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Semi–Topological Approach to Root Images of Median Filters

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Abstract

Median filters are frequently used in signal analysis because they are robust edge–preserving smoothing filters. Since median filters are nonlinear filters, the tools of linear theory are not applicable to them. One approach to deal with nonlinear filters consists in investigating their root images (fixed elements or signals transparent to the filter). Whereas for one–dimensional median filters the set of all root signals can be completely characterized, this is not true for higher dimensional filters.

Tyan (1981) and Döhler (1989) proposed a method for construction of small root images for two–dimensional median filters. Although the Tyan–Döhler construction is valid for a wide class of median filters, their arguments were not correct and their assertions do not hold universally. In this paper we give a rigorous treatment for the construction of Tyan and Döhler. Moreover, the approach is generalized to the d–dimensional case.

1 Introduction

A d–dimensional *discrete signal* (see [15]) is a mapping $f : \mathbb{Z}^d \longrightarrow Y$ where $\mathbb{Z}^d$ is the d–dimensional *discrete space* which is the set of all points in $\mathbb{R}^d$ having integer coordinates. Usually Y is the set of real numbers or more generally an ordered set. We denote by $\mathcal{S}$ the set of all discrete signals. Let B be a finite subset of $\mathbb{Z}^d$. The *neighborhood* of a point $x \in \mathbb{Z}^d$ is the set of all points $x + u$, $u \in B$. Denote by $\#B$ the number of elements of B. For an integer k with $1 \leq k \leq \#B$ the *k–th rank order filter* $\rho_k : \mathcal{S} \longrightarrow \mathcal{S}$ is defined in the following way:

For $x \in \mathbb{Z}^d$ sort (e.g. in ascending order) the values $f(x+u)$, $u \in B$. Then $(\rho_k f)(x)$ is the kth element in the sorted sequence.

For $k = 1$ we get the *minimum filter* and for $k = \#B$ the *maximum filter* which are relevant in mathematical morphology. If $\#B$ is odd, then $k = \dfrac{\#B+1}{2}$ yields the *median filter* $\mu : \mathcal{S} \longrightarrow \mathcal{S}$ which is the main subject of this paper.

We first collect some of the properties of the median filter (see e.g. [20, Chapter 4] for details and precise formulations):

- μ is a nonlinear smoothing filter.
- The median filter preserves edge sharpness better than the mean filter.
- The median value is the best estimator with respect to the 1–norm.
- The most striking property of the median filter is its “robustness”. Specifically, it is insensitive to “outliers” in the data.
- The output of a rank order filter has the same data type as the input data. For example, if the input values $f(x)$ are of data type Byte, then the output values $(\rho_k f)(x)$ have type Byte, too.
- Rank order filters can be implemented very efficiently on signal processing hardware since their evaluation needs only very simple arithmetic.

- The investigation of convergence properties of iterated median filters becomes interesting in the context of smoothing algorithms for curve evolution. Merriman, Bence and Osher [18] proposed a method for calculating mean curvature flow which was based on iterated median filtering. This approach was mathematically justified by Evans [8] and extended by Guichard and Morel [13].

There exist numerous applications of rank order filters. Specifically in digital image processing they have certain advantages over linear filters. Numerous classes of related filters were developed for different applications in this area [19]. One very important application is compression of digital images. For a more recent publication on this application see e.g. [7]. For such applications, properties of two and higher dimensional filters become important.

Whereas for linear filters one has a very powerful theory (Fourier transform, $\mathcal{Z}$–transform, zero– and pole plans etc.), these theoretical tools are not applicable to nonlinear filters. Therefore one tries to model the properties of nonlinear filters according to linear theory. One tool for understanding nonlinear filters are fixed points or "root images" of them. These are signals which are not changed by the filter under investigation. The set of all such signals is the nonlinear analog of the passband of linear filters. In mathematical morphology we can decompose a function into a root image with respect to opening (or closing) and a signal which vanishes under opening (or closing) [4]. Such an analog of "orthogonal decomposition" of a signal is not obviously possible in the rank order filter context. However, we can try to decompose a signal into a suitable root signal and a remainder signal. There is the hope that the root signal carries the "most important" information about the given signal and the remainder signal provides detail information.

1.1 Historical Sketch

The median filter was invented in 1977 by J. W. Tuckey [25].

In 1954 E. N. Gilbert [11] had shown — in a different context — that for rank order filters it is sufficient to know their effect on binary signals. The so called "stacking principle" allows (under certain conditions) some sort of basis decomposition of a signal into binary signals. The importance of Gilbert's result rests in the fact that binary functions can be interpreted as characteristic functions of sets. Thus rank order filters can be treated in a geometric language as mappings between sets in $\mathbb{Z}^d$.

In 1981 Neal C. Gallagher, Jr. and Gary L. Wise [9] proved a characterization theorem for root signals of one–dimensional median filters. In 1991 Zi–Jun Gan and Ming Mao provided a complete convergence theory for one–dimensional iterated median filters. This theory states that by iterated application of a median filter one has either convergence in a "weak" sense to a root signal or else the iterates oscillate — under very special conditions — between two limit signals. E. Goles and J. Olivos [12] proved in 1981 that independently of the dimension by iterated application of a median filter to a finite set after a finite number of iterations either a root signal is obtained or else the sequence of iterates becomes periodic with period 2.

In 1981 S. G. Tyan [26] published an article on deterministic properties of such filters. In this article he proposed a method for constructing root signals of two–dimensional median filters for binary signals. For a neighborhood (or "window") which is "centrally symmetric" and contains the zero element, Tyan's construction of root images or "objects" is as follows:

> For each window, consider those directed line segments (spokes) which emanate from the center (base) and end at a point (tip) on the boundary. Each of these line segments

> can also be identified by its angle θ, $0 \leq \theta \leq 360^o$ with respect to a segment joining the center, say $(0,0)$ and the point $(1,0)$. Then one can easily recognize that each object has a boundary which is piecewise linear and is made up of those spokes in the descending or ascending order of θ, each with its base connected to the tip of its predecessor. Then the object is the convex set in $\mathbb{Z}^2$ with the above constructed boundary.
>
> ...
>
> However, it seems that it is true provided that the window A satisfies the following;
>
> $$\lim_{n\to\infty} A_n = \mathbb{Z}^2.$$
>
> Where $A_{n+1} = A_n + A$ ...We can consider a window A as *degenerate* if it fails to satisfy the above condition, ...

Tyan did not attempt to prove anything, he only gave five examples. Hans–Ulrich Döhler [2] provided in 1989 a first attempt to prove validity of Tyans construction. In his proof he invested two very clever ideas, however, the proof was neither professional nor correct. The main results in Döhler's paper are:

> ...our approach is restricted on a special root signal, which we call the smallest surviving object (SSO) of a median filter. This SSO is a convex digital polygon without holes.
>
> ...
>
> **Proposition 3.** *The contour of the SSO results from the linkage of all possible straight line segments that connect the center of the window with some other lattice point within the window. This linkage must be done in such a manner that all straight line segments are sorted with respect to their slopes. If there are segments with the same slope, only the longest of them is to be considered.*

It can be shown by means of counterexamples that the construction of Tyan and Döhler does not always result in a root image (see Figure 2).

The aim of this paper is to prove some results concerning Tyan–Döhler root images. Moreover, the construction process of these root images is generalized to the d–dimensional case. In [3] it was shown that for "normal" neighborhoods in $\mathbb{R}^d$ the Tyan–Döhler root image is always convex and indeed a root image. It is also smallest at least among all convex root images.

1.2 Notation

For a subset $S \subseteq \mathbb{Z}^d$ denote by $\complement S = \mathbb{Z}^d \setminus S$ the set theoretic complement (with respect to $\mathbb{Z}^d$) of S.

$\mathcal{P}\mathbb{Z}^d$ is the set of all subsets of $\mathbb{Z}^d$.

For two subsets S_1 and S_2 the set operation

$$S_1 \oplus S_2 := \{x_1 + x_2 \mid x_1 \in S_1,\ x_2 \in S_2\}$$

is called the *Minkowski–addition* of S_1 and S_2.

Θ_d is the zero element of $\mathbb{R}^d$ or $\mathbb{Z}^d$, respectively.

Let $B = \{u_1, u_2, \cdots, u_n\}$ with $u_i \neq u_j$ for $i \neq j$ be a nonempty finite *zero neighborhood* in $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$ is $B(x) = x + B = \{x\} \oplus B$ the *neighborhood* of x.

B is called a *centrally symmetric* neighborhood, if $u \in B$ implies $-u \in B$. If B is centrally symmetric and contains Θ_d then $\#B$ is odd.

We need a convergence concept for sets:

Definition 1.1 *A sequence $\{S_r\}_{r=0}^{\infty}$ of sets in $\mathbb{Z}^d$* converges weakly *to a set S if for any bounded set $C \subseteq \mathbb{Z}^d$ there exists a number r_0 such that*

$$S_r \cap C = S \cap C \qquad \text{for all} \quad r \geq r_0.$$

Notation: $S_r \longrightarrow S$.

In a certain special case weak convergence can be guaranteed:

Lemma 1.1 *Let $\{S_r\}_{r=0}^{\infty}$ be a monotonically decreasing sequence of sets (i. e. $S_{r+1} \subseteq S_r$ for all r). Then*

$$S_r \longrightarrow \bigcap_{r=0}^{\infty} S_r.$$

Let $\{S_r\}_{r=0}^{\infty}$ be a monotonically increasing sequence of sets i. e. $S_r \subseteq S_{r+1}$ for all r). Then

$$S_r \longrightarrow \bigcup_{r=0}^{\infty} S_r.$$

Proof Let $\overline{S} = \bigcap_{r=0}^{\infty} S_r$. For each $x \in \mathbb{Z}^d$ is either $x \in \overline{S}$ or else there is an r_0 with $x \notin S_{r_0}$. By monotonicity we have $x \notin S_r$ for all $r \geq r_0$.

Given a bounded set C. There are only finitely many points $x \in C$ having the second property, which means that from a certain r_0 on $x \notin S_r$. Choosing the maximal such r_0 we get the assertion that for this maximal r_0 we have $x \in \overline{S} \Longleftrightarrow x \in S_r$ whenever $r \geq r_0$.

The second assertion is obtained by duality. $\square$

Finally we introduce the concept of semi–topology (see [16, 17]):

Definition 1.2 *Given a set $\mathcal{X}$. The system $\mathcal{T}$ of subsets of $\mathcal{X}$ is a* semi–topology *for $\mathcal{X}$ if*

1. *$\emptyset \in \mathcal{T}$ and $\mathcal{X} \in \mathcal{T}$,*
2. *The union of any system of elements of $\mathcal{T}$ belongs to $\mathcal{T}$.*

The elements of $\mathcal{T}$ are termed semi–open sets *or* open sets *for short. A set whose complement is open is called a* closed set.

Remark 1.1 *There are mainly three approaches for dealing with rank order filters. These three approaches are essentially equivalent:*

- *The theory of complete lattices (Heijmans [14], Ronse [22]),*
- *Mathematical morphology (Serra [23]),*
- *The semi–topological approach (Latecki [16, 17], Eckhardt [4]).*

1.3 Definitions and Simple Properties

Definition 1.3 *Given a set $S \subseteq \mathbb{Z}^d$ and a number $k \in \{1, 2, \cdots \#B\}$, the k–*th rank order filter *$\rho_k = \rho_k^B$ is a mapping $\rho_k : \mathcal{P}\mathbb{Z}^d \longrightarrow \mathcal{P}\mathbb{Z}^d$ which is defined by*

$$\rho_k(S) = \left\{x \in \mathbb{Z}^d \mid \#(B(x) \cap S) \geq k\right\}.$$

The median filter *$\mu = \mu_B$, which is the main topic of this paper is a special rank order filter. It is usually defined for neighborhoods with an odd number of elements as the $\frac{\#B+1}{2}$–th rank order filter, $\mu(S) = \rho_{\frac{\#B+1}{2}}(S)$. We therefore have*

$$x \in \mu(S) \qquad \Longleftrightarrow \qquad \#(B(x) \cap S) \geq \frac{\#B+1}{2}.$$

Remark 1.2 *If we introduce the* characteristic function *$\chi_S : \mathbb{Z}^d \longrightarrow \{0, 1\}$ for a set S by*

$$\chi_S(x) = \begin{cases} 1 & if \quad x \in S \\ 0 & if \quad x \notin S \end{cases}$$

then the definition given here coincides with the definition given above in the introduction.

Lemma 1.2 *If $S_r \longrightarrow S$ then $\rho_k(S_r) \longrightarrow \rho_k(S)$.*

That means that the mapping ρ_k is continuous with respect to weak convergence.

Proof Let C be a bounded subset of $\mathbb{Z}^d$. We put $C_1 = C \oplus B$. Since C_1 is bounded, there exists an r_1 such that $S_r \cap C_1 = S \cap C_1$ for all $r \geq r_1$.

For $x \in C$ is by construction $U(x) \subseteq C_1$, consequently $x \in \rho_k(S_r) \Longleftrightarrow \#(U(x) \cap S_r) \geq k \Longleftrightarrow \#(U(x) \cap S) \geq k \Longleftrightarrow x \in \rho_k(S)$, that means $\rho_k(S_r) = \rho_k(S)$. $\square$

1.3.1 Examples

There are some neighborhoods which are often used for multi–dimensional rank order filtering. Here we restrict ourselves on two–dimensional filters. In the following pictures the sets under consideration are denoted "•", other points in $\mathbb{Z}^2$ by "·". If certain points are emphasized, other symbols will be used which are explained if they occur.

An often used neighborhood in $\mathbb{Z}^2$ is the 4–*neighborhood* (*cross–neighborhood*, *rook's neighborhood*) (the origin in $\mathbb{Z}^2$ is emphasized):

$$B^{(4)} = \begin{array}{ccc} \cdot & \bullet & \cdot \\ \bullet & \odot & \bullet \\ \cdot & \bullet & \cdot \end{array}$$

The 8–*neighborhood* (*king's neighborhood*) is

$$B^{(8)} = \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \odot & \bullet \\ \bullet & \bullet & \bullet \end{array}$$

Sometimes the 6–*neighborhood* is used. It is the natural neighborhood of the space

$$\mathbb{Z}_6^2 = \left\{ n_1 \cdot \begin{pmatrix} 1 \\ \\ 0 \end{pmatrix} + n_2 \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \;\middle|\; n_1, n_2 \in \mathbb{Z} \right\}.$$

In $\mathbb{Z}^2$ this neighborhood is not invariant with respect to the natural motions of $\mathbb{Z}^2$

$$B^{(6)} = \begin{array}{ccc} \cdot & \bullet & \bullet \\ \bullet & \odot & \bullet \\ \bullet & \bullet & \cdot \end{array}$$

In picture processing also the 12–*neighborhood* is used [5, 6]:

$$B^{(12)} = \begin{array}{ccccc} \cdot & \cdot & \bullet & \cdot & \cdot \\ \cdot & \bullet & \bullet & \bullet & \cdot \\ \bullet & \bullet & \odot & \bullet & \bullet \\ \cdot & \bullet & \bullet & \bullet & \cdot \\ \cdot & \cdot & \bullet & \cdot & \cdot \end{array}$$

Rank order filters with neighborhood $B^{(m)}$ are denoted $\rho_k^{(m)}$, $k = 1, 2, \cdots, m+1$, $m = 4, 6, 8, 12$.

Example 1.1 *Consider the set $S \subseteq \mathbb{Z}^2$ in the upper left subpicture of Figure 1. The effect of application of the rank order filters $\rho_k^{(4)}$ to the set S is demonstrated in the other subpictures of this Figure. Points of S which do not belong to the set obtained by filtering are denoted by $\odot$ and points which belong to the filtered set but not to S are marked $*$.*

Example 1.2 *In Figure 2 a counterexample for the Tyan–Döhler construction is given. The neighborhood used in this example (knight's neighborhood) is nondegenerate in the sense of Tyan. It is well–known that each two points on a checkerboard can be joined by a sequence of knight's moves. Straightforward application of the construction yields a set which is not a root image but contains a root image.*

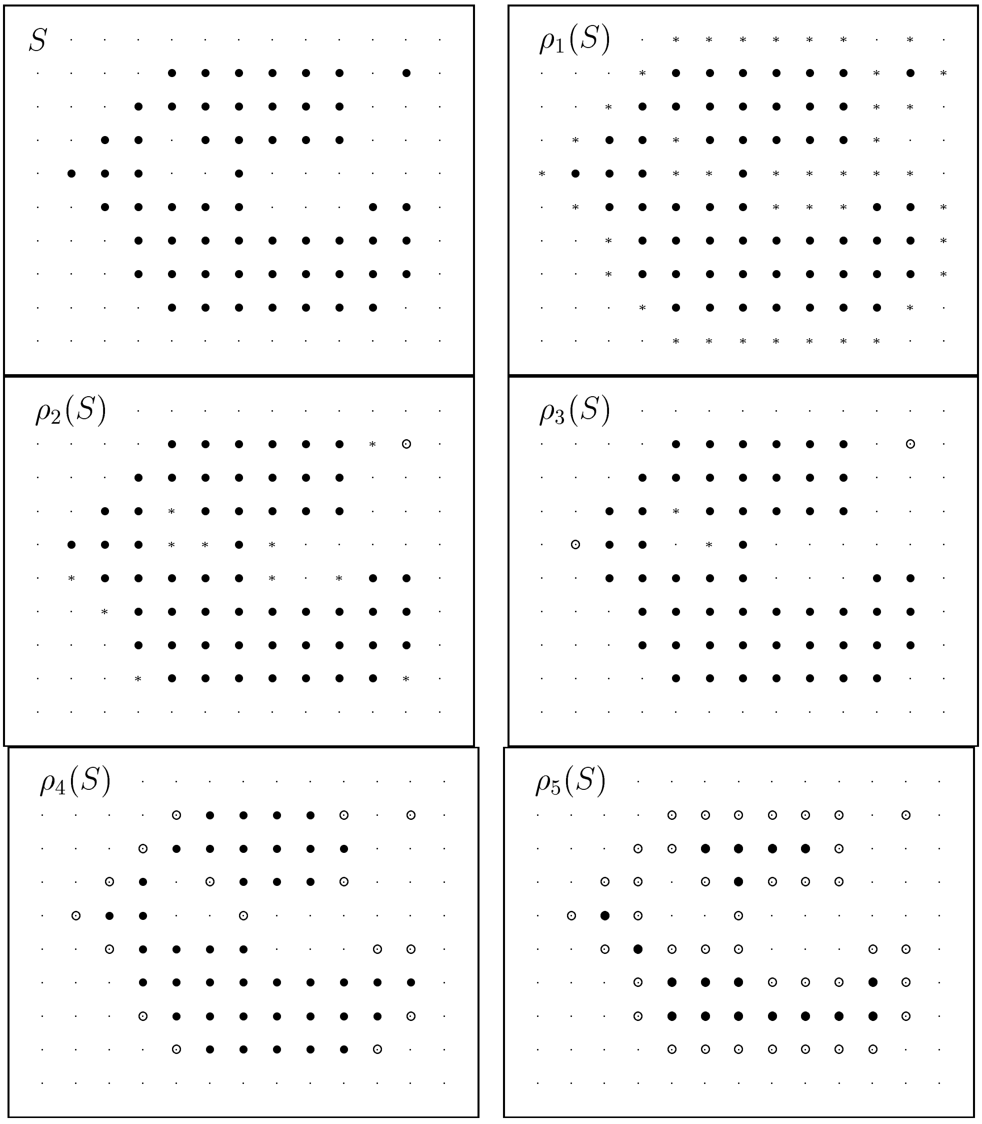


Figure 1: Application of rank order filters $\rho_k^{(4)}$.

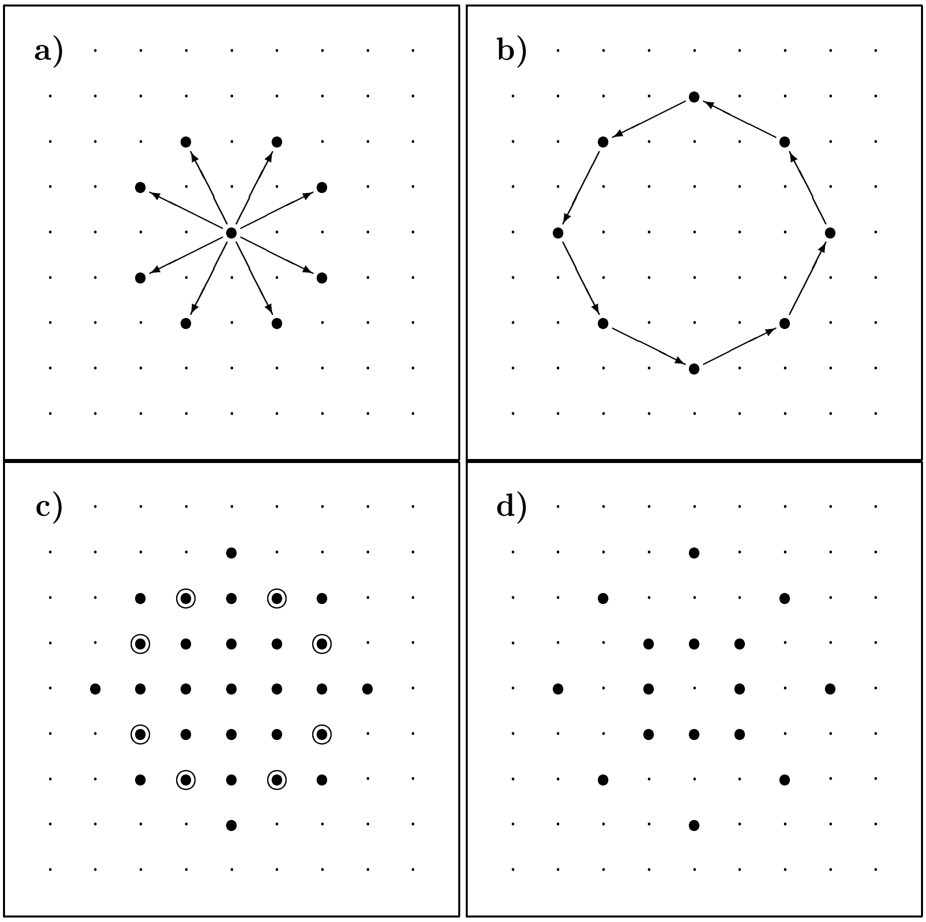


Figure 2: The Tyan–Döhler construction for the knight's neighborhood.

a) Knight's neighborhood.
b) Tyan–Döhler construction of the boundary of an "object".
c) The convex hull of the constructed object is not a root image. The encircled points have five neighbors not in the convex hull and thus they do not belong to the median filtered image.
d) Root image contained in the Tyan–Döhler set.

2 Semi–Topological Formulation

In this Section we are developing the semi–topological formulation of our problem. Specifically, we will define the interior and the closure of a digital set with respect to a certain semi–topology related to rank order filters. It will turn out that for the median filter the closure of the interior of a set S as well as the interior of the closure of S are root images (Theorem 2.1). It can even be shown that these sets are exactly all root images of the median filter (Corollary 2.1). Moreover, these root images are in some sense "close" to the set S which means that the difference set has empty interior (Corollary 2.2). This means that we get a decomposition of a set into a root image and a "thin" set which contains "less important information". This corresponds to the orthogonal decomposition of a signal into a main signal carrying "essential" information and a detail signal in linear theory.

Definition 2.1 *For an index $k \in \{1, 2, 3, \cdots, \#B\}$ the* complementary index *is $k^* = \#B + 1 - k$. The filter ρ_{k^*} is the* dual *filter to ρ_k. (see [14, p. 98]).*

Lemma 2.1

1. *$\rho_{k^*}(S) = \mathbb{C}\rho_k(\mathbb{C}S)$ for all $S \subseteq \mathbb{Z}^d$.*
2. *μ is* self–dual, *i.e.*

$$\mu(S) = \mathbb{C}(\mu(\mathbb{C}S)).$$

3. *ρ_k $(k = 1, 2, \cdots, \#B)$ is* isotone, *i. e. for $S_1, S_2 \in \mathbb{Z}^d$, $S_1 \subseteq S_2$ is*

$$\rho_k(S_1) \subseteq \rho_k(S_2).$$

4. *$\rho_{k+1}(S) \subseteq \rho_k(S)$ for each $S \subseteq \mathbb{Z}^d$ and any $k = 1, 2, \cdots, \#B - 1$.*
5. *For any system $\{S_\alpha\}$ of sets one has*

$$\rho_k\left(\bigcap_\alpha S_\alpha\right) \subseteq \bigcap_\alpha \rho_k(S_\alpha)$$

and

$$\bigcup_\alpha \rho_k(S_\alpha) \subseteq \rho_k\left(\bigcup_\alpha S_\alpha\right).$$

Proof

1. $x \in \mathbb{C}\rho_k(\mathbb{C}S) \iff x \notin \rho_k(\mathbb{C}S) \iff \#(B(x) \cap \mathbb{C}S) < k \iff \#(B(x) \cap S) > n + 1 - k \iff x \in \rho_{n+2-k}(S) \iff x \in \rho_{k^*}(S)$.

The other assertions are clear. $\square$

Lemma 2.2 *Given a system $\{S_\alpha\}$ of sets with $S_\alpha \subseteq \rho_k(S_\alpha)$ for all α.*

Then also

$$\bigcup_\alpha S_\alpha \subseteq \rho_k\left(\bigcup_\alpha S_\alpha\right).$$

Given a system $\{S_\alpha\}$ of sets with $\rho_k(S_\alpha) \subseteq S_\alpha$ for all α.

Then also

$$\rho_k\left(\bigcup_\alpha S_\alpha\right) \subseteq \bigcup_\alpha S_\alpha.$$

Proof We prove only the first assertion, the other follows by dualization. Let $x \in \bigcup S_\alpha$. Then $x \in S_{\alpha_0}$ for any α_0. In $B(x)$ are at least k elements of S_α, hence also at least k elements of $\bigcup S_\alpha$. This proves the first assertion. $\square$

The assertions of the Lemma justify the following definition:

Definition 2.2 *The semi–topology $\mathcal{T}_k$ on $\mathbb{Z}^d$ is the collection of all sets S with $S \subseteq \rho_k(S)$.*

*A set $S \in \mathcal{T}_k$ is termed k–*open*. A set $S \in \mathcal{T}_k$ whose complement is k–open is termed a k–*closed *set.*

We denote the set of all k–open subsets of $\mathbb{Z}^d$ by $\mathcal{O}_k$, the set of all k–closed subsets of $\mathbb{Z}^d$ by $\mathcal{F}_k$.

The set of all root images of ρ_k is the set $\mathcal{O}_k \cap \mathcal{F}_k$.

Remark 2.1 *In lattice theory the set $\mathcal{O}_k$ is called the* domain of extensivity *of ρ_k (*$\mathrm{Ext}(\rho_k)$*), and $\mathcal{F}_k$ is the* domain of antiextensivity *of ρ_k (*$\mathrm{Antext}(\rho_k)$*) and $\mathcal{O}_k \cap \mathcal{F}_k$ is the* invariance domain *of ρ_k (*$\mathrm{Inv}(\rho_k)$*) (see [22, p. 23] or [14, Chapter 6.1]).*

We state two rather trivial properties of these topologies in the next Lemma:

Lemma 2.3

1. *The closed sets in the topology $\mathcal{T}_k$ are exactly the sets S with $\rho_{k^*}(S) \subseteq S$.*
2. *Each $(k+1)$–open set is k–open. That means, the topology $\mathcal{T}_k$ is* finer *than the topology $\mathcal{T}_{k+1}$ or the topology $\mathcal{T}_{k+1}$ is* coarser *than the topology $\mathcal{T}_k$.*

Proof 1. This is an immediate consequence of Lemma 2.1, property 1.

2. Clear. $\square$

Definition 2.3 *Let $S \subseteq \mathbb{Z}^d$ be a set. The (k–)* closure $\mathrm{cl}_k S$ *of S is the smallest k–closed set containing S.*

The (k–) interior $\mathrm{int}_k S$ *is the largest k–open set contained in S.*

It is a direct consequence of the definition that these two concepts are well–defined.

Example 2.1 *Let B be a zero neighborhood in $\mathbb{Z}^d$, $\Theta_d \in B$. We investigate some special rank order filters ρ_k.*

For $k = 1$ is $\rho_1(S) = \{x \mid \#(B(x) \cap S) \geq 1\}$, specifically $S \subseteq \rho_1(S)$, that means that each set is 1–open. Hence each set is also 1–closed. In the terminology of mathematical morphology (for centrally symmetric B) the set $\rho_1(S)$ is termed the dilation *of S.*

*For $k = 2$ is $\rho_2(S) = \{x \mid \#(B(x) \cap S) \geq 2\}$. A point $x \in S$ is called a B–*isolated point *if $B(x) \cap S = \{x\}$. A set is 2–open if and only if it does not contain any B–isolated points.*

A point is termed a 1–hole *if it is an isolated point of the complement of S. 1–holes play a special role in thinning of binary images [5, 6].*

A set is 2–closed if and only if it contains no 1–holes.

$\mathrm{int}_2\, S$ *is the set obtained from S by removing all isolated points from it.* $\mathrm{cl}_2\, S$ *is the set obtained from S by filling all its 1–holes.*

One has

$$\mathrm{cl}_2\, \mathrm{int}_2\, S = \mathrm{int}_2\, \mathrm{cl}_2\, S$$

for any set S.

Lemma 2.4 *If S is open then the same is true for $\rho_k(S)$.*

If S is closed then the same is true for $\rho_{k^*}(S)$.

In other words: ρ_k maps the set $\mathcal{O}_k$ into itself and ρ_{k^*} maps the set $\mathcal{F}_k$ into itself.

Proof 1. From $S \subseteq \rho_k(S)$ we get $\rho_k(S) \subseteq \rho_k(\rho_k(S))$, hence $\rho_k(S)$ is k–open.

2. From $\rho_{k^*}(S) \subseteq S$ we get $\rho_{k^*}(\rho_{k^*}(S)) \subseteq \rho_{k^*}(S)$, Hence $\rho_{k^*}(S)$ is k–closed. □

Lemma 2.5 *It is always*

$$\mathrm{int}_k\, S \subseteq S \cap \rho_k(S) \subseteq S \cup \rho_{k^*}(S) \subseteq \mathrm{cl}_k(S).$$

Proof 1. By $\mathrm{int}_k\, S \subseteq S$ and

$$\mathrm{int}_k\, S \subseteq \rho_k(\mathrm{int}_k\, S) \subseteq \rho_k(S).$$

the left side of the inclusion is derived.

2. By duality, $S \subseteq \mathrm{cl}_k\, S$ and

$$\rho_{k^*}(S) \subseteq \rho_{k^*}(\mathrm{cl}_k\, S) \subseteq \mathrm{cl}_k\, S.$$

This proves the remaining inclusions. □

We now define recursively a sequence of sets:

For a given set S we put $\underline{S}^{(0)} = S$ and

$$\underline{S}^{(r+1)} = \underline{S}^{(r)} \cap \rho_k \left(\underline{S}^{(r)}\right)$$

for $r = 0, 1, 2, \cdots$.

In Heijmans' book the limit of the sequence $\{\underline{S}^{(r)}\}$ in the case of the median filter is called the *median opening* [14, Example 13.36].

Lemma 2.6 *The sequence $\{\underline{S}^{(r)}\}$ has the properties*

1. $\underline{S}^{(r+1)} \subseteq S$ *for all* r.
2. $\underline{S}^{(r+1)} \subseteq \underline{S}^{(r)}$ *and* $\underline{S}^{(r+1)} \subseteq \rho_k(\underline{S}^{(r)})$.
3. $\underline{S}^{(r+1)} \subseteq \bigcap_{i=0}^{r} \rho_k \left(\underline{S}^{(i)}\right)$.
4. $\operatorname{int}_k S \subseteq \underline{S}^{(r)}$ *for* $r = 0, 1, 2, \cdots$.

Proof The first two assertions are direct consequences of the definition of $\underline{S}^{(r)}$.

3. By Part 1 of the Lemma is $\underline{S}^{(r+1)} \subseteq \underline{S}^{(r)}$, hence $\underline{S}^{(r+1)} \subseteq \underline{S}^{(i)}$ for $i = 0, 1, \cdots, r$. $\underline{S}^{(r+1)} \subseteq \rho_k(\underline{S}^{(r)}$ (Part 1) implies $\underline{S}^{(r+1)} \subseteq \rho_k(\underline{S}^{(i)})$ for $i = 0, 1, \cdots, r$, and this implies the assertion.

4. $\operatorname{int}_k S \subseteq S$ and

$$\operatorname{int}_k S \subseteq \rho_k(\operatorname{int}_k S) \subseteq \rho_k(S),$$

hence $\operatorname{int}_k(S) \subseteq S \cap \rho_k(S) = \underline{S}^{(1)}$. The assertion is obtained by induction. □

Lemma 2.7 *For a set $S \subseteq \mathbb{Z}^d$ let $\underline{S}^{(r)}$ be defined iteratively as above. Then*

$$\underline{S}^{(r)} \longrightarrow \operatorname{int}_k S$$

in the sense of weak convergence (Lemma 1.1).

Proof From monotonicity of the sequence $\left\{\underline{S}^{(r)}\right\}$ we get by Lemma 1.1 weak convergence to $\widetilde{S} := \bigcap_{r=0}^{\infty} \underline{S}^{(r)}$. This means that for any bounded set $C \subseteq \mathbb{Z}^d$ there is a number r_0 such that $\underline{S}^{(r)} \cap C = \widetilde{S} \cap C$ for all $r \geq r_0$. In particular, $\underline{S}^{(r+1)} \cap C = \underline{S}^{(r)} \cap C$ for all $r \geq r_0$. Consequently, for all such r

$$\underline{S}^{(r+1)} \cap C = \underline{S}^{(r)} \cap \rho_k(\underline{S}^{(r)}) \cap C = \underline{S}^{(r)} \cap C$$

This implies $\underline{S}^{(r)} \cap C \subseteq \rho_k(\underline{S}^{(r)}) \cap C$ for all $r \geq r_0$.

Now, $\widetilde{S} = \bigcap_{\nu=0}^{\infty} \underline{S}^{(\nu)} \subseteq \underline{S}^{(r)}$ for all r, hence

$$\widetilde{S} \cap C \subseteq \underline{S}^{(r)} \cap C \subseteq \rho_k \left(\underline{S}^{(r)}\right) \cap C.$$

Therefore, for $r \geq r_0$

$$\widetilde{S} \cap C = \underline{S}^{(r)} \cap C \subseteq \rho_k\left(\underline{S}^{(r)}\right) \cap C.$$

By Lemma 1.2 for all sufficiently large r

$$\rho_k\left(\underline{S}^{(r)}\right) \cap C = \rho_k\left(\widetilde{S}\right) \cap C.$$

This implies

$$\widetilde{S} \cap C \subseteq \rho_k\left(\widetilde{S}\right) \cap C.$$

Thus it is proved that the set $\widetilde{S}$ is k–open. Furthermore, (Lemma 2.6, properties 1 and 4) $\widetilde{S} \subseteq S$ and $\mathrm{int}_k\, S \subseteq \widetilde{S}$. Since $\mathrm{int}_k\, S$ is the largest open set $\subseteq S$, we have $\mathrm{int}_k\, S = \widetilde{S}$. □

Dualization yields the following assertions:

For a given set S we put $\overline{S}^{(0)} = S$ and

$$\overline{S}^{(r+1)} = \overline{S}^{(r)} \cup \rho_k\left(\overline{S}^{(r)}\right)$$

for $r = 0, 1, 2, \cdots$.

Lemma 2.8 *The sequence* $\{\overline{S}^{(r)}\}$ *has the properties*

1. $S \subseteq \overline{S}^{(r)}$ *for all* r.
2. $\overline{S}^{(r)} \subseteq \overline{S}^{(r+1)}$ *and* $\rho_k(\overline{S}^{(r)}) \subseteq (\overline{S}^{(r+1)})$.
3. $\bigcup_{i=0}^{r} \rho_k\left(\overline{S}^{(i)}\right) \subseteq \overline{S}^{(r+1)}$.
4. $\overline{S}^{(r)} \subseteq \mathrm{cl}_k\, S$ *for* $r = 0, 1, 2, \cdots$.

Lemma 2.9 *For a set* $S \subseteq \mathbb{Z}^d$ *let* $\overline{S}^{(r)}$ *be defined iteratively as above. Then* $\overline{S}^{(r)} \longrightarrow \mathrm{cl}_k\, S$.

Remark 2.2 *Both iteration processes provide by Lemma 2.7 (or Lemma 2.9, respectively) for a bounded set* S *a constructive method for determining* $\mathrm{int}_k\, S$ *(or* $\mathrm{cl}_k\, S$*, respectively).*

Remark 2.3 *In* 4*–topology* $x \in \mu(S) = \rho_3(S)$ *for* $x \in S$*, means that* x *has at least two direct neighbors in* S*.* $x \notin \mathrm{int}_3\, S$ *means for* $x \in S$*, that* x *has at most one direct neighbor in* S*. Consequently, in* 4*–topology always* $\mathrm{int}_3\, S = S \cap \mu(S)$ *and* $S \cup \mu(S) = \mathrm{cl}_3\, S$*.*

Given any set $S \subseteq \mathbb{Z}^d$, we can easily assign to this set two root images of the median filter which are in some sense “close” to this set.

Theorem 2.1 *Assume that $\mathbb{Z}^d$ is equipped with the semi–topology associated to the median filter (i.e. $k = \frac{\#B+1}{2}$).*

1. *If S is (k–) open then $\operatorname{cl} S$ is a root image.*
2. *If S is (k–) closed then $\operatorname{int} S$ is a root image.*
3. *For any set $S \subseteq \mathbb{Z}^d$ the sets $\operatorname{cl}_k \operatorname{int}_k S$ and $\operatorname{int}_k \operatorname{cl}_k S$ are root images.*

Proof Assume that S is k–open. Then by Lemma 2.8 the iteration process $\overline{S}^{(0)} = S$, $\overline{S}^{(r+1)} = \overline{S}^{(r)} \cup \mu(\overline{S}^{(r)})$ yields a sequence converging to $\operatorname{cl}_k S$. Since S is open, $\overline{S}^{(r)} \subseteq \mu(\overline{S}^{(r)})$, hence $\overline{S}^{(r+1)} = \mu(\overline{S}^{(r)})$.

Let C be a bounded subset of $\mathbb{Z}^d$, $C' = C \oplus B$. Since the sequence $\{\overline{S}^{(r)}\}$ converges monotonically to $\operatorname{cl} S$, there exists a number r_0 such that

$$\overline{S}^{(r)} \cap C' = \operatorname{cl} S \cap C' \qquad \text{for all} \quad r \geq r_0.$$

If $x \in C$ then $B(x) \subseteq C'$ by construction of C'. Since in C' the sets $\overline{S}^{(r)}$ and $\operatorname{cl} S$ coincide, we have

$$\mu\left(\overline{S}^{(r)}\right) \cap C = \mu\,(\operatorname{cl} S) \cap C \qquad \text{for all} \quad r \geq r_0.$$

Since this holds for all bounded C, $\lim_{r\to\infty} \overline{S}^{(r)} = \operatorname{cl} S$.

The second assertion holds by duality, the third one is a consquence of the first two assertions. $\square$

In general topology so–called *regular* sets play a special role. These are sets having the property that $S = \operatorname{int} \operatorname{cl} S$ or $S = \operatorname{cl} \operatorname{int} S$. This concept is due to Stone [24] and it has many applications (see [21] and [14, Example 7.9]). For an application in our context, see [1].

The following Corollary is an immediate consequence of the last Lemma:

Corollary 2.1 *For the topology associated to the median filter (i.e. $k = \frac{\#B+1}{2}$) the following equivalence holds*

$$\left.\begin{array}{ll} S = \operatorname{int}_k \operatorname{cl}_k S & \textit{or} \\ S = \operatorname{cl}_k \operatorname{int}_k S & \end{array}\right\} \quad \Longleftrightarrow \quad S \text{ is a root image of } \mu.$$

In the following Lemmas some properties of the interior and the closure operators are collected.

Lemma 2.10 *Always is* $\operatorname{int} \operatorname{int} S = \operatorname{int} S$.

Proof Let $S_0 = \operatorname{int} S$ and $S_{r+1} = S_r \cap \mu(S_r)$ for $r = 0, 1, 2, \cdots$. Then

$$S_1 = S_0 \cap \mu(S_0) = \operatorname{int} S \cap \mu(\operatorname{int} S) = \operatorname{int} S,$$

analogously $S_r = \operatorname{int} S$ which proves the assertion. $\square$

Lemma 2.11 *For each set* $S \subseteq \mathbb{Z}^d$ *holds* $\operatorname{int}(S \setminus \operatorname{int} S) = \emptyset$ *and* $\operatorname{int}(\operatorname{cl} S \setminus S) = \emptyset$.

Proof Let $S_0 = S \setminus \operatorname{int} S$ and $S_{r+1} = S_r \cap \mu(S_r)$ for $r = 0, 1, 2, \cdots$ furthermore $\widetilde{S}_0 = S$ and $\widetilde{S}_{r+1} = \widetilde{S}_r \cap \mu(\widetilde{S}_r)$ for $r = 0, 1, 2, \cdots$. Then

$$\begin{aligned} S_1 &= S_0 \cap \mu(S_0) = S \cap \mathbb{C}\operatorname{int} S \cap \mu\,(S \cap \mathbb{C}\operatorname{int} S) \subseteq S \cap \mathbb{C}\operatorname{int} S \cap \mu(S) \cap \mu\,(\mathbb{C}\operatorname{int} S) = \\ &= S \cap \mathbb{C}\operatorname{int} S \cap \mu(S) \cap \underbrace{\mathbb{C}\mathbb{C}\mu(}_{=\mu}\mathbb{C}\operatorname{int} S) = S \cap \mathbb{C}\operatorname{int} S \cap \mu(S) \cap \underbrace{\mathbb{C}\mu(\operatorname{int} S)}_{\subseteq \mathbb{C}\operatorname{int} S} \subseteq \\ &\subseteq S \cap \mu(S) \cap \mathbb{C}\operatorname{int} S = S \cap \mu(S) \setminus \operatorname{int} S, \end{aligned}$$

hence $S_1 \subseteq \widetilde{S}_1 \setminus \operatorname{int} S$. If $S_r \subseteq \widetilde{S}_r \setminus \operatorname{int} S$ then also

$$\begin{aligned} S_{r+1} &= S_r \cap \mu(S_r) \subseteq \widetilde{S}_r \setminus \operatorname{int} S \cap \mu\left(\widetilde{S}_r \setminus \operatorname{int} S\right) \subseteq \widetilde{S}_r \cap \mathbb{C}\operatorname{int} S \cap \mu\left(\widetilde{S}_r\right) \cap \mu\,(\mathbb{C}\operatorname{int} S) \subseteq \\ &\subseteq \widetilde{S}_{r+1} \cap \mathbb{C}\operatorname{int} S. \end{aligned}$$

From $\widetilde{S}_r \longrightarrow \operatorname{int} S$ we conclude $S_r \longrightarrow \emptyset$.

Analogously the proof of the second assertion is carried out. Let $S_0 = \operatorname{cl} S \setminus S$ and $S_{r+1} = S_r \cap \mu(S_r)$ for $r = 0, 1, 2, \cdots$ furthermore $\widetilde{S}_0 = S$ and $\widetilde{S}_{r+1} = \widetilde{S}_r \cup \mu(\widetilde{S}_r)$ for $r = 0, 1, 2, \cdots$. Then

$$\begin{aligned} S_1 &= \operatorname{cl} S \cap \mathbb{C}S \cap \mu\,(\operatorname{cl} S \cap \mathbb{C}S) \subseteq \operatorname{cl} S \cap \mathbb{C}S \cap \underbrace{\mu\,(\operatorname{cl} S)}_{\subseteq \operatorname{cl} S} \cap \underbrace{\mu\,(\mathbb{C}S)}_{=\mathbb{C}\mu(S)} \subseteq \\ &\subseteq \operatorname{cl} S \cap \mathbb{C}S \cap \mathbb{C}\mu(S) = \operatorname{cl} S \cap \mathbb{C}(S \cup \mu(S)) = \operatorname{cl} S \setminus \widetilde{S}_1 \end{aligned}$$

and so on. $\square$

Corollary 2.2 *For any set* S *the following identities are true*

1. $\operatorname{int}(S \setminus \operatorname{cl}\operatorname{int} S) = \emptyset$,
2. $\operatorname{int}(S \setminus \operatorname{int}\operatorname{cl} S) = \emptyset$,
3. $\operatorname{int}(\operatorname{cl}\operatorname{int} S \setminus S) = \emptyset$,
4. $\operatorname{int}(\operatorname{int}\operatorname{cl} S \setminus S) = \emptyset$.

Proof The first two assertions are a consequence of the trivial inclusion $S \subseteq \operatorname{cl} S$, the latter ones follow from the second assertion of the Lemma. $\square$

Lemma 2.12 *We have*

$$\operatorname{cl}\operatorname{int}\operatorname{cl} S \subseteq \operatorname{cl} S \qquad \text{and} \qquad \operatorname{int} S \subseteq \operatorname{int}\operatorname{cl}\operatorname{int} S$$

and

$$\operatorname{cl}\operatorname{int} S = \operatorname{cl}\operatorname{int}\operatorname{cl}\operatorname{int} S \qquad \text{and} \qquad \operatorname{int}\operatorname{cl} S = \operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} S.$$

Proof The first assertion follows by application of the cl–operator on $\operatorname{int}\operatorname{cl} S \subseteq \operatorname{cl} S$, the second by application of the int–operator on $\operatorname{int} S \subseteq \operatorname{cl}\operatorname{int} S$.

Applying again the int–operator, we get

$$\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} S \subseteq \operatorname{int}\operatorname{cl} S.$$

Analogously we get by applying the cl–operator on the set $\operatorname{cl} S$

$$\operatorname{int}\operatorname{cl} S \subseteq \operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} S,$$

consequently

$$\operatorname{int}\operatorname{cl}\operatorname{int}\operatorname{cl} S = \operatorname{int}\operatorname{cl} S.$$

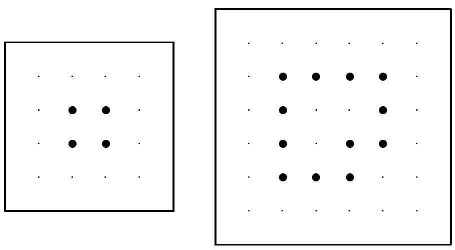
Analogously the last assertion. $\square$

3 Existence of Finite k–Open Sets

The empty set and the whole space $\mathbb{Z}^d$ are k–open for all k, $1 \leq k \leq \#B$. For $k = 1$ all sets are k–open (if $\Theta_d \in B$). It is also possible to find "large" k–open sets since the set $\mathcal{O}_k$ is stable with respect to intersections. However, "small" finite nontrivial k–open sets are of importance. In favourable cases a semi–topology can be described by means of its smallest open sets and in very lucky cases there exist finite point bases of a semi–topology. In the case of the topologies $\mathcal{T}_k$ the situation is certainly more complicated as is illustrated by the following example.

Example 3.1 *Consider the cross median filter which is based on the neighborhood $B^{(4)}$. In the following picture the left set is without any doubt a "smallest" open set and even a root image in the median–filter topology $\mathcal{T}_3$. It is indeed the Tyan–Döhler set for the cross median filter. If any point of this set is removed from it then the remaining set is no longer a 4–open set.*

The right set is also a "smallest" set in the same sense, however, the left set is not contained in the right one. Moreover, an infinite number of sets can be constructed in a similar manner as the right set and none of them is contained in the other. That means, there is no finite base for the topology $\mathcal{T}_3$.



We start with a negative assertion.

Lemma 3.1 *1. If $\Theta_d \in B$ then there are no finite nonempty k–open sets for $k > \frac{\#B+1}{2}$.*

2. If $\Theta_d \notin B$ then there are no finite nonempty k–open sets for $k > \frac{\#B}{2}$.

Proof Given a bounded digital set S. Since B is finite, there exists a vector $x^* \in \mathbb{R}^d$ such that $\langle x^*, u_j \rangle \neq 0$ for all $u_j \in B$, $u_j \neq \Theta_d$ ($\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^d$). Let $B^+ = \{u \in B \mid \langle x^*, u \rangle > 0\}$ and $B^- = \{u \in B \mid \langle x^*, u \rangle < 0\}$ and (without loss of generality) $\#B^- \leq \#B^+$. Choose $x_0 \in S$ so that $\langle x^*, x_0 \rangle \geq \langle x^*, x \rangle$ for all $x \in S$. Such an x_0 exists by finiteness of S. Then (by $B^+(x_0) \cap S = \emptyset$)

$$\#(B(x_0) \cap S) = \begin{cases} \#(B^-(x_0) \cap S) + 1 \leq \#B^- + 1 \leq \dfrac{\#B + 1}{2} & \text{if} \quad \Theta_d \in B, \\ \#(B^-(x_0) \cap S) \leq \#B^- \leq \dfrac{\#B}{2} & \text{if} \quad \Theta_d \notin B, \end{cases}$$

hence in both cases $x_0 \notin \rho^{(k)}(S)$, therefore S is not open. □

Corollary 3.1 *1. If $\Theta_d \in B$ then there are no finite nonempty (k–) root images for $k \neq \frac{\#B+1}{2}$.*

2. If $\Theta_d \notin B$ then there are no finite nonempty (k–) root images at all.

Proof Let S be a k–root image, i. e. $\rho^{(k)}(S) = S$. Furthermore, $\rho^{(k)}(\mathbb{C}S) = \mathbb{C}S$ implies $\rho^{(k^*)}(S) = S$. Therefore is S k^*–open.

1. If $\Theta_d \in B$ then S k–open implies $k \leq \frac{\#B+1}{2}$ and S k^*–open implies $k^* = \#B + 1 - k \leq \frac{\#B+1}{2}$ or $k \geq \frac{\#B+1}{2}$. Hence, root images are only possible for $k = \frac{\#B+1}{2}$, which implies that $\#B$ is odd.

2. If $\Theta_d \notin B$ then S k–open implies $k \leq \frac{\#B}{2}$. S k^*–open implies $k^* = \#B + 1 - k \leq \frac{\#B}{2}$ or $k \geq \frac{\#B}{2} + 1$. Hence, there is no such k. □

Lemma 3.2 *If B is not centrally symmetric then there are no (nontrivial) root images at all.*

Proof The Corollary implies that for $\Theta_d \notin B$ there are no nontrivial root images anyway. Therefore we can assume $\Theta_d \in B$.

B is centrally symmetric if and only if for each $x^* \neq \Theta_d$

$$\#\{u \in B \mid \langle x^*, u \rangle > 0\} = \#\{u \in B \mid \langle x^*, u \rangle < 0\}.$$

Therefore, if B is not centrally symmetric, there exists an $x^* \neq \Theta_d$ such that $\langle x^*, u \rangle \neq 0$ for all $u \in B \setminus \{\Theta_d\}$ and, if $B^+ = \{u \in B \mid \langle x^*, u \rangle > 0\}$ and $B^- = \{u \in B \mid \langle x^*, u \rangle < 0\}$ then $\#B^- > \#B^+$ (without loss of generality). We choose $x_0 \in S$ such that $\langle x^*, x_0 \rangle \geq \langle x^*, x \rangle$ for all $x \in S$. Then $\langle x^*, x_0 + u \rangle > \langle x^*, x \rangle$ for all $u \in B^+$, hence $x_0 + u \notin S$ for all $u \in B^+$. We obtain $\#(B(x_0) \cap S) = \#(B^- \cap S) \leq \#B^- + 1$. Since $\#B = \#B^- + \#B^+ + 1 > 2 \cdot \#B^- + 1$, we get $\#B^- < \frac{\#B}{2} - \frac{1}{2}$, consequently $\#(B(x_0) \cap S) < \frac{\#B+1}{2}$, hence S is not a root image. □

Definition 3.1 *A* cycle *is a sequence $\{u_{i_1}, u_{i_2}, \cdots, u_{i_\ell}\}$ of elements in B such that there are positive integers α_j, $j = 1, 2, \cdots, \ell$ with*

$$\sum_{j=1}^{\ell} \alpha_j u_{i_j} = \Theta_d. \tag{3.1}$$

Lemma 3.3 *The sequence* $\{u_{i_1}, u_{i_2}, \cdots, u_{i_\ell}\}$ *is a cycle if and only if there are real numbers* $\lambda_1, \lambda_2, \cdots, \lambda_\ell$ *such that*

$$\sum_{j=1}^{\ell} \lambda_j u_{i_j} = \Theta_d, \qquad \lambda_j > 0 \quad \text{for } j = 1, 2, \cdots, \ell, \qquad \sum_{j=1}^{\ell} \lambda_j = 1.$$

Proof 1. If there are no such numbers λ_j then there is no nontrivial solution of (3.1) in nonnegative real numbers and thus also no nontrivial solution in nonnegative integers.

2. Assume that there are real numbers $\lambda_1, \lambda_2, \cdots, \lambda_\ell$ with $\lambda_j > 0$ for $j = 1, 2, \cdots, \ell$ and $\sum_{j=1}^{\ell} \lambda_j = 1$ and $\sum_{j=1}^{\ell} \lambda_j u_{i_j} = \Theta_d$. It is possible (for example by Gaussian elimination) to eliminate dependent variables. Approximating the independent variables suitably by rational numbers, the dependent variables remain positive and are rational. Multiplying by a suitable integer one gets an integer solution. □

The cycles which can be formed by the elements of B are of course not uniquely determined.

Lemma 3.4 *If there are* $k-1$ *cycles having no elements* $u_j \in B$ *in common, then there exists a bounded nontrivial* k*–open set.*

Proof Assume that the equation (3.1) has a nontrivial solution. Let $x_0 \in \mathbb{Z}^d$. We construct a chain $v_0, v_1, v_2, \cdots$, of vectors with $v_0 = x_0$ and $v_{i+1} = v_i + u_{i_j}$ for all j. We require that each u_{i_j} occurs exactly $\nu\alpha_j$ times in the construction. By (3.1) the chain of the v_i is closed. By choosing $S = \{v_i\}$ we get a 2–open set.

Assume that there is a second cycle. We generate from the set S a second set by shifting the set S along the second cycle (in morphological terminology: the Minkowski addition of S with the second cycle is performed). The resulting set has the property that both cycles pass through each of its points, hence it is 3–open.

Continuing the process with all cycles, we get a k–open set. □

For 2–open sets we can prove the converse of the Lemma:

Lemma 3.5 *There exists a* 2*–open set if and only if there exists a cycle.*

Proof Assume that the Diophantine equation (3.1) has only the trivial nonnegative integer solution. Then $\Theta_d \notin \mathrm{conv}\{u_{i_1}, u_{i_2}, \cdots, u_{i_\ell}\}$. By the Separation Theorem for Convex Sets [27, Chapters II, V] there exists a vector $x^* \in \mathbb{R}^d$, $x^* \neq \Theta_d$, such that $\langle u_{i_j}, x^* \rangle > 0$ for all j. Choose $x_0 \in S$ such that $\langle x_0, x^* \rangle \geq \langle x, x^* \rangle$ for all $x \in S$. Then $\langle x_0 + u_{i_j}, x^* \rangle > \langle x, x^* \rangle$ for all $x \in S$, consequently $x_0 + u_{i_j} \notin S$ for all j. □

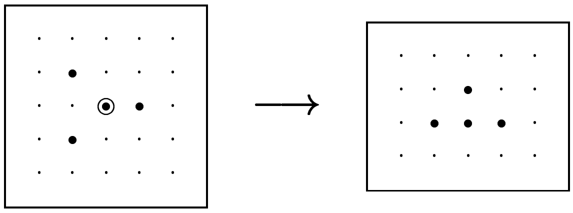
Remark 3.1 *For a centrally symmetric neighborhood* B *(containing* Θ_d*) there are always* $\frac{\#B-1}{2}$ *cycles* $\{u_i, -u_i\}$*. Therefore for such a neighborhood (and only for such a neighborhood) there exist always nontrivial* $\frac{\#B+1}{2}$*–open finite sets.*

Example 3.2 *We consider in* $\mathbb{Z}^2$ *the neighborhood*

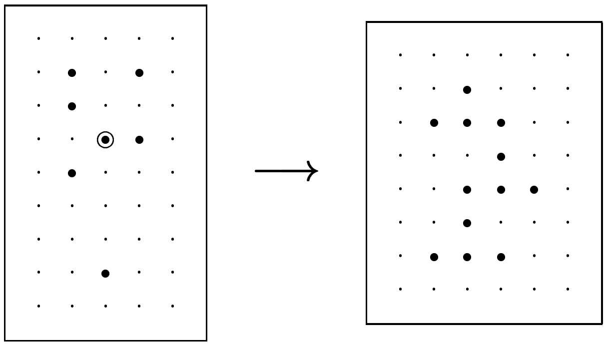
$$B = \{\Theta_2, u_1, u_2, u_3\}$$

with $u_1 = (1, 0)$*,* $u_2 = (-1, 1)$*,* $u_2 = (-1, -1)$*.*

By construction of Lemma 3.4 we get the 2*–open set in the right image.*

$\boldsymbol{B} =$ 

Example 3.3 *Consider the neighborhood B in the following picture. This neighborhood is the union of the neighborhood of the last example and another neighborhood. Both neighborhoods allow cycles. Therefore a 3–open set can be constructed as in Lemma 3.4.*

$\boldsymbol{B} =$ 

4 Construction of k–Open Sets, Centrally Symmetric Case

We now assume that B is centrally symmetric and $\Theta_d \in B$.

Definition 4.1 *A digital set S is termed a B–*path *if for each element $x \in S$*

$$2 \leq \#(B(x) \cap S) \leq 3$$

and if for at most two elements (the end points*) $\#(B(x) \cap S) = 2$. A bounded B–path with $\#(B(x) \cap S) = 3$ for all x is a* closed B–path.

Remark 4.1 *For centrally symmetric B (having more than three elements) each B–path is 2–open.*

Each B–path without end points (i.e. a closed B–path or an infinite B–path) is 3–open.

Lemma 4.1 *Given any set S. Then the set*

$$\delta_u(S) := S \oplus \{\Theta_d, u\} = S \cup (S + u)$$

is 2–open for each $u \in B \setminus \{\Theta_d\}$

Proof $x \in \delta_u(S)$ means either $x \in S$ and then $x + u \in \delta_u(S)$, or else $x \notin S$, then there exists an $x' \in S$ such that $x' + u \in S$. Since B is centrally symmetric, $x - u \in \delta_u(S)$ and $-u \in B$. Hence $\delta_u(S)$ is 2–open. $\square$

Lemma 4.2 *Given a k–open set S and any neighborhood $\widetilde{B}$. Let $\delta(S) = S \oplus \widetilde{B}$ be the (usual) dilation of S with the structuring element $\widetilde{B}$.*

Then $\delta(S)$ is also a k–open set.

Proof Let $x \in \delta(S)$. If x in S, then clearly $\#(B(X) \cap S) \geq k$. If $x \notin S$, then there exist an $x' \in S$ and a $u \in \widetilde{B}$ such that $x = x' + u$. Since S is k–open, there are k pairwise different elements $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ of $\widetilde{B} \setminus \{\Theta_d\}$ such that $x' + u_{i_j} \in S$ for all j. Consequently, $x + u_{i_j} \in \delta(S)$ for all j, hence $\delta(S)$ is also k–open. □

From these observations we get a very simple method for construction of a k–open set (in case of a centrally symmetric neighborhood containing Θ_d) for $k \leq \frac{\#B+1}{2}$. Assume that a digital set S is given and elements $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ of $B \setminus \{\Theta_d\}$, such that $u_{i_j} \neq u_{i_\ell}$ and $u_{i_j} \neq -u_{i_\ell}$ for $j, \ell = 1, 2, \cdots k$.

Algorithm $\mathcal{A}_k(S; u_{i_1}, u_{i_2}, \cdots, u_{i_k})$

Start Choose any subset $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ of $B \setminus \{\Theta_d\}$, such that $u_{i_j} \neq u_{i_\ell}$ and $u_{i_j} \neq -u_{i_\ell}$ for $j, \ell = 1, 2, \cdots k$.
Let $S_0 := S$.

Iteration Let $S_j = \delta_{u_{i_j}}(S_{j-1})$ for $j = 1, 2, \cdots, k$.

Result $\mathcal{A}_k(S; u_{i_1}, u_{i_2}, \cdots, u_{i_k}) := S_k$

Lemma 4.3 *Given $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ in $B \setminus \{\Theta_d\}$, such that $u_{i_j} \neq u_{i_\ell}$ and $u_{i_j} \neq -u_{i_\ell}$ for $j, \ell = 1, 2, \cdots k$.*

Then $\mathcal{A}_k(S; u_{i_1}, u_{i_2}, \cdots, u_{i_k})$ is a $k+1$–open set.

4.1 Construction of Tyan–Döhler Sets

Let $B = \{\Theta_d, u_1, u_2, \cdots, u_n\}$ be a centrally symmetric neighborhood ($u_i \neq \Theta_d$ for all $i = 1, 2, \cdots, n$ and $u_i \neq u_j$ for $i \neq j$).

Let furthermore $\{v_1, v_2, \cdots, v_m\}$ be a *complete system of generators of B*, which means that $m = \frac{n}{2}$, and for each $i = 1, 2, \cdots, n$ there is a $j \in \{1, 2, \cdots, m\}$, such that $u_i = v_j$ or $u_i = -v_j$. Conversely, for each $j \in \{1, 2, \cdots, m\}$ there is an $i \in \{1, 2, \cdots, n\}$, such that $v_j = u_i$.

Definition 4.2 *The* Tyan–Döhler set *belonging to the complete system of generators $\{v_1, v_2, \cdots, v_m\}$ is the set $B_0 = \mathcal{A}_m(\{x_0\}; v_1, v_2, \cdots, v_m)$ for a given point $x_0 \in \mathbb{Z}^d$.*

Lemma 4.4 *Let $\{i_1, i_2, \cdots, i_m\}$ be a permutation of the indices $\{1, 2, \cdots, m\}$. The set of all paths*

$$x_0, x_0 + v_{i_1}, x_0 + v_{i_1} + v_{i_2}, \cdots x_0 + v_{i_1} + v_{i_2} + \cdots + v_{i_m} \tag{4.2}$$

covers B_0 completely.

Proof

1. Without loss of generality let $x_0 = \Theta_d$. Choose $x_1 \in B_0$. By construction there are directions $v_{i_1}, v_{i_2}, \cdots, v_{i_\ell}$ in B ($\ell \le m$, v_{i_j} pairwise different), such that $v_{i_1} + v_{i_2} + \cdots + v_{i_\ell} = x_1$. We can complete the set $\{v_{i_j}\}$ to get a complete system of generators which shows that each $x_1 \in B_0$ is on a path as given above. This implies that B_0 is contained in the union of all these paths.

2. Assume that x_1 is contained in a path of the form (4.2), say $x_1 = v_{i_1} + v_{i_2} + \cdots + v_{i_\ell}$. Since the composition of dilations is commutative, we can permute the directions v_{i_j} such that the indices are sorted in ascending order. This implies that x_1 wil be met by one of the dilations generating B_0, hence $x_1 \in B_0$. □

Lemma 4.5 *Let* $\{v_1, v_2, \cdots, v_m\}$ *be a complete system of generators of* B *and let* $B_0 = \mathcal{A}_m(\{x_0\}; v_1, v_2, \cdots, v_m)$.

Given $x_1 \in B_0$*, there exists a complete system* $\{w_1, w_2, \cdots, w_m\}$ *of generators of* B *such that* $B_0 = \mathcal{A}_m(\{x_1\}; w_1, w_2, \cdots, w_m)$.

Proof Assume that $x_1 = x_0 + v_{i_1}$. Then we choose the complete system of generators $-v_{i_1}$, v_{i_2}, $\cdots$, v_{i_m}. The path $x_0 + v_{i_1}, x_0 + v_{i_1} + v_{i_2}, \cdots, x_0 + v_{i_1} + v_{i_2} \cdots, +v_{i_m}$ is a subpath of $x_0, x_0 + v_{i_1}, x_0 + v_{i_1} + v_{i_2}, \cdots, x_0 + v_{i_1} + v_{i_2} \cdots, +v_{i_m}$ and thus it is completely contained in B_0. We add the point $x_0 + v_{i_1} + v_{i_2} \cdots, +v_{i_m} - v_{i_1}$. Also this point is in B_0 (Sorting yields the point $x_0 + v_{i_2} + v_{i_3} \cdots, +v_{i_m}$, and this point is by Lemma 4.4 in B_0).

Assume that x_2 is a point of the permuted sequence. If for the construction of x_2 the direction $-v_{i_1}$ was not used, everything is clear.

Otherwise, if the direction v_{i_1} was used for construction of x_2, then the remaining directions lead from x_0 to $x_2 \in B_0$.

In the same way one argues for other starting points $x_1 \in B_0$. □

Remark 4.2 *In* $\mathbb{Z}^2$*, the construction yields the Tyan–Döhler set whenever the latter is convex. This can be easily seen by choosing generators which are ordered by their slopes.*

5 B–Convexity

Throughout this section we assume that B is centrally symmetric and $\Theta_d \in B$.

Definition 5.1 *A digital set* S *is* B–convex *if the following inclusion is valid for each* $u \in B$

$$x + tu \in S \text{ for any } t > 0 \quad \text{and} \quad x - su \in S \text{ for any } s < 0 \qquad \Longrightarrow \qquad x \in S.$$

Remark 5.1 *Instead of* B*–convexity it is sufficient to require* local B–convexity*: For each* $u \in B$

$$x \pm u \in S \qquad \Longrightarrow \qquad x \in S.$$

Obviously, each digitally convex *set (i.e.* $S = \mathbb{Z}^d \cap \operatorname{conv} S$, conv *denoting the usual convex hull in* $\mathbb{R}^d$*) is* B*–convex for any centrally symmetric* B *containing* Θ_d.

Lemma 5.1 *Any B–convex set is $\frac{\#B+1}{2}$–closed.*

Proof If $x \notin S$ then, by B–convexity, $x \pm u$ for $u \in B$ cannot both belong to S. Hence at least one half of the elements $x + u$, $u \in B$, belong not to S. Therefore, $x \notin \mu(S)$. $\square$

Remark 5.2 *If the Tyan–Döhler set B_0 is not convex, then we know that between B_0 and* conv B_0 *there is a root image. This can be found by iteratively applying μ to both sets. Since B_0 is open and* conv B_0 *is closed, both iterates converge towards a root image.*

Remark 5.3 *Döhler [2] was the first one who emphasized the role of convexity in the construction of Tyan–Döhler sets.*

6 Conclusions

It was shown that the Tyan–Döhler construction for "small" root images of median filters can be rigorously defined and generalized to the nonconvex and higher–dimensionbal cases. This was achieved by reformulating the problem in a semi–topological context. As a by–product insights were obtained about the decomposition of a set into a related root image and a "thin" reimaining set (Lemma 2.11).

There remain, however, open problems. We list some of them:

- Under what conditions will the process in Section 4.1 indeed yield a root image? A partial answer was given in [3].
- Are the root images obtained by the Tyan–Döhler construction indeed "smallest" as claimed by Döhler [2]? In the convex case this can be proved.
- How does a basis of the semi–topology $\mathcal{T}_{\frac{\#B+1}{2}}$ look like? For example, in the topology corresponding to the cross median filter, do the closed 4–curves which are topologically open (as sketched in Example 3.1) constitute a basis of $\mathcal{T}_3$?
- Sets which can be represented as unions of Tyan–Döhler sets are open. The question is, whether they have other distinctive properties among open sets. The semi–topology generated by all unions of translates of B_0 has character 1. This implies that it has many attractive properties [17, 4] which make such sets attractive for investigation.

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