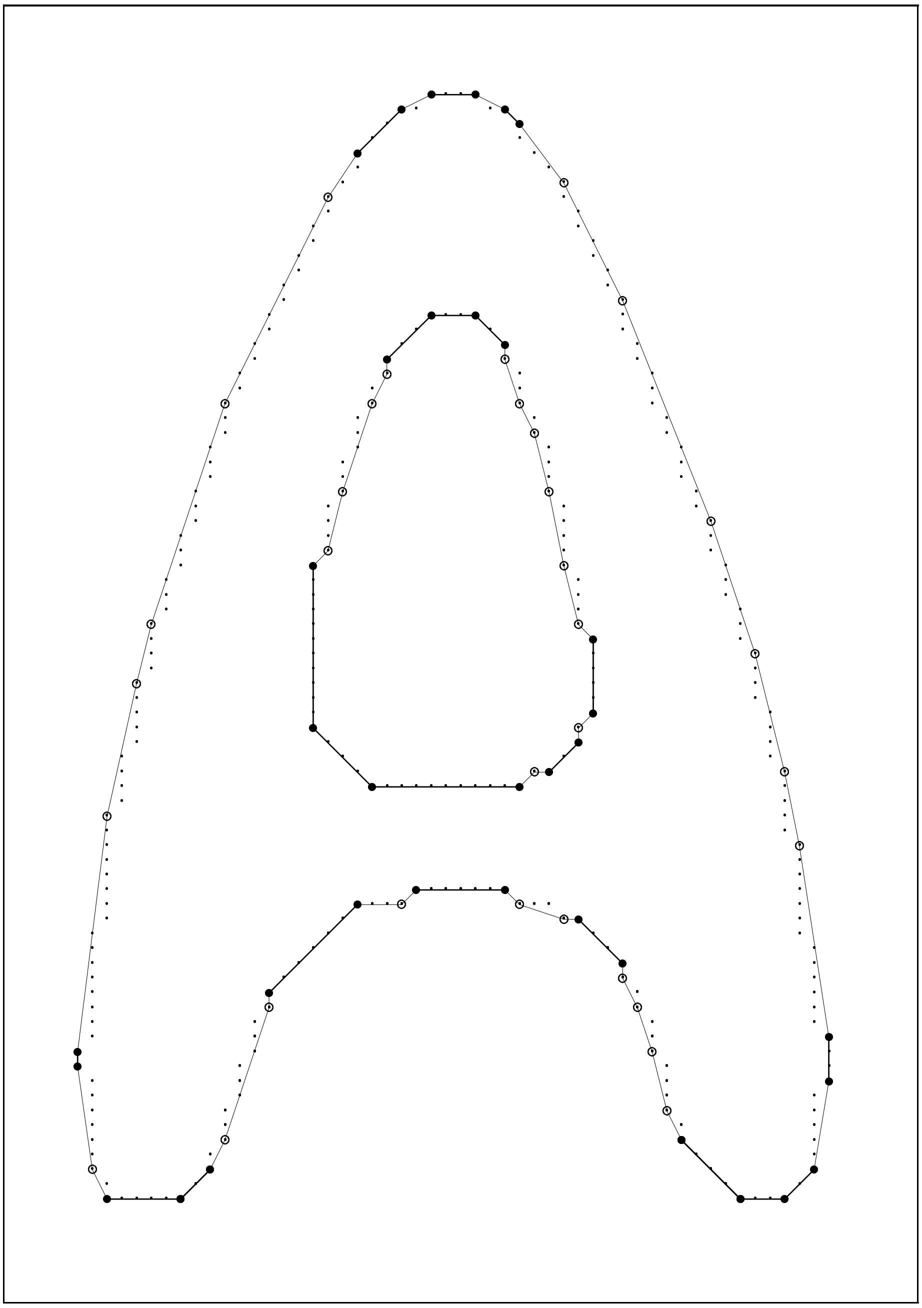

Figure 12: Polygonal representation of Letter 'A'.

The points of the borders of the set are marked $\cdot$. Points of the border which correspond to Scherl's descriptors are marked $\bullet$ and the descriptor tangents are indicated by thick lines. Exposed points of the convex and the concave parts of the border are indicated by $\circ$.